The role of integrability by compensation in conformal geometric analysis.

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I Wente's inequalities and integrability by compensation.

The *Integrability by Compensation* is an improvement in the a-priori rate of integrability of a function due to special cancellation, compensation, phenomena. It was probably first discovered by Henry C. Wente in [Wen] in his work on constant mean curvature surfaces.

Given two functions a and b in the Sobolev Space $W^{1,2}(\omega,\mathbb{R})$ of L^2 functions on a domain ω of $\mathbb{R}^2$ whose distributional derivatives are also in L^2, the jacobian function

$$\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial a}{\partial y}\frac{\partial b}{\partial x} \tag{I.1}$$

is a-priori only in $L^1(\omega)$. The observation made by H.C. Wente was that the convolution of this function together with the Green Kernel $\log|x|$ of the Laplacian is in the Sobolev Space $W^{1,2}_{loc}(D^2)$ and also in $L^\infty_{loc}(D^2)$. This realizes an improvement of the a-priori integrability properties given by classical singular integral theory. Indeed the convolution of an L^1 function together with the Green Kernel $\log|x|$ is a-priori only in the space of Bounded Mean Oscillations BMO and that the derivatives of such a convolution are a-priori only in the weak L^2 space - or Marcinkiewicz space - $L^{2,\infty}$ (see [Ste]). Recall the definition of the $L^{2,\infty}$ norm of measurable function on a domain Ω :

$$\|f\|_{L^{2,\infty}(\omega)}=\sup_{\lambda>0}\ \lambda\ |\{x\in\omega\ ;\ |f(x)|\geq\lambda\}|^{\frac{1}{2}}\quad .$$

This improvement of intergrability, or regularity, is due to the special algebraic structure of the quadratic nonlinearity (I.1) which is of jacobian type.

Later on, in [Ta1], Luc Tartar wrote a new proof of Wente's result using an argument which permitted to improve the gain of integrability obtained by Wente. Tartar indeed established that the Fourier transform of the convolution between the jacobian (I.1) and the Green function $\log|x|$ was in a strictly smaller space than L^2 : the Lorentz Space $L^{2,1}$ dual to the weak L^2 space $L^{2,\infty}$. We recall now a characterization of the $L^{2,1}$ space : a measurable function f on Ω is in the Lorentz space $L^{2,1}$ if and only if

$$\int_0^{+\infty}|\{x\in\Omega\ ;\ |f(x)|\geq\lambda\}|^{\frac{1}{2}}\ d\lambda<+\infty$$

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A systematic presentation of Lorentz spaces can be found in [Ta3]. In [Hel] section 3.4, Frédéric Hélein presents another argument by Luc Tartar, based on the use of an interpolation result of Jacques-Louis Lions for bilinear operators, which shows that derivatives of the convolution between the Green Kernel $\log|x|$ and the jacobian (I.1) are themselves in the Lorentz space $L^{2,1}$. This permits in particular to recover the full result of Wente since functions on a 2-dimensional domain whose gradients are locally in $L^{2,1}$ are continuous.

In [Mu], Stefan Müller proved, under the additional assumption that the jacobian (I.1) has a sign, that, still assuming that a and b are in $W^{1,2}$, this jacobian is in a smaller space than L^1 : the Orlicz space $L^1 \log L^1$. As a consequence, using the classical theory of Calderon Zygmund operators, see [Ste], one then obtain that the convolution between the Green Kernel $\log|x|$ and the jacobian (I.1) is locally in the Sobolev Space $W^{2,1}$ which permits in particular, using Lorentz-Sobolev embeddings- see [Ta2] and [Ta3], to recover Wente's and Tartar's results under this sign assumption. Later on, Ronald Coifman, Pierre-Louis Lions, Yves Meyer and Stephen Semmes were able to drop the sign assumption made by Müller and proved that, under the assumption that ∇a and ∇b are in L^2 only, the jacobian (I.1) is in the local Hardy space $\mathcal{H}^1_{loc}$. Recall that, for positive functions, the local Hardy space $\mathcal{H}^1_{loc}$ coincides with the Orlicz space $L^1 \log L^1$. Using the Fefferman-Stein characterization of Hardy, under the only assumption that ∇a and ∇b are in L^2, one deduces that the convolution between the Green Kernel $\log|x|$ and the jacobian (I.1) is in the Sobolev space $W^{2,1}$.

These improvements in integrability or regularity were originally obtained together with estimates. We sumarize then the previous discussion and give the corresponding estimates in the following theorem

Theorem I.1 *[Wente 1969, Tartar 1983, Müller 1989, Coiffman-Lions-Meyer-Semmes 1990] Let ω be a bounded regular domain of $\mathbb{R}^2$. Let a and b be two measurable functions on ω whose gradients are in $L^2(\omega)$. Then there exists a unique solution φ in $W^{1,2}(\omega)$ to*

$$\begin{cases} -\Delta\varphi = \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x} \qquad in\ \omega \\[2ex] \varphi = 0 \qquad on\ \partial\omega \end{cases} \tag{I.2}$$

Moreover there exists a constant $C(\omega)$ such that

$$\|\varphi\|_{L^\infty(\omega)} + \|\nabla\varphi\|_{L^{2,1}(\omega)} + \|\nabla^2\varphi\|_{L^1(\omega)} \le C(\omega)\ \|\nabla a\|_{L^2}\ \|\nabla b\|_{L^2} \quad . \tag{I.3}$$

In particular φ is continuous in ω. □

A consequence of the previous theorem was obtained by Fabrice Bethuel in [Be] using a duality argument and is another remarquable result in the theory of integrablity by compensation (see also a presentation of this result in [Hel]) .

Theorem I.2 *[Bethuel 1992] Let ω be a bounded regular domain of $\mathbb{R}^2$. Let a and b be two measurable functions on ω. Assume that the distributional derivatives ∇a and ∇b are respectively in $L^2(\omega)$ and $L^{2,\infty}(\omega)$. Then there exists a* *unique solution φ in $W^{1,2}(\omega)$ to*

$$\left\{\begin{array}{l} -\Delta\varphi = \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x} \qquad \textit{in } \omega \\[4mm] \varphi = 0 \qquad \textit{on } \partial\omega \end{array}\right. \tag{I.4}$$

Moreover there exists a constant $C(\omega)$ such that

$$\|\nabla\varphi\|_{L^2(\omega)} \le C(\omega)\ \|\nabla a\|_{L^2(\omega)}\ \|\nabla b\|_{L^{2,\infty}(\omega)} \quad . \tag{I.5}$$

□

II Hildebrandt's conjecture on critical points to conformally invariant Lagrangians.

II.1 Conformally invariant quadratic coercive lagrangians in 2 dimensions.

Due to the important role they play in physics and geometry, the analysis of critical points to conformally invariant lagrangians has raised a special interrest in the mathematical community since at least the early 50's and in particular under the impulsion of Charles B. Morrey. Because of the richness of it's conformal group, the dimension 2 should maybe be first looked at in priority.

