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ASYMPTOTIC ANALYSIS FOR THE GINZBURG-LANDAU EQUATIONS

Mini-Course given by

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January 1997

Section 1: Introduction

We consider a domain $\Omega \subset \mathbb{R}^3$, filled by a superconducting element. Superconductivity corresponds to the formation of electron pairs, called Cooper pairs, and is represented by a complex function u on Ω, such that $|u|^2$ equals the density of Cooper pairs. By applying an external magnetic field H_{ext}, the superconductor reacts with producing a magnetic field h. One may choose a vectorfield A on Ω such that curl$A = h$ and the superconductor's state is fully characterized by the couple (u, A), or more precisely by an equivalence class for the relation

$$(u, A) \sim (ue^{i\varphi}, A + \nabla\varphi)\,, \tag{1}$$

where φ is a real valued function on Ω. Under the influence of the external field, the superconductor can roughly get into two different states, depending on the intensity of H_{ext}:

- $|u|^2 \equiv 1$ and $h \equiv 0$: superconducting state.
- $|u|^2 \equiv 0$ and $h \equiv H_{ext}$: normal state.

For superconducting elements of type II (see below), on may observe two critical values $H_{c_1} < H_{c_2}$ such that

- for $H_{ext} < H_{c_1}$ the superconductor is in superconducting state ,
- for $H_{ext} > H_{c_2}$ the superconductor is in normal state.

In between H_{c_1} and H_{c_2}, there is a transition phase or mixed state, which is characterized by the formation of filament like non-superconduction zones, called vortices, in the middle of superconducting areas.

Let ξ denote the characteristic width of each filament and λ the characteristic distance between two filaments. The parameter $\kappa := \frac{\lambda}{\xi}$ then characterizes the nature of the superconductor and in particular superconductors of type II, considered above, are those with κ larger than some given critical value. Here we are interested in large κ and will write $\kappa = \frac{1}{\varepsilon}$. Empirical observation yields:

$$H_{c_1} \approx \frac{\varepsilon}{2} \log \frac{1}{\varepsilon}$$

Moreover the induced field h verifies an equation, called London equation, of the form

$$-\lambda^2 \triangle h + h = \varphi_0 \sum \delta_{vortex} ,$$

where δ_{vortex} is a Dirac mass in each vortex. Finally if Ω is a cylinder, with an external field H_{ext} directed parallelly to the axis, the filaments are also parallel to the axis and looking at a slice of Ω, we get a regular pattern of vortices, called Abrikosov lattice.
For further details on the physical aspects of these problems, see [16].

The action functional put forth by Ginzburg and Landau to describe the free energy of this system is

$$G_\varepsilon(u, A) := \int_\Omega \left(|\varepsilon \nabla u - iAu|^2 + (1 - |u|^2)^2 + |\mathrm{curl} A|^2 \right) dx \tag{2}$$

and for the total energy

$$G_\varepsilon^{H_{ext}}(u, A) := G_\varepsilon(u, A) - 2 \int \mathrm{curl} A \cdot H_{ext} \, dx \, .$$

These expressions actually are invariant under the equivalence relation given above in (1); this invariance is known as gauge invariance.

The related mathematical problem, which we are going to consider is the following: Let $(u_\varepsilon, A_\varepsilon)$ be a minimizing couple for $G_\varepsilon^{H_{ext}}$ (existence of such a couple in an adequate function space is a standard problem, see [7]), for an exterior field H_{ext} close to $H_{c_1} \approx \frac{1}{2} \varepsilon \log \frac{1}{\varepsilon}$. We then let ε tend to 0, in order to amplify the discontinuity produced by a vortex and the goal is to study the asymptotic behaviour of the couple $(u_\varepsilon, A_\varepsilon)$, with the hope of detecting some loss of compactness in the given function space, as ε tends to 0 , describing the creation of vortices.
This problem, in such generality, as it is explained in [6], is mathematically quite difficult, and not entirely solved so far. In their initial work, F.Bethuel, H.Brezis, and F.Hélein (cf.[4]) study a simplified model, with neither magnetic field, nor gauge invariance, where the effect of the exterior magnetic field is simulated by the prescription of some vorticity at the boundary (non-vanishing degree for boundary data

mapping into S^1). Precisely, as ε tends to zero, they study the behavior of the critical points of the energy

$$E_\varepsilon(u;\Omega) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \right) dx$$

among the maps

$$u \in W^{1,2}_g(\Omega,\mathbb{C}) := \{u \in W^{1,2}(\Omega,\mathbb{C}) \ \mid \ u = g \text{ on } \partial\Omega\}$$

for a smooth boundary data $g : \partial\Omega \to S^1$ having a degree $\deg g = d > 0$. They give a complete description of this behavior (see Theorem 1 and 2 in part II) which correspond to an answer to a first question about the relation between vorticity and the actual formation of vortices:

$$\text{vorticity} \Longrightarrow \text{formation of vortices}$$

This first problem and its solution is offered in the second section of this mini-course. In the third section, we consider a similar problem but including the induced magnetic field $h = \ \text{curl}\ A$, Precisely we minimize the functional

$$G_\varepsilon(u,A) := \frac{1}{2}\int_\Omega \left(\left|dA\right|^2 + \left|\nabla u - iAu\right|^2 + \frac{1}{2\varepsilon^2}\left(1-|u|^2\right)^2 \right) dx$$

among couples (u, A) in the space

$$V := \Big\{ (u,A) \ \in W^{1,2}(\Omega,\mathbb{C}) \times W^{1,2}(\Omega,\mathbb{R}^2) \,\Big|\, |u| = 1 \text{ on } \partial\Omega,$$

$$\deg(u,\partial\Omega) = d, < iu, \tau\cdot\nabla_A u >= J \ \text{ on } \partial\Omega \Big\}$$

for a given degree $d > 0$ and a given regular function J on $\partial\Omega$ which corresponds to the tangential current. We give an answer to the corresponding question (see theorem 18):

$$\text{vorticity} \Longrightarrow \begin{cases} \text{formation of vortices and induction of a magnetic field} \\ + \\ \text{the induced magnetic field verifies the London equation}\,. \end{cases}$$

Moreover we sketch the solution of the problem

$$\text{exterior field } \geqslant H_{c_1} \Longrightarrow \text{vorticity}\,,$$

under some additional assumptions (see the remark following theorem 18). Let us mention a recent work of S. Serfaty (see [17]) where she proves the stability of solutions of the complete Ginzburg-Landau problem with external magnetic field, which have more and more vortices as this external magnetic field increases starting from the critical value H_{c_1}.

In this paper we restrict ourselves to the static problem in dimension 2. Let us just give some references for the corresponding static problem in higher dimension [15], [12] and the corresponding heat-flow problem [9], [10], [8].

Section 2: Asymptotic behaviour of critical points of the Ginzburg-Landau functional without magnetic field

Consider the Ginzburg Landau functional

$$E_\varepsilon(u) := E_\varepsilon(u;\Omega) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right) dx$$

defined for maps

$$u \in W^{1,2}_g(\Omega,\mathbb{C}) := \{u \in W^{1,2}(\Omega,\mathbb{C}) \ | \ u = g \,\text{on}\, \partial\Omega\}$$

with smooth boundary data $g : \partial\Omega \to S^1$ and fixed degree $\deg g = d > 0$, for a smooth, bounded, simply connected, starshaped domain Ω in $\mathbb{R}^2$. In this context starshaped means that there is some $\alpha > 0$ such that

$$x \cdot \nu \geqslant \alpha > 0, \ \forall x \in \partial\Omega,$$

ν being the outward unit normal to $\partial\Omega$.
The goal of this lecture is to analyse the asymptotic behaviour as $\varepsilon \to 0$ of minimizers as well as of critical points of E_ε in $W^{1,2}_g(\Omega,\mathbb{C})$. Throughout the general reference is [4].

Theorem 1 *Let $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}}$ be a sequence of minimizers of E_{ε_n} in $W^{1,2}_g(\Omega,\mathbb{C})$, with $\varepsilon_n \to 0$, as $n \to \infty$.*
Then there is a subsequence, still denoted by $\{u_{\varepsilon_n}\}$, there are exactly $d = deg\, g$ points $a_1,\dots,a_d$ in Ω such that

$$u_{\varepsilon_n} \to u_\star \quad in \ C^k_{loc}\big(\Omega \setminus \{a_1,\dots,a_d\}\big)\,, \quad \forall k \in \mathbb{N}\,, \tag{3}$$

$$in \ C^{1,\alpha}_{loc}\big(\overline{\Omega} \setminus \{a_1,\dots,a_d\}\big)\,, \quad for\, 0 < \alpha < 1\,. \tag{4}$$

where $u_\star : \Omega \setminus \{a_1,\dots,a_d\} \to S^1$ is the following harmonic map

$$u_\star(z) = \prod_{j=1}^{d} \frac{z-a_j}{|z-a_j|}\exp(i\varphi)$$

with $\Delta\varphi = 0$ on Ω, $u_\star = g$ on $\partial\Omega$.
Moreover, there is a function ("the renormalized energy"), depending only on Ω and g

$$W : \Omega^d \to \mathbb{R}\,,$$

such that $(a_i)_{i=1}^d$ minimizes W in Ω^d.

The previous result has been extended to the case where Ω is only simply connected and not necessary starshaped by M. Struwe in [16]. The result on the asymptotic behaviour of critical points, which are not necessarily minimizers, is similar but the limiting map need not have degree 1 in each singularity (vortex) and there is no precise information on the number N of singularities, for which one merely obtains an upper bound.

Theorem 2 *Let $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}}$ be a sequence of critical points of E_{ε_n} in $W^{1,2}_g(\Omega,\mathbb{C})$, with $\varepsilon_n \to 0$, as $n\to\infty$.*
Then there is a subsequence, still denoted by $\{u_{\varepsilon_n}\}$, there are N points $a_1,\dots,a_N$ in Ω, N integers $d_1,...,d_N$

$$u_{\varepsilon_n} \to u_\star \quad in \;\; C^k_{loc}\big(\Omega \setminus \{a_1,\dots,a_N\}\big)\,, \quad \forall k\in\mathbb{N}\,, \tag{5}$$

$$in \;\; C^{1,\alpha}_{loc}\big(\overline{\Omega} \setminus \{a_1,\dots,a_N\}\big)\,, \quad for\; 0<\alpha<1\,. \tag{6}$$

where $u_\star : \Omega\setminus\{a_1,\dots,a_N\} \to S^1$, is the following harmonic map

$$u_\star(z) = \prod_{j=1}^{d}\left(\frac{z-a_j}{|z-a_j|}\right)^{d_j}\exp(i\varphi)$$

with $\Delta\varphi = 0$ on Ω, $u_\star = g$ on $\partial\Omega$.
Moreover there is a function ("the renormalized energy"), depending only on Ω N and g,

$$W : \Omega^N \times \mathbb{Z}^N \to \mathbb{R}\,,$$

such that $(a_i)_{i=1}^N$ is a critical point of W in Ω^N.

The complete definition of W is given in the step 11 of the proof of theorem 1 and 2 bellow. Let us make a digression to the questions which arise from the results stated just above. Observe that once you know the d_j and the a_j you know the limit $u_\star$. So the limiting problem $\varepsilon = 0$ is included in the finite dimensional one which consists of finding the critical points of W. Now it would be interresting to understand what happens just before $\varepsilon = 0$ (i.e. ε small but different from zero) : How many critical points of E_ε do exist ? In [1] and [2] critical points of E_ε which are not necessary minimizers are found by the mean of topological methods. But in a more systematic approach we can ask the question of understanding the number of familly of critical points of E_ε wich converges to a $u_\star$ constructed from a <u>given</u> (a_j,d_j), critical point of W. The first result in this approach was given by F.H. Lin and T.C. Lin in [11] : they prove that, in the case where all the d_j are equal to $+1$, if (a_j) is a non degenerate critical point of W, there exists at least a familly of critical points u_ε of E_ε which converges to the corresponding $u_\star$. In [13] this result is extended to the case where $d_j = \pm 1$. Finally the problem of describing exactly the number of branch of solution converging to a given $u_\star$ has only be solved, untill now, in the particular case where we restrict ourselves to minimizers (see [14]). The general situation is still far from being understood especially when the multiplicity of the limiting vortices are different from ± 1.
In the sequel, we will sketch the main ideas of the proof of theorems 1 and 2, which consist of eleven steps:
Step 1 : The Euler-Lagrange Equation.
The Euler-Lagrange equations for critical points of $E_\varepsilon(u)$ are:

$$\begin{cases} -\triangle u \;=\; \dfrac{1}{\varepsilon^2}\, u\,(1-|\,u\,|^2) & \text{in } \Omega\,, \\ \qquad\qquad u = g & \text{on } \partial\Omega\,. \end{cases} \tag{7}$$

Step 2 : $L^\infty-$ estimate for u:

Lemma 3 *Let u be a solution of (7), then* $\|u\|_{L^\infty(\Omega)} \leqslant 1$.

