Bergman-Bourgain-Brezis-type Inequality

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May 4, 2021

Abstract

In this note, we prove a fractional version in 1-D of the Bourgain-Brezis inequality [1]. We show that such an inequality is equivalent to the fact that a holomorphic function $f \colon \mathbb{D} \to \mathbb{C}$ belongs to the Bergman space $\mathcal{A}^2(\mathbb{D})$, namely $f \in L^2(\mathbb{D})$, if and only if

$$\|f\|_{L^1 + H^{-1/2}(S^1)} := \limsup_{r \to 1^-} \|f(re^{i\theta})\|_{L^1 + H^{-1/2}(S^1)} < +\infty.$$

Possible generalisations to the higher-dimensional torus are explored.

Keywords. Fractional Laplacian, holomorphic functions, Hardy spaces, Bergman spaces.
MSC 2000. 26A33, 32A10, 32A35, 32A36

Contents

1 Introduction

In his pioneering work [12], Riesz studied fine properties of the so-called Hardy spaces $\mathcal{H}^p(\mathbb{D})$, which are the spaces of holomorphic functions [1] $f \colon \mathbb{D} \to \mathbb{C}$ such that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < +\infty \tag{1}$$

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[1] In 1915 Hardy observed that if f is holomorphic in $\mathbb{D}$ then $r \mapsto M(r) = \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ is a nondcreasing function.

for $p > 0$. Under condition (1), it is known that $f(e^{i\theta})$ exists and

$$\lim_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p d\theta = 0 \quad \text{as well as} \quad \lim_{r \to 1^-} f(re^{i\theta}) = f(e^{i\theta}), \tag{2}$$

for almost every θ, (see e.g. Section 4 [12]). For $f \in \mathcal{H}^p(\mathbb{D})$, one defines $\|f\|_{\mathcal{H}^p(\mathbb{D})} := \|f\|_{L^p(S^1)}$.

We can independently consider holomorphic functions in $L^2(\mathbb{D})$ which corresponds to the well-known Bergman space $\mathcal{A}^2(\mathbb{D})$ [2], see e.g [6].

The connection between Hardy spaces and the Bergman space $\mathcal{A}^2(\mathbb{D})$ is given by the embedding $\mathcal{H}^1(\mathbb{D}) \hookrightarrow \mathcal{A}^2(\mathbb{D})$ together with the estimate

$$\|f\|_{L^2(\mathbb{D})} \le C\|f\|_{\mathcal{H}^1(\mathbb{D})} := \|f\|_{L^1(S^1)}. \tag{3}$$

In the case $\lim_{r \to 1^-} \|f(re^{i\theta})\|_{H^{-1/2}(S^1)} < +\infty$, then, by definition, the following inequality holds as well:

$$\|f\|_{L^2(\mathbb{D})} \le C\|f\|_{H^{-1/2}(S^1)} := \lim_{r \to 1^-} \|f(re^{i\theta})\|_{H^{-1/2}(S^1)}. \tag{4}$$

In this note, we prove the following combination of (3) and (4):

Theorem 1. *Let $f \colon \mathbb{D} \to \mathbb{C}$ be an analytic function. Then f belongs to the Bergman space $\mathcal{A}^2(\mathbb{D})$ if and only if*

$$\|f\|_{L^1 + H^{-1/2}(S^1)} := \limsup_{r \to 1^-} \|f(re^{i\theta})\|_{L^1 + H^{-1/2}(S^1)} < +\infty.$$

Moreover, it holds

$$\|f\|_{L^2(\mathbb{D})} \le C\|f\|_{L^1 + H^{-1/2}(S^1)}. \tag{5}$$

Lastly, in section 7, we provide a proof of the inequalities (3) and (4).

This type of inequalities takes its roots in the pioneering work [1], where Bourgain and Brezis proved the following striking result:

Theorem 2 (Lemma 1 in [1])**.** *Let u be a 2π-periodic function in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} u\, dx = 0$, and let $\nabla u = f + g$, where $f \in \dot{W}^{-1,\frac{n}{n-1}}(\mathbb{R}^n)$[3] and $g \in L^1(\mathbb{R}^n)$ are 2π-periodic vector-valued functions. Then*

$$\|u\|_{L^{\frac{n}{n-1}}} \le c \left(\|f\|_{\dot{W}^{-1,\frac{n}{n-1}}} + \|g\|_{L^1} \right). \tag{6}$$

By duality, this implies the following corollary:

Corollary 1 (Theorem 1 in [1])**.** *For every 2π-periodic function $h \in L^n(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} h = 0$, there exists a 2π-periodic $v \in \dot{W}^{1,n} \cap L^\infty(\mathbb{R}^n)$ satisfying*

$$\operatorname{div} v = h \quad in\ \mathbb{R}^n$$

and

$$\|v\|_{L^\infty} + \|v\|_{\dot{W}^{1,n}} \le C(n)\|h\|_{L^n}. \tag{7}$$

One of the main result of this note is a fractional type Bourgain-Brezis inequality on the circle S^1 and on T^n. More precisely, we have the following:

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[2] We recall that $\mathcal{A}^2(\mathbb{D}) := \{f \colon \mathbb{D} \to \mathbb{C} : f \text{ holomorphic and } \|f\|_{L^2(\mathbb{D})} < +\infty\}$

[3] For $1 < p < +\infty$, we will denote by $\dot{W}^{1,p}(\mathbb{R}^n)$ the homogeneous Sobolev space defined as the space of $f \in L^1_{loc}(\mathbb{R}^n)$ such that $\nabla f \in L^p(\mathbb{R}^n)$ and by $\dot{W}^{-1,p'}(\mathbb{R}^n)$ the corresponding dual space (p' is the conjugate of p). Every function $f \in \dot{W}^{-1,p'}(\mathbb{R}^n)$ can be represented as $f = \sum_{i=1}^n \partial_{x_i} f^j$ with $f^j \in L^{p'}(\mathbb{R}^n)$.

Theorem 3. *Let $u \in \mathcal{D}'(S^1)$ be such that $(-\Delta)^{\frac{1}{4}}u, \mathcal{R}(-\Delta)^{\frac{1}{4}}u \in \dot{H}^{-\frac{1}{2}}(S^1) + L^1(S^1)$.[4] Then $u - \fint_{S^1} u \in L^2_*(S^1)$ and the following estimate holds true:*

$$\left\| u - \fint_{S^1} u \right\|_{L^2} \leq C \left(\|(-\Delta)^{1/4}u\|_{\dot{H}^{-1/2}(S^1)+L^1(S^1)} + \|\mathcal{R}(-\Delta)^{1/4}u\|_{\dot{H}^{-1/2}(S^1)+L^1(S^1)} \right), \qquad (8)$$

for some $C > 0$ independent of u.

Theorem 4. *Let $u \in \mathcal{D}'(T^n)$ be complex-valued and such that:*

$$(-\Delta)^{\frac{n}{4}}u, \mathcal{R}^j(-\Delta)^{\frac{n}{4}}u \in (L^1 + \dot{H}^{-\frac{n}{2}})(T^n), \quad \forall j \in \{1, \ldots n\}.$$

Then we have $u - \fint_{T^n} u\,dx \in L^2_*(T^n)$ with

$$\left\| u - \fint_{T^n} u\,dx \right\|_{L^2} \leq C \left(\|(-\Delta)^{n/4}u\|_{L^1+\dot{H}^{-n/2}(T^n)} + \sum_{j=1}^n \|\mathcal{R}^j(-\Delta)^{n/4}u\|_{L^1+\dot{H}^{-n/2}(T^n)} \right), \qquad (9)$$

for some $C > 0$ independent of u.

The second main result is the equivalence between Theorems 1 and 3, establishing the connection between fractional Bourgain-Brezis inequalities and Bergman spaces. It would be interesting to investigate a similar connection in dimensions $n \geq 2$.

We would like to add some comments about Bourgain-Brezis' inequality (6). The inequality (6) in its general form is of interest in the study of the PDE $\operatorname{div} Y = f$ for $f \in L^n_*(T^n)$, where for finite $p \geq 1$, $L^p_*(T^n)$ denotes the Banach subspace of L^p-functions with vanishing mean over the torus. Precisely, in [1] the authors found that Y can be chosen to be continuous and in $\dot{W}^{1,n}(T^n)$, a result which is non-trivial due to the fact that $\dot{W}^{1,n}(T^n)$ does not continuously embed into $L^\infty(T^n)$. The key ingredient in the proof is a duality argument based on an estimate similar to (6) and some general results from functional analysis regarding closedness properties of the image space. This motivates the general interest in inequalities of the same type, as improved regularity results in limit cases can be invaluable. Indeed, later, such estimates have been considered and extended in different directions. In [2] Bourgain and Brezis showed how Theorem 1 in [1] is closely connected to a remarkable property concerning differential forms with coefficients in the critical Sobolev space $W^{1,n}(T^n)$ and they got new regularity results for the Hodge decomposition. In [9] Maz'ya extended the inequality (6) on the Sobolev space $H^{1-\frac{n}{2}}(\mathbb{R}^n)$ leading to a different existence result for the PDE $\operatorname{div} Y = f$. Finally in [10] Mironescu unified in 2 dimensions the two different approaches in [1] and in [9] by using elementary properties of fundamental solutions of the biharmonic operator. In [4], the first two authors of the current paper provide an alternative proof of (6) in 2 dimensions without assuming the periodicity of the function u. The proof is related to some compensation phenomena observed first in [5] in the analysis of 2-dimensional perfect incompressible fluids and then also applied by Rivière in [13] in the analysis of *isothermic surfaces*. For an overview of the results in the literature regarding variations of Theorem 2 and Corollary 1, we refer for instance to the interesting paper by Van Schaftingen [16].

As seen in [4], the inequality (6) also represents the first key ingredient in the study of the regularity of $L^2(\mathbb{D}, \mathbb{R}^n)$ solutions u to a linear elliptic system of the following form

$$\operatorname{div}(S\,\nabla u) \quad = \quad \sum_{j=1}^n \operatorname{div}(S_{ij}\,\nabla u^j) = \sum_{j=1}^n \sum_{\alpha=1}^2 \frac{\partial}{\partial x_\alpha}\left(S_{ij} u^j_{x_\alpha}\right) = 0, \qquad (10)$$

where S is a $W^{1,2}(\mathbb{D})$ symmetric $n \times n$ matrix, such that $S^2 = id_n$.

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[4]We denote by $\mathcal{R}$ and $\mathcal{R}^j$ the Riesz transform respectively on S^1 and with respect to the x^j variable on T^n, for $j \in \{1, \ldots, n\}$ and by $\dot{H}^{-\frac{n}{2}}(T^n)$ the space of $f \in \mathcal{D}'(S^1)$ such that $f = (-\Delta)^{n/4}g$, with $g \in L^2(T^n)$. Recall that $L^2_*(S^1) := \{u \in L^2(S^1): \ \fint_{S^1} u = 0\}$.

We would like to mention some results on Riesz potentials showing that the 1-dimensional case plays a particular role in the L^1-estimates for Riesz potentials. More precisely, one can deduce from the results in [15] that for all $0 < \alpha < 1$, we have:

$$\|I_\alpha u\|_{L^{\frac{1}{1-\alpha}}} \leq C(\|\mathcal{R}u\|_{L^1} + \|u\|_{L^1}), \tag{11}$$

for all u in the Hardy space $\mathcal{H}^1(\mathbb{R})$. It follows in particular that:

$$\|u\|_{L^{\frac{1}{1-\alpha}}} \leq C(\|\mathcal{R}(-\Delta)^{\alpha/2}u\|_{L^1} + \|(-\Delta)^{\alpha/2}u\|_{L^1}).$$

In [14], the authors show that if $N \geq 2$ and $0 < \alpha < N$, then there is a constant $C = C(\alpha, N) > 0$ such that

$$\|I_\alpha u\|_{L^{\frac{N}{N-\alpha}}} \leq C\|\mathcal{R}u\|_{L^1} \tag{12}$$

for all $u \in C_c^\infty(\mathbb{R}^N)$, such that $\mathcal{R}u \in L^1(\mathbb{R}^N)$. The estimate (12) is however false in 1-D, as seen in [14].