Let first consider the Dirichlet Energy for functions u from a 2-dimensional domain ω into $\mathbb{R}$

$$L(u) := \int_\omega |\nabla u|^2(x,y)\ dx\wedge dy \quad .$$

This is maybe the most simple example of a 2-dimensional conformally invariant Lagrangian. Indeed, let ϕ be a conformal transformation on $\mathbb{C}$, it satisfies

$$\left\{\begin{array}{l} \left|\dfrac{\partial\phi}{\partial x}\right| = \left|\dfrac{\partial\phi}{\partial y}\right| \quad , \\[4mm] \dfrac{\partial\phi}{\partial x}\cdot\dfrac{\partial\phi}{\partial y} = 0 \quad , \\[4mm] det\,\nabla\phi \ge 0 \quad \text{and} \quad \nabla\phi \ne 0 \quad . \end{array}\right. \tag{II.1}$$

(In other words ϕ is an holomorphic function). Then, for any u in $W^{1,2}(\omega,\mathbb{R})$ the following holds

$$L(u) = L(u\circ\phi) = \int_{\phi^{-1}(\omega)} |\nabla(u\circ\phi)|^2(x,y)\ dx\wedge dy \quad . \tag{II.2}$$

Critical points to L for any perturbation of the form $u + t\chi$, where χ is an arbitrary smooth compactly supported function on ω, are the harmonic functions satisfying

$$\Delta u = 0 \quad \text{dans } \omega \quad . \tag{II.3}$$

Among the analysis questions related to that functional come first the regularity issues, uniqueness questions for fixed given boundary data, questions regarding the shape of the solution - possible symmetries - ...etc. In that elementary situation one can observe that most of these priority questions regarding the analysis of solution to (II.3) can be solved by the mean of the maximum principle.

The previous problem can be generalized in the following ways. We can first extend L to maps taking values into an arbitrary Euclidian space $\mathbb{R}^n$ as follows

$$L(u) := \int_\omega |\nabla u|^2(x,y)\ dx\wedge dy = \int_\omega \sum_i |\nabla u^i|^2\ dx\wedge dy \quad .$$

where u^i are the coordinates of u. This is again an example of conformally invariant Lagrangian (satisfying (II.2)) and it's critical points are again solving equation (II.3) coordinate by coordinate.

Consider now a metric $g(X) = (g_{ij}(X))_{1\le i,j\le n}$ on $\mathbb{R}^n$ whose C^1-norm is assumed to be bounded ($\|g\|_{C^1(\mathbb{R}^n)} < +\infty$). The Dirichlet Lagrangian corresponding to that metric for maps from ω into $(\mathbb{R}^n, g)$ reads

$$L_g(u) := \int_\omega |\nabla u|^2_g(x,y)\ dx\wedge dy = \int_\omega \sum_{i,j} g_{ij}(u(x,y))\nabla u^i(x,y)\cdot\nabla u^j(x,y)\ dx\wedge dy \quad .$$

It is not difficult to check that this Lagrangian is again conformally invariant. Critical points to this lagrangian for perturbation of the form $u + t\psi$, where ψ is an arbitrary map in $C^\infty_0(\omega,\mathbb{R}^n)$ satisfy the following Euler Lagrange equation

$$\Delta u^i + \Gamma^i_{kj}(u)\nabla u^k\cdot\nabla u^j = 0 \quad , \tag{II.4}$$

where $\Gamma^i_{jk}(X)$ are the Christoffel symbols of the metric g at the point $X = (x^1\cdots x^n) \in \mathbb{R}^n$: $\Gamma^i_{jk}(X) = 1/2\ g^{il}(\partial_{x^k} g_{jl} + \partial_{x^j} g_{kl} - \partial_{x^l} g_{jk})$ where (g^{ij}) is the inverse matrix to (g_{ij}). This equation can then be understood as being the natural generalization of (II.3) when we consider a non necessarily trivial metric g on $\mathbb{R}^n$. This equation is called the *harmonic map equation* from ω into $(\mathbb{R}^n, g)$.

The harmonic map equation is related to the following geometrical problem : if we assume that u is a solution to (II.4) and if we additionally assume that u is a *conformal immersion* then one shows that u is a *minimal immersion* from ω into the riemannian manifold $(\mathbb{R}^n, g)$ (the mean curvature of $u(\omega)$ in $(\mathbb{R}^n, g)$ is zero).

Again, for solutions to this harmonic map equations the priority analysis questions remain the same : regularity, uniqueness, symmetry...etc as for the solutions to (II.2) or (II.3). However, due to the non-linearity in equation (II.4), one cannot work with each coordinate independently from the others and the maximum principle cannot be applied a-priori. Equation (II.4) belongs to the familly of elliptic systems with *quadratic growth*, also called elliptic systems of*natural growth*, of the form

$$\Delta u = f(u,\nabla u) \quad , \tag{II.5}$$

where $f(X,p)$ is an arbitrary continuous function for which there exist $c_0 > 0$ et $c_1 > 0$ satisfying

$$\forall X\in\mathbb{R}^n \quad \forall p\in\mathbb{R}^n\otimes\mathbb{R}^2 \qquad f(X,p) \le c_1|p|^2 + c_0 \quad . \tag{II.6}$$

This equation is *critical* in 2 dimension for the $W^{1,2}-$norm : indeed this assumption implies that the non-linearity $f(u,\nabla u)$ is in $L^1(\omega)$. Injecting this information into (II.5) gives back that u is in $W^{1,p}_{loc}$ for any $p < 2$. We are then almost "back on our feet" by getting as an outcome of this previous bootstrapping operation almost the assumption we started from which was u in $W^{1,2}$. This justifies then the qualification for this equation of being *critical* in 2-dimension relative to the $W^{1,2}$ norm (If we would instead have gained information in this bootstrapping operation, the equation would have been called *subcritical*, whereas a substancial loss of information would have made the equation *supercritical*).

Another exemple of conformally invariant Lagrangian in 2-dimension is the following : Let $\Lambda = \Lambda_{ij}(X)\ dx^i \wedge dx^j$ be a 2-form on $\mathbb{R}^n$ that we assume to be bounded in C^1 ($\|\Lambda\|_{C^1(\mathbb{R}^n)} < +\infty$). For any map u in $W^{1,2}(\omega, \mathbb{R}^n)$ from a 2-dimensional domain ω into $\mathbb{R}^n$ we introduce the following lagrangian

$$L^\Lambda(u) := \int_\omega |\nabla u|^2(x,y)\ dx \wedge dy + u^*\Lambda \quad , \tag{II.7}$$

where $u^*\Lambda$ denotes the pull-back of Λ by u :

$$u^*\Lambda = \sum_{ij} \Lambda_{ij}(u)\ (\partial_x u^i \partial_y u^j - \partial_y u^i \partial_x u^j)\ dx \wedge dy \quad .$$