Proof
Note that by elliptic regularity and a standard boot-strap argument, weak $W^{1,2}$ solutions are smooth. We thus may take the scalar product with u in (7) and deduce:

$$-\frac{1}{2}\triangle|u|^2 = -|\nabla u|^2 - < \triangle u, u > = -|\nabla u|^2 + \frac{1}{\varepsilon^2}|u|^2(1-|u|^2),$$

and

$$\frac{1}{2}\triangle(|u|^2-1) - (\frac{|u|^2}{\varepsilon^2})(|u|^2-1) \geqslant 0.$$

Since $|u| = |g| = 1$ on $\partial\Omega$, the claim follows by applying the maximum principle to $|u|^2-1$. □

Step 3 : L^∞ estimate for the gradient.

Lemma 4 *Solutions of (7) satisfy* $\|\nabla u\|_\infty \leqslant \frac{C}{\varepsilon}$.

Proof
In [3] the following interpolation inequality is proved:

Lemma 5 *Assume $u \in L^\infty(\Omega)$ satisfies $\triangle u \in L^\infty(\Omega)$ for some smooth, open $\Omega \subset\subset \mathbb{R}^N$, then:*

i) $$|\nabla u|^2(x) \leqslant C\left(\|\triangle u\|_{L^\infty}\|u\|_{L^\infty} + \frac{\|u\|^2_{L^\infty}}{dist^2(x,\partial\Omega)}\right), \quad \forall x \in \Omega.$$

ii) *If in addition $u = 0$ on $\partial\Omega$, then:*

$$\|\nabla u\|^2_{L^\infty} \leqslant C\left(\|\triangle u\|_{L^\infty}\|u\|_{L^\infty}\right).$$

Now the result follows by equation (7) and lemma 3, with a constant depending on g. □

Step 4 : Pohozaev identity

Lemma 6 *Every critical point u of E_ε in $W^{1,2}_g(\Omega,\mathbb{C})$ satisfies*

$$\frac{1}{\varepsilon^2}\int_\Omega (1-|u|^2)^2\,dx = -\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\frac{\partial u}{\partial\tau}\tau\cdot x\,do + \frac{1}{2}\int_{\partial\Omega} x\cdot\nu\,(|\frac{\partial u}{\partial\tau}|^2 - |\frac{\partial u}{\partial\nu}|^2)\,do.$$

Here we only need the assumption that Ω is smooth and bounded.

Proof
In the sequel, we will drop the index ε. The lemma follows from a Pohozaev identity, which is obtained by multiplying equation (7) by $\sum_{i=1}^{2} x_i \frac{\partial u}{\partial x_i}$ and integrating over Ω, noting that

$$\mathrm{div}\big(\nabla u^i\, x_j \partial_j u^i\big) \;=\; \triangle u^i\, x\cdot\nabla u^i + |\nabla u|^2 + \frac{1}{2}\, x\cdot\nabla\big(|\nabla u|^2\big)\,,$$

$$u\,(\,1-|u|^2\,) \;=\; -\frac{1}{2}\, x\cdot\nabla(\,1-|u|^2\,)^2\,,$$

and

$$\frac{\partial u}{\partial\nu}\,(x\nabla u) \;=\; (\frac{\partial u}{\partial\nu})^2\,\nu\cdot x + \frac{\partial u}{\partial\nu}\frac{\partial u}{\partial\tau}\,\tau\cdot x\,,$$

where ν and τ denote the outward unit normal and a unit tangent vector to $\partial\Omega$, we obtain:

$$\frac{1}{\varepsilon^2}\int_\Omega (\,1-|u|^2\,)^2\,dx \;-\; \frac{1}{2\varepsilon^2}\int_{\partial\Omega} x\cdot\nu\,\underbrace{(\,1-|u|^2\,)^2}_{=0\text{ on }\partial\Omega}\,do$$

$$=\;-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\cdot x\nabla u\,do + \frac{1}{2}\int_{\partial\Omega} x\cdot\nu\,|\nabla u|^2\,do$$

so this yields the result. $\square$

Corollary 7 *There is a constant C independent of ε, such that for every critical point u_ε of E_ε in $W_g^{1,2}(\Omega,\mathbb{C})$, we have:*

$$\int_\Omega \frac{1}{\varepsilon^2}\,(\,1-|u|^2\,)^2\,dx \;\leqslant\; C\,.$$

Here we essentially need Ω to be starshaped.

Proof
By Young's inequality and the previous lemma, we obtain

$$\Big|\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\frac{\partial u}{\partial\tau}\,\tau\cdot x\;do\Big| \;\leqslant\; \delta^{-1}\int_{\partial\Omega}|x|^2|\frac{\partial u}{\partial\tau}|^2\;do + \delta\int_{\partial\Omega}|\frac{\partial u}{\partial\nu}|^2\,do\,,$$

and since Ω is starshaped, we may choose δ such that $0<\delta<\alpha\leqslant x\cdot\nu\,,\,\forall x\in\Omega$ and thus we obtain

$$\frac{1}{\varepsilon^2}\int_\Omega(\,1-|u|^2\,)^2\,dx \;\leqslant\; C\int_{\partial\Omega}|\frac{\partial g}{\partial\tau}|^2\,do\,.$$

$\square$

Step 5: A remark.

From equation (7) we see that $\triangle u\parallel u$, which is equivalent to

$$\sum_{i=1}^{2}\partial_i\,(u\wedge\partial_i u)\;=:\;\mathrm{div}(u\wedge\nabla u)=0.$$

In particular, if $|u| \geqslant \frac{1}{2}$ locally on some simply connected subdomain $\tilde{\Omega} \subset \Omega$, we may write $u = \rho e^{i\varphi}$, where φ satisfies the following elliptic equation:

$$\operatorname{div}(\,\rho^2 \nabla\varphi\,) \;=\; 0,$$

with $\frac{1}{2} \leqslant \rho \leqslant 1$.
Actually for $u = \rho e^{i\varphi}$ equation (7) transforms into the system

$$\operatorname{div}(\rho^2 \nabla\varphi) = 0 \tag{8}$$

$$-\triangle\rho + \rho|\nabla\varphi|^2 \;=\; \frac{1}{\varepsilon^2}\rho(1-\rho^2)\,,$$

(cf [4] p. 109) and since $u \wedge \nabla u \;=\; \rho^2\nabla\varphi$ this may be written as

$$\operatorname{div}(u \wedge \nabla u) = 0 \tag{9}$$

$$-\triangle|u| + \frac{1}{|u|^3}|u \wedge \nabla u|^2 \;=\; \frac{1}{\varepsilon^2}|u|(1-|u|^2)\,.$$

Note that the first equation is independent of ε and will be preserved under weak $W^{1,p}$-limits $\lim\limits_{\varepsilon\to 0} u_\varepsilon = u_*$ for solutions u_ε of (7). In view of the elliptic equations (8), the limit u_* can be expected to be regular in the part of the domain where $|u| \geqslant \frac{1}{2}$. The problem is now to locate the part of the domain where (8) is degenerate, i.e. where $|u_\varepsilon| < \frac{1}{2}$.

Step 6 : Locating the " bad set" , where $|u| < \frac{1}{2}$.

Lemma 8 *Let u_ε denote a minimizer of E_ε in $W^{1,2}_g(\Omega,\mathbb{C})$.*
There are constants $N \in \mathbb{N}$, $\lambda > 0$ depending only on Ω and g, such that for each $\varepsilon > 0$ there are $x_1^\varepsilon, \ldots, x_{N_\varepsilon}^\varepsilon \in \Omega$ and $N_\varepsilon \leqslant N$ with

$$\mathfrak{B}_\varepsilon := \{\, x \in \Omega \;\mid\; |u_\varepsilon| < \frac{1}{2}\,\} \;\subset\; \bigcup_{i=1}^{N_\varepsilon} B_{\lambda\varepsilon}(x_j^\varepsilon)\,,$$

where the balls $B_{\lambda\varepsilon}(x_j^\varepsilon)$ are mutually disjoint.

Proof
In the sequel we omit the index ε for u. Suppose $|u|(y) < \frac{1}{2}$ for some $y \in \Omega$. Since $\|\nabla u\|_{L^\infty} \leqslant \frac{C}{\varepsilon}$, there is some $\lambda_0 > 0$ independent of ε such that

$$|u(x)| \leqslant \frac{3}{4} \quad \text{ on } B_{\lambda_0\,\varepsilon}(y)\,,$$

thus

$$\frac{1}{\varepsilon^2}\,(1-|u|^2)^2 \;\geqslant\; C_1 \quad \text{ on } B_{\lambda_0\,\varepsilon}(y)\,,$$

and

$$\frac{1}{\varepsilon^2}\int_{B_{\lambda_0\,\varepsilon}(y)}(1-|u|^2)^2\,dx \geqslant C_0 > 0\,,$$

C_0 being independent of ε.
Vitali's covering theorem applied to the cover $\{\,B_{\lambda_0\,\varepsilon}(x)\ \mid\ x\in\mathfrak{B}_\varepsilon\,\}$ gives us a countable subset $\{x_j^\varepsilon\}_{j\in J_\varepsilon}\subset\mathfrak{B}_\varepsilon$ such that

$$B_{\lambda_0\varepsilon}(x_j^\varepsilon)\cap B_{\lambda_0\varepsilon}(x_i^\varepsilon)=\emptyset\ \text{ for }\ i\neq j\,,$$

$$\mathfrak{B}_\varepsilon\subset\bigcup_{i\in J_\varepsilon}B_{5\lambda_0\varepsilon}(x_j^\varepsilon)\,.$$

From the global bound in Step 4, we deduce:

$$C_0\,|J_\varepsilon|\ \leqslant\ \sum_{j\in J_\varepsilon}\int_{B_{\lambda_0\,\varepsilon}(x_j^\varepsilon)}\frac{1}{\varepsilon^2}(1-|u|^2)^2\,dx\ \leqslant\ \frac{1}{\varepsilon^2}\int_\Omega(1-|u|^2)^2\,dx\ \leqslant\ C\,,$$

i.e. $|J_\varepsilon|\ \leqslant\ N$ independently of ε.
Set $\tilde\lambda:=5\lambda_0$.
Furthermore one may chose a subset $J'\subset J$ and a constant $\lambda\geqslant\tilde\lambda$, such that

$$|x_i-x_j|\geqslant 8\lambda\varepsilon\,,\ \forall i,j\in J'\,,i\neq j\ \text{ and }\ \bigcup_{i\in J}B_{\tilde\lambda\varepsilon}(x_j^\varepsilon)\subset\bigcup_{i\in J'}B_{\lambda\varepsilon}(x_j^\varepsilon)\,.$$

Indeed we may proceed by induction on $\mathrm{card}(J)=|J|$: If there are x_i,x_j with $|x_i-x_j|\leqslant 8\tilde\lambda\varepsilon$,
set $J':=J\smallsetminus\{j\}$ and $\lambda:=9\tilde\lambda$.
In particular we obtain a covering of $\mathfrak{B}_\varepsilon$ by disjoint balls. □

Step 7: Convergence of the "bad set" to limiting singularities.
From step 6 we obtained a covering

$$\mathfrak{B}_\varepsilon\subset\bigcup_{i=1}^{N_\varepsilon}B_{\lambda\varepsilon}(x_j^\varepsilon)$$

with mutually disjoint balls.
Since $0\leqslant N_\varepsilon\leqslant N$, we may select a converging subsequence N_{ε_n}, which must be stationary for n sufficiently large and we may assume $N_{\varepsilon_n}\equiv N$. Extracting a converging subsequence from

$$(x_1^{\varepsilon_n},\ldots,x_N^{\varepsilon_n})_{n\in\mathbb{N}}\subset\bar\Omega^N\,,$$

we get a covering of $\mathfrak{B}_{\varepsilon_n}$ by N disjoint balls, with converging centers

$$x_j^{\varepsilon_n}\to a_j\ \ (n\to\infty)\,,\ \ 1\leqslant j\leqslant N\,.$$

It might happen that $a_j\in\partial\Omega$ for some j. In [4] they actually prove that the a_j lie in Ω.

Step 8: Convergence to a limiting map, away from the singularities.