The inequality (8) generalizes the inequality (11) in the case $\alpha = 1/2$ and the counter-example in [14] for the estimate (12) in 1-D shows that the estimate (8) is in some sense optimal.

We finally point out that it may be interesting to investigate a possible generalization of Theorem 4 in the framework of nonlocal operators on differential forms as it has been done in [2].

The paper is organized as follows: In section 2, we recall the definitions of the fractional Laplacian on the unit circle and on the torus. In section 3, we provide two distinct proofs of Theorem 3. In section 4, we establish the equivalence of Theorem 1 and Theorem 3. In section 5, we provide a short introduction of Clifford algebras and we extend the fractional Bourgain-Brezis inequality using Clifford algebras to the n-dimensional torus T^n. In section 6, we prove existence results for certain fractional PDE-operators in the same spirit as Corollary 1. In section 7, we provide a proof of the inequalities (3) and (4) for the reader's convenience.

2 Fractional Laplacian on the unit circle and on the torus

Before we enter the discussion and the proofs of the main results, let us recall a few notions essential in our later arguments. We mainly focus on fractional Laplacians, fractional Sobolev.

Throughout this note, we shall denote by T^n the torus of dimension $n \in \mathbb{N}$. This means:

$$T^n = \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}} = \mathbb{R}^n/(2\pi\mathbb{Z})^n \tag{13}$$

where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. We denote by $\mathcal{D}(T^n) := C^\infty(T^n)$ the Fréchet space of smooth functions on T^n and by $\mathcal{D}'(T^n)$ its topological dual. The natural duality paring is denoted by $\langle \cdot, \cdot \rangle$.

For $u \in \mathcal{D}'(T^n)$ and $m \in \mathbb{Z}^n$, we define the Fourier coefficients of u as follows:

$$\widehat{u}(m) := \frac{1}{(2\pi)^n} \int_{T^n} u(x)e^{-i\langle m, x \rangle}dx = \left\langle u, e^{-i\langle m, \cdot \rangle} \right\rangle. \tag{14}$$

The Fourier coefficients completely determine u as a distribution on T^n and convergence in the sense of distributions obviously implies convergence of the Fourier coefficients. Notice that, for all $u \in \mathcal{D}'(T^n)$, there exists some $N > 0$ such that $|\hat{u}(m)| \lesssim (1 + |m|)^N$. Moreover, we recall that $v \in C^\infty(T^n)$ if and only if the Fourier coefficients $\hat{v}(m)$ have rapid decay, i.e. $\sup_m(1 + |m|)^N |\hat{v}(m)| < \infty$ for all $N > 0$.

Given $s \in \mathbb{R}$, we define the non-homogeneous and homogeneous Sobolev spaces respectively by

$$H^s(T^n) := \left\{ v \in \mathcal{D}'(T^n) : \|v\|_{H^s}^2 := \sum_{k \in \mathbb{Z}^n} (1 + |k|^2)^s |\hat{v}(k)|^2 < \infty \right\},$$

and

$$\dot{H}^s(T^n) := \left\{ v \in \mathcal{D}'(T^n) : \|v\|_{\dot{H}^s}^2 := \sum_{k \in \mathbb{Z}^n} |k|^{2s} |\hat{v}(k)|^2 < \infty \right\},$$

where $\mathcal{D}'(T^n)$ is again the space of distributions on T^n. Notice that if $s = 0$, we have $L^2(T^n) = H^0(T^n)$ and $L^2_*(T^n) \simeq \dot{H}^0(T^n)$.

An important family of operators throughout our considerations are the so-called *fractional Laplacians*. Let $s > 0$ be real, then we define for $u : T^n \to \mathbb{C}$ smooth the s-Laplacian of u by the following multiplier property:

$$\widehat{(-\Delta)^s}u(m) = |m|^{2s}\hat{u}(m), \quad \forall m \in \mathbb{Z}^n. \tag{15}$$

Clearly, this definition can immediately be extended to the spaces $H^s(T^n)$ or even $\mathcal{D}'(T^n)$ as a multiplier operator on Fourier coefficients, possibly defining merely a distribution on T^n. Finally, we recall the definition of the j-Riesz transform on T^n as a multiplier operator:

$$\mathcal{R}^j u(x) = \sum_{m \in \mathbb{Z}^n} i \frac{m^j}{|m|} \hat{u}(m) e^{i\langle m, x \rangle}, \quad \forall x \in T^n. \tag{16}$$

In particular, in the case $n = 1$, we have:

$$\mathcal{R}u(x) = \sum_{m \in \mathbb{Z}} i \, \mathrm{sign}(m) \hat{u}(m) e^{im \cdot x}, \quad \forall x \in S^1. \tag{17}$$

3 Fractional Bourgain-Brezis inequality on the unit circle $\mathbf{S}^1$

In this section, we provide two distinct proofs of Theorem 3. The first proof is in the spirit of the one presented in [1], while the second one is inspired by that in [4] and is based on some particular compensation phenomena. We assume for simplicity that u is real valued (the proof for complex-valued function is completely analogous, see Remark 1).

First, we would like to observe that if $u \in C^\infty(S^1)$, then by definition:

$$\left\| u - \fint_{S^1} u \right\|_{L^2} \leq C \|(-\Delta)^{1/4}u\|_{\dot{H}^{-1/2}(S^1)}. \tag{18}$$

On the other hand, as we have already observed in the introduction, we also have[5]:

$$\left\| u - \fint_{S^1} u \right\|_{L^2(S^1)} \leq C \left(\|(-\Delta)^{1/4}u\|_{L^1(S^1)} + \|\mathcal{R}(-\Delta)^{1/4}u\|_{L^1(S^1)} \right) \simeq \|(-\Delta)^{1/4}u\|_{\mathcal{H}^1(S^1)} \tag{19}$$

3.1 A first proof of Theorem 3

Let us suppose that $u \in C^\infty(S^1)$, $\fint_{S^1} u = 0$. We assume for simplicity that u is real-valued, see Remark 1 for the complex-valued case. The proof below follows the main arguments of the original proof by Bourgain and Brezis. We write:

$$\begin{cases} (-\Delta)^{1/4}u = f^1 + g^1 \\ \\ \mathcal{R}(-\Delta)^{1/4}u = f^2 + g^2 \end{cases} \tag{20}$$

---

[5]Actually, an even sharper inequality than (19) holds true with $L^2(S^1)$ being replaced by the smaller Lorentz space $L^{2,1}(S^1)$.

where $f^1, f^2 \in \dot{H}^{-1/2}(S^1)$, $g^1, g^2 \in L^1(S^1)$. We set $u = \sum_{n \in \mathbb{Z}^*} u_n e^{in\theta}$. Since u is real-valued, it holds $\bar{u}_n = u_{-n}$. We see:

$$\sum_{n \in \mathbb{Z}^*} |u_n|^2 \;\; = \;\; \sum_{n \in \mathbb{Z}^*} |n|^{1/2} u_n \frac{u_{-n}}{|n|^{1/2}} = \sum_{n \in \mathbb{Z}^*} \frac{f_n^1 + g_n^1}{|n|^{1/2}} u_{-n}, \tag{21}$$

$$\sum_{n \in \mathbb{Z}^*} \frac{f_n^1 u_{-n}}{|n|^{1/2}} \;\; \leq \;\; \left[\sum_{n \in \mathbb{Z}^*} \frac{|f_n^1|^2}{|n|} \right]^{1/2} \left[\sum_{n \in \mathbb{Z}^*} |u_n|^2 \right]^{1/2}, \tag{22}$$

$$\sum_{n \in \mathbb{Z}^*} \frac{g_n^1 u_{-n}}{|n|^{1/2}} \;\; = \;\; \sum_{n > 0} \frac{g_n^1 u_{-n}}{|n|^{1/2}} + \sum_{n < 0} \frac{g_n^1 u_{-n}}{|n|^{1/2}}. \tag{23}$$

Observe that by definition of the Riesz transform:

$$\mathcal{R}(-\Delta)^{1/4} u \;\; = \;\; i \left[- \sum_{n < 0} |n|^{1/2} u_n e^{in\theta} + \sum_{n > 0} |n|^{1/2} u_n e^{in\theta} \right]. \tag{24}$$

Therefore:

$$u_n = \begin{cases} \dfrac{f_n^2 + g_n^2}{-i|n|^{1/2}} & \text{if } n < 0 \\[2ex] \dfrac{f_n^2 + g_n^2}{i|n|^{1/2}} & \text{if } n > 0 \end{cases} \tag{25}$$

By combining (23) and (25), we obtain:

$$\sum_{n \in \mathbb{Z}^*} \frac{g_n^1 u_{-n}}{|n|^{1/2}} \;\; = \;\; \sum_{n > 0} g_n^1 \frac{f_{-n}^2 + g_{-n}^2}{-i|n|} + \sum_{n < 0} g_n^1 \frac{f_{-n}^2 + g_{-n}^2}{i|n|}. \tag{26}$$

Let us estimate the different parts of the sum (26) individually:

1. We first estimate

$$\sum_{n \in \mathbb{Z}^*} \text{sign}(n) \frac{g_n^1 \, f_{-n}^2}{|n|} \;\; = \;\; \sum_{n \in \mathbb{Z}^*} \text{sign}(n) \frac{|n|^{1/2} u_n - f_n^1}{|n|^{1/2}} \frac{f_{-n}^2}{|n|^{1/2}}$$

$$\leq \;\; \left(\sum_{n \in \mathbb{Z}^*} |u_n|^2 \right)^{1/2} \left(\sum_{n \in \mathbb{Z}^*} \frac{|f_n^2|^2}{|n|} \right)^{1/2} + \left(\sum_{n \in \mathbb{Z}^*} \frac{|f_n^1|^2}{|n|} \right)^{1/2} \left(\sum_{n \in \mathbb{Z}^*} \frac{|f_n^2|^2}{|n|} \right)^{1/2}$$

$$\leq \;\; \|u\|_{L^2} \|f^2\|_{\dot{H}^{-1/2}} + \|f^1\|_{\dot{H}^{-1/2}} \|f^2\|_{\dot{H}^{-1/2}}. \tag{27}$$

2. It remains to estimate

$$\sum_{n \in \mathbb{Z}^*} \text{sign}(n) \frac{g_n^1 \, g_{-n}^2}{i|n|}.$$

For this purpose, we consider the following operator:

$$\mathbf{A} \colon L^1(S^1) \times L^1(S^1) \;\; \to \;\; \mathbb{C}, \qquad (g^1, g^2) \mapsto \sum_{n \in \mathbb{Z}^*} \text{sign}(n) \frac{g_n^1 \, g_{-n}^2}{i|n|}.$$

Claim 1. The operator $\mathbf{A}$ is continuous, i.e. we have the following estimate:

$$|\mathbf{A}(g^1, g^2)| \leq C \|g^1\|_{L^1} \|g^2\|_{L^1}. \tag{28}$$

Proof of Claim 1. It is sufficient to prove the claim in the case where g^1 and g^2 are arbitrary Dirac-delta measures.[6] Therefore, we consider $g^1 = \sum_{i \in I} \lambda_i \delta_{a_i}$ and $g^2 = \sum_{j \in J} \mu^j \delta_{b^j}$. We have $\|g^1\|_{\mathcal{M}(S^1)} = \sum_{i \in I} |\lambda_i|$, $\|g^2\|_{\mathcal{M}(S^1)} = \sum_{j \in J} |\mu^j|$. By bilinearity, we deduce:

$$
\begin{aligned}
|\mathbf{A}(g^1, g^2)| &= |\mathbf{A}(\sum_{i \in I} \lambda_i \delta_{a_i}, \sum_{j \in J} \mu^j \delta_{b^j})| \\[2mm]
&\leq \sum_{i \in I, j \in J} |\lambda_i||\mu^j||\mathbf{A}(\delta_{a_i}, \delta_{b^j})| \\[2mm]
&\leq \sup_{(a,b) \in S^1 \times S^1} |\mathbf{A}(\delta_a, \delta_b)| \sum_{i \in I} |\lambda_i| \sum_{j \in J} |\mu^j| \\[2mm]
&= \sup_{(a,b) \in S^1 \times S^1} |\mathbf{A}(\delta_a, \delta_b)| \|g^1\|_{\mathcal{M}(S^1)} \|g^2\|_{\mathcal{M}(S^1)}.
\end{aligned}
\tag{29}
$$

If $\sup_{(a,b) \in S^1 \times S^1} |\mathbf{A}(\delta_a, \delta_b)| < +\infty$, then the claim holds for linear combinations of Dirac measures. By a density argument, we get claim 1 for arbitrary $g^1, g^2 \in L^1(S^1)$. Hence, claim 1 is a consequence of the following:

Claim 2. $\sup_{(a,b) \in S^1 \times S^1} |\mathbf{A}(\delta_a, \delta_b)| < +\infty$.