Critical points of L^Λ for the above mentioned perturbations satisfy the following Euler-Lagrange equation

$$\forall i = 1 \cdots n \qquad \Delta u^i = 2H^i(u)(\partial_x u, \partial_y u) \quad , \tag{II.8}$$

where H^i are the 2-forms on $\mathbb{R}^n$ given by

$$\forall X \in \mathbb{R}^n \quad \forall\, U, V, W \in \mathbb{R}^n \quad d\Lambda(X)(U,V,W) = 4 \sum_{i=1}^n U^i\, H^i(X)(V,W) \quad . \tag{II.9}$$

As an illustration, in the particular case of $n = 3$, there exists a continuous function H such that $d\Lambda(X) = 4H(X)\, dx^1 \wedge dx^2 \wedge dx^3$. The equation (II.8) becomes then

$$\Delta u = 2H(u)\ \partial_x u \wedge \partial_y u \quad . \tag{II.10}$$

Again if u is a conformal immersion the equation (II.10) admits the following interpretation : $u(\omega)$ is a surface whose mean-curvature at $u(x,y)$ is $H(u(x,y))$. This equation is then naturally called *prescribed mean curvature equation*. We observe that under the assumption that the C^1 norm of Λ is finite, the *prescribed mean curvature equation* is again in the form (II.5)-(II.6) which is the quadratic growth form.

Finally, combining L_g and L^Λ, for an arbitrary choice of metric g on $\mathbb{R}^n$ and an arbitrary choice of a 2-form Λ on $\mathbb{R}^n$, both bounded in $C^1(\mathbb{R}^n)$, we obtain a conformally invariant problem which is a generalization of all the previous cases. Preciselly, for any function u from an open set ω of $\mathbb{R}^2$ into $(\mathbb{R}^n, g)$, we introduce the quantity

$$L_g^\Lambda(u) := \int_\omega g_{ij} \nabla u^i \cdot \nabla u^j\ dx \wedge dy + u^*\Lambda \quad . \tag{II.11}$$

The critical points of L_g^Λ satisfy the following Euler-Lagrange equation

$$\forall i = 1 \cdots n \qquad \Delta u^i + \Gamma^i_{jk} \nabla u^j \cdot \nabla u^k = 2H^i(u)(\partial_x u, \partial_y u) \quad . \tag{II.12}$$

This equation is called the *prescribed mean curvature equation in* $(\mathbb{R}^n, g)$, where H^i are the 2-forms on $\mathbb{R}^n$ satisfying

$$\forall X \in \mathbb{R}^n \quad \forall\, U, V, W \in \mathbb{R}^n \quad d\Lambda(X)(U,V,W) = 4\sum_{i,j=1}^{n} g_{ij} U^i\, H^j(X)(V,W) \quad . \tag{II.13}$$

It has been proved by M.Grüter (see also [Kp]), that the Familly of Lagrangians L^Λ_g covers all possible coercive conformally invariant Lagrangian having a quadratic growth. Precisely, we have the following result.

Theorem II.1 *[Gr1] Let $l(X,p)$ be a function from $\mathbb{R}^n \times \mathbb{R}^{2n}$ into $\mathbb{R}$. Assume that l is C^1 with respect to the first variable and C^2 with respect to the second one. Assume moreover that l satisfies the following coercivity-quadratic growth assumption :*

$$\exists C > 0 \quad s.t. \quad \forall X \in \mathbb{R}^n \quad \forall p \in \mathbb{R}^{2n} \qquad C^{-1}|p|^2 \le l(X,p) \le C|p|^2 \quad . \tag{II.14}$$

Let $\mathcal{L}$ be the lagrangian of density l defined for the maps u in $W^{1,2}$ from a domain ω of $\mathbb{R}^2$ into $\mathbb{R}^n$ and given by :

$$\mathcal{L}(u) = \int_\omega l(u, \nabla u)(x,y)\ dx \wedge dy \quad . \tag{II.15}$$

Assume finally that $\mathcal{L}$ is conformally invariant : for every positive conformal map ϕ from $\mathbb{C}$ into $\mathbb{C}$ (that is ϕ satisfy (II.1))

$$\mathcal{L}(u \circ \phi) = \int_{\phi^{-1}(\omega)} l(u\circ\phi, \nabla(u\circ\phi))(x,y)\ dx \wedge dy = \mathcal{L}(u) \quad . \tag{II.16}$$

Then there exists a metric g which is C^1 on $\mathbb{R}^n$ and a 2-form Λ which is also C^1 on $\mathbb{R}^n$ such that

$$\mathcal{L} = L^\Lambda_g \quad . \tag{II.17}$$

□

Untill now we were restricting ourselves to maps from a domain of $\mathbb{C}$ into a 1-chart manifold : $(\mathbb{R}^n, g)$. More generally we can introduce the Sobolev space $W^{1,2}(\omega, N^n)$ of maps from a domain ω into an arbitrary riemannian manifold (N^n, g). Assuming this manifold is compact without boundary (which is the assumption we shall make from now on) the definition goes as follows :

Using Nash's theorem, the manifold can be isometrically embedded in an euclidian space $\mathbb{R}^K$. The Sobolev space $W^{1,2}(\omega, N^n)$ is then the subspace of maps u in $W^{1,2}(\omega, \mathbb{R}^K)$ taking values almost everywhere into N^n. Let Λ be a C^1 2-form on N^n, for any u in $W^{1,2}(\omega, N^n)$

$$L^\Lambda(u) = \int_\omega |\nabla u|^2(x,y)\ dx \wedge dy + u^*\Lambda \quad . \tag{II.18}$$

The critical points of L^Λ in $W^{1,2}(\omega, N^n)$ are defined in the following way : Denote by π_N be the orthogonal projection on N^n which, to every point in a sufficiently small neighborhood of N^n, assignes the nearest point on N^n. For a sufficiently small neighborhood of N^n π_N is a smooth map. A map u in

$W^{1,2}(\omega, N^n)$ is said to be a critical point of L^λ if for any ψ in $C_0^\infty(\omega, \mathbb{R}^K)$ we have

$$\frac{d}{dt} L^\Lambda(\pi_N(u + t\psi))_{t=0} = 0 \quad . \tag{II.19}$$

This condition (II.19) is satisfied for every ψ in $C_0^\infty(\omega, \mathbb{R}^K)$ if and only if u is a solution to the following Euler-Lagrange Equation

$$\Delta u + A(u)(\nabla u, \nabla u) = H(u)(\nabla^\perp u, \nabla u) \tag{II.20}$$

where we are using the following notation : $A(X)$ is the second fundamental form at the point X in N^n issued from the embedding of N^n into $\mathbb{R}^K$ which, to a pair of vectors in $T_X N^n$ assignes a vector perpendicular to $T_X N^n$. $A(u)(\nabla u, \nabla u)$ at the point (x, y) of ω is precisely the vector of $\mathbb{R}^K$ given by

$$A(u)(\nabla u, \nabla u)(x, y) := A(u(x, y))(\partial_x u, \partial_x u) + A(u(x, y))(\partial_y u, \partial_y u) \quad .$$

We used moreover the notation $\nabla^\perp u$ for the $\pi/2-$rotation of the gradient of u : $\nabla^\perp u = (-\partial_y u, \partial_x u)$. $H(u)(\nabla^\perp u, \nabla u)$ at the point (x, y) of ω is then the following vector of $\mathbb{R}^K$:

$$\begin{aligned} H(u)(\nabla^\perp u, \nabla u)(x, y) \quad &:= H(u(x, y))(\partial_x u, \partial_y u) - H(u(x, y))(\partial_y u, \partial_x u) \\ &= 2H(u(x, y))(\partial_x u, \partial_y u) \quad , \end{aligned}$$

where $H(X)$ is the skew-symmetric 2-form of $T_X N^n$ taking values into $T_X N^n$ and given by

$$\forall\, U, V, W \ \in T_X N^n \qquad d\Lambda(U, V, W) := \langle U, H(X)(V, W) \rangle \quad .$$

($<\,,\,>$ is the scalar product in $\mathbb{R}^K$).