There are two different approaches, depending on whether we consider minimizers or merely critical points of E.

i) For minimizers one may derive lower and upper bounds for the energy from which we deduce a uniform upperbound for the energy outside the singular set.

ii) For critical points in general we prove uniform $W^{1,p}$ estimates, for $p < 2$.

The second method is more general, so we will merely give a sketch of the first and develop the second one.

<u>Sketch of the first method:</u>

• On the one hand, we have:

Lemma 9 *There are constants ε_0, $C > 0$, such that for any minimizer u_ε of E_ε in $W^{1,2}_g(\Omega, \mathbb{C})$, with $\varepsilon \leqslant \varepsilon_0$ we have*

$$E_\varepsilon(u_\varepsilon) \leqslant 2\pi d \log(\frac{1}{\varepsilon}) + C\,.$$

This can be seen by using test functions of the following form: Fix $d = \deg g$ distinct points $a_1, \dots, a_d$ in Ω such that $B_{2\varepsilon}(a_i) \cap B_{2\varepsilon}(a_j) = \emptyset$ for $i \neq j$ and set

$$\bar{u}_\varepsilon(z) := \prod_{k=1}^{d} \frac{z - a_j}{|z - a_j|}\, e^{i\varphi(z)} \quad \text{on} \quad \Omega \setminus \bigcup_{k=1}^{d} B_{2\varepsilon}(a_k)\,, \tag{10}$$

where φ is a harmonic function on Ω chosen in such a way that $\bar{u}_\varepsilon = g$ on $\partial\Omega$. Further set

$$\bar{u}_\varepsilon(z) := \varrho\left(\frac{z - a_j}{2\,\varepsilon}\right) \frac{z - a_j}{|z - a_j|}\, e^{iH_j(z)} \quad \text{on} \quad B_{2\varepsilon}(a_j)\,,\ \ (1 \leqslant j \leqslant d)\,, \tag{11}$$

where $\varrho \in C^\infty([0,1])$ $\varrho(1) = 1$ and $\varrho \equiv 0$ on $[0, \frac{1}{2}]$ and H_j is a harmonic function determined by

$$e^{iH_j(z)} = \frac{|z - a_j|}{z - a_j}\, \bar{u}_\varepsilon(z) \quad \text{on} \quad \partial B_{2\varepsilon}(a_j)\,,\ \ (1 \leqslant j \leqslant d)\,.$$

Noting that

$$\int_{B_R(0)\setminus B_r(0)} \left|\nabla\left(\frac{x}{|x|}\right)\right|^2 dx = 2\pi \log\left(\frac{R}{r}\right), \tag{12}$$

we see that each vortex a_j creates an energy $2\pi\log(\frac{1}{\varepsilon})$ plus constants independent of ε. Indeed, according to (12), (11) produces an energy $\backsim \log(\frac{2\varepsilon}{\varepsilon}) + C$, but the term $2\pi\log(2\varepsilon)$ cancels with the corresponding term of the energy of (10) (cf [4] theorem III.1, p.44).

• On the other hand, there is an optimal lower bound for the energy around the vortices (this is the more difficult part to prove):

Lemma 10 *For $\delta > 0$ there is a constant C_δ such that for (sub-)sequence of minimizers as in step 7, satisfies:*

$$\int_{\bigcup_{k=1}^{N} B_\delta(x_j^{\varepsilon_n})} \left(|\nabla u|^2 + \frac{1}{2\varepsilon_n^2}\,(\,1-|u|^2\,)^2 \right) dx \;\geqslant\; 2\pi d\,\log(\frac{1}{\varepsilon_n}) - C_\delta$$

(cf [4] theorems V.2 and V.3 . They also show that there are exactly d vortices, which is actually a consequence of lemma 9 and 10 .)

• Subtracting both estimates yields a bound on the energy away from the singularities, uniformly in ε, and ensures weak $W^{1,2}_{loc}(\Omega\setminus\{a_1,\dots,a_d\})$ convergence to a limiting map.

<u>The second method, $W^{1,p}$ estimate method for critical points of E_ε:</u>

Lemma 11 *For any critical point u_ε of E_ε and any $p<2$, $\|u_\varepsilon\|_{W^{1,p}(\Omega)}$ is uniformly bounded with respect to ε.*

Proof
Let $\tilde\Omega := \tilde\Omega_\varepsilon := \Omega\setminus\bigcup_{j=1}^{N_\varepsilon}\overline{B_{\lambda\varepsilon}(x_j^\varepsilon)}$ for a covering $\{B_{\lambda\varepsilon}(x_j^\varepsilon)\}_{j=1}^{N_\varepsilon}$ of $\mathfrak{B}_\varepsilon$ as in step 6. On $\tilde\Omega$ we have $|u|\geqslant\frac{1}{2}$. Now we are tempted to write $u=|u|e^{i\varphi}$ and use the elliptic equation $\operatorname{div}(|u|^2\nabla\varphi)=0$ (cf. step 5) which yields appropriate estimates. But u does not admit such an expression globally, since $\tilde\Omega$ is not simply connected. Note that locally for $u=|u|e^{i\varphi}$:

$$\sum_{i=1}^{2}(u\wedge\partial_i u)\,dx_i =: u\wedge du = |u|^2 d\varphi \quad\text{and}\quad d\big(\frac{1}{|u|^2}u\wedge du\big) \;=\; d\big(\frac{u}{|u|}\wedge\frac{du}{|u|}\big) \;=\; 0\,,$$

here $\wedge$ denotes the vector product in $\mathbb{R}^2$.
But $\frac{u}{|u|}\wedge\frac{du}{|u|}$ is not exact in $\tilde\Omega$ because

$$\int_{\partial\Omega}\frac{u}{|u|}\wedge\frac{du}{|u|} \;=\; \int_{\partial\Omega} g\wedge dg \;=\; 2\pi d\,.$$

The idea is to subtract the "topologically non trivial part" from $\frac{u}{|u|}\wedge\frac{du}{|u|}$ in order to obtain an exact form which will satisfy the elliptic equation. We thus need some kind of Hodge decomposition. We present this Hodge decomposition for arbitrary dimensions, which can be usefull for Ginzburg-Landau problems in higher dimensions, in particular in dimension 3.

Hodge decomposition

First observe that the topological part of $\frac{u}{|u|} \wedge \frac{du}{|u|}$ is "finite":

$$\int_{\partial B_{\lambda\varepsilon}(x_j^\varepsilon)} \frac{u}{|u|} \wedge \frac{du}{|u|} \;=\; 2\pi d_j^\varepsilon \tag{13}$$

and from step 3, we know $\|\nabla u\|_\infty \leqslant \frac{C}{\varepsilon}$, which implies that $|d_j^\varepsilon|$ must be uniformly bounded.
In order to obtain a decomposition

$$\frac{u}{|u|} \wedge \frac{du}{|u|} \;=\; \frac{1}{|u|^2}\, d^*\psi + dH\,, \tag{14}$$

for some 2-form ψ and some 0-form H, where the "topological part" $d^*\psi$ should possess as little energy as possible, we consider a solution ψ of the following minimization problem

$$\begin{aligned} \text{Min } \{ \int_{\tilde\Omega} \tfrac{1}{|u|^2} |d^*\psi - u \wedge du|^2 dx \;\text{ for }\; \psi \in W^{1,2}(\Lambda^2\tilde\Omega), \\ d(*\psi)\big|_{\partial\tilde\Omega} = d_T\big(*\psi\big|_{\partial\tilde\Omega}\big) = 0 \,\} \end{aligned} \tag{15}$$

for $\psi \in W^{1,2}(\Lambda^2\tilde\Omega)$, which projects to $\psi\big|_{\partial\tilde\Omega} \in W^{\frac{1}{2},2}(\Lambda^2\partial\tilde\Omega)$. The index T denotes restriction of the considered operation or form to the tangential components of $\partial\tilde\Omega$. Observe that unlike the usual Hodge decomposition, we do not separate a purely harmonic part, which contains the topological information. As we will see below, this information is contained in $d^*\psi$. Such a decomposition is always possible on any manifold "whose topology comes from the boundary"(cf (20) below), typically an open set in $\mathbb{R}^n$.
In the sequel we treat the 2-dimensional problem, the same approach might yield analogous results in higher dimensions. In particular in dimension 2, the boundary constraint implies $*\psi$ = constant on each connected component of $\partial\tilde\Omega$ and so we may choose

$$*\,\psi = 0 \;\text{ on } \partial\Omega\,, \tag{16}$$

$$*\psi = c_j^\varepsilon \;\text{ on } \partial B_{\lambda\varepsilon}(x_j^\varepsilon) \;\text{ for } 1 \leqslant j \leqslant N_\varepsilon\,.$$

Now we claim that for minimizers ψ of this problem

$$\frac{1}{|u|^2}\, d^*\psi - \frac{u}{|u|} \wedge \frac{du}{|u|}$$

is exact.
Indeed the (weak) variational equations for ψ read as follows:

$$\int_{\tilde\Omega} < d^*\xi\,,\, \frac{1}{|u|^2}\big(d^*\psi - u\wedge du\big) > \; dx \;=\; 0 \tag{17}$$

for any smooth 2-form ξ satisfying $d_T(*\xi) = 0$ on $\partial\tilde{\Omega}$.
Choosing a 2-form ξ with compact support in $\tilde{\Omega}$, we deduce:

$$d\Big(\frac{1}{|u|^2}\, d^*\psi - (\frac{u}{|u|} \wedge \frac{du}{|u|})\Big) \;=\; 0 \;\text{ in }\; \tilde{\Omega} \tag{18}$$

and since $\frac{u}{|u|} \wedge \frac{du}{|u|}$ is closed, also

$$d\Big(\frac{1}{|u|^2}\, d^*\psi\Big) \;=\; 0 \;\text{ in }\; \tilde{\Omega}. \tag{19}$$

Moreover combining (17) and (19) it follows:

$$\int_{\partial\tilde{\Omega}} *\xi \wedge \frac{1}{|u|^2}\big(d^*\psi - u \wedge du\big) \;=\; \int_{\partial\tilde{\Omega}} *\xi \wedge d\big(\frac{1}{|u|^2}\, d^*\psi\big)\, dx \underbrace{=0}_{by(19)},$$

for any 2-form ξ satisfying $d_T(*\xi) = 0$ on $\tilde{\partial\Omega}$. Now this implies that

$$\beta := \frac{1}{|u|^2}\big(d^*\psi - u \wedge du\big)$$

is exact. Indeed note that in dimension 2, $(*\xi)$ is a function. Choosing for $*\xi$ the characteristic function of a given connected component of $\partial\tilde{\Omega}$, we see that

$$\int_C \beta \;=\; 0\,, \quad \text{for every connected component C of } \partial\tilde{\Omega} \tag{20}$$

and since every closed path in $\tilde{\Omega}$ is homotopic to an integral sum of connected components of $\partial\tilde{\Omega}$, the Poincaré dual

$$\gamma \to \int_\gamma \beta \quad \big(\gamma \text{ a Lipschitz-representent of } H^1_{\text{sing}}(\tilde{\Omega})\big)$$

is identically zero and thus the DeRahm-class of β must be zero.
We thus obtain a decomposition as in (14) and in the sequel we will derive estimates for $d^*\psi$, dH and $|u|$.