Proof of Claim 2. For $g^1 = \delta_a$ and $g^2 = \delta_b$, we have $g_n^1 = e^{ina}$ and $g_n^2 = e^{inb}$. In this case, we observe:

$$
\begin{aligned}
\mathbf{A}(\delta_a, \delta_b) &= \sum_{n \in Z^*} \text{sign}(n) \frac{g_n^1 g_{-n}^2}{i|n|} \\[2mm]
&= \sum_{n \in \mathbb{Z}^*} \text{sign}(n) \frac{e^{in(a-b)}}{i|n|} = 2 \sum_{n > 0} \frac{\sin(n(a-b))}{n} < +\infty.^7
\end{aligned}
\tag{30}
$$

This proves claim 2 and from (30), we can deduce claim 1 as well.

By combining (21)-(29) we get

$$
\begin{aligned}
\|u\|_{L^2}^2 &\lesssim \|u\|_{L^2} \left(\|f^1\|_{\dot{H}^{-1/2}} + \|f^2\|_{\dot{H}^{-1/2}} \right) + \|f^1\|_{\dot{H}^{-1/2}} \|f^2\|_{\dot{H}^{-1/2}} + C\|g^1\|_{L^1} \|g^2\|_{L^1} \\[2mm]
&\lesssim \frac{1}{2}\|u\|_{L^2}^2 + \frac{1}{2} \left(\|f^1\|_{\dot{H}^{-1/2}}^2 + \|f^2\|_{\dot{H}^{-1/2}}^2 \right) + \|f^1\|_{\dot{H}^{-1/2}} \|f^2\|_{\dot{H}^{-1/2}} + C\|g^1\|_{L^1} \|g^2\|_{L^1} \\[2mm]
&\lesssim \frac{1}{2}\|u\|_{L^2}^2 + \left(\|f^1\|_{\dot{H}^{-1/2}}^2 + \|f^2\|_{\dot{H}^{-1/2}}^2 \right) + \frac{1}{2} \left(\|g^1\|_{L^1}^2 + \|g^2\|_{L^1}^2 \right).
\end{aligned}
\tag{31}
$$

This estimate permits us to conclude the proof of Theorem 3. Since f^1, f^2, g^1, g^2 were arbitrary, one can deduce (8). In the general case where $u \in \mathcal{D}'(S^1)$, one argues by approximation (see section 3.2 for further details). $\qquad\square$

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[6] We recall that the linear span of Dirac measures is dense in the space of Radon measures $\mathcal{M}(S^1)$ equipped with the weak-* topology.

[7] The value of such a series is deduced from the Fourier series of $f(x) = \frac{x}{2\pi}$ for $0 < x < 2\pi$ and $f(x + 2\pi) = f(x)$.

3.2 A second proof of Theorem 3

As in the first proof, we will show the following: Let $u \in \mathcal{D}'(S^1)$ be such that:

$$(-\Delta)^{\frac{1}{4}} u = f^1 + g^1 \tag{32}$$

$$\mathcal{R}(-\Delta)^{\frac{1}{4}} u = f^2 + g^2, \tag{33}$$

where $f^1, f^2 \in \dot{H}^{-\frac{1}{2}}(S^1)$ and $g^1, g^2 \in L^1(S^1)$. Under these conditions, we prove:

$$u - \int_{S^1} u\,dx \in L^2_*(S^1) = \left\{ u \in L^2(S^1) : \ \fint_{S^1} u = 0 \right\}, \tag{34}$$

together with the following estimate:

$$\left\| u - \int_{S^1} u\,dx \right\|_{L^2} \le C \left(\|f^1\|_{\dot{H}^{-\frac{1}{2}}} + \|f^2\|_{\dot{H}^{-\frac{1}{2}}} + \|g^1\|_{L^1} + \|g^2\|_{L^1} \right), \tag{35}$$

where $C > 0$ is independent of f^1, f^2, g^1, g^2 and u. We may assume for simplicity that u is real-valued (see Remark 1 for the complex-valued case).

Firstly, observe that it suffices to consider the case:

$$\int_{S^1} u\,dx = 2\pi \cdot \hat{u}(0) = 0, \tag{36}$$

by merely changing u by a constant. Similarly, by the conditions in (32) and (33), we see that f^j, g^j have vanishing integral over S^1 and consequently vanishing Fourier coefficient for $n = 0$.[8] For now, let us assume that u, f^j, g^j are all smooth on S^1. The general case can be dealt with using convolution with an appropriate smoothing kernel and approximation arguments as specified at the end of the proof.

First, let us define the following operators on $\mathcal{D}'(S^1)$:

$$Dv := (-\Delta)^{\frac{1}{4}} \big(Id + \mathcal{R} \big) v \tag{37}$$

$$\overline{D}v := (-\Delta)^{\frac{1}{4}} \big(Id - \mathcal{R} \big) v, \tag{38}$$

for every $v \in \mathcal{D}'(S^1)$. Consequently, using (32) and (33), we have:

$$Du = f^1 + f^2 + g^1 + g^2 = f + g \tag{39}$$

$$\overline{D}u = f^1 - f^2 + g^1 - g^2 = \tilde{f} + \tilde{g}. \tag{40}$$

Let us calculate the Fourier multipliers associated with $D, \overline{D}$. For every $n \in \mathbb{Z}$, we have:

$$\mathcal{F}\big(Dv \big)(n) = |n|^{\frac{1}{2}} (1 + i\operatorname{sign}(n)) \hat{v}(n) \tag{41}$$

$$\mathcal{F}\big(\overline{D}v \big)(n) = |n|^{\frac{1}{2}} (1 - i\operatorname{sign}(n)) \hat{v}(n) \tag{42}$$

Claim 1: Given $f \in \dot{H}^{-1/2}(S^1)$, there is a real-valued function $F \in L^2_*(S^1)$[9], such that $DF = f$.
Proof of the Claim 1 In order to solve $DF = f$, we should have:

$$\hat{F}(n) = \frac{1}{1 + i\operatorname{sign}(n)} \frac{\hat{f}(n)}{\sqrt{|n|}}, \quad \text{if } n \ne 0 \tag{43}$$

---

[8] It would be possible to treat f^j, g^j with non-vanishing integral, i.e. treat the case $(-\Delta)^{1/4}u, \mathcal{R}(-\Delta)^{1/4}u \in L^1 + H^{-1/2}(S^1)$ by reducing to vanishing Fourier coefficient at $n = 0$: We have by the conditions $\widehat{f^j}(0) = -\widehat{g^j}(0)$. Note that $|\widehat{g^j}(0)| \lesssim \|g^j\|_{L^1}$. Note that $\|f^j\|^2_{H^{-1/2}} \simeq |\widehat{f^j}(0)|^2 + \|\tilde{f}^j\|^2_{\dot{H}^{-1/2}}$, where $\tilde{f}^j$ denotes the corrected f^j with vanishing 0th Fourier coefficient. Thus, we could reduce to the case of vanishing integral.

[9] The observation that F may be chosen real-valued is due to $\widehat{F}(-n) = \overline{\widehat{F}(n)}$ for all n.

Using the fact that the L^2-norm of F can be characterized in terms of the l^2-norm of the Fourier coefficients, we obtain:

$$\|F\|_{L^2}^2 = \sum_{n \neq 0} \frac{1}{|1 + i\operatorname{sign}(n)|^2} \frac{|\hat{f}(n)|^2}{|n|}$$

$$\leq \sum_{n \neq 0} \frac{|\hat{f}(n)|^2}{|n|}$$

$$= \|f\|_{\dot{H}^{-\frac{1}{2}}}^2, \tag{44}$$

where we used the definition of the $\dot{H}^{-\frac{1}{2}}$-norm. Observe that a converse inequality could be obtained along the same lines. $\qquad\square$

Next, by defining $\tilde{u} := u - F$, we observe that due to (39):

$$D\tilde{u} = g. \tag{45}$$

Let now $w \in \mathcal{D}'(S^1)$ real-valued be such that $Dw = \tilde{u}$ and $\hat{w}(0) = 0$. Once more, existence of such a distribution w is easily deduced using Fourier coefficients. We would like to emphasise at this point that due to the assumed smoothness of u, f^j, g^j, w is smooth as well, as is F.

By (45), we thus notice:

$$D^2w = g. \tag{46}$$

Going over to Fourier coefficients, we see that for every $n \in \mathbb{Z}^*$:

$$\mathcal{F}(D^2w)(n) = (1 + i\operatorname{sign}(n))^2|n|\hat{w}(n) = 2i\operatorname{sign}(n)|n|\hat{w}(n) = 2in\hat{w}(n) = \hat{g}(n), \tag{47}$$

or by rearranging:

$$\hat{w}(n) = -\frac{i}{2}\frac{\hat{g}(n)}{n}. \tag{48}$$

Next, we are going to find a suitable distribution K with coefficients $\hat{K}(n) = -\frac{i}{2n}$, in order to express w as a convolution of g with K. To this end, let us consider the function $k : [-\pi, \pi] \to \mathbb{R}$ defined by:

$$k(x) = \begin{cases} x + \pi, & \text{if } x < 0 \\ x - \pi, & \text{if } x > 0 \end{cases} \tag{49}$$

By slight abuse of notation, let us identify k with its 2π-periodic extension, therefore $k : S^1 \to \mathbb{R}$. We calculate the Fourier coefficients of k: If $n = 0$, it is obvious due to anti-symmetry that $\hat{k}(0) = 0$. Otherwise, we have $n \neq 0$ and so by using integration by parts:

$$\hat{k}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} k(x)e^{-inx}dx$$

$$= \frac{1}{2\pi} \left(\int_{-\pi}^{0} (x + \pi)e^{-inx}dx + \int_{0}^{\pi} (x - \pi)e^{-inx}dx \right)$$

$$= \frac{1}{2\pi} \int_{0}^{\pi} (\pi - x)e^{inx} - (\pi - x)e^{-inx}dx$$

$$= \frac{i}{\pi} \int_{0}^{\pi} (\pi - x)\sin(nx)dx$$

$$= \frac{i}{n} - \frac{i}{\pi} \int_{0}^{\pi} \frac{\cos(nx)}{n}dx = \frac{i}{n}. \tag{50}$$

Consequently, observe that $K = -\frac{1}{2}k$ precisely yields the desired distribution. Let us notice that K is therefore bounded and measurable on S^1, thanks to the explicit formula for k.