In the particular case where $\Lambda = 0$ the equation is reduced to

$$\Delta u + A(u)(\nabla u, \nabla u) = 0 \quad , \tag{II.21}$$

which is the *harmonic map equation* into N^n.

$W^{1,2}$-solutions (in the distributional sense) to (II.21) are called *weakly harmonic maps.*

We observe that the Euler Lagrange equations of the form (II.20) belong again to the elliptic systems of quadratic growth given by (II.5)-(II.6).

II.2 Regularity issues for critical points to conformally invariant lagrangians.

We have seen in the previous subsection that Euler-Lagrange equations of conformally coercive lagrangians with quadratic growth are elliptic systems of the form (II.5)-(II.6).

First one can observe that the $W^{1,2}$ solutions to these systems are not necessarily regular in dimension strictly larger than one.

Take for instance the following function

$$u(x, y) = \log \log \frac{1}{|(x, y)|} \in W^{1,2}(\omega, \mathbb{R}) \quad , \tag{II.22}$$

where ω is the disk of center 0 and radius $1/2$. One can easily check that it is a $W^{1,2}$ solution to the following elliptic equation with quadratic growth (for the most elementary non-trivial non linearity $f(X,p)$ satisfying (II.6))

$$-\Delta u = |\nabla u|^2 \quad . \tag{II.23}$$

It has moreover been proved by Frehse, [Fre], that this equation is variational and is the Euler-Lagrange equation of the following (non conformally invariant) lagrangian

$$L(u) = \int_\omega \left(1 + \frac{1}{1 + e^{12\,u}\ (\log 1/|(x,y)|)^{-12}}\right)\ |\nabla u|^2(x,y)\ dx\wedge dy \quad . \tag{II.24}$$

Hence even the variational nature of equation (II.23) does not prevent the existence of non-smooth solution to it.

Restricting ourselves to scalar equations of the form (II.5) for arbitrary smooth non-linearities f satisfying (II.6), then there is a classical result by Ladyzenskaya and Uraltseva which asserts that bounded $W^{1,2}$ solutions to such scalar equations are Hölder continuous.

This last result does not extend to systems of the form (II.5)-(II.6) as the following exemple of Frehse shows :

$$u(x,y) = \left(\sin\log\log\frac{1}{|(x,y)|}, \cos\log\log\frac{1}{|(x,y)|}\right) \in W^{1,2}(\omega,\mathbb{R}^2)\cap L^\infty \quad , \tag{II.25}$$

is a non continuous solution to the following elliptic system of quadratic growth :

$$\left\{\begin{array}{l} -\Delta u^1 = (u^1+u^2)\ |\nabla u|^2 \\ \\ -\Delta u^2 = (u^2-u^1)\ |\nabla u|^2 \quad . \end{array}\right. \tag{II.26}$$

When a system of the form (II.5)-(II.6) however is conformally invariant, S.Hildebrandt has formulated the following conjecture (which also appears in the works of E.Heinz in the particular case of the prescribed mean curvature equation see [Hei1], [Hei2] and [Hei3])

Conjecture 1 *[Hil1] [Hil2] The maps from $\mathbb{R}^2$ into $\mathbb{R}^n$ which are critical points to continuously differentiable, coercive, conformally invariant functionals are continuous.*

Remark II.1 *From classical results of Hildebrandt-Widman, [HiW], and Jost-Karcher, [JoK], the continuous critical points to continuously differentiable, coercive, conformally invariant functionals are in fact Hölder continuous $C^{0,\alpha}$ for some $\alpha > 0$. Once this Hölder continuity is known, using classical bootstrap arguments, stronger assumptions on the regularity of the functional can be transfered to the solution.*

In other words, according to conjecture 1, $W^{1,2}$-maps which are critical points of L_g^Λ, for some arbitrary choice of a C^1-bounded metric g and a C^1-bounded 2-form Λ , are continuous and hence Hölder continuous. We prove this conjecture in [Ri1] and we extend the result to the critical points to continuously differentiable, coercive, conformally invariant functionals of maps from a domain in $\mathbb{R}^2$ into an arbitrary closed C^2 submanifold of an euclidian space. We deduce in fact this result from an even more general regularity result regarding solutions to Schrödinger type systems with antisymmetric potentials as it is described in the following section.

III Conservation laws for solutions to Schrödinger systems with antisymmetric potentials and the resolution of Hildebrandt's conjecture

As explained above, considering a solution u to an elliptic system of the form (II.12), the maximum principle cannot be applied to each component u^i one by one, independently of the others due to the vectorial nature of the problem. Progresses for proving conjecture 1 came slowly until the resolution provided in [Ri1]. This resolution is based on the discovery of conservation laws which permit to write systems of the type (II.12) in divergence form.

Let us now describe these progresses.

The first result chronologically which comes to reinforce this conjecture is a result by C.B. Morrey in [Mo] which states that minimums to functionals of the form L^Λ_g are Hölder continuous. Then in the early 80's, M. Grüter proves that solutions to (II.10) which are also conformal (i.e. for which the Hopf differential is identically 0 : $\Phi(u) = |\partial_x u|^2 - |p_y u|^2 - 2i\partial_x u \cdot \partial_y u \equiv 0$) are continuous and hence Hölder continuous. In [Sc1], R.Schoen proved that maps from a domain of $\mathbb{R}^2$ into N, a C^2 submanifold of $\mathbb{R}^n$, which are critical points of L^Λ, for an arbitrary choice of a 2-form Λ in $C^1(\wedge^2 N)$, and which are also *stationary* (i.e. the Hopf differential Φ is holomorphic) are Hölder continuous. In 1990, F.Hélein, [Hel], used conservation laws, for proving that *weakly harmonic maps* from a domain of $\mathbb{R}^2$ into the unit sphere of $\mathbb{R}^n$ are analytic. In the following years, using the ideas developped by F.Hélein, F.Bethuel and J.M. Ghidaglia proved in [Be], [BeG1] and [BeG2] that $W^{1,2}$-critical points to L^Λ from a domain of $\mathbb{R}^2$ into $\mathbb{R}^3$ are Hölder continuous under various strong additional assumptions on Λ (i.e. the Lipschitz norm of $d\Lambda$ is uniformly bounded, $d\Lambda$ only depends on 2 variables...etc).