Estimates on $d^*\psi$ for $p < \frac{n}{n-1}$ (for the case n=2).
We have

$$d\big(\frac{1}{|u|^2}\, d^*\psi\big) \;=\; 0 \;\text{ in }\tilde{\Omega}\,,$$

$$\int_{\partial B_{\lambda\varepsilon}(x_j^\varepsilon)} \frac{u}{|u|} \wedge \frac{du}{|u|} \;=\; \int_{\partial B_{\lambda\varepsilon}(x_j^\varepsilon)} \frac{1}{|u|^2}\, d^*\psi \;=\; 2\pi d_j^\varepsilon\,, \tag{21}$$

$$\int_{\partial\Omega} \frac{1}{|u|^2}\, d^*\psi \;=\; 2\pi d\,.$$

Combining (21) and (16), this yields in dimension 2 for the function $\varphi = *\psi$

$$\operatorname{div}\big(\frac{1}{|u|^2}\nabla\varphi\big) \;=\; 0\,, \qquad \text{in } \tilde{\Omega}\,,$$

$$\int_{\partial B_{\lambda\,\varepsilon}(x_j^\varepsilon)} \frac{1}{|u|^2}\,\frac{\partial\varphi}{\partial\nu}\,do \;=\; 2\pi\,d_j^\varepsilon\,, \tag{22}$$

$$\varphi = 0 \qquad \text{on } \partial\Omega\,,$$

$$\varphi = c_j^\varepsilon \qquad \text{on } \;\partial B_{\lambda\varepsilon}(x_j^\varepsilon)\;\text{ for } 1 \leqslant j \leqslant N_\varepsilon\,.$$

Note that this is an $\varepsilon-$ approximation of the following problem:

$$\triangle\tilde{\psi} \;=\; 2\pi\sum_{j=1}^{N} d_j^\varepsilon\,\delta_{x_j^\varepsilon} \;\text{ in }\; \Omega\,,$$

$$\tilde{\psi} \;=\; 0 \;\text{ in }\; \partial\Omega$$

and since the Green-function in $\mathbb{R}^2$ is $W^{1,p}$ for $p < 2$ we have:

$$\|\tilde{\psi}\|_{W^{1,p}} \;\leqslant\; C\,\sum_{j=1}^{N}|d_j^\varepsilon|\,. \tag{23}$$

We actually obtain the following similar estimate for a solution φ:

Lemma 12 *Let φ be a solution of (22), with $\frac{1}{2} \leqslant |u| \leqslant 1$ on $\tilde{\Omega} \;=\; \Omega\setminus\bigcup_{j=1}^{N}\, B_{\lambda\,\varepsilon}(x_j^\varepsilon)$, for some smooth u. Then*

$$\|\nabla\varphi\|_{L^p} \;\leqslant\; C$$

where the constant C only depends on λ, Ω and $(d_j^\varepsilon)_j$, and $1 < p < 2$.

Proof

For fixed h and some constant c consider the weak solution

$$\zeta \in V^p := \{\,\xi := (\xi_1, \xi_2) \quad | \;\; \xi_i \in H^{1,p}(\tilde{\Omega}, \mathbb{C})\,,\; \xi = c \;\text{ on }\; \partial B_{\lambda\varepsilon}(x_j^\varepsilon)$$

$$\text{for }\; 1 \leqslant j \leqslant N_\varepsilon \;\; , \xi = 0 \;\text{ on }\; \partial\Omega\,\}$$

of

$$\int_{\tilde{\Omega}} \big(\frac{1}{|u|^2}\,\nabla\zeta\big)\,\nabla\alpha\,dx \;=\; \int_{\tilde{\Omega}} h\,\nabla\alpha\,dx\,, \quad \forall\alpha \in V^q\,. \tag{24}$$

where q is the Hölder conjugate of p: $\frac{1}{p} + \frac{1}{q} = 1$. We have:

$$\|\zeta\|_{L^\infty} \;\leqslant\; C_q(u)\,\|h\|_{L^q}\,.$$

The proof can be found in [4], Lemma X.8, p.117 and uses a method due to G. Stampacchia, which consists in testing with $(\zeta - k)^+$. where k is any real constant. Now for fixed $h = (h_1, h_2)$, $h_i \in L^q(\Omega, \mathbb{C})$, test equation (22) with ζ solution of (24) in order to obtain

$$0 \;=\; -\int_{\tilde\Omega} \frac{1}{|u|^2}\,\nabla\varphi\cdot\nabla\zeta\,dx \;+\; \int_{\partial\tilde\Omega} \frac{1}{|u|^2}\,\nabla\varphi\cdot\nu\,\zeta\,do$$

i.e.

$$\int_{\tilde\Omega} \frac{1}{|u|^2}\,\nabla\varphi\cdot\nabla\zeta\,dx \;=\; 2\pi\sum_{j=1}^{N_\varepsilon} d_j^\varepsilon\;\zeta/_{\left(\partial B_{\lambda\,\varepsilon}(x_j^\varepsilon)\right)}$$

On the other hand inserting φ in equation (24) yields

$$\int_{\tilde\Omega} \frac{1}{|u|^2}\,\nabla\varphi\cdot\nabla\zeta\,dx \;=\; \int_{\tilde\Omega} \nabla\varphi\cdot h\,dx$$

so

$$\Big|\int_{\tilde\Omega} \nabla\varphi\cdot h\,dx\Big| \;\leqslant\; 2\pi\sum_{j=1}^{N_\varepsilon} |d_j^\varepsilon|\,\|\zeta\|_{L^\infty}\,, \tag{25}$$

$$\leqslant\; 2\pi\sum_{j=1}^{N_\varepsilon}\; C_q(u)\;|d_j^\varepsilon|\;\|h\|_{L^q}\,. \tag{26}$$

Now by duality:

$$\|\nabla\varphi\|_{L^p} \;\leqslant\; C_q(u)\;\sum_{j=1}^{N_\varepsilon}\,|d_j^\varepsilon|\,.$$

□

Thus as expected in (23) the solution $\psi = *\varphi$ of (22) satisfies

$$\|\nabla\psi\|_{L^p} \leqslant 2\pi\, C_p(u)\,\sum_j |d_j^\varepsilon|$$

and by (13) the right hand side is bounded independently of ε.

* * *

Let us digress before finishing the $W^{1,p}$ estimates for $|u|$.
Consider a similar problem in higher dimensions: Let Ω be a smooth, simply connected, bounded domain of $\mathbb{R}^n$ and let u be a map from Ω into $\mathbb{C}$, such that

$$\|\nabla u\|_\infty \;\leqslant\; \frac{C}{\varepsilon}\,,$$

$$|u| \;\geqslant\; \frac{1}{2}\;\text{ on }\;\tilde\Omega_\varepsilon := \Omega\smallsetminus\bigcup_{i=1}^{N_\varepsilon} B_\varepsilon(x_i)\,, \tag{27}$$

$$\text{where } N_\varepsilon \leqslant \frac{C}{\varepsilon^{n-2}}.$$

$d^*\psi$ given by (15) satisfies

$$d\left(\frac{1}{|u|^2}d^*\psi\right) = 0$$

$$\text{and for any } \Gamma, \text{ regular curve in } \tilde{\Omega}_\varepsilon : \tag{28}$$

$$\int_\Gamma \frac{1}{|u|^2}d^*\psi = \int_\Gamma \frac{1}{|u|^2}u \wedge du = 2\pi \ \deg{}_\Gamma(\frac{u}{|u|})$$

$$d_T(*\psi)\big|_{\partial\tilde{\Omega}} = 0$$

Thus ψ is an "$\varepsilon-$approximation" of a Green function $\tilde{\psi}$ associated to a Dirac mass along an $n-2$ dimensional manifold Γ_ε with multiplicity given by the degree of u around each part of this manifold. Because of (27) the total mass of this current is uniformly bounded independently of ε and we have L^p estimates for $d^*\tilde{\psi}$ independently of ε for $p < \frac{n}{n-1}$. The question is whether we also have L^p estimates for $d^*\psi$ itself,the solution of (28), under the hypothesis (27) independently of ε for $p < \frac{n}{n-1}$. This is still an open question.

* * *

Estimates for ∇H:
In the decomposition (14) above the 0-form H is given by

$$dH = \frac{1}{|u|^2}u \wedge du - \frac{1}{|u|^2}d^*\psi \tag{29}$$

Since $u||\triangle u$ we have $d^*(u \wedge du) = 0$ (cf step 5) and thus

$$d^*(|u|^2 dH) = 0$$

Furthermore

$$\int_{\partial B_{\lambda\varepsilon}(x_j)} |u|^2 \frac{\partial H}{\partial \nu}\, do = 0 \quad \text{ by (21)}$$

and also

$$\frac{\partial H}{\partial \nu} = \frac{1}{|u|^2}\, u \wedge \frac{\partial u}{\partial \nu} \quad \text{ on } \partial B_{\lambda\varepsilon}(x_j)$$

by (29) and the boundary constraint for ψ.

Since $u \in L^\infty(\tilde{\Omega}, \mathbb{C})$ and $|u| \geqslant \frac{1}{2}$, this implies

$$\int_{\tilde{\Omega}} |\nabla H|^2 dx \leqslant C$$

(for a proof see [4] lemma X.9, p.120).

Estimates for $\nabla|u|$:

Following [4] one establishes the following bound

$$\int_{\tilde\Omega} |\nabla \psi|^2 dx \leqslant C\left(\log(\frac{1}{\varepsilon})+1\right)$$

(cf [4] lemma X.10, p.122) .

Moreover, multiplying the variational equation for $|u|$ by $(|u|-1)$ (cf step 5 (9)) one computes

$$\int_{\tilde\Omega} \big|\nabla|u|\big|^2 dx \leqslant C_1 + C_2\int_{\tilde\Omega}\big(|\nabla\psi|^2+|\nabla H|^2\big)dx \leqslant C\big(\log(\frac{1}{\varepsilon})+1\big) \tag{30}$$

(for the exact computation see [4], lemma X.12, p.123).

Now setting $S := S_\varepsilon := \{x\in\tilde\Omega \mid 1-|u(x)|^2 \geqslant \varepsilon^\beta\}$ for some $\beta\in]0,1[$ and using the upper bound $\frac{1}{\varepsilon^2}\int_\Omega (1-|u|^2)^2\,dx \leqslant C$ from lemma 6, we obtain for $1<p<2$

$$\int_S \big|\nabla|u|\big|^p dx \leqslant (\int_{\tilde\Omega}\big|\nabla|u|\big|^2dx)^{p/2}\ |S|^{(1-p/2)}\ \leqslant\ \tilde C\,\big(\log(\frac{1}{\varepsilon})+1\big)^{p/2}\ \varepsilon^{(2-2\beta)(1-p/2)}$$

by (30), i.e.

$$\int_{S=\{x\in\tilde\Omega/|u(x)|^2\geqslant 1-\varepsilon^\beta\}} \big|\nabla|u|\big|^p dx \leqslant C\varepsilon^\alpha$$

for some $\alpha=\alpha(\beta), \beta\in]0,1[$, $1<p<2$ and ε sufficiently small.
Finally, multiplying equation (8) by $1-\bar\rho$, where $\bar\rho=\max\{|u|^2, 1-\varepsilon^\beta\}$ we deduce

$$\int_{\Omega\setminus S}\big|\nabla|u|\big|^2dx \leqslant C_1\varepsilon^\beta(\int_{\tilde\Omega}|u\wedge du|^2+C_2) \leqslant C_3\varepsilon^\beta(\int_{\tilde\Omega}|\nabla\psi|^2+|\nabla H|^2+C_2) \leqslant C\varepsilon^\beta(\log(\frac{1}{\varepsilon})+1)$$

(cf [4] p.124 (102)-(103)) and this finally yields a uniform bound on $\|\nabla|u|\|_{L^p(\tilde\Omega)}$ for $1<p<2$.

Final estimates for u:

Now writing $u=|u|e^{i\varphi}$ locally and noting that $u\wedge\nabla u = |u|^2\nabla\varphi$, we see that

$$|\nabla u| \leqslant \big|\nabla|u|\big| + \frac{1}{|u|}|u\wedge\nabla u| \leqslant \big|\nabla|u|\big| + 2\Big(|\nabla\psi|+|\nabla H|\Big)$$

since $\frac{1}{2}\leqslant|u|\leqslant 1$ on Ω. Thus by combining the estimates on $\nabla|u|, \nabla\psi$ and ∇H, we obtain the uniform bound

$$\|u_\varepsilon\|_{W^{1,p}(\tilde\Omega)} \leqslant C_p \quad \text{for } 1<p<2. \tag{31}$$

Finally, by $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)}\leqslant\frac{C}{\varepsilon}$ and $\|u\|_{L^\infty}\leqslant 1$ from lemma 3 and 4, it follows

$$\|u_\varepsilon\|_{W^{1,2}(B_{\lambda\varepsilon}(x_j^\varepsilon))} \leqslant \tilde C \quad \text{for } 1\leqslant j\leqslant N_\varepsilon$$

and so (31) actually holds for Ω instead of $\tilde{\Omega}$. □

Step 9: Stronger convergence of $\{u_{\varepsilon_n}\}$ in $K \subset\subset \Omega \setminus \{a_j\}_{j=1}^N$

"Standard" elliptic estimates, derived from the equations for $|u|$, ψ and H, imply strong $W^{1,2}_{loc}(\Omega \setminus \{a_1, \dots, a_N\})$-convergence of a subsequence u_{ε_n} to some $u_* \in W^{1,2}_g(\Omega \setminus \{a_1, \dots, a_N\}, S^1)$ and using ideas from [3] one obtains convergence in

$$C^k_{loc}(\Omega \setminus \{a_1, \dots, a_N\}) \quad \forall k \in \mathbb{N}$$

and

$$C^{1,\alpha}_{loc}(\bar{\Omega} \setminus \{a_1, \dots, a_N\}) \quad \text{for } 0 < \alpha < 1.$$

(cf [4] Thm. X.2 p.127 and Thm. X.3 p.130)

Step 10: The limit u_* is a harmonic map from $\Omega \setminus \{a_1, \dots, a_N\}$ into S^1.