Using (48) and the convolution formula for Fourier coefficients, namely:

$$\widehat{f * g}(n) = 2\pi \hat{f}(n)\hat{g}(n), \quad \forall n \in \mathbb{Z}, \tag{51}$$

it is clear that:

$$w = \frac{1}{2\pi} K * g. \tag{52}$$

From (52), we obtain by using Young's inequality on S^1:

$$\|w\|_{L^\infty} \le \frac{1}{2\pi}\|K\|_{L^\infty}\|g\|_{L^1} = \frac{1}{4}\|g\|_{L^1}. \tag{53}$$

To conclude the first part of the proof, let us observe the following[10]:

$$\begin{aligned}
\int_{S^1} (u - F)^2 dx &= \int_{S^1} Dw(u - F)dx \\
&\simeq \sum_{n \in \mathbb{Z}} \widehat{Dw}(n)\widehat{u - F}(-n) \\
&= \sum_{n \in \mathbb{Z}} |n|^{\frac{1}{2}}(1 + i\,\text{sign}(n))\hat{w}(n)\widehat{u - F}(-n) \\
&= \sum_{n \in \mathbb{Z}} \hat{w}(n) \cdot |n|^{\frac{1}{2}}(1 - i\,\text{sign}(-n))\widehat{u - F}(-n) \\
&\simeq \int_{S^1} w\overline{D}(u - F)dx \\
&= \int_{S^1} w\overline{D}u\,dx - \int_{S^1} w\overline{D}F dx \\
&= \int_{S^1} w\tilde{g}dx + \int_{S^1} w\tilde{f}dx - \int_{S^1} w\overline{D}F dx
\end{aligned} \tag{54}$$

where we used the Fourier representation of the distribution $u - F$ to justify the second equation. Observe that this enables us to estimate:

$$\left|\int_{S^1} w\tilde{g}dx\right| \le \|w\|_{L^\infty}\|\tilde{g}\|_{L^1} \le \frac{1}{4}\left(\|g^1\|_{L^1} + \|g^2\|_{L^1}\right)^2, \tag{55}$$

and:

$$\left|\int_{S^1} w\overline{D}F dx\right| \le \|w\|_{\dot{H}^{\frac{1}{2}}}\|\overline{D}F\|_{\dot{H}^{-\frac{1}{2}}} \le C\|u - F\|_{L^2}\|f\|_{\dot{H}^{-\frac{1}{2}}}. \tag{56}$$

The remaining summand may be estimated completely analogous to (56). Notice that we used the explicit definition of the norms of Sobolev spaces with negative exponents and the Fourier multipliers to obtain (56), see (44) for the main ideas. Using (55) and (56) yields:

$$\begin{aligned}
\|u - F\|_{L^2}^2 &\le \frac{1}{4}\left(\|g^1\|_{L^1} + \|g^2\|_{L^1}\right)^2 + 2C\|u - F\|_{L^2}\|f\|_{\dot{H}^{-\frac{1}{2}}} \\
&\le \frac{1}{4}\left(\|g^1\|_{L^1} + \|g^2\|_{L^1}\right)^2 + \frac{1}{2}\|u - F\|_{L^2}^2 + \frac{4C^2}{2}\|f\|_{\dot{H}^{-\frac{1}{2}}}^2,
\end{aligned} \tag{57}$$

---

[10]This will actually be the first and only point in the proof where we use the fact that u is real-valued in a meaningful way. See Remark 1 for an extension to complex-valued distributions.

using the arithmetic-geometric mean inequality. Note that the factor $2C$ is due to estimate (56) also applying to the integral of $w\tilde{f}$. By absorbing the L^2-norm of $u - F$, we arrive at:

$$\|u - F\|_{L^2}^2 \leq \frac{1}{2}\left(\|g^1\|_{L^1} + \|g^2\|_{L^1}\right)^2 + 4C^2\|f\|_{\dot{H}^{-\frac{1}{2}}}^2$$
$$\leq \max\{\frac{1}{2}, 4C^2\}\left(\|g^1\|_{L^1} + \|g^2\|_{L^1} + \|f^1\|_{\dot{H}^{-\frac{1}{2}}} + \|f^2\|_{\dot{H}^{-\frac{1}{2}}}\right)^2. \tag{58}$$

Consequently, by estimating the L^2-norm of F by the $\dot{H}^{-\frac{1}{2}}$-norm of f^1, f^2 using (44), we immediately conclude:

$$\|u\|_{L^2} \leq \tilde{C}\left(\|g^1\|_{L^1} + \|g^2\|_{L^1} + \|f^1\|_{\dot{H}^{-\frac{1}{2}}} + \|f^2\|_{\dot{H}^{-\frac{1}{2}}}\right). \tag{59}$$

The constant $\tilde{C} > 0$ appearing in the estimate is independent of u, f^j, g^j.

Now, for a general distribution $u \in \mathcal{D}'(S^1)$ with $\hat{u}(0) = 0$, let us observe that if we convolute u with a smooth function φ, the resulting distribution $\varphi * u$ will be a smooth function as well (in the sense of regular distributions). By a direct computation, (32) and (33) will continue to hold true if we replace u, f^j, g^j by their corresponding convolutions with φ. This is an immediate consequence of the fact that the operators $(-\Delta)^{\frac{1}{4}}, \mathcal{R}$ are Fourier multipliers as well as the linearity of convolutions. Choosing φ to be supported on arbitrarily small neighbourhoods of the neutral element in S^1 (i.e. an approximation of the identity φ_ε) ensures that the convolutions of φ with f^j, g^j converge in the respective norms as we collapse the support of φ (i.e. let the parameter ε in φ_ε tend to 0) and the approximations of u converge in the distributional sense. As a result, we obtain uniform bounds in the respective spaces. This results in an uniform L^2-bound for u convoluted with φ_ε independent of ε, which can be seen to imply $u \in L_*^2(S^1)$ by using a weak-L^2-convergent subsequence. The estimate follows by the lower semi-continuity of the norm. This concludes our proof. □

Remark 1. *Before we enter the discussion of applications and later a generalisation of Theorem 3, let us quickly discuss the assumption that u is real-valued. In fact, this is merely used at a single point in the proof, namely in (54). However, if we proceed similar to the proof of the generalised result in section 5, i.e. we use:*

$$\int_{S^1} |u - F|^2 dx = \int_{S^1} (u - F) \cdot \overline{u - F} dx = \int_{S^1} Dw \cdot \overline{u - F} dx \sim \sum_{n \in \mathbb{Z}} \widehat{Dw}(n) \cdot \overline{\widehat{u - F}(n)}, \tag{60}$$

we can easily avoid the use of properties of real-valued distributions. The remainder of the proof follows then completely analogous, i.e. we can remove the assumption of u being real-valued effortlessly. Indeed, this slight generalization will be key to our applications to Bergman spaces below.

4 Fractional Bourgain-Brezis inequality in the Bergman space $\mathcal{A}^2(\mathbb{D})$

We start with the **Proof of Theorem 1.**
Let us consider an analytic function $f \colon \mathbb{D} \to \mathbb{C}$ such that $\limsup_{r \to 1^-} \|f(re^{i\theta})\|_{L^1 + H^{-1/2}(S^1)} < +\infty$. Now let us write $f(z) = \sum_{n \geq 0} f_n z^n$ and $u(e^{i\theta}) = \sum_{n \geq 1} \frac{f_n}{\sqrt{n}} e^{in\theta}$. We first observe that $f - f(0) = \sum_{n \geq 1} f_n z^n$.

1. Assume first that $f(e^{i\theta}) = g(e^{i\theta}) + h(e^{i\theta})$ with $g \in L^1(S^1)$ and $h \in H^{-1/2}(S^1)$. Then:

$$f(e^{i\theta}) - f(0) = g - \fint_{S^1} g + h - \widehat{h}(0).$$

Note that $h - \widehat{h}(0) \in \dot{H}^{-1/2}(S^1)$ with the norm being controlled by $\|h\|_{H^{-1/2}(S^1)}$. We observe that, using the explicit definitions of the norm:

$$\left\| g - \fint_{S^1} g \right\|_{L^1(S^1)} + \|h - \widehat{h}(0)\|_{\dot{H}^{-1/2}(S^1)} \lesssim \|g\|_{L^1(S^1)} + \|h\|_{H^{-1/2}(S^1)}$$

Therefore, we may conclude by taking the infimum over all such g, h:

$$\|f - f(0)\|_{L^1 + \dot{H}^{-1/2}(S^1)} \leq C \|f\|_{L^1 + H^{-1/2}(S^1)}. \tag{61}$$

Assume therefore first that

$$f(e^{i\theta}) - f(0) = \sum_{n \geq 1} f_n e^{in\theta} \in L^1 + \dot{H}^{-1/2}(S^1).$$

In this case, we get $(-\Delta)^{1/4} u = f - f(0) \in L^1 + \dot{H}^{-1/2}(S^1)$. Additionally, we observe that since u contains only positive frequencies, we trivially have $\mathcal{R}(-\Delta)^{1/4} u \in L^1 + \dot{H}^{-1/2}(S^1)$ as well with

$$\|\mathcal{R}(-\Delta)^{1/4} u\|_{L^1 + \dot{H}^{-1/2}(S^1)} = \|(-\Delta)^{1/4} u\|_{L^1 + \dot{H}^{-1/2}(S^1)}.$$

From the inequality (8), observing that $\fint_{S^1} u = 0$, we deduce that

$$\|u\|_{L^2(S^1)} \leq C \|(-\Delta)^{1/4} u\|_{(L^1 + \dot{H}^{-1/2})(S^1)} = C \|f - f(0)\|_{L^1 + \dot{H}^{-1/2}(S^1)} \leq C' \|f\|_{L^1 + H^{-1/2}(S^1)},$$

where we used (61). Hence $\sum_{n>0} \frac{f_n}{n} e^{in\theta} \in H^{1/2}(S^1)$ and $g(z) = \sum_{n>0} \frac{f_n}{n} z^n \in H^1(\mathbb{D})$. We have $g'(z) = \sum_{n \geq 0} f_{n+1} z^n \in L^2(\mathbb{D})$ and

$$\begin{aligned} \|f(z) - f(0)\|_{L^2(\mathbb{D})} &= \|z g'(z)\|_{L^2(\mathbb{D})} \\ &\leq C \|g\|_{H^{1/2}(S^1)} = C \|u\|_{L^2(S^1)} \\ &\leq C \|f\|_{L^1 + H^{-1/2}(S^1)}. \end{aligned} \tag{62}$$

The desired estimate follows by the triangle inequality, if we can show:

$$|f(0)| \leq C \|f\|_{L^1 + H^{-1/2}(S^1)}$$

To achieve this, let us decompose $f = g + h$ with $g \in L^1(S^1)$ as well as $h \in H^{-1/2}(S^1)$. Then we denote as usual the Fourier coefficients of g, h by g_n, h_n for all $n \in \mathbb{Z}$ and define:

$$G(z) := \sum_{n \geq 0} g_n z^n + \sum_{n < 0} g_n \overline{z}^{|n|}, \quad H(z) := \sum_{n \geq 0} h_n z^n + \sum_{n < 0} h_n \overline{z}^{|n|}.$$

By the summability properties, these define harmonic functions on $\mathbb{D}$ having boundary values g, h respectively. By comparison of the coefficients, we also observe:

$$f(z) = G(z) + H(z),$$

in particular for $z = 0$. Moreover, by the mean value property of harmonic functions over the boundary of the disc, we can deduce:

$$|G(0)| \lesssim \|g\|_{L^1(S^1)}.$$

Using the mean value property over the entire disc, we similarily see by Hölder's inequality:

$$|H(0)| \lesssim \|H\|_{L^1(\mathbb{D})} \lesssim \|H\|_{L^2(\mathbb{D})}.$$

It is easy to verify by a direct computation analogous to the same characterisation of the norm in $\mathcal{A}^2(\mathbb{D})$ that:

$$\|H\|_{L^2(\mathbb{D})}^2 \sim \sum_{n \in \mathbb{Z}} \frac{|h_n|^2}{|n| + 1} \leq \|h\|_{H^{-1/2}(S^1)}.$$