By the mean of a technic he introduced, the *Moving Frame Technic*, F. Hélein succeeded in 1991 to prove the continuity, and hence the Hölder continuity, of weakly harmonic maps between a 2-dimensional domain and an arbitrary C^2 closed submanifold of $\mathbb{R}^K$. This technic, to which an important part of the book [Hel] is devoted, consists in assigning to any weakly harmonic map u into N^n a well chosed "lifting" into the frame bundle of N^n which is nothing but a section of the frame bundle associated to the weak vector bundle $u^{-1}TN$. A refined analysis of the formulation of the harmonic map equation written relative to this frame permits to establish the regularity of weakly harmonic maps.

Finally, by adapting F.Hélein's method, P.Choné proved the continuity, and hence the Hölder continuity, of $W^{1,2}$ solutions to (II.20) under the assumption that N is C^3 and that Λ is $W^{2,\infty}$. The attempt to extend the Bethuel-Hélein-Choné approach, which requires the assumptions that N is C^3 and that Λ is

$W^{2,\infty}$, to the Hildebrant's conjecture setting, which corresponds to the assumptions N is C^2 and Λ is $W^{1,\infty}$, meets the following fundamental difficulty : the assumptions N is C^3 and Λ is $W^{2,\infty}$ imply immediately that the map $H\circ u$ is in $W^{1,2}\cap L^\infty$ exactly like u itself. From the weaker set of assumptions however, N is C^2 and Λ is $W^{1,\infty}$, one can only deduce *a-priori* that $H\circ u$ is a measurable bounded map when u is in $W^{1,2}$; u cannot transmit anymore it's regularity at the differential level ($\nabla u\in L^2$) to $H\circ u$. Hence the progresses were stuck at this point untill the work of the author,[Ri1], that we shall describe now.

Let us first recall the approach introduced by F.Hélein in order to prove the regularity of harmonic maps from a domain ω of $\mathbb{R}^2$ into the unit sphere of $\mathbb{R}^{n+1}$. In this particular case the Euler-Lagrange Equation associated to the energy L writes

$$\Delta u+u|\nabla u|^2=0 \tag{III.1}$$

(when u is smooth this equation is equivalent to the fact that Δu is parallel to u assuming $|u|\equiv 1$ of course). It has been observed by J. Shatah that, if u is a $W^{1,2}$ solution of (III.1) then, for every pair $1\le i,j\le n+1$ we have

$$div\left(u^i\nabla u_j-u_j\nabla u^i\right)=0 \quad . \tag{III.2}$$

From Poincaré lemma, we get the existence of a function $B^i_j\in W^{1,2}(\omega)$ such that

$$\nabla^\perp B^i_j=u^i\nabla u_j-u_j\nabla u^i \tag{III.3}$$

Going back to (III.1) and observing that $\sum_j u_j\ \nabla u^j=0$ we can make then appear B^i_j in that equation as follows : for any $1\le i\le n$

$$\begin{aligned}-\Delta u^i &=\sum_{j=1}^{n+1}u^i\nabla u_j\cdot\nabla u^j\\ &=\sum_{j=1}^{n+1}\left[u^i\nabla u_j-u_j\nabla u^i\right]\cdot\nabla u^j\\ &=\sum_{j=1}^{n+1}\nabla^\perp B^i_j\cdot\nabla u^j\end{aligned} \tag{III.4}$$

Introducing the notation $\nabla^\perp B$ for the vector field taking values into matrices given by $\nabla^\perp B:=(\nabla^\perp B^i_j)$, (III.1) becomes finally equivalent to

$$-\Delta u=\nabla^\perp B\cdot\nabla u \quad . \tag{III.5}$$

We then observe that the non-linearity in the R.H.S. of (III.5) has a particular structure : for every i it is a sum of the jacobians

$$\nabla^\perp B^i_j\cdot\nabla u^j=\partial_x u^j\partial_y B^i_j-\partial_y u^j\partial_x B^i_j \quad .$$

when j varies between 1 and $n+1$. We can now use the theory of integrability by compensation presented in section 1 of the present paper. Precisely we apply theorem I.1 to (III.5) and we deduce that u is continuous and hence, from remark II.1, Hölder continuous and in $W^{1,p}$ for some $p>2$. A classical bootstrapping argument permits to deduce that u is C^∞. Finally, classical results on non-linear PDE (see [Mo]) imply that u is real analytic.

As soon as one deforms the target N^n, originally S^n, into a “non perfectly round sphere”, even slightly, the attempt to prove the continuity of the solutions to (II.21) using the previous direct approach a-priori fails in this general situation.

In order to overcome this difficulty, F.Hélein introduced the *moving frame technic* we mentionned above. Beside the fact that it is relatively indirect and requires sophisticated result on the isometric embedding of submanifold into parallelisable ones, one of the disadvantage of this technic for proving the regularity of weakly harmonic maps is that it does not provide *expectable* estimates such as $W^{2,1}$ estimates of the solution. It is explained in [LiR] how such kind of estimates are useful in the analysis of the loss of compactness of sequences of solutions. $W^{2,1}-$estimates are easily obtained in the case where N is a round sphere, it would be unnatural that such an estimate suddently disappears as soon as on deforms slightly the round sphere. Hence the *moving frame technic* could be not the ”canonical” one and one should look for some alternative approach to F.Hélein’s result. Bearing this goal in mind, a legitimate question to address is :

”what really remains *special* in the non-linearity of (II.21) when the target is not a round sphere anymore ? ” .

Let us first consider the case when N^n is an oriented closed hypersurface of $\mathbb{R}^K$ (i.e. $K = n+1$). The harmonic map equation in that case writes

$$\Delta u + \nu\ \nabla\nu \cdot \nabla u = 0 \quad , \tag{III.6}$$

where ν denotes the composition of u with the unit normal vector field to N^n. Using coordinates in $\mathbb{R}^K$ (III.6) means

$$\forall i = 1 \cdots n+1 \qquad \Delta u^i + \sum_{j=1}^{n+1} \nu^i\ \nabla\nu_j \cdot \nabla\nu^j = 0 \quad . \tag{III.7}$$

Inspired by the approach we had above for the round sphere, it is tempting to substract to $\nu^i\ \nabla\nu_j$ the quantity $\nu_j\nabla\nu^i$. This is in fact possible since the sums $\sum_{j=1}\nu_j\partial_x u_j$ and $\sum_{j=1}\nu_j\partial_y u_j$ are both zero : in one hand the vector ν is normal to the tangent space to N^n in the other hand $\partial_x u$ and $\partial_y u$ are both tangent to N^n. Hence we obtain an equation which is very reminicent to the form obtained for equation (III.4) :

$$\forall i = 1 \cdots n+1 \qquad -\Delta u^i = \sum_{j=1}^{n+1} \left[\nu^i\ \nabla\nu_j - \nu_j\ \nabla\nu^i\right] \cdot \nabla\nu^j \quad . \tag{III.8}$$

There is no reason however for the divergence of the vector-field $\nu^i\ \nabla\nu_j - \nu_j\nabla\nu^i$ to be zero and to exhibit a *grad-curl* situation which would permit to write the right-hand-side of (III.8) as a linear combination of jacobians.