Indeed $\triangle u_{\varepsilon_n} \wedge u_{\varepsilon_n} = 0$ implies $\operatorname{div}(u_{\varepsilon_n} \wedge \nabla u_{\varepsilon_n}) = 0$ and we may pass to the limit in $W^{1,p}(\Omega)(1 < p < 2)$ in order to obtain

$$\operatorname{div}(u_* \wedge \nabla u_*) = 0 \quad \text{a.e. in } \Omega. \tag{32}$$

Moreover, by the estimate of lemma 6 we see that

$$|u_*| = 1 \quad \text{a.e. in } \Omega. \tag{33}$$

By results of L. Almeida (cf [Alm]), equation (32),(33) are not sufficient to conclude that u_* is a strong harmonic map but the additional regularity from step 9 actually implies

$$-\triangle u_* = u_* |\nabla u_*|^2 \quad \text{a.e. in } \tilde{\Omega} := \Omega \setminus \{a_1, \dots, a_n\}$$

i.e. u_* is a smooth harmonic map from $\tilde{\Omega}$ to S^1.

Step 11: The vortices $\{a_i\}_{i=1}^N$ are critical points of some renormalized energy W.

1) *Definition of W:*

Consider the solution $\Phi = \Phi_{b,d}$ of

$$\triangle \Phi = \sum_{j=1}^N 2\pi \cdot d_j \, \delta_{b_j} \quad \text{in } \Omega, \tag{34}$$

$$\frac{\partial \Phi}{\partial \nu} = g \wedge \frac{\partial g}{\partial \tau} \quad \text{on } \partial\Omega, \tag{35}$$

for mutually distinct points $b_j \in \Omega$, $d_j \in \mathbb{Z}$ $(1 \leqslant j \leqslant \mathbb{N})$ with $\sum_{j=1}^N d_j = d$ fixed.

Φ is unique up to a constant and we may normalize by $\int_{\partial\Omega} \Phi \, do = 0$.

Note that the functions

$$S_j(x) = \Phi(x) - d_j \cdot \log|x - b_j| \tag{36}$$

are harmonic on $B_\rho(b_j)$ for ρ sufficiently small and further that

$$\int_{\partial B_\rho(b_j)} \Phi \frac{\partial \Phi}{\partial \nu} do \;=\; \int_{\partial B_\rho(b_j)} (\frac{\partial S_j}{\partial \nu} + \frac{d_j}{\rho})(S_j + d_j \log \rho) \, do \tag{37}$$

$$= 2\pi \, d_j^2 \, \log \rho + \int_{\partial B_\rho(b_j)} \frac{\partial S_j}{\partial \nu} S_j \, do + d_j \, \frac{1}{\rho} \int_{\partial B_\rho(b_j)} S_j \, do + d_j \, \log \rho \int_{\partial B_\rho(b_j)} \frac{\partial S_j}{\partial \nu} \, do$$

$$= 2\pi \, d_j^2 \, \log \rho + \int_{B_\rho(b_\rho)} |\nabla S_j|^2 dx + 2\pi \, d_j \, S_j(b_j) + d_j \, \log \rho \int_{B_\rho(b_j)} \underbrace{\triangle S_j}_{=0} \, dx \, .$$

We then compute for $\tilde{\Omega} := \tilde{\Omega}_{b,\rho} = \Omega \setminus \sum_{j=1}^N B_\rho(b_j)$

$$\int_{\tilde{\Omega}} |\nabla \Phi|^2 dx = \int_{\partial\Omega} \Phi \frac{\partial \Phi}{\partial \nu} do - \sum_{j=1}^N \int_{\partial B_\rho(b_j)} \frac{\partial \Phi}{\partial \nu} do$$

$$= \sum_{j=1}^N 2\pi \, d_j^2 \, \log(\frac{1}{\rho}) + C + O(\rho)$$

where $C = C(b,d)$ is a constant independent of ρ and $O(\rho)$ a function such that $|O(\rho)| \leqslant$const.$\cdot\rho$, for ρ close to 0.

Remark: There is a unique harmonic map $u = u^{b,d}$: $\Omega \setminus \{b_1, \dots, b_N\} =: \tilde{\Omega}_b \to S^1$ associated to $\Phi_{b,d}$ defined by (34), $u = u^{b,d}$ being determined by

$$\begin{cases} u \wedge \frac{\partial u}{\partial x_1} & = -\frac{\partial \Phi_{b,d}}{\partial x_2} \\ u \wedge \frac{\partial u}{\partial x_2} & = \frac{\partial \Phi_{b,d}}{\partial x_1} \end{cases} \quad \text{in } \tilde{\Omega}_b$$

(cf [4] p. 10) and we have

$$\operatorname{div}(u \wedge \nabla u) = \triangle \Phi_{b,d} \, , \quad \deg(u, b_i) = d_i \, ,$$

$$|\nabla u| = |\nabla \Phi|$$

and thus

$$\int_\Omega |\nabla u|^2 \, dx = \int_\Omega |\nabla \Phi|^2 \, dx.$$

This motivates the following

Definition:

$$W(b_1, \dots, b_N, d_1, \dots, d_N) := W(b,d) := \lim_{\rho \to 0} \Big(\int_{\tilde{\Omega}_{b,\rho}} |\nabla \Phi|^2 \, dx - 2\pi \sum_{j=1}^N d_j^2 \log \frac{1}{\rho} \Big)$$

which is welldefined for $b_1, \dots, b_N \in \Omega$ mutually distinct and $d_1, \dots, d_N \in \mathbb{Z}$.

2) The vortex configuration $\{a_i\}_{i=1}^N$ is a critical point of W for fixed $d = \deg g$.

For fixed $b \in \Omega^N, d \in \mathbb{Z}^N$ set

$$u_{b,d}(z) := \prod_{j=1}^{N} \Big(\frac{z-b_j}{|z-b_j|}\Big)^{d_j} \cdot e^{i\Phi(z)} \tag{38}$$

where $\triangle\Phi = 0$ in Ω and such that

$$u_{b,d} = g \quad \text{on } \partial\Omega.$$

Lemma 13 *For fixed $d \in \mathbb{Z}^N$, the point $b \in \Omega^N$ is a critical point of $W_d(b) := W(b,d)$*
if and only if
for each $j \in \{1, \dots, N\}$, $u_{b,d}$ may be written as

$$u_{b,d}(z) = \Big(\frac{z-b_j}{|z-b_j|}\Big)^{d_j} e^{iH_j(z)} \tag{39}$$

with

$$\nabla H_j(b_j) = 0 \ \ \text{and} \ \ \triangle H_j = 0 \tag{40}$$

in a sufficiently small neighbourhood of b_j in Ω.

For the proof see [4] corollary VIII.1, p. 85, and for the definition of W see theorem I.7, p. 20.

Actually one proves the following

$$\begin{aligned} DW_d(b) &= -2\pi \Big[d_j \Big(\frac{\partial S_j}{\partial x_1}(b_j), \frac{\partial S_j}{\partial x_2}(b_j) \Big) \Big]_{j=1,\dots,N} \\ &= 2\pi \Big[d_j \Big(-\frac{\partial H_j}{\partial x_2}(b_j), \frac{\partial H_j}{\partial x_1}(b_j) \Big) \Big]_{j=1,\dots,N} \end{aligned}$$

for S_j as in (36) and H_j as in (39). (cf [4] theorem VIII.3, p. 84).

In the sequel we will sketch the proof of the following result:

Theorem 14 *The limit map $u_* = \lim_{n\to\infty} u_{\varepsilon_n}$ from the previous steps with singularities $a = (a_1, \dots, a_N)$ satisfies*

i) $u_* = u_{a,d}$, where $d_i = \deg(u_*, a_i) = \deg(u_*, \partial B_\rho(a_i))$ ($u_{a,d}$ defined as in (38)).

ii) u_* admits local expressions around the a_i's as in Lemma 13, (39).

Corollary 15 *The vortex configuration of the limit map u_* is a critical point of the renormalized energy W.*

Sketch of the proof of theorem (14):
i) Both the limit map u_* and $u_{a,d}$ defined by (38) are smooth harmonic maps from

$$\tilde{\Omega} = \Omega \setminus \{a_1, \dots, a_N\} \quad \text{into } S^1$$

equal to g on $\partial\Omega$ and $\deg(u_*, a_i) = \deg(u_{a,d}, a_i)$ by definition.

Now we
Claim
There is a function ψ such that

$$\begin{aligned} \triangle\psi = 0 \qquad & \text{in } \tilde{\Omega} = \Omega \setminus \{a_1, \dots, a_N\} \\ \psi = 0 \qquad & \text{on } \partial\Omega \end{aligned} \tag{41}$$

and

$$u_* = e^{i\psi}\, u_{a,d}\,. \tag{42}$$

Indeed both u_* and $u_{a,d}$ satisfy the equation

$$\triangle u = -u|\nabla u|^2 \quad \text{in } \tilde{\Omega}.$$

Writing locally $u_* = e^{i\varphi_*}$, $u_{a,d} = e^{i\varphi}$, this is (locally) equivalent to the linear equation

$$\triangle\varphi \;=\; \triangle\varphi_* = 0 \quad \text{in } \tilde{\Omega}$$

and in particular since $\varphi_* = \varphi + \psi$, ψ (locally defined) as in (42) must also be harmonic in $\tilde{\Omega}$. Now since $\deg(u_*, a_j) = \deg(u_{a,d}, a_j)$, $u_{a,d}^{-1} \cdot u_* = e^{i\psi}$ has degree 0 around the a_j's and a continuous ψ satisfying (41) and (42) defined on all of $\tilde{\Omega}$ may be found.

(For a more rigid and computational proof of the claim see [4] Theorem I.5, p. 11).

From step 8 we know that $u_* \in W^{1,1}(\Omega)$ and since $u_{a,d} \in W^{1,1}(\Omega)$, using $|u_*| = |u_{a,d}| \equiv 1$ on $\tilde{\Omega}$, we deduce $\nabla\psi \in L^1(\Omega)$, by applying ∇ to (41).

Now $\triangle\psi = \operatorname{div}\nabla\psi = 0$ on $\Omega \setminus \{a_1, \dots, a_N\}$, so $\operatorname{spt}(\triangle\psi) \subset \{a_1, \dots a_N\}$ as a distribution and therefore

$$\triangle\psi = \sum_{i=1}^{N} c_i \delta_{a_i} + \sum_{1 \leqslant i \leqslant N,\, j=1,2} c_{ij}\, \partial_j \delta_{a_i}, \tag{43}$$

$\triangle\psi$ being a distribution of order 1.

First note that the constants c_{ij} must be zero, since $\nabla\psi \in L^1(\Omega)$ implies $\operatorname{div}\nabla\psi \in (W^{1,\infty}(\Omega))^*$, whereas $\partial_j\delta_{a_i} \notin (W^{1,\infty}(\Omega))^*$. Thus

$$\psi = \sum_{j=1}^{N} c_j \log|x - a_j| + \chi \tag{44}$$

and

$$u_* = u_{a,d}\, e^{i\sum_j c_j \log|x-a_j|} e^{i\chi}\,,$$

where χ is a smooth harmonic function on Ω.