In conclusion, we have:

$$|f(0)| \leq C \left(\|g\|_{L^1(S^1)} + \|h\|_{H^{-1/2}(S^1)} \right).$$

By taking the infimum over g, h such that $f = g + h$ we get

$$|f(0)| \leq C\|f\|_{L^1 + H^{-1/2}(S^1)} \tag{63}$$

By combining (62) and (63), we obtain the desired estimate:

$$\|f\|_{L^2(\mathbb{D})} \leq C\|f\|_{L^1 + H^{-1/2}(S^1)}. \tag{64}$$

2. In the general case when $\limsup_{r \to 1^-} \|f(re^{i\theta})\|_{L^1 + H^{-1/2}(S^1)} < +\infty$, we consider for every $0 < r < 1$ the function $f_r(z) = f(rz) \in C^\infty(\bar{B}(0,1))$. We can apply (62) to f_r and obtain that

$$\|f_r\|_{L^2(\mathbb{D})} \leq C\|f_r\|_{L^1 + H^{-1/2}(S^1)}. \tag{65}$$

Since by assumption $\limsup_{r \to 1^-} \|f(re^{i\theta})\|_{L^1 + H^{-1/2}(S^1)} < +\infty$, we deduce that

$$\sup_{0 < r < 1} \|f_r\|_{L^2(\mathbb{D})} < +\infty. \tag{66}$$

The inequality implies that actually $f \in L^2(\mathbb{D})$ as well as[11] and

$$\|f\|_{L^2(\mathbb{D})} \leq C\|f\|_{L^1 + H^{-1/2}(S^1)}. \tag{69}$$

Conversely, let $f \colon \mathbb{D} \to \mathbb{C}$ be in $\mathcal{A}^2(\mathbb{D}^2)$. We write $f(z) = \sum_{n=0}^\infty a_n z^n$. We prove the following:
Claim: $\limsup_{r \to 1^-} \|f(re^{i\theta})\|_{L^1 + H^{-1/2}(S^1)} < +\infty$.
Proof of the claim. We show that $\limsup_{r \to 1^-} \|f(re^{i\theta})\|_{H^{-1/2}(S^1)} < +\infty$. For every $0 < r < 1$, we set $f_r(z) = f(rz) \in C^\infty(\bar{B}(0,1))$. Since $f \in L^2(\mathbb{D})$, we have

$$\limsup_{r \to 1^-} \|f_r\|_{L^2(\mathbb{D})} = \|f\|_{L^2(\mathbb{D})}. \tag{70}$$

Moreover

$$\|f_r\|_{H^{-1/2}(S^1)}^2 = \sum_{n \geq 0} \frac{|f_n|^2}{1 + n} r^{2n} \tag{71}$$

---

[11]Let $f(z) = \sum_{n \geq 0} f_n z^n$ We observe that

$$\begin{aligned}
\|f_r\|_{L^2(\mathbb{D})}^2 &= \int_0^1 \int_0^{2\pi} |f(\rho r e^{i\theta})|^2 \rho d\theta d\rho \\
&= 2\pi \int_0^1 \sum_{n=0}^\infty |a_n|^2 r^{2n} \rho^{2n+1} d\rho = 2\pi \sum_{n=0}^\infty \frac{|a_n|^2 r^{2n}}{2n + 2}.
\end{aligned} \tag{67}$$

and similarily

$$\|f\|_{L^2(\mathbb{D})}^2 = 2\pi \sum_{n=0}^\infty \frac{|a_n|^2}{2n + 2}. \tag{68}$$

From (66) and extracting a weakly convergent subsequence which by convergence of the Fourier coefficients must have limit f, it follows that $\|f\|_{L^2(\mathbb{D})}^2 < +\infty$ and Abel's Theorem on power series yields that

$$\lim_{r \to 1^-} \|f_r\|_{L^2(\mathbb{D})}^2 = \|f\|_{L^2(\mathbb{D})}^2.$$

and

$$\int_{\mathbb{D}} |f_r|^2 = 2\pi \int_0^1 \sum_{n\geq 0} |f_n|^2 r^{2n} s^{2n} s\,ds \simeq \sum_{n\geq 0} \frac{|f_n|^2}{2n+2} r^{2n}$$

$$\simeq \frac{1}{2} \sum_{n\geq 0} \frac{|f_n|^2}{n+1} r^{2n} \simeq \|f_r\|_{H^{-1/2}(S^1)}. \tag{72}$$

By combining (70) and (72), we get that

$$\limsup_{r\to 1^-} \|f_r\|_{H^{-1/2}(S^1)} \lesssim \|f\|_{L^2(\mathbb{D})} < +\infty. \tag{73}$$

We conclude the proof. $\square$

Next we show that Theorem 1 is actually equivalent to Theorem 3.

Proposition 4.1. *Theorem 1 implies Theorem 3. Therefore, they are equivalent.*

Proof. We have already seen in the proof of Theorem 1 that Theorem 3 implies the fact that a holomorphic function with the property that $\limsup_{r\to 1^-} \|f(re^{i\theta})\|_{L^1+H^{-1/2}(S^1)} < +\infty$ is in $L^2(\mathbb{D})$, namely it belongs to the Bergman space $\mathcal{A}^2(\mathbb{D})$.

Conversely, let us consider $u \in C^\infty(S^1)$ such that $(-\Delta)^{1/4}u, \mathcal{R}(-\Delta)^{1/4}u \in L^1 + \dot{H}^{-1/2}(S^1)$. We assume that $\int_0^{2\pi} u(e^{i\theta})d\theta = 0$. We decompose $u = u^+ + u^-$, where

$$u^+ = \sum_{n>0} u_n e^{in\theta}, \quad u^- = \sum_{n<0} u_n e^{in\theta}.$$

Let us first consider u^+. By assumption we have $\sum_{n\geq 1} n^{1/2} u_n e^{in\theta} = 1/2((-\Delta)^{1/4}u - i\mathcal{R}(-\Delta)^{1/4}u) \in L^1 + \dot{H}^{-1/2}(S^1)$. Let $f(z) = \sum_{n\geq 1} n^{1/2} u_n z^n$ be the harmonic extension of $v = (-\Delta)^{1/4}u^+$ in $\mathbb{D}$. From Theorem 1, it follows that $f^+ = \sum_{n>0} n^{1/2} u_n z^n \in L^2(\mathbb{D})$ and

$$\|f^+\|_{L^2(\mathbb{D})} \leq C\|f^+\|_{L^1+H^{-1/2}(S^1)} \leq \|(-\Delta)^{1/4}u^+\|_{L^1+\dot{H}^{-1/2}(S^1)}.$$

Switching to the homogeneous Sobolev space is possible, as we have $\dot{H}^{-1/2}(S^1) \subset H^{-1/2}(S^1)$ continuously embedded. Since $f^+(z) = \sum_{n>0} n^{1/2} u_n z^n \in L^2(\mathbb{D})$, it follows that $\sum_{n>0} \frac{u_n}{n^{1/2}} z^n \in H^1(\mathbb{D})$ and therefore $\sum_{n>0} \frac{u_n}{n^{1/2}} e^{in\theta} \in \dot{H}^{1/2}(S^1)$. Hence $u^+ \in L^2(S^1)$ with

$$\|u^+\|_{L^2(S^1)} \leq C\|(-\Delta)^{1/4}u^+\|_{L^1+\dot{H}^{-1/2}(S^1)}$$
$$\leq C\left(\|(-\Delta)^{1/4}u\|_{L^1+\dot{H}^{-1/2}(S^1)} + \|\mathcal{R}(-\Delta)^{1/4}u\|_{L^1+\dot{H}^{-1/2}(S^1)}\right). \tag{74}$$

The same arguments hold for u^-. We conclude the proof. $\square$

5   The Bourgain-Brezis Inequality on the Torus $T^n, n \geq 2$

In this section, we are going to prove Theorem 4 which generalises the result from Theorem 3 to domains of dimension $n \geq 2$. To achieve this while retaining the general structure of the proof, we first have to determine the right set of conditions and the appropriate domain. Observe that it is clear, due to the proof for S^1 heavily relying on Fourier series, that the natural domain for such a generalisation is the torus T^n. In investigating generalisations of the proof, we have to focus on two aspects: Clifford algebras and boundedness of the kernel. In the first part of the proof we introduce complex Clifford algebras and show how to generalize the argument presented in section 3.2. The results and properties of Clifford algebras are due to [7] and [8] and are briefly discussed

in section 5.1 below. In the second part of the proof we show that the kernel used is actually bounded, following an argument presented in [1, p.405-406]. In the case $n = 1$, we have seen that k has an explicit description as a sawtooth function. In higher dimensions, unfortunately, we are not aware of an explicit formula for the kernel. However, due to some estimates on alternating sums, we can remedy this lack of explicit representation and derive the crucial properties abstractly.

5.1 A short introduction to Clifford algebras

The material covered here is due to [7] and [8] and we refer to them for further details on the topics introduced. For the remainder of this subsection, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ denote a scalar field and V a finite dimensional $\mathbb{K}$-vector space. Let $Q : V \to \mathbb{K}$ be a map, such that:

1.) For all $\lambda \in \mathbb{K}$ and $v \in V$, we have: $Q(\lambda v) = \lambda^2 \cdot Q(v)$.

2.) The map $B(v, w) := \frac{1}{2}\big(Q(v + w) - Q(v) - Q(w)\big)$ defines a $\mathbb{K}$-bilinear map on $V \times V$.

Such a Q will be called a *quadratic form* and the pair (V, Q) a *quadratic space*. Standard examples include real vector spaces equipped with scalar products, but not complex vector spaces with scalar products due to complex anti-linearity in the second argument. Inspired by this example, we say that a basis $e_1, \ldots, e_n$ of a quadratic space (V, Q) is *B-orthonormal*, if for all $j \in \{1, \ldots, n\}$, we have $|Q(e_j)| = 1$ as well as:

$$B(e_j, e_k) = 0, \quad \forall j \neq k \in \{1, \ldots, n\}. \tag{75}$$

Given such a quadratic space (V, Q), we call a pair $(\mathcal{A}, \nu)$ a *Clifford algebra* for (V, Q), if the following holds, see [7, p.8, (2.1)]:

i.) $\mathcal{A}$ is an associative algebra with unit 1 and $\nu : V \to \mathcal{A}$ is $\mathbb{K}$-linear and injective.

ii.) $\mathcal{A}$ is generated as an algebra by $\nu(V)$ and $\mathbb{K} \cdot 1$.

iii.) For every $v \in V$, we have: $\nu(v)^2 = -Q(v) \cdot 1$

An important immediate corollary of the definition is the following commutation relation:

$$\nu(v)\nu(w) + \nu(w)\nu(v) = -2B(v, w) \cdot 1, \quad \forall v, w \in V. \tag{76}$$

Thus, pairs of <u>orthogonal</u> vectors with respect to B anti-commute as elements in $\mathcal{A}$. We usually omit explicitly mentioning ν and therefore identify v with $\nu(v)$, which is justified due to ν being injective.

For the remainder of the section, let us focus on (V, Q) non-degenerate, i.e. for all $v \in V$, there is a $w \in V$, such that $B(v, w) \neq 0$. In this case, there actually exists a basis $e_1, \ldots, e_n$, where $n = \dim_{\mathbb{K}} V$, orthonormal with respect to B and, consequently, such that:

$$e_j e_k + e_k e_j = \pm 2\delta_{jk} \cdot 1, \quad \forall j, k \in \{1, \ldots, n\}, \tag{77}$$

(see e.g. Theorem 1.5 in [7]). The signs are determined by the signature of the quadratic form Q and may vary for different choices j, k. Provided $\mathbb{K} = \mathbb{C}$, we may assume that all signs are the same, see [7].