The main contribution in [Ri1] was to observe that the issue of getting the regularity of u solving (II.21) is not related exclusively to the fact that the divergence of $u^i\nabla u_j - u_j\ \nabla u^i)$ is zero but the continuity of the solution is also independentely a consequence of the <u>antisymmetry</u> of the matrix $(u^i\ \nabla u_j - u_j\ \nabla u^i)_{ij}$ which is still present for general targets. This antisymmetry in the non-linearity of the equation is much more robust than the *grad-curl* structure which disappears as soon as the target is not a round sphere anymore. Precisely, one of the main results in [Ri1] is the following :

Theorem III.1 *[Ri1] Let ω be a domain in $\mathbb{R}^2$, let u be a map in $W^{1,2}(\omega,\mathbb{R}^K)$ and $\Omega=(\Omega^i_j)_{1\le i,j\le K}$ a vector-field in $L^2(\omega)$ taking values into antisymmetric matrices (c.a.d. $\Omega\in L^2(\omega,so(K)\otimes\wedge^1\mathbb{R}^2)$). Assuming that u satisfies the following elliptic system*

$$-\Delta u=\Omega\cdot\nabla u \quad \textit{dans} \quad \mathcal{D}'(\omega) \quad , \tag{III.9}$$

which means using coordinates $-\Delta u^i=\sum_{j=1}^K\Omega^i_j\cdot\nabla u^j$, then u is Hölder continuous, $C^{0,\alpha}$, in ω for some $\alpha>0$. □

First we observe that this result can be applied to (III.8) by taking $\Omega^i_j=\nu^i\ \nabla\nu_j-\nu_j\ \nabla\nu^i$. This observation goes even further as we are explaining now.

We proved in [Ri1] that theorem III.1 can be applied to any Euler-Lagrange equation of conformally invariant Lagrangians which are coercive with a quadratic dependance in ∇u. Precisely we have

Theorem III.2 *Let ω be an open set of $\mathbb{R}^2$, let N^n be a C^2 orientable closed submanifold of $\mathbb{R}^K$, let Λ be a C^1 2-form on N^n and let u be a map in $W^{1,2}(\omega,N^n)$ which is a critical point to L^Λ (satisfying equation (II.20)), then there exists an L^2 vector-field Ω on ω taking values into the space of $K\times K$ antisymmetric matrices (i.e. $\Omega\in L^2(\omega,so(K)\otimes\wedge^1\mathbb{R}^2)$) such that u satisfies (III.9). u is then an Hölder continuous map on ω.* □

This proves Hildebrandt's conjecture.

For instance a uniformly bounded function H on $\mathbb{R}^3$ being given, we consider u a $W^{1,2}$ solution to (II.10) and we introduce the following vector-field in $L^2(\omega)$ taking value into $so(3)$:

$$\Omega:=H(u)\begin{pmatrix}0 & \nabla^\perp u^3 & -\nabla^\perp u^2\\ -\nabla u^3 & 0 & \nabla^\perp u^1\\ \nabla^\perp u^2 & -\nabla^\perp u^1 & 0\end{pmatrix} \tag{III.10}$$

it is then easy to check that u satisfies (III.9) and hence from theorem III.1 that u is Hölder continuous.

Some ideas behind the proof of theorem III.1 :

The proof is based on 3 main ingredients

i) The use of non-linear Hodge decomposition.

ii) The existence of conservation laws.

iii) The use of integrability by compensation theory.

A naive approach in order to try to prove theorem III.1 would be to proceed first to the L^2-Hodge decomposition of Ω and to obtain the existence of maps P and ξ, both in $W^{1,2}_{loc}(\omega,M_K(\mathbb{R}))$ and satisfying

$$\Omega=\nabla P+\nabla^\perp\xi \quad \text{in } \omega \quad . \tag{III.11}$$

(III.9) becomes then $-\Delta u = \nabla P \cdot \nabla u + \nabla^\perp \xi \cdot \nabla u$. The quantity $\nabla^\perp \xi \cdot \nabla u$ is a linear combination of jacobians of maps in $W^{1,2}$. Such a quantity is then favorable in view of applying Wente's result in section 1for getting the continuity of u. The other term $\nabla P \cdot \nabla u$ however is a scalar product of gradients and does not possess the same features as jacobian like quantities. This prevent the direct application of Wente's result in this approach.

The main idea in [Ri1] was then to replace the previous **linear Hodge decomposition** of Ω by the following **non-linear Hodge decomposition** issued from non-abelian Gauge theory : we look for P in $W^{1,2}_{loc}(\omega, SO(K))$ and ξ in $W^{1,2}_{loc}(\omega, so(K))$ satisfying

$$\Omega = P^{-1}\nabla P + P^{-1}\nabla^\perp \xi P \quad \text{ dans } \omega \tag{III.12}$$

Ω is then interpreted as being the expression of an L^2 connection of an $SO(K)$ bundle over ω with respect to some trivialization. $\nabla^\perp \xi$ is then the expression of the same connection after the change of trivialization corresponding to the multiplication by the rotation P. This particular form, $\nabla^\perp \xi$, of this new expression of the connection is divergence free and is called *Coulomb Gauge* of the connection given by Ω. The substantial advantage of (III.12) compare to (III.11), despite the non-linear nature of the decomposition, relies on the fact that the antisymmetric structure of Ω is exploided by beeing "integrated" and this integration operation, modulo the curl part $\nabla^\perp \xi$, generates a rotation valued map : P (roughly speaking the primitive of an antisymmetric matrix valued map is a $SO(K)-$valued map).

The algebraic fact that the matrix valued map P in (III.12) takes values into $SO(K)$ gives "for free" an L^∞ estimate on P which a-priori could not be deduced simply from the fact that P is in $W^{1,2}$ (this last information providing only a BMO estimate on P). This "little gain" is one of the "pivot" on which the proof of theorem III.1 is based. The existence of P and ξ satisfying the non-linear Hodge decomposition (III.12) respectively in the spaces $W^{1,2}_{loc}(\omega, SO(K))$ and $W^{1,2}_{loc}(\omega, so(K))$ is given, under the assumption that the L^2 norm of Ω is small enough, by adapting to our situation a classical work of K.Uhlenbeck [Uhl].

The second main ingredient in the proof of theorem III.1 is the discovery of conservation laws associated to the equation (III.9). This conservation laws permit to write the equation in divergence form. Precisely a second main result of [Ri1] is the following theorem whose proof is based on a direct computation.