Actually we have $c_i = 0$ too, which implies theorem 14 i). The proof is quite lenghty and in the sequel we will merely sketch the basic ideas, the main reference being [4] theorem VII.1 and VII.4.
For a sequence $u_{\varepsilon_n} \to u_*$ as in step 9 define the Hopf differentials

$$\omega_n := |\frac{\partial u_{\varepsilon_n}}{\partial x}|^2 - |\frac{\partial u_{\varepsilon_n}}{\partial y}|^2 - 2i\frac{\partial u_{\varepsilon_n}}{\partial x}\cdot\frac{\partial u_{\varepsilon_n}}{\partial y}, \tag{45}$$

the dot denoting the real scalar product of vectors.
From the variational equations (7) one deduces

$$\frac{\partial \omega_n}{\partial \bar z} = \frac{\partial}{\partial z}\left(\frac{1}{2\varepsilon_n^2}(1 - |u_{\varepsilon n}|^2)^2\right).$$

Further (cf [4] Lemma VII.1)

$$\frac{1}{2\varepsilon_n^2}\big(1 - |u_{\varepsilon n}|^2\big)^2 \rightharpoonup \sum_{j=1}^{N} m_j \delta_{a_j}$$

weakly as a Radon measure.
It follows (cf [4] p. 67-69)

$$\omega_n \to \omega_* = \beta + 2\alpha \quad \text{in } C^k_{loc}(\Omega \setminus \{a_1, \dots, a_N\}) \tag{46}$$

where $\alpha_* = -\sum_j \frac{m_j}{\pi(z-a_j)^2}$ and β is some holomorphic function on Ω.
On the other hand, from the definition (45) we obtain

$$\omega_n \to \omega_* = |\frac{\partial u_*}{\partial x}|^2 - |\frac{\partial u_*}{\partial y}|^2 - 2i\frac{\partial u_*}{\partial x}\cdot\frac{\partial u_*}{\partial y} \quad \text{in } C^k_{loc}\big(\Omega \setminus \{a_1, \dots, a_N\}\big). \tag{47}$$

Since we have for $u = e^{i\varphi}$ there holds

$$|\frac{\partial u}{\partial x}|^2 - |\frac{\partial u}{\partial y}|^2 - 2i\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} = \left(\frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y}\right)^2,$$

(46), (47) combined with (44) yield an equation that both implies $c_j = 0$ (cf [4] p. 70) and produces an expression of u_* as stated in theorem (14) ii) (cf [4] p. 70).

Section 3: The gauge invariant Dirichlet problem for the Ginzburg Landau functional with magnetic field.

The goal of this lecture is to develop the same kind of analysis as in the previous chapter for the functional

$$G_\varepsilon(u,A) := \frac{1}{2}\int_\Omega \left(|dA|^2 + |\nabla u - iAu|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \right) dx \tag{48}$$

where Ω is a 2-dimensional, smooth, bounded, simply connected domain of $\mathbb{R}^2$, u is a complex-valued function $u \in W^{1,2}(\Omega,\mathbb{C})$ and the "magnetic field" A is a real-valued one-form over Ω, also considered as a vector-valued function $A \in W^{1,2}(\Omega,\mathbb{R}^2)$. Throughout the main reference is [5]. Observe that the above functional differs from the original one by a scaling factor ε^2 after substituting A by $1/\varepsilon A$.
G_ε is invariant under the action of gauge transforms

$$s : \Omega \longrightarrow S^1 \subset \mathbb{C}$$
$$x \longmapsto s(x) = e^{i\varphi(x)}$$

given by

$$s^*u(x) := s(x)u(x) \qquad (\text{ complex multiplication })$$

and

$$s^*A := A - i\, s^{-1}ds = A + d\varphi\,.$$

Therefore we should not impose $u = g$ on $\partial\Omega$ for some given $g : \partial\Omega \to S^1$ as boundary constraint, since this breaks gauge invariance. On the other hand $u = g$ implies $\nabla u\cdot\tau = \nabla g\cdot\tau$, for the unit tangent vectorfield τ on $\partial\Omega$ and if we choose

$$\nabla = \nabla_A := \nabla - iA$$

this new constraint actually is gauge invariant. In our problem we prescribe $|u| = 1$ as well as the vorticity $\deg(u,\partial\Omega)$ on the boundary, which are gauge invariant quantities. Moreover we impose

$$< iu, (\tau\cdot\nabla u - i\tau A u) > = < iu, \tau\cdot\nabla_A u > \overset{!}{=} J \quad \text{ on } \partial\Omega,$$

where J is some given real-valued function on $\partial\Omega$ and $< u, v >:= \mathrm{Re}\,(u\bar{v})$ is the real scalar product. The one form $j :=< iu, \nabla_A u >$ has some physical significance. Note that locally on the boundary we have $u = e^{i\varphi}$ and

$$< iu, \nabla_A u >= \mathrm{Re}(i\bar{u}\,\nabla_A u) = iu^{-1}\,\nabla_A u = -(\nabla\varphi - A)$$

since φ and A are real-valued.
We will look for minimizers of G_ε in the following class:

$$V := \Big\{ (u,A) \in W^{1,2}(\Omega,\mathbb{C})\times W^{1,2}(\Omega,\mathbb{R}^2) \,\big|\, |u| = 1 \text{ on } \partial\Omega, \\ \deg(u,\partial\Omega) = d,\ < iu, \tau\cdot\nabla_A u >= J \text{ on } \partial\Omega \Big\} \tag{49}$$

Theorem 16 *There is a minimizer $(u_\varepsilon, A_\varepsilon) \in V$ of G_ε. It may be chosen in such a way that*

$$d * \bar{A}_\varepsilon = 0 \quad \text{in } \Omega\,, \quad \bar{A}_\varepsilon \cdot \nu = 0 \quad \text{on } \partial\Omega\,.$$

Proof
By gauge invariance, if $\{(u_n, A_n)\}_n \in V$ is a minimizing sequence for G_ε, then so is $\{(u_n + e^{i\varphi_n}, A_n + d\varphi_n)\}_n \in V$ for any $\varphi_n \in W^{2,2}_{loc}(\Omega, \mathbb{R})$, with $\nabla\varphi_n \in W^{1,2}(\Omega, \mathbb{R})$. In order to obtain adequate bounds, we will choose particular representatives in the gauge classes

$$[(u_n, A_n)] := \{ (e^{i\varphi} u_n, A_n + d\varphi) \,\big|\, \varphi \in \mathfrak{G} \}$$

$$\text{with} \quad \mathfrak{G} := \{ \varphi \in W^{2,2}_{\text{loc}}(\Omega, \mathbb{R}) \mid \nabla\varphi \in W^{1,2}(\Omega, \mathbb{R}) \}$$

of a minimizing sequence $\{(u_n, A_n)\}_n$.

Lemma 17 *Coulomb gauge*
For each $(u, A) \in W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{R}^2)$, there is $(\bar{u}, \bar{A}) \in [(u, A)]$ such that

$$\begin{cases} d * \bar{A} = 0 & \text{in } \Omega\,, \\ \bar{A} \cdot \nu = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{50}$$

where ν denotes the exterior unit normal vectorfield on $\partial\Omega$.

Proof of the lemma
Consider $\xi \in W^{2,2}(\Omega, \mathbb{R})$ such that

$$\triangle\xi = *dA \quad \text{in } \Omega\,, \tag{51}$$

$$\xi = 0 \quad \text{on } \partial\Omega\,.$$

Set $\bar{A} := *d\xi$, then $d * \bar{A} = d^2\xi = 0$ in Ω, $\bar{A} \cdot \nu = \frac{\partial\xi}{\partial\tau} = 0$ on $\partial\Omega$ and finally $d(A - \bar{A}) = dA - *\triangle\xi = 0$ in Ω, thus since Ω is simply connected there is a function ψ such that $A - \bar{A} = d\psi$. Note that $\triangle\psi = d^* A \in L^2$ and so $\psi \in W^{2,2}_{loc}(\Omega, \mathbb{R})$. Now for $\bar{u} = e^{-i\psi} u$, we have $(u, A) \simeq (\bar{u}, \bar{A})$ with (50).
QED lemma

In order to prove the existence result of theorem 16, we consider a minimizing sequence $\{(u_n, A_n)\}_n$ in V for $\inf\limits_V G_\varepsilon$. For each A_n, choose ξ_n as in (51)

$$\triangle\xi_n = *dA_n \quad \text{in } \Omega\,,$$

$$\xi_n = 0 \quad \text{on } \partial\Omega\,.$$

Since

$$\int_\Omega |{*dA_n}|^2 \, dx \;=\; \int_\Omega |dA_n|^2 \, dx \;\leqslant\; G_\varepsilon(u_n, A_n) \;<\; C$$

by the Calderon-Zygmund inequality, a subsequence also denoted $\{\xi_n\}$ converges weakly in $W^{2,2}$ and thus $\bar{A}_n = *d\xi_n$ converges weakly in $W^{1,2}$. The uniform bound on $\|\bar{A}_n\|_{W^{1,2}}$ together with

$$\int_\Omega \left|\nabla \bar{u}_n - \, i\bar{A}_n\bar{u}_n\right|^2 dx \leqslant C_\varepsilon$$

and

$$\int_\Omega \left|\bar{u}_n\right|^4 dx \leqslant C'_\varepsilon$$

imlpies $\|\nabla \bar{u}_n\|_{L^2} \leqslant C''_\varepsilon$, so $\{\bar{u}_n\}$ also converges weakly in $W^{1,2}$. By weak lower semi-continuity of G_ε with respect to these norms, the weak limit actually is a minimizer.

This minimizer lies in V and satisfies (50), because all these constraints are preserved under weak $W^{1,2}(\Omega)$ and $W^{\frac{1}{2},2}(\partial\Omega)$ limits, respectively strong $L^p(\Omega)$ and $L^p(\partial\Omega)$ limits for $1 < p < \infty$. $\square$

Theorem 18 *Consider a sequence $\varepsilon_n \underset{n\to\infty}{\to} 0$ and corresponding minimizers $(u_{\varepsilon_n}, A_{\varepsilon_n})$ of G_{ε_n} in V.*
Then there is a subsequence still denoted by $\{\varepsilon_n\}$ and d points $\{a_1, \ldots, a_d\} \subset \Omega$ such that

$$h_{\varepsilon_n} := *dA_{\varepsilon_n} \underset{n\to\infty}{\to} h_* \quad \text{in} \; \begin{cases} W^{1,p}(\Omega) & \text{for } 1 \leqslant p < 2\,, \\ C^k_{loc}(\Omega \setminus \{a_1, \ldots, a_d\}) & \text{for all } k \in \mathbb{N}\,, \end{cases}$$

where h_* satisfies the London equation

$$-\triangle h_* + h_* = 2\pi \sum_{j=1}^{d} \delta_{a_j} \quad \text{in } \Omega, \tag{52}$$

$$\frac{\partial h_*}{\partial \nu} = -J \quad \text{on } \partial\Omega.$$

Moreover the configuration $(a_1, \ldots, a_d)$ minimizes a function $W : \Omega^d \to \mathbb{R}$, which is regular on $\Omega \setminus \triangle$ for $\triangle := \{\, (a, \ldots, a) \in \Omega^d \,\big|\, a \in \Omega \,\}$ and has the following form

$$W(a_1, \ldots, a_d) = 2\pi \sum_{j\neq k} \log\left(\frac{1}{|a_j - a_k|}\right) + R(a_1, \ldots, a_d) \tag{53}$$

where R is regular (C^∞) on Ω^d.
Further each class $[(u_{\varepsilon_n}, A_{\varepsilon_n})]$ admits a representative $(\bar{u}_n, \bar{A}_n)$ such that

$$\bar{u}_n \to u_* \;, \bar{A}_n \to A_* \quad \text{in } C^k_{loc}(\Omega \setminus \{a_1, \ldots, a_d\}) \;, \forall k \in \mathbb{N}$$

and the limiting sections satisfy

$$\nabla^*_{A_*}\nabla_{A_*} u_* = u_* \left|\nabla_{A_*} u_*\right|^2,$$

i.e. u_* is A_*-harmonic.