It can be shown that every Clifford algebra has $\mathbb{K}$-dimension at most 2^n. If the dimension is equal to 2^n, the Clifford algebra is called *universal*.[12] An important result in [7, Thm. 2.7] states that there always exists a universal Clifford algebra for any given quadratic space. Moreover, there exist explicit descriptions of all universal Clifford algebras up to isomorphisms in terms of matrices,

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[12] This definition is justified, as universal Clifford algebras $\mathcal{A}$ have an extension property for linear maps from V to any Clifford algebra respecting the characteristic multiplication relation in $\mathcal{A}$, see [7].

see [7].

To conclude this brief treatment of Clifford algebras, let us provide an explicit example: Let $V = \mathbb{C}^n$, $\mathbb{K} = \mathbb{C}$ and define Q as follows:

$$Q(z_1, \ldots, z_n) := \sum_{j=1}^n z_j^2, \quad \forall (z_1, \ldots, z_n) \in \mathbb{C}^n. \tag{78}$$

It is clear that (V, Q) is a non-degenerate quadratic space, as B is the standard scalar product up to a complex conjugation in the second argument. In this case, the standard basis $e_1, \ldots, e_n$ already is B-orthonormal. Thus, we have:

$$e_j e_k + e_k e_j = -2\delta_{jk} \cdot 1, \quad \forall j, k \in \{1, \ldots, n\}. \tag{79}$$

The universal Clifford algebra is then spanned by the finite products e_α of the basis elements, where $\alpha \subset \{1, \ldots, n\}$ is an ordered subset and we define:

$$e_\alpha = \prod_{j \in \alpha} e_j$$

In particular, $e_\emptyset = 1$ by definition. It can be seen that every complex universal Clifford algebra associated with a non-degenerate quadratic space of dimension n is isomorphic to this one, see [7] and the definition of universal Clifford algebra presented there.

Lastly, let us introduce a few definitions from Chapter 1, Section 7 in [7]: We may identify the universal Clifford algebra $\mathcal{A}$ as a vector space with $\mathbb{K}^{2^n}$, if $\dim_{\mathbb{K}} V = n$. This allows us to generalize the natural scalar product-induced norm on $\mathbb{K}^{2^n}$ to the Clifford algebra and we shall denote this norm by $\|\cdot\|$. Moreover, there is a notion of conjugation on Clifford algebras defined by:

$$\overline{e_{j_1} \ldots e_{j_k}} := (-1)^k Q(e_{j_1}) \ldots Q(e_{j_k}) \cdot e_{j_k} \ldots e_{j_1} = (-1)^{\frac{k(k+1)}{2}} Q(e_{j_1}) \ldots Q(e_{j_k}) \cdot e_{j_1} \ldots e_{j_k}, \tag{80}$$

and extending linearly. If $\mathbb{K} = \mathbb{C}$, we also conjugate the complex coefficients in the usual manner, i.e. we extend complex anti-linearly. We highlight the following key property of the conjugation:

$$\overline{xy} = \overline{y} \cdot \overline{x}, \quad \forall x, y \in \mathcal{A}. \tag{81}$$

This is due to the inversion of factors in (80). We emphasise that the definition in (80) is precisely made with the identity below in mind:

$$\overline{e_{j_1} \ldots e_{j_k}} \cdot e_{j_1} \ldots e_{j_k} = 1. \tag{82}$$

The following property will be useful later as well: Let $x \in \mathcal{A}$ be given and denote by P_0 the linear projection of an element in the Clifford algebra to the coefficient associated with the neutral element 1. More precisely, $P_0 : \mathcal{A} \to \mathbb{K}$ is the following linear map:

$$P_0 \left(\sum_\alpha x_\alpha e_\alpha \right) = x_\emptyset$$

We have by a direct computation:

$$\begin{aligned} P_0(\overline{x}x) &= \sum_{\alpha \subset \{1, \ldots, n\}} \overline{x_\alpha} x_\alpha \\ &= \|x\|^2, \end{aligned} \tag{83}$$

where we wrote explicitly $x = \sum_{\alpha \subset \{1, \ldots, n\}} x_\alpha e_\alpha$ with $x_\alpha \in \mathbb{K}$. It suffices to observe that $e_\alpha \cdot e_\beta$ has non-vanishing contribution in the $e_\emptyset = 1$-direction, if and only if $\alpha = \beta$. The formula then follows.

5.2 Proof of Theorem 4.

Let us first note that, if we take T^n with $n \geq 2$, there are n different Riesz transforms, one for each basis direction. This suggests that the right conditions should involve some restriction on each of the Riesz transforms. In addition, considering the symbol of D^2, we see that we rely some cancellation property stemming from the complex nature of i. [13] Therefore, a natural way to obtain a generalisation would involve Clifford algebras to include sufficiently many anticommuting complex units.

Firstly, it is immediate that the same simplifications as in the case $n = 1$ apply here. So we may assume $\hat{u}(0) = 0$. Throughout most of this proof, the coefficient $m = 0$ will be implicitly omitted, as it will be vanishing for all functions/distributions considered. Moreover, the reduction to smooth functions applies equally well in this case. Therefore, we may assume without loss of generality that u, f^j, g^j are all smooth.

The heart of the argument lies in the correct definition of D and $\overline{D}$ on T^n. As mentioned in the introduction of the current section, Clifford algebras and their set of complex units actually provide the desired framework. Let $\mathbb{C}_n$ denote the universal complex Clifford algebra associated with the quadratic space $(\mathbb{C}^n, Q)$, where:

$$Q(z_1, \ldots, z_n) := -\sum_{j=1}^n z_j^2, \quad \forall (z_1, \ldots, z_n) \in \mathbb{C}^n. \tag{84}$$

We emphasise that the particular choice of Q is at odds with usual conventions for complex Clifford algebras, but using our quadratic form, we obtain the appropriate basis commutation relations while remaining isomorphic to the usual convention. One could reduce to the usual defining quadratic form by choosing $i \cdot e_j$ instead of the standard basis e_j throughout our proof. In fact, the main reason why we decided to use our convention is to use the Riesz operators in their usual form.

Observe that we then have, for the standard basis denoted by $e_1, \ldots, e_n$:

$$e_j e_k + e_k e_j = 2\delta_{jk}, \quad \forall j, k \in \{1, \ldots, n\}, \tag{85}$$

simply by the definition of Clifford algebras and the quadratic form Q. We define now for any $v \in C^\infty(T^n, \mathbb{C})$:

$$Dv = \Delta^{\frac{n}{4}}(Id + \sum_{j=1}^n e_j \mathcal{R}^j)v \tag{86}$$

$$\overline{D}v = \Delta^{\frac{n}{4}}(Id - \sum_{j=1}^n e_j \mathcal{R}^j)v. \tag{87}$$

We emphasise the similarity with [7, (5.14)] used in the context of Hardy spaces. The crucial observation for our purposes is the following multiplier property for Fourier series for every $m \in \mathbb{Z}^n$:

$$\mathcal{F}(Dv)(m) = |m|^{\frac{n}{2}}\big(1 + \sum_{j=1}^n e_j \cdot i\frac{m^j}{|m|}\big)\mathcal{F}(v)(m) \tag{88}$$

$$\mathcal{F}(\overline{D}v)(m) = |m|^{\frac{n}{2}}\big(1 - \sum_{j=1}^n e_j \cdot i\frac{m^j}{|m|}\big)\mathcal{F}(v)(m), \tag{89}$$

where $|m|$ denotes the Euclidean norm on $\mathbb{Z}^n$. We highlight that at this point, we know that Du and $\overline{D}u$ are functions in $L^1 + \dot{H}^{-\frac{n}{2}}(T^n, \mathbb{C}_n)$. Completely analogous to the proof of Theorem 3,

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[13]This refers to the property $i^2 = -1$ which was key to reduce the multiplier of D^2 to a simpler form.

we may find $F \in L^2$ (due to the invertibility of non-zero vectors $v \in \mathbb{R}^n$ in $\mathbb{C}_n$[14]). To be precise, observe that if $DF = f$, f and g are defined to satisfy $Du = f + g$ by splitting the terms f^j, g^j in the natural way, then:

$$\forall m \neq 0: \quad |m|^{\frac{n}{2}}\big(1 + \sum_{j=1}^{n} e_j \cdot i\frac{m^j}{|m|}\big)\widehat{F}(m) = \widehat{f}(m), \tag{90}$$

which may be rewritten as:

$$\widehat{F}(m) = \frac{1}{2|m|^{\frac{n}{2}}}\big(1 - \sum_{j=1}^{n} e_j \cdot i\frac{m^j}{|m|}\big)\widehat{f}(m), \tag{91}$$

by using the multiplication relations and associativity on $\mathbb{C}_n$. To conclude that $F \in L^2$, it suffices to check summability of the Fourier coefficients:

$$\sum_{m \in \mathbb{Z}^n \setminus \{0\}} \|\widehat{F}(m)\|^2 = \sum_{m \neq 0} \Big\|\frac{1}{2|m|^{\frac{n}{2}}}\big(1 - \sum_{j=1}^{n} e_j \cdot i\frac{m^j}{|m|}\big)\widehat{f}(m)\Big\|^2$$

$$\lesssim \sum_{m \neq 0} \frac{1}{|m|^n}\|\widehat{f}(m)\|^2$$

$$\lesssim \|f\|_{\dot{H}^{-\frac{n}{2}}}^2 < +\infty. \tag{92}$$

We mention here that the characterisations for regularity and integrability carry over without problem, even if we use Clifford algebra-valued functions by verifying componentwise regularity.

Consequently, as in the case $n = 1$, we may define $\tilde{u} = u - F$ and observe that $D\tilde{u} =: g \in L^1$. Solving $Dw = \tilde{u}$ in the sense of distributions leaves us with $D^2 w = g$.

The key point behind the second proof of Theorem 3 lies in the fact, that D^2 has an inverse given by the convolution with a bounded function. By a direct computation, we arrive at the following expression for the multiplier associated with D^2:

$$\mathcal{F}(D^2 w)(m) = |m|^n\big(1 + \sum_{j=1}^{n} e_j \cdot i\frac{m^j}{|m|}\big)^2 \mathcal{F}(w)(m) = 2i \cdot |m|^n\big(\sum_{j=1}^{n} e_j\frac{m^j}{|m|}\big)\mathcal{F}(w)(m), \tag{93}$$

for every $m \in \mathbb{Z}^n$. Observe that we used the fact that the complex unit i of $\mathbb{C}$ commutes with all e_j (as the Clifford algebra is a complex algebra) and that:

$$(i \cdot e_j)^2 = i^2 \cdot e_j^2 = i^2 = -1. \tag{94}$$

Let us identify $m = \sum m^j e_j$, i.e. we consider the vector $m \in \mathbb{Z}^n \subset \mathbb{C}^n$ as an element in $\mathbb{C}_n$. Therefore, (93) becomes:

$$\mathcal{F}(D^2 w)(m) = 2i|m|^{n-1} \cdot m\mathcal{F}(w)(m), \quad \forall m \in \mathbb{Z}^n. \tag{95}$$

As stated before, all vectors $\mathbb{R}^n \subset \mathbb{C}_n$ are invertible due to:

$$z^2 = -Q(z), \quad \forall z \in \mathbb{C}^n. \tag{96}$$

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[14]Observe that for real vectors in $\mathbb{R}^n$, we find $m^2 = |m|^2$. For general vectors in $\mathbb{C}^n$, this fails, as can be seen in the counterexample:

$$(e_1 + ie_2)^2 = 0$$

So, for the real vector m, we have due to $m \cdot m = -Q(m)$:

$$m^{-1} = \frac{m}{|m|^2}, \quad \forall 0 \neq m \in \mathbb{Z}^n. \tag{97}$$

This means that $D^2 w = g$ can be restated as:

$$\mathcal{F}(w)(m) = \frac{1}{2i} \cdot \frac{m}{|m|^{n+1}} \mathcal{F}(g)(m), \tag{98}$$

for every $0 \neq m \in \mathbb{Z}^n$.