Theorem III.3 *Let ω be a domain of $\mathbb{R}^2$, let $\Omega = (\Omega^i_j)_{1\le i,j\le K}$ be a vector-field in $L^2(\omega)$ taking values into antisymmetric matrices (i.e. $\Omega \in L^2(\omega, so(K) \otimes \wedge^1\mathbb{R}^2)$) and let A and B respectively in $W^{1,2}(\omega, Gl_K(\mathbb{R}))$ and $W^{1,2}(\omega, M_K(\mathbb{R})$ ($Gl(K)$ denotes the group of real invertible $K \times K$-matrices). We assume that A, B and Ω verify the equation*

$$\nabla A - A\Omega = \nabla^\perp B \tag{III.13}$$

where $A\Omega$ is the matrix multiplication of A and Ω. Then u in $W^{1,2}(\omega, \mathbb{R}^K)$ solves (III.9) if and only if

$$div(A\nabla u + B\nabla^\perp u) = 0 \quad . \tag{III.14}$$

$\square$

In the particular case of weakly harmonic maps into the sphere S^{K-1} for instance, (III.5) is equivalent to (III.14) by taking

$$\left\{\begin{array}{l} A = id_K = (\delta^i_j)_{1\le i,j\le K} \\ \\ \nabla^\perp B^i_j = u^i\nabla u_j - u_j \nabla u^i \end{array}\right. \tag{III.15}$$

In the general case, the local existence of A and B respectively in $W^{1,2}(\omega, Gl_K(\mathbb{R}))\cap L^\infty$ and $W^{1,2}(\omega, M_K(\mathbb{R}))$ is established in [Ri1] by the mean of the non-linear Hodge decomposition of Ω given by (III.12), assuming then that the L^2 norm of Ω is small enough (this last fact is always possible simply by localizing in space). We additionally obtain from our approach that A^{-1} is in $W^{1,2}\cap L^\infty$. The continuity of u is then a consequence of the following argumentation : we have

$$\left\{\begin{array}{l} div(A\nabla u) = \nabla^\perp B\cdot\nabla u \\ \\ rot(A\nabla u) = \nabla^\perp A\cdot \nabla u \end{array}\right. \tag{III.16}$$

Now comes the 3rd ingredient in the proof of theorem III.1 : The use of *Integrability by Compensation Theory* .

Precisely, we observe that the 2 right-hand-sides of the two equations of (III.16) are made of sums of jacobians of functions in $W^{1,2}$. Applying then Wente's estimates of section 1 we deduce that $A\nabla u$ is in $W^{1,1}$. Since A^{-1} is in $W^{1,2}\cap L^\infty$ we deduce that ∇u is in $W^{1,1}$ and hence that u is in $W^{2,1}$. By a classical Sobolev continuous injection result we then obtain that u is continuous.

A direct proof of the Hölder continuity of the solution u to (III.9)

For getting the Hölder continuity of u we can proceed as follows. First, for an arbitrary choice of point $x_0\in\omega$ and $0<r<dist(x_0,\partial\omega)$ such that $\int_{B_r(x_0)}|\nabla B|^2+|\nabla A|^2<\varepsilon$, where ε will be chosed small enough later, we take C in $W^{1,2}(B_r(x_0))$ satisfying

$$\left\{\begin{array}{l} \Delta C = div(A\nabla u) \qquad \text{in } B_r(x_0) \\ \\ C=0 \qquad \text{on } \partial B_r(x_0) \end{array}\right. \tag{III.17}$$

From Poincaré lemma there exists D in $W^{1,2}(B_r(x_0))$ satisfying

$$A\nabla u = \nabla C+\nabla^\perp D \quad . \tag{III.18}$$

We can choose D to have average 0 on $B_r(x_0)$. Classical elliptic estimates imply

$$\int_{B_r(x_0)}|\nabla C|^2+|\nabla D|^2\le c_0\int_{B_r(x_0)}|\nabla u|^2 \tag{III.19}$$

for some universal constant c_0. Let χ be a smooth cut-off function equal to 1 on $B_{1/2}(0)$ and supported in $B_1(0)$. Denote $\chi_r:=\chi(r^{-1}(\cdot-x_0))$. Using the embedding theorem of $W^{1,2}(B_r)$ into BMO It is not difficult to check that there exists a universal constant c_1 such that

$$\begin{array}{l} \|\chi_r C\|_{BMO(B_r(x_0))}\le c_1\ \|\nabla C\|_{L^2(B_r(x_0))} \\ \\ \|\chi_r D\|_{BMO(B_r(x_0))}\le c_1\ \|\nabla D\|_{L^2(B_r(x_0))} \end{array} \tag{III.20}$$

Then we multiply the first equation of (III.16) by $\chi_r C - \overline{C}_r$ and the second by $\chi_r D - \overline{D}_r$ where $\overline{C}_r$ and $\overline{D}_r$ are respectively the averages of $\chi_r C$ and $\chi_r D$ on the annulus $B_r(x_0) \setminus B_{r/2}(x_0)$. We integrate by part in the left hand sides of both equalities. Then, after applying the *integrability by compensation theory* of section 1 which gives the fact that the jacobians in the right-hand of (III.16) are in Hardy, the duality between Hardy and BMO and Cauchy-Schwartz and Poincaré inequalities we finally obtain an inequality of the form

$$\begin{aligned}
&\int_{B_{r/2}(x_0)} |\nabla C|^2 + |\nabla D|^2 \\
&\quad \le c_2\ \varepsilon\ \left[\int_{B_r(x_0)} |\nabla u|^2\right]^{\frac{1}{2}}\ \left[\|\chi_r C\|_{BMO(B_r(x_0))} + \|\chi_r D\|_{BMO(B_r(x_0))}\right] \\
&\quad +c_2 \int_{B_r(x_0)\setminus B_{r/2}(x_0)} |\nabla C|^2 + |\nabla D|^2
\end{aligned} \tag{III.21}$$

Combining this last inequality together with (III.20) we obtain, for ε chosen small enough, the existence of a constant c_3 independent of r and x_0 such that

$$\int_{B_{r/2}(x_0)} |\nabla u|^2 \le c_3 \int_{B_r(x_0)\setminus B_{r/2}(x_0)} |\nabla u|^2 \quad . \tag{III.22}$$

This last inequality implies a Morrey type decrease for the Dirichlet energy of u of the form $\int_{B_r(x_0)} |\nabla u|^2 \le c\ r^\gamma$. This last inequality valid uniformly for x_0 in any compact subset of ω implies the Hölder continuity of u (see for instance [Gi]).

Remarks on existing and possible generalizations

Theorem III.1 can be extended to solution to equation (III.9) in higher dimension see [RiSt].

In [LaR], using a similar approach as the one presented above but in a slightly more complex setting, T.Lamm and the author have proved the Hölder continuity of any 4 dimensional $W^{2,2}$ solutions to the following type of 4-th order systems

$$\Delta^2 u + \Delta(V \cdot \nabla u) + div(v\ \nabla u) + \Omega \cdot \nabla u = 0$$

where V, v and Ω are arbitrary potentials respectively in $W^{1,2}$, L^2 and $W^{-1,2}$ and where Ω is assumed to be antisymmetric. This type of equation includes for instance the intrinsic and extrinsic bi-harmonic map equations.

It is natural to believe that a general result exists for $m-$th order linear systems in m dimension whose 1st order potential is antisymmetric.