Remark:
We would like to point out the link between this result and the problems presented in the introduction. The Dirichlet boundary conditions and in particular the boundary constraint $|u| \equiv 1$ are not meaningfull for the underlying physics. Nonetheless the preceding result yields a rigorous description of the mechanism

$$\text{vorticity} \Longrightarrow \begin{cases} \text{formation of vortices and induction of a magnetic field} \\ + \\ \text{the induced magnetic field verifies the London equation}\,, \end{cases}$$

which actually is the first question addressed in the introduction. The second question, which is still left to be understood, may be formulated in the following way:

$$\Big\{ \text{exterior field } H_{ext} \geqslant \text{critical value } H_{c_1} \sim \kappa^{-1}(\frac{1}{2}\log(\kappa) + c_1) \Big\} \Longrightarrow \text{vorticity}\,.$$

A rigorous mathematical theory for the spontanious apparition of vorticity, for the minimizers of $F_{\kappa,H_{ext}}(u,A)$, when applying an external field $H_{ext} \sim H_{c_1}$, is still to be found and this seems to be quite difficult. Still the preceding result gives some light on the phenomenon. Actually the creation of vortices is coupled to an effect in the vicinity of the boundary, called Meissner effect, which implies that (u,A) is not superconducting close to the boundary. We now assume that $\partial\Omega$ is not the real boundary of the superconductor, but simply the delimitation of some interior sample-domain, far away from the real boundary of the superconductor,such that we can ignore the Meissner effect. It then becomes physically relevant to prescribe $|u| \equiv 1$ on $\partial\Omega$. Moreover we may assume there is a tangent current $(iu, \nabla_A u)\cdot\tau \equiv J$ on $\partial\Omega$ which is independent of ε, but has <u>free vorticity d.</u> We then mimimize

$$F_{\kappa,H_{ext}}(u,A) \;=\; \int_\Omega \big(|\nabla_A u|^2 + \frac{\kappa^2}{2}(1-|u|^2)^2 + |dA|^2 \big)\, dx - 2\int_\Omega dA\cdot H_{ext}$$

for the preceding constraints. Choosing $J \equiv 0$, in order to simplify the presentation, we obtain

$$\int_\Omega dA\cdot H_{ext} \;=\; H_{ext}\cdot \int_\Omega dA \;=\; \int_{\partial\Omega} A\cdot\tau \;=\;=\; 2\pi d\, H_{ext}\,.$$

Setting as before $\kappa = \frac{1}{\varepsilon}$, we thus minimize

$$F_{\varepsilon,H_{ext}}(u,A) \;=\; G_\varepsilon(u,A) - 4\pi d\, H_{ext}\,.$$

The asymptotic developpement of Theorem 18 then yields

$$\begin{aligned} F_{\varepsilon,H_{ext}}(u,A) &= 2\pi\log\frac{1}{\varepsilon} + W(a) + c\,d + \delta(\varepsilon) - 4\pi H_{ext} \\ &= 2\pi\big(\log\frac{1}{\varepsilon} - 2H_{ext}\big) + W(a) + c\,d + \delta(\varepsilon) \end{aligned}$$

We easily see that there is a value c_1, such that for $c \leqslant c_1$ and $H_{ext} = \frac{1}{2}\log\frac{1}{\varepsilon} + c$, it is better to have $d = 0$ (and in the same time $W(a) = 0$), whereas for $c > c_1$ this is

not the case anymore. We can also determine the optimal vorticity as a function of c.
The above argument and the use of theorem 18 is rigorous, if we assume that d is a free parameter, which is merely bounded by a given constant for $\varepsilon \to 0$. It would be interesting to prove this result without the assumption that d is bounded.

Sketch of the proof of theorem (18):
We will follow the same approach as in the case without magnetic field, but the proof is not quite the same and also works for non starshaped domains. Actually this method could also be used to treat the problem without magnetic field and implies F. Bethuel, H. Brezis and H. Hélein result for arbitrary domains which was the result established by M. Struwe in [18]. The idea consists in combining the Pohozaev identity on balls of radius ε^α with the η-compactness lemma, offered below and the global bound on the energy.

Step 1 : The Euler-Lagrange equations. The Euler-Lagrange equations for the critical points (u, A) of G_ε are

$$\nabla_A^* \nabla_A u = \frac{1}{\varepsilon^2} u (1 - |u|^2) \quad \text{in } \Omega \,, \tag{54}$$

$$- d^* dA = < iu, \nabla_A u >=: j \quad \text{in } \Omega \,, \tag{55}$$

$$\frac{\partial h}{\partial \nu} = -j \cdot \tau = -J \quad \text{on } \partial\Omega \,. \tag{56}$$

Actually in Coulomb gauge (56) follows from (54), (55) since the latter equations then become elliptic and solutions are smooth up to the boundary. Now since (56) is gauge invariant, by transforming back, we see that it holds in arbitrary gauge.

Here $\nabla_A^* \nabla_A = -\sum_k \big(\frac{\partial}{\partial x_k} - iA\big)\big(\frac{\partial}{\partial x_k} - iA\big)$ and as before $h := *dA$. Note that (56) may be written as $-d^* h = j$, if we set $d^* h := \frac{\partial h}{\partial x_2}\, dx_1 - \frac{\partial h}{\partial x_1}\, dx_2$.

Step 2: L^∞ bound on u.

Lemma 19 *Solutions of the Euler-Lagrange equations satisfy* $\|u_\varepsilon\|_{L^\infty} \leqslant 1$.

This follows as in the previous chapter from the maximum principle applied to

$$\frac{1}{2} \triangle |u_\varepsilon| = -\frac{1}{\varepsilon^2} |u_\varepsilon|^2 (1 - |u_\varepsilon|^2) + \big|\nabla_{A_\varepsilon} u_\varepsilon\big|^2 .$$

Step 3: Global bound on the Energy.

Lemma 20 *For minimizers $(u_\varepsilon, A_\varepsilon)$ of G_ε in V we have*

$$G(u_\varepsilon, A_\varepsilon) \leqslant 2\pi \log\big(\frac{1}{\varepsilon}\big) + C \,, \tag{57}$$

which is obtained by evaluating $G_\varepsilon(v_\varepsilon, 0)$ for v_ε a minimizer of the functional E_ε considered in chapter 1, for some boundary value g solving $< ig, \tau \cdot \nabla g >= J$ and $|g| = 1$ on $\partial\Omega$.

Step 4: A remark

From the previous step we obtain the bounds

$$\int_\Omega |h_\varepsilon|^2\, dx \;\leqslant\; C \log\big(\frac{1}{\varepsilon}\big)\,, \tag{58}$$

$$\int_\Omega \big|\nabla_{A_\varepsilon} u_\varepsilon\big|^2\, dx \;\leqslant\; C \log\big(\frac{1}{\varepsilon}\big)\,. \tag{59}$$

We may nonetheless obtain better local bounds on A. In a way A does not have local effects, but rather global ones: it resorbs the degree of u prescribed at the boundary.
Indeed, set $r = \varepsilon^\alpha$ and choose Coulomb gauge on a ball B_r:

$$\begin{cases} d * \bar{A}_\varepsilon = 0 & \text{in } B_r\,, \\ \bar{A}_\varepsilon \cdot \nu = 0 & \text{on } \partial B_r\,, \end{cases} \tag{60}$$

i.e. as in the proof of lemma 17 set $\bar{A}_\varepsilon := *d\xi$ for ξ given by

$$\triangle \xi = h_\varepsilon \quad \text{in } B_r\,, \quad \xi = 0 \quad \text{on } \partial B_r\,.$$

Now

$$\int_{B_r} |A_\varepsilon|^2\, dx \;\leqslant\; C_1 \int_{B_r} |\nabla \xi|^2\, dx \;\leqslant\; C_2 \int_{B_r} |h_\varepsilon|^2\, dx \;\leqslant\; C_3\, r^2 \,\log\big(\frac{1}{\varepsilon}\big)\,, \tag{61}$$

so

$$\int_{B_{\varepsilon^\alpha}} |A_\varepsilon|^2\, dx \;\leqslant\; C\, \varepsilon^{2\,\alpha} \log\big(\frac{1}{\varepsilon}\big) \underset{\varepsilon \to 0}{\longrightarrow} 0\,. \tag{62}$$

Suppose we have a vortex of degree 1 in $x_0 \in \Omega$, i.e. $u_\varepsilon(x_0) = 0$ and $|u_\varepsilon| > 0$ on ∂B_r, with $\int_{\partial B_r} \frac{1}{|u|^2} \big(u \wedge du\big) \;=\; 2\pi$.

Then one may show that $\int_{B_{\varepsilon^\alpha}(x_0)} \big|\nabla u\big|^2\, dx \;\geqslant\; C_\alpha\, \log\big(\frac{1}{\varepsilon}\big)\,,$

whereas $\int_{B_{\varepsilon^\alpha}(x_0)} |A|^2\, dx \;\sim\; \varepsilon^{2\alpha}\, \log\big(\frac{1}{\varepsilon}\big)\,.$ (cf. [5] proposition IV.3)

Step 5: Global estimates

Lemma 21 *For minimizers $(u_\varepsilon, A_\varepsilon)$ of G_ε in V we have*

$$\|\nabla_{A_\varepsilon} u_\varepsilon\|_{L^\infty} \;\leqslant\; \frac{C}{\varepsilon}\,. \tag{63}$$

In particular, this implies

$$\|\nabla|u_\varepsilon|\|_{L^\infty} \leqslant \frac{C}{\varepsilon} \quad \text{and} \quad \|\nabla h_\varepsilon\|_{L^\infty} \leqslant \frac{C}{\varepsilon}\,,$$

from equation (55) .

Proof
(63) essentially follows from a scaling argument, the bounds of step 3 and 4 and the fact that in Coulomb gauge on the unit ball, i.e. for

$$d^*A = 0 \text{ in } B_1 \quad \text{and} \quad A\cdot\nu = 0 \quad \text{on} \quad \partial B_1\,.$$

we have

$$\int_{B_1} \left|\nabla A\right|^2 dx \,+\, \int_{\partial B_1} |A|^2\, dx \,=\, \int_{B_1} \left|dA\right|^2 dx \,=\, \int_{B_1} |h|^2\, dx\,. \tag{64}$$

Indeed let (u, A) be a minimizer of G_ε in V in Coulomb gauge (cf. theorem 17), where for simplicity we drop the indices ε. Fix $x_0 =: 0 \in \Omega$ and define the rescaled solutions

$$\tilde{u}(x) := u(\varepsilon x), \quad \tilde{A} := \varepsilon A(\varepsilon x) = \sum_{i=1,2} \varepsilon A_i(\varepsilon x) dx_i$$

for $x \in B_2 := B_2(0)$.
Then $\left|\nabla_{\tilde{A}}\tilde{u}\right|(x) = \varepsilon\left|\nabla_A u\right|(\varepsilon x), \quad |\tilde{h}(x)| = \varepsilon^2 |h(\varepsilon x)|$. Now (58),(61) and (64) yield

$$\int_{B_2} \left(|\tilde{A}|^2 + \left|\nabla\tilde{A}\right|^2\right) dx \,\leqslant\, C \int_{B_2} |\tilde{h}|^2\, dx \,\leqslant\, C\,\varepsilon^2 \log\left(\frac{1}{\varepsilon}\right)$$

and moreover, keeping in mind that we chose Coulomb gauge, the Euler-Lagrange equations for the scaled solutions on B_2 read

$$-\triangle\tilde{u} = \tilde{u}\,(1 - |\tilde{u}|^2) - i\tilde{A}^2\tilde{u} + 2i\tilde{A}\nabla\tilde{u}\,,$$

$$-\triangle\tilde{A} = \varepsilon^2 < i\tilde{u}, \nabla_{\tilde{A}}\tilde{u} > \,.$$

From the estimate $\|\tilde{u}\|_{L^\infty} \leqslant 1$, elliptic regularity and the usual boot-strap arguments, one concludes

$$\|\nabla\tilde{u}\|_{L^\infty(B_{\frac{3}{2}})} \leqslant C\,,$$

for a constant C independent of ε. Now scaling back and covering $\bar{\Omega}$ with a finite number of balls of radius $\frac{3}{2}$, combined with boundary-regularity yields $\|\nabla u\|_{L^\infty} \leqslant \frac{C}{\varepsilon}$ and also $\|\nabla_A u\|_{L^\infty} \leqslant \frac{C}{\varepsilon}$.
(cf. [5] proposition II.6) □

Step 6: Pohozaev identity
In the non gauge invariant case of the previous chapter, we obtained the Pohozaev identity by multiplying the Euler-Lagrange equations by $\sum_{i=1,2} x_i \frac{\partial u}{\partial x_i} = \mathfrak{L}_{r\frac{\partial}{\partial r}} u$, where $\mathfrak{L}$ is the Lie-derivative. This is equivalent to saying that u is a critical point

with respect to variations induced by translations in the domain, i.e. perturbations of the form:

$$u_t(x) := u\big(x + (x - x_0)t\big).$$

Now we also set: $A_t(x) := A\big(x + (x - x_0)t\big)$ and if (u, A) is a critical point of $G_\varepsilon(u, A) = \int_\Omega g_\varepsilon(u, A)\,dx$, then we have

$$\frac{d}{dt}\bigg|_{t=0} \int_G g_\varepsilon(u, A)\,dx = 0 \quad \forall G \subset\subset \Omega.$$

After a translation, we may assume $0 = x_0 \in \Omega$ and find

Lemma 22 *Let (u, A) be a critical point of $G_\varepsilon(u, A)$ we have*

$$\begin{aligned}\int_G \left(\frac{1}{\varepsilon^2}(1 - |u|^2)^2 - 2h^2\right) dx = &\int_{\partial G} (x\cdot\nu)\left(\frac{1}{2\varepsilon^2}(1 - |u|^2)^2 - h^2\right) do \\ &+ \int_{\partial G}\left(\big|\nabla_A u\cdot\tau\big|^2 - \big|\nabla_A u\cdot\nu\big|^2\right) do \\ &- 2\int_{\partial G}(x\cdot\tau)\left(\tau\cdot\nabla_A u, \nu\cdot\nabla_A u\right) do.\end{aligned} \tag{65}$$

Step 7: Local estimates

Lemma 23 *Let $0 < \alpha < 1$, $x_0 \in \Omega$, $(u_\varepsilon, A_\varepsilon)$ a minimizer of G_ε in V and $h_\varepsilon = *dA_\varepsilon$. Then*

$$\int_{B_{\varepsilon^\alpha}(x_0)\cap\Omega} |h_\varepsilon|^2\,dx \leqslant C\,\varepsilon^\alpha \log\left(\frac{1}{\varepsilon}\right) \quad \forall\, 0 < \alpha < 1 \tag{66}$$

and

$$\int_{B_{\varepsilon^\alpha}(x_0)\cap\Omega} \frac{1}{\varepsilon^2}(1 - |u_\varepsilon|^2)^2\,dx \leqslant C_\alpha, \tag{67}$$

where C_α depends on α as well as d, J and Ω.