For now, let us assume that a bounded function K on the torus exists, such that:

$$\widehat{K}(m) = \frac{1}{2i} \cdot \frac{m}{|m|^{n+1}}, \quad \forall m \in \mathbb{Z}^n \setminus \{0\}. \tag{99}$$

In this case, we may check using Fourier coefficients that (keeping in mind that the order of factors in the convolution matters for products in Clifford algebras):

$$w = \frac{1}{(2\pi)^n} K * g \tag{100}$$

Thus, we have the following inequality:

$$\|w\|_{L^\infty} \lesssim \|K\|_{L^\infty} \|g\|_{L^1}. \tag{101}$$

This is an immediate consequence of the definition, Minkowski's inequality and continuity of the Clifford multiplication in the Clifford algebra norm.
Moreover, we may deduce:

$$
\begin{aligned}
\|u - F\|_{L^2}^2 &= \int_{T^n} P_0\big(\overline{(u-F)} \cdot (u-F)\big) dx \\
&= P_0\Big(\int_{T^n} \overline{(u-F)} \cdot (u-F) dx \Big) \\
&\leq \Big\| \int_{T^n} \overline{(u-F)} \cdot (u-F) dx \Big\| \\
&= \Big\| \int_{T^n} \overline{Dw} \cdot (u-F) dx \Big\| \\
&= \Big\| \int_{T^n} \Big(\sum_m \overline{\widehat{Dw}(m)} e^{-i\langle m, x\rangle} \Big) \cdot \Big(\sum_{\tilde{m}} \widehat{u-F}(m) e^{i\langle \tilde{m}, x\rangle} \Big) dx \Big\| \\
&\simeq \Big\| \sum_m \overline{\widehat{Dw}(m)} \cdot \widehat{u-F}(m) \Big\| \\
&= \Big\| \sum_m \overline{|m|^{\frac{n}{2}} \Big(1 + \sum_{j=1}^n e_j \cdot i \frac{m^j}{|m|} \Big) \widehat{w}(m)} \cdot \widehat{u-F}(m) \Big\| \\
&= \Big\| \sum_m \overline{\widehat{w}(m)} \cdot |m|^{\frac{n}{2}} \overline{\Big(1 + \sum_{j=1}^n e_j \cdot i \frac{m^j}{|m|} \Big)} \widehat{u-F}(m) \Big\| \\
&= \Big\| \sum_m \overline{\widehat{w}(m)} \cdot |m|^{\frac{n}{2}} \Big(1 - \sum_{j=1}^n e_j \cdot i \frac{m^j}{|m|} \Big) \widehat{u-F}(m) \Big\| \\
&= \Big\| \sum_m \overline{\widehat{w}(m)} \cdot \widehat{D(u-F)}(m) \Big\| \\
&\simeq \Big\| \int_{T^n} \overline{w} \cdot \overline{D}(u-F) dx \Big\|. \tag{102}
\end{aligned}
$$

Observe that in the first inequality, we used that the norm squared of $u - F$ actually appears as the coefficient associated with 1 in the product $\overline{u - F} \cdot (u - F)$. In addition, the conjugation in the ninth line can easily deduced from our definition in the preliminary section of the paper, see (80). The remainder of the argument then follows completely analogous to the $1D$-proof, up to the obvious modifications. Again, simple considerations show that we even have the following inequality:

$$\left\| u - \fint_{T^n} u\,dx \right\|_{L^2} \lesssim \sum_{j=0}^{n} \left\| (-\Delta)^{\frac{n}{2}} \mathcal{R}^j u \right\|_{\dot{H}^{-\frac{n}{2}} + L^1}, \tag{103}$$

where $\mathcal{R}_0 = Id$.

To complete the proof in the same way as for Theorem 3, we still need to find a bounded kernel K satisfying:

$$\widehat{K}(m) = \frac{1}{2i} \cdot \frac{m}{|m|^{n+1}}, \quad \forall 0 \neq m \in \mathbb{Z}^n. \tag{104}$$

This is the purpose of the next subsection, so we may conclude the proof of Theorem 4 at this point. $\qquad\square$

5.3   Boundedness of the Kernel

Lastly, let us find an appropriate kernel. We first notice that due to linearity, symmetry and the splitting into different directions, it is enough to find a bounded function k, such that:

$$\widehat{k}(m) = \frac{m_1}{|m|^{n+1}}, \quad \forall 0 \neq m \in \mathbb{Z}^n. \tag{105}$$

Consequently, we want to study the boundedness of the following conditionally convergent series:

$$k(x) = \sum_{m \in \mathbb{Z}^n \setminus \{0\}} \frac{m_1}{|m|^{n+1}} e^{i\langle m, x \rangle}. \tag{106}$$

Let us fix some notation. We usually identify $m \in \mathbb{Z}^n$ with $m = (m_1, \tilde{m})$, where $\tilde{m} \in \mathbb{Z}^{n-1}$. We will sometimes use the same notation for $x \in \mathbb{R}^n$. Moreover, for any m, we define $m' = (-m_1, \tilde{m})$. This allows us to immediately see:

$$\widehat{k}(m') = -\widehat{k}(m), \quad \forall m \in \mathbb{Z}^n \setminus \{0\}. \tag{107}$$

This observation enables us to rewrite (106) as follows:

$$k(x) = 2i \cdot \sum_{m_1 > 0} \sum_{\tilde{m} \in \mathbb{Z}^{n-1}} \frac{m_1}{|m|^{n+1}} \sin(m_1 x_1) e^{i\langle \tilde{m}, \tilde{x} \rangle}. \tag{108}$$

The strategy of the proof is based on [1, p.405-406]. Thus, the main point is to split the sum into partial sums involving m_1 and $|\tilde{m}|$ being comparable to some dyadic 2^{k_1} and $2^{\tilde{k}}$ respectively. Then, we distinguish $k_1 \leq \tilde{k}$ and $k_1 \geq \tilde{k}$ to conclude. Thus, we consider the following sum derived from (108):

$$|k(x)| \leq \sum_{k_1 \geq 0} \sum_{\tilde{k} \geq 0} \left| \sum_{m_1 \sim 2^{k_1}} \sum_{|\tilde{m}| \sim 2^{\tilde{k}}} \frac{m_1}{|m|^{n+1}} \sin(m_1 x_1) e^{i\langle \tilde{m}, \tilde{x} \rangle} \right|. \tag{109}$$

Let us mention an uniform estimate for fixed $k_1, \tilde{k}$. To achieve this, we distinguish two cases: $k_1 \geq \tilde{k}$ and $k_1 < \tilde{k}$. We shall need the following estimate that can be found in [1, (4.22)]:

$$\left| \sum_{\ell \in I} \sin(\ell x) \right| \lesssim 4^k |x| \wedge \frac{1}{|x|}, \tag{110}$$

for every $k \in \mathbb{N}$, $x \in S^1$ and subinterval $I \subset [2^{k-1}, 2^k]$. Here, $\wedge$ denotes the minimum of two functions. Let us provide the argument in a more abstract manner: Consider a finite sum of the form:

$$\sum_{m_1} \sum_{\tilde{m}} a_{m_1} b_{\tilde{m}} c_{m_1, \tilde{m}}. \tag{111}$$

Observe that the summands in (109) inside the absolute value clearly have this form. Let us denote by A_{m_1} the partial sum of all a_l up to the m_1-th element. In the case of (109), this would be a sum of $\sin(lx)$ over an interval with l comparable to 2^{k_1}, hence we may use the bound (110). Therefore, we may rewrite (111) as:

$$\sum_{m_1} \sum_{\tilde{m}} a_{m_1} b_{\tilde{m}} c_{m_1, \tilde{m}} = \sum_{m_1} \sum_{\tilde{m}} (A_{m_1} - A_{m_1 - 1}) b_{\tilde{m}} c_{m_1, \tilde{m}}$$

$$= \sum_{m_1} \sum_{\tilde{m}} A_{m_1} b_{\tilde{m}} (c_{m_1, \tilde{m}} - c_{m_1 + 1, \tilde{m}}), \tag{112}$$

which, in the case of (109), can be estimated using the bound on sums of sinus functions in (110), the boundedness of the $b_{\tilde{m}}$ which are merely $e^{i\langle \tilde{m}, \tilde{x} \rangle}$ and finally the estimate:

$$\left| \frac{m_1}{|m|^{n+1}} - \frac{m_1 + 1}{((m_1 + 1)^2 + |\tilde{m}|^2)^{\frac{n+1}{2}}} \right| \lesssim \frac{1}{|m|^{n+1}}. \tag{113}$$

We mention the slight imprecision, as in (112), the extremal partial sums A_l require further attention. However, in the case we are considering, similar techniques can be applied (since we no longer sum over m_1) and we omit further details.

Therefore, we arrive at the following estimate:

$$\left| \sum_{m_1 \sim 2^{k_1}} \sum_{|\tilde{m}| \sim 2^{\tilde{k}}} \frac{m_1}{|m|^{n+1}} \sin(m_1 x_1) e^{i\langle \tilde{m}, \tilde{x} \rangle} \right| \lesssim 2^{k_1} \left(2^{k_1} |x_1| \wedge \frac{1}{2^{k_1} |x_1|} \right) \left\| \frac{1}{|m|^{n+1}} \right\|_{l^1(m_1 \sim 2^{k_1}, |\tilde{m}| \sim 2^{\tilde{k}})} \tag{114}$$

If $k_1 \geq \tilde{k}$, we may simplify (109) using (114) as follows:

$$|k(x)| \lesssim \sum_{k_1 \geq 1} 2^{\tilde{k}(n-1)} 2^{k_1} \frac{1}{2^{k_1(n+1)}} \cdot 2^{k_1} \left(2^{k_1} |x_1| \wedge \frac{1}{2^{k_1} |x_1|} \right)$$

$$\leq \sum_{k_1 \geq 0} 2^{k_1} |x_1| \wedge \frac{1}{2^{k_1} |x_1|} \lesssim C < \infty, \tag{115}$$

which can be easily bounded by the definition of the minimum.

If $\tilde{k} > k_1$, we find:

$$\sum_{k_1 \geq 0} \sum_{\tilde{k} \geq 0} \left| \sum_{m_1 \sim 2^{k_1}} \sum_{|\tilde{m}| \sim 2^{\tilde{k}}} \frac{m_1}{|m|^{n+1}} \sin(m_1 x_1) e^{i\langle \tilde{m}, \tilde{x} \rangle} \right|$$

$$\lesssim \sum_{k_1} \sum_{\tilde{k} > k_1} 4^{k_1} \left(2^{k_1} |x_1| \wedge \frac{1}{2^{k_1} |x_1|} \right) \cdot \frac{1}{2^{k_1(n+1)}} \sum_{|\tilde{m}| \sim 2^{\tilde{k}}} \frac{1}{(1 + \frac{|\tilde{m}|^2}{2^{2k_1}})^{\frac{n+1}{2}}}$$

$$\lesssim \sum_{k_1 \geq 0} \frac{2^{k_1(n-1)} 4^{k_1}}{2^{k_1(n+1)}} \left(2^{k_1} |x_1| \wedge \frac{1}{2^{k_1} |x_1|} \right)$$

$$\leq \sum_{k_1 \geq 0} 2^{k_1} |x_1| \wedge \frac{1}{2^{k_1} |x_1|} \leq C < \infty, \tag{116}$$

where we estimated the sum over $\tilde{m}, \tilde{k}$ by a dominating integral. This shows that $k(x)$ is actually bounded and possesses the required Fourier coefficients, hence adding the last ingredient missing in our proof of Theorem 4.

6 Existence Result for a certain Fractional PDE

Similar to [1], the estimates in Theorem 3 and 4 may be used to derive existence results for a particular differential operator. However, before turning to the PDE itself, let us briefly provide an alternative formulation of our main theorems for a more general class of distributions:

Theorem 5. *Let $u \in \mathcal{D}'(T^n, \mathbb{C}_n)$ be $\mathbb{C}_n$-valued and assume that:*

$$Du, \overline{D}u \in \dot{H}^{-\frac{n}{2}} + L^1(T^n, \mathbb{C}_n). \tag{117}$$

Here, D and $\overline{D}$ are the operators defined in the proof of Theorem 4. Then $u \in L^2(T^n, \mathbb{C}_n)$ and we have the following estimate:

$$\left\| u - \int_{T^n} u \, dx \right\|_{L^2} \lesssim \|Du\|_{\dot{H}^{-\frac{n}{2}} + L^1} + \|\overline{D}u\|_{\dot{H}^{-\frac{n}{2}} + L^1}. \tag{118}$$

This result is an immediate corollary of the proof of Theorem 4, as we always work with Du and $\overline{D}u$ rather than the $\mathcal{R}^j(-\Delta)^{\frac{n}{2}}$. The possibility to generalise to Clifford algebra-valued distributions follows directly, as all arguments involved behave well with respect to the Clifford algebra product. One could also rewrite the estimate by separating the identity operator from the Riesz operators.

Let us now turn to the existence result. We would like to consider the following problem:

$$g = (-\Delta)^{\frac{n}{4}} f_0 + \sum_{j=1}^{n} (-\Delta)^{\frac{n}{4}} \bar{\mathcal{R}}^j f^j, \tag{119}$$

where $g \in L_*^2(T^n) = \left\{ u \in L^2(T^n) : \fint_{T^n} u = 0 \right\}$.[15] Obviously, the PDE admits solutions $f_0, \dots, f_n$ in $\dot{H}^{\frac{n}{2}}(T^n)$. Again, using Sobolev embeddings, it is also clear that there is a-priori no way to deduce that the f^j may be chosen to be bounded or even continuous. We shall remedy this apparent lack of regularity:

Corollary 2. *Let $g \in L_*^2(T^n)$. Then there exist $f_0, \dots, f_n \in \dot{H}^{\frac{n}{2}} \cap C^0(T^n)$, such that (119) holds.*

Proof of Corollary 2. The proof is completely analogous to the one in [1, Proof of Theorem 1]: Let us define the following operator:

$$T : \bigoplus_{j=0}^{n} \dot{H}^{\frac{n}{2}} \cap C^0(T^n) \to L_*^2(T^n), \quad T(u_0, \dots, u_n) := (-\Delta)^{\frac{n}{4}} u_0 + \sum_{j=1}^{n} (-\Delta)^{\frac{n}{4}} \bar{\mathcal{R}}^j u^j. \tag{120}$$

It is clear that T is a bounded, linear operator. Moreover, we have that its dual operator is given by:

$$T^* : L_*^2(T^n) \to \bigoplus_{j=0}^{n} \dot{H}^{-\frac{n}{2}} + \mathcal{M}(T^n), \quad T^*(v) := \left((-\Delta)^{\frac{n}{4}} v, \mathcal{R}^1 (-\Delta)^{\frac{n}{4}} v, \dots, \mathcal{R}_n (-\Delta)^{\frac{n}{4}} v \right). \tag{121}$$

Here, $\mathcal{M}(T^n)$ denotes the collection of Radon measures on T^n. As in [1, (4.3)], it can be easily seen (using convolutions) that:

$$\| \cdot \|_{\dot{H}^{-\frac{n}{2}} + \mathcal{M}} = \| \cdot \|_{\dot{H}^{-\frac{n}{2}} + L^1} \quad \text{on } \dot{H}^{-\frac{n}{2}} + L^1(T^n). \tag{122}$$

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[15]The conjugate operator $\bar{\mathcal{R}}^j$ appears due to the duality used in the proof. This ensures, that we can apply the result in Theorem 4. It is simpel to see that by suitably exchanging $\mathcal{R}^j$ by $\bar{\mathcal{R}}^j$ throughout the proof of Theorem 4, the same inequality can be obtained for the dual operators and thus yields the same result as in Corollary 2 for the usual Riesz operators.

Therefore, we know by (103) that:

$$\|u\|_{L^2} \lesssim \|T^* u\|_{\bigoplus \dot{H}^{-\frac{n}{2}} + \mathcal{M}(T^n)}. \tag{123}$$

This implies that T is surjective (see Theorem 2.20 in [3]). The open mapping Theorem yields that there is $C > 0$ such that $B^{L^2_*}(0, C) \subseteq T(B^E(0, 1))$, where $E = \bigoplus_{j=0}^n \dot{H}^{\frac{n}{2}} \cap C^0(T^n)$. Therefore, for every $g \in L^2_*(S^1)$, there are $(f_0, \dots, f_n) \in E$ such that $(-\Delta)^{1/4} f_0 + \sum_{i=1}^n (-\Delta)^{1/4} \bar{\mathcal{R}}^j f^j = g$ and

$$\sum_{j=0}^n \|f^j\|_{\dot{H}^{\frac{n}{2}} \cap L^\infty} \leq C \|g\|_{L^2}, \tag{124}$$

for some fixed $C > 0$. This concludes the proof. $\qquad\square$

Using Corollary 2, we may derive the following simple result:

Corollary 3. *Let* $f \in \dot{H}^{\frac{n}{2}}(T^n)$. *Then there exist* $f_0, \dots, f_n \in \dot{H}^{\frac{n}{2}} \cap C^0(T^n)$ *as well as a smooth function* $\varphi \in C^\infty(T^n)$, *such that:*

$$f = \varphi + \sum_{j=0}^n \mathcal{R}^j f^j. \tag{125}$$

Proof of Corollary 3. Take $g = (-\Delta)^{\frac{n}{4}} f \in L^2_*(T^n)$. By Corollary 2, we see that there exist $f_0, \dots, f_n \in \dot{H}^{\frac{n}{2}} \cap C^0(T^n)$, such that (119) is satisfied. Therefore, we know:

$$(-\Delta)^{\frac{n}{4}} \left(f - \sum_{j=0}^n \mathcal{R}^j f^j \right) = 0. \tag{126}$$

But this implies that the difference lies in the kernel of $(-\Delta)^m$, where m is the smallest integer larger or equal than $\frac{n}{4}$. Thus the difference is smooth, leading to the desired decomposition. $\quad\square$

7 Appendix

In this section, we provide for the reader's convenience a proof of the two inequalities (3) and (4), since the authors have not found a precise reference in the literature.

1. Assume first that $f(z) = \sum_{n \geq 0} a_n z^n$ is an analytic function such that $\lim_{r \to 1^-} \|f(re^{i\theta})\|_{L^1(S^1)} < +\infty$. Let $h \in L^1(S^1)$ be such that $\lim_{r \to 1^-} \|f(re^{i\theta}) - h\|_{L^1(S^1)} = 0$. We set $g(z) = \sum_{n \geq 0} \frac{a_n}{n+1} z^{n+1}$. We observe that $g'(z) = f(z)$. From our hypothesis, we have $\lim_{r \to 1^-} \|g'(re^{i\theta})\|_{L^1(S^1)} < +\infty$. Observe that this implies that $\lim_{r \to 1^-} (\|\partial_\theta g(re^{i\theta})\|_{L^1(S^1)} + \|\partial_r g(re^{i\theta})\|_{L^1(S^1)}) < +\infty$. Define $g_r(z) = g(rz)$ for $0 < r < 1$. Since g is harmonic in $\mathbb{D}$, we have

$$
\begin{aligned}
0 &= \int_{\mathbb{D}} (\Delta g_r \bar{g}_r + g_r \Delta \bar{g}_r) dx = \int_{\partial \mathbb{D}} (\partial_r g_r \cdot \bar{g}_r + \partial_r \bar{g}_r g_r) d\sigma - 2 \int_{\mathbb{D}} |\nabla g_r|^2 dx \\
&= \int_{\partial \mathbb{D}} (\partial_r g \cdot \bar{g} + \partial_r \bar{g} g) d\sigma - \int_{\mathbb{D}} |g_r'|^2 dx. \tag{127}
\end{aligned}
$$

We first have (observe that $\fint_{S^1} g_r = 0$)

$$\|g_r\|_{L^\infty(S^1)} \lesssim \|\partial_\theta g_r\|_{L^1(S^1)} \tag{128}$$

and from (127) it follows that

$$\|f_r\|_{L^2(\mathbb{D})} \simeq \|g_r'\|_{L^2(\mathbb{D})} \lesssim \|g_r\|_{L^\infty(S^1)} \|\partial_r g\|_{L^1(S^1)} \lesssim \|g_r'\|_{L^1(S^1)}^2. \tag{129}$$

We let $r \to 1$ in (129) and get

$$\|f\|_{L^2(\mathbb{D})} \lesssim \|h\|_{L^1(S^1)}. \tag{130}$$

2. Assume now that $f(z) = \sum_{n\geq 0} a_n z^n$ is an analytic function such that:

$$\lim_{r\to 1^-} \|f(re^{i\theta})\|_{H^{-1/2}(S^1)} < +\infty.$$

Claim. Assume $a_0 = 0$ in the power series above. Then the series $\sum_{n\geq 1} \frac{|a_n|^2}{n} < +\infty$ and

$$\sum_{n\geq 1} \frac{|a_n|^2}{n} = \lim_{r\to 1^-} \|f(re^{i\theta})\|^2_{\dot{H}^{-1/2}(S^1)}.$$

Proof of the claim. We set $A = \lim_{r\to 1^-} \|f(re^{i\theta})\|^2_{H^{-1/2}(S^1)}$. We observe that:

$$\|f(re^{i\theta})\|^2_{\dot{H}^{-1/2}(S^1)} = \sum_{n\geq 1} \frac{|a_n|^2 r^{2n}}{n}.$$

For every $N > 1$, we have

$$
\begin{aligned}
A \geq \lim_{r\to 1^-} \sum_{n=1}^{N} \frac{|a_n|^2 r^{2n}}{n} &= \sum_{n=1}^{N} \lim_{r\to 1^-} \frac{|a_n|^2 r^{2n}}{n} \\
&= \sum_{n=1}^{N} \frac{|a_n|^2}{n}.
\end{aligned}
\tag{131}
$$

By letting $N \to +\infty$, we get $\sum_{n=1}^{\infty} \frac{|a_n|^2}{n} < +\infty$ and by Abel's theorem on power series, we deduce that the norms converge

$$\lim_{r\to 1^-} \sum_{n>0} \frac{|a_n|^2 r^{2n}}{n} = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n}.$$

Therefore, $f(e^{i\theta}) \in \dot{H}^{-1/2}(S^1)$ and $\lim_{r\to 1^-} \|f(re^{i\theta}) - f(e^{i\theta})\|_{\dot{H}^{-1/2}(S^1)} = 0$, by observing that the convergence holds weakly and the norms converge, which is an equivalent characterisation for convergence with respect to the norm in Hilbert spaces. This proves the claim.

Consider the function $g_r(z) = \sum_{n\geq 0} \frac{a_n}{n+1}(rz)^{n+1}$. In this case we have $g_r \in \dot{H}^{1/2}(S^1)$. We have $r \cdot f_r(z) = g'_r(z)$. Since g is harmonic in $\mathbb{D}$ we have

$$\|f_r\|_{L^2(\mathbb{D})} \simeq \|\nabla g_r\|_{L^2(\mathbb{D})} \lesssim \|g_r\|_{\dot{H}^{1/2}} \equiv \|f_r\|_{H^{-1/2}} \tag{132}$$

We let $r \to 1$ in (132) and get

$$\|f\|_{L^2(\mathbb{D})} \lesssim \|f\|_{H^{-1/2}(S^1)}. \tag{133}$$

Both inequalities (3) and (4) have been proved.

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