Remark: We choose ε^α because it is the largest scale for which the Pohozaev identity yields a bound of the form $\int_{B_{\varepsilon^\alpha}} \frac{1}{\varepsilon^2}(1 - |u_\varepsilon|^2)^2\,dx \leqslant C_\alpha$. In particular this implies, as in the previous chapter, that the number of bad discs B_{ε^α} is finite, which will be discussed in step 8.

Proof
In the sequel we will drop the index ε. Noting that h is a function, we deduce from (55) and (56)

$$|dh| = \big|\nabla h\big| \leqslant \big|\nabla_A u\big| \tag{68}$$

Indeed since in Coulomb gauge the variational equations (54), (55) are elliptic, weak solutions are actually smooth and the equations hold pointwise. Now since the above quantities are gauge invariant, (68) holds for weak solutions in any gauge.
Combining with (59) we obtain

$$\int_{B_{\varepsilon^\alpha}} |\nabla h|^2 \, dx \leqslant C \log\left(\frac{1}{\varepsilon}\right)$$

and from (58) we have

$$\int_{B_{\varepsilon^\alpha}} |h|^2 \, dx \leqslant C \log\left(\frac{1}{\varepsilon}\right).$$

Using the Sobolev injections $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, $1 \leqslant p \leqslant \infty$ and Hölder's inequality, we deduce

$$\int_{B_{\varepsilon^\alpha}} |h|^2 \, dx \leqslant (2\pi\varepsilon^{2\alpha})^{\frac{2}{q}} \left(\int_{B_{\varepsilon^\alpha}} |h|^p \, dx\right)^{\frac{2}{p}} \leqslant C_p \, (\varepsilon^\alpha)^{\frac{4}{q}} \int_{B_{\varepsilon^\alpha}} \left(|\nabla h|^2 + |h|^2\right) dx$$

with $\frac{1}{p} + \frac{1}{q} = 1$, which gives the first result by setting $p = 4$.
For the second estimate, let $0 < \alpha < 1$ and write

$$g_\varepsilon(x) := g_\varepsilon\big(u(x), A(x)\big) := \frac{1}{2}\left(|dA|^2(x) + |\nabla_A u|^2(x) + \frac{1}{2\varepsilon^2}(1 - |u(x)|^2)^2\right)$$

for the energy density. Then

$$\int_{B_{\varepsilon^{2\alpha}} \setminus B_{\varepsilon^\alpha}} g_\varepsilon(x) \, dx = \int_{\varepsilon^\alpha}^{\varepsilon^{\frac{\alpha}{2}}} \frac{1}{r}\left(r \int_{\partial B_r} g_\varepsilon(r\omega) \, d\omega\right) dr \leqslant \log\big(\frac{1}{\varepsilon}\big).$$

Thus there is some $\rho_\varepsilon \in [\varepsilon^\alpha, \varepsilon^{\frac{\alpha}{2}}]$, such that

$$\rho_\varepsilon \int_{\partial B_{\rho_\varepsilon}} g_\varepsilon(\rho_\varepsilon \omega) \, d\omega \leqslant C_\alpha,$$

where the constant C_α depends on α, J, d but not on ε. Combining the Pohozaev inequality (65) on B_{ρ_ε} with the previous estimate (66), we obtain

$$\int_{B_{\varepsilon^\alpha}} \frac{1}{\varepsilon^2} \left(1-|u|^2\right)^2 dx \leqslant \int_{B_{\rho_\varepsilon}} \frac{1}{\varepsilon^2} \left(1-|u|^2\right)^2 dx \leqslant C\left(\varepsilon^\alpha \log\big(\frac{1}{\varepsilon}\big) + \rho \int_{\partial B_{\rho_\varepsilon}} g_\varepsilon \, do\right) \leqslant C_\alpha.$$

Note that in the case $B_{\rho_\varepsilon} \cap \partial\Omega \neq \emptyset$, one should integrate over $B_{\rho_\varepsilon} \cap \Omega$ and make use of the boundary data. □

Step 8: The η-compactness lemma

This lemma roughly says, that if we don't have enough energy on a ball, then $|u|$ is larger than $1/2$ on the ball of half of the radius. Now this implies some compacteness properties for $\{u_\varepsilon\}_{\varepsilon>0}$ on this ball. The η-compactness property is also one of the key ingredients for studying similar asymptotic phenomena for minimizers of the Ginzburg-Landau Functional in dimension larger than 2, but in higher dimensions the η-compactness lemma is much more delicate to establish (see [15] and [12] on this subject).

Lemma 24 *There is a constant $\eta > 0$, such that, for all minimizers $(u_\varepsilon, A_\varepsilon)$ of G_ε in V, all $x_0 \in \Omega$ and all $\rho \geqslant \varepsilon$*

$$\int_{B_\rho(x_0)\cap\Omega} g_\varepsilon(u_\varepsilon, A_\varepsilon)\, dx \;\leqslant\; \eta\, \log\big(\frac{\rho}{\varepsilon}\big)$$

$$\Downarrow$$

$$|u_\varepsilon| \;\geqslant\; \frac{1}{2} \quad in\, B_{\frac{\rho}{2}(x_0)} \cap \Omega\,.$$

Consequences of the η-compactness lemma:

Let $0 < \alpha < 1$. We call $B_r(x)$, for $x \in \Omega$ a *bad ball* , if there is some $y \in B_{\frac{r}{2}}(x)$, such that $|u|(y) < \frac{1}{2}$. Here and in the following B_r or $B_r(x)$ actually stands for $B_r(x)\cap\Omega$, for some $x \in \Omega$. If $B_r(x) \cap \partial\Omega \neq \emptyset$ some care is required and the boundary data should be used.
Choose a covering by balls $\{B_{\varepsilon^\alpha/2}(x^i)\}_{i\in I_\varepsilon}$ of $\bar\Omega$ and set $J_\varepsilon := \{\, i \in I_\varepsilon \mid B_{\varepsilon^\alpha}(x^i) \text{ is bad } \}$. By the η-compactness lemma we have

$$C\, \log\big(\frac{1}{\varepsilon}\big) \;\geqslant\; \sum_{j\in J_\varepsilon} \int_{B_{\varepsilon^\alpha}(x^j)} g_\varepsilon(u_\varepsilon, A_\varepsilon)\, dx \;>\; \eta\, \log\big(\frac{\varepsilon^\alpha}{\varepsilon}\big)\cdot(\sharp J_\varepsilon) \;=\; (\sharp J_\varepsilon)\,\eta\,(1-\alpha)\, \log\big(\frac{1}{\varepsilon}\big)\,.$$

So $\sharp J_\varepsilon \;\leqslant\; C_\alpha$, i.e. the number of bad balls of radius ε^α for $0 < \alpha < 1$ is uniformly bounded with respect to ε.
On the other hand, from lemma 21 and lemma 23, we know that

$$\|\nabla|u_\varepsilon|\|_{L^\infty} \;\leqslant\; \frac{C}{\varepsilon}\,, \tag{69}$$

and

$$\int_{B_{\varepsilon^\alpha}} \frac{1}{\varepsilon^2}\left(1-|u|^2\right)^2 dx \;\leqslant\; C_\alpha\,. \tag{70}$$

This yields a uniform bound on the number of bad balls of radius $\lambda\,\varepsilon$, for a constant $0 < \lambda < 1$ independent of ε. Indeed, by (69), there is a $\lambda \in]0,1[$, such that $|u| \leqslant \frac{3}{4}$ on $B_{\lambda\varepsilon}(y)$ if $|u(y)| < \frac{1}{2}$, thus

$$\int_{B_{\lambda\varepsilon}(y)} \frac{1}{\varepsilon^2}\left(1-|u|^2\right)^2 dx \geqslant \frac{\pi^2\lambda^2}{16} > 0\,,$$

and so the number of bad balls of radius $\lambda\varepsilon$ contained in some bad ball of radius ε^α is bounded independly of ε by (70).

Proof of the η-compactness lemma in dimension 2.
First note that it suffices to prove $|u(x_0)| \geqslant \frac{1}{2}$.
Now by lemma 23 , $\rho \leqslant \varepsilon^\alpha$ implies

$$\int_{B_\rho} |h|^2 dx \;\leqslant\; \varepsilon^\alpha \log\frac{1}{\varepsilon} \;\underset{\varepsilon \to 0}{\longrightarrow}\; 0 \,.$$

By assumption we have

$$\int_\varepsilon^\rho \frac{1}{r}\left(r \int_{\partial B_r} g_\varepsilon(u_\varepsilon, A_\varepsilon) d\sigma \right) dr \;\leqslant\; \eta \log\frac{\rho}{\varepsilon}$$

and so there is some $r_0 \in [\varepsilon, \rho]$ such that

$$r_0 \int_{\partial B_{r_0}} \left(\left|\nabla_A u\right|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 + |dA|^2 \right) dx \;\leqslant\; \eta \tag{71}$$

Combining (65) and (71), we obtain

$$\int_{B_{r_0}} \frac{1}{\varepsilon^2}(1-|u|^2)^2 \, dx \;\leqslant\; C\,\eta \,.$$

Using $\|\nabla |u|\|_{L^\infty} \leqslant \frac{C}{\varepsilon}$, this yields $|u(x_0)| > \frac{1}{2}$ for η sufficiently small. $\square$

Step 9 : $\quad W^{1,p}$ estimates for $h = dA$.

$h := dA$ satisfies

$$d\big(\frac{1}{|u|^2}\, d^* h \big) \;+\; h = 0 \;\;\text{ on }\;\; \tilde{\Omega} := \Omega \setminus \bigcup_{j \in J} B_\varepsilon(x_j) \,, \tag{72}$$

where $\{B_\varepsilon(x_j)\}_{j \in J}$ is a finite cover of the bad set. (cf. step 8)
Indeed on $\tilde{\Omega}$ we have $|u| \geqslant \frac{1}{2}$ and locally we may write $u = |u| e^{i\varphi}$. Then

$$< iu, \nabla_A u >= Re(iu \cdot \overline{\nabla_A u}) = |u|^2\,(d\varphi - A)$$

and

$$d < i\frac{u}{|u|^2}, D_A u >= -dA = -h \,.$$

Applying now d to (55), we obtain

$$-dd^* h \;=\; \big(|u|^2 \;< i\frac{u}{|u|^2}, \nabla_A u > \big) \;=\; \frac{1}{|u|^2} d\big(|u|^2\big)(-d^* h) - |u|^2 h \,,$$

so

$$\frac{1}{|u|^2} dd^* h - \frac{2d|u|}{|u|^3} d^* h + h \;=\; 0 \,,$$

which is equivalent to (72). Now this equation for h is very similar to that obtained for ψ and H in the case without magnetic field, considered in the previous chapter.

But here we did not have to make use of a Hodge decomposition, since h turns out to be the right variable to work with. Note that the Dirichlet boundary condition for ψ is now replaced by a Neuman boundary condition

$$\frac{\partial h}{\partial \nu} = -J \quad \text{on} \quad \partial\Omega\,.$$

$W^{1,p}$ estimates are obtained by similar methods and the other steps too can be developped mutatis mutandi as in the case without magnetic field.

Acknowledgements

These notes arose from a mini-course given by the author at the ETH-Zurich on January 1997. He would like to express his thanks to the Forschungsinstitute für Mathematik at the ETH for its hospitality. The author is also most grateful to Michael Struwe for his attention and their enriching discussions. These lecture notes were taken by Paul Harpes, who actively participated in clarifying the exposition by his remarks and pertinent suggestions; he should be thanked here.

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