It would be interresting to study in which way theorem III.1 extends to degenerate elliptic operators. For instance Let u be a $W^{1,m}$ map from a domain in $\mathbb{R}^m$ into $\mathbb{R}^K$ satisfying a system of the form

$$-div(|\nabla u|^{m-2}\nabla u) = \Omega \cdot |\nabla u|^{m-2}\nabla u \quad . \tag{III.23}$$

Assuming $\Omega \in L^m$ is antisymmetric, is it true that u is continuous ?

IV The role of Integrability by compensation in the analysis of Willmore surfaces.

IV.1 Analysis questions related to Willmore surfaces.

For a given oriented surface Σ and a smooth positive immersion $\vec{\Phi}$ of Σ into the Euclidian space $\mathbb{R}^m$, for some $m \geq 3$, we introduce first the Gauss map $\vec{n}$ from Σ into $Gr_{m-2}(\mathbb{R}^m)$, the grassmanian of oriented $m-2-$planes of $\mathbb{R}^m$, which to every point x in Σ assigns the unit $m-2$-unit vector defining the $m-2-$plane $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$ orthogonal to the oriented tangent space $T_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$. This map $\vec{n}$ from Σ into $Gr_{m-2}(\mathbb{R}^m)$ defines a projection map $\pi_{\vec{n}}$: for every vector ξ in $T_{\vec{\Phi}(x)}(\mathbb{R}^m)$ $\pi_{\vec{n}}(\xi)$ is the orthogonal projection of ξ onto $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$. Let then $\vec{B}_x$ be the second fundamental form of the immersion $\vec{\Phi}$ of Σ. $\vec{B}_x$ is a symmetric bilinear form on $T_x\Sigma$ with values into $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$. $\vec{B}_x$ is given by $\vec{B}_x = \pi_{\vec{n}} \circ d^2\vec{\Phi}$. By the mean of the ambiant scalar product in $\mathbb{R}^m$, which induces a metric g on Σ, we define the trace of $\vec{B}_x$, $tr(\vec{B}_x)$, which is a vector in $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$ given by $tr(\vec{B}_x) = \vec{B}_x(e_1, e_1) + \vec{B}_x(e_2, e_2)$ where (e_1, e_2) is an arbitrary orthonormal basis of $T_x\Sigma$. The mean curvature vector $\vec{H}(x)$ at x of the immersion by $\vec{\Phi}$ of Σ is with theses notations the vector in $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$ given by

$$\vec{H}(x) = \frac{1}{2} tr(\vec{B}_x) \quad .$$

In the case where $m = 3$, $\vec{H}(x)$ is the product of the mean value $H = 1/2(\kappa_1 + \kappa_2)$ of the principal curvatures κ_1, κ_2 of the surface at $\vec{\Phi}(x)$ by $\vec{n}$, the unit normal vector.

The so called *Willmore Functional* is then the following Lagrangian

$$W(\vec{\Phi}(\Sigma)) = \int_\Sigma |\vec{H}|^2 \ d\,vol_g \quad . \tag{IV.1}$$

where $d\,vol_g$ is the area form of the metric g induced on $\vec{\Phi}(\Sigma)$ by the canonical metric on $\mathbb{R}^m$.

This lagrangian has been apparently first considered in the early 20th century in various works by Thomsen [Tho], Schadow and a bit later by Blaschke [Bla]. It has been reintroduced and more systematically studied in the framework of the conformal geometry of surfaces in space by Willmore in 1965 [Wil1]. Beyond conformal geometry this lagrangian plays an important role in various areas in science such as molecular biology, where it has been considered as a surface energy for lipid bilayers known as *Helfrich Model* [Hef], such as non-linear elasticity in solid mechanics where it arises as limiting energy in thin plate theory (see [FJM] for instance) or even in general relativity where the lagrangian (IV.1) is the main term in the so called *Hawking quasi local mass* (see [Haw], [HI])...etc.

One of the reason for the genericity of this lagrangian is maybe the property discovered by White [Whi] for $m = 3$ and proved by B.Y. Chen [Che] for arbitrary m which says that the functional remains unchanged under the action of a conformal diffeomorphism of $\mathbb{R}^m$ (and even under conformal changes of metric of the ambiant space). Precisely the following theorem holds

Theorem IV.1 [Conformal invariance of Willmore Functional] *[Whi], [Che]. Let $\vec{\Phi}$ be a smooth immersion of an oriented closed surface Σ in $\mathbb{R}^m$. Let Ψ be a conformal diffeomorphism of $\mathbb{R}^m \cup \{+\infty\}$. Then the following holds*

$$W(\Psi \circ \vec{\Phi}) = W(\vec{\Phi}) \quad .$$

We are interrested in this section in the critical points of (IV.1) for perturbations of the form $\vec{\Phi} + t\vec{\xi}$ where $\vec{\xi}$ is an arbitrary smooth map from Σ into $\mathbb{R}^m$. These critical points are the so called *Willmore surfaces*. Because of the invariance of the lagrangian under the action of conformal transformations of $\mathbb{R}^m$, images of Willmore surfaces by such conformal transformations are still Willmore. Examples of Willmore surfaces are minimal surfaces for which $\vec{H} \equiv 0$ and which realize then absolut minimum of W. Willmore surfaces satisfy an Euler-Lagrange equation discovered by Willmore for $m = 3$ (though it was apparently known by it's predecessors on the subject Thomsen, Schadow and Blaschke in the twenties) and was established in it's full generality, for arbitrary m, by Weiner in [Wei]. Before presenting the equation we need the following notations : Consider for every vector $\vec{w}$ in $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$ the symmetric endomorphism $A_x^{\vec{w}}$ of $T_x\Sigma$ satisfying for every pair of vectors $\vec{X}$ and $\vec{Y}$ in $T_x\Sigma$ the identity $g(A_x^{\vec{w}}(\vec{X}), \vec{Y}) = B_x(\vec{X}, \vec{Y}) \cdot \vec{w}$, where $\cdot$ denotes the standard scalar product in $\mathbb{R}^m$. The map which to $\vec{w}$ assigns the symmetric endomorphism $A_x^{\vec{w}}$ of $T_x\Sigma$ for the scalar product g is an homomorphism that we denote A_x from $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$ into $S_g\Sigma_x$, the linear space of symmetric endomorphisms from $T_x\Sigma$ with respect to g. Denote $\tilde{A}_x$ the endomorphism of $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$ obtained by composing the transpose ${}^tA_x$ of A_x with A_x : $\tilde{A}_x = {}^tA_x \circ A_x$. Let $(\vec{e}_1, \vec{e}_2)$ be an orthonormal basis of $T_x\Sigma$ and let $\vec{L}$ be a vector in $N_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$, we have that $\tilde{A}(\vec{L}) = \sum_{i,j} \vec{B}(\vec{e}_i, \vec{e}_j)\ \vec{B}(\vec{e}_i, \vec{e}_j) \cdot \vec{L}$

With these notations, $\vec{\Phi}$ is a smooth Willmore immersion if and only if it solves the following Euler-Lagrange equation

$$\Delta_\perp \vec{H} - 2|\vec{H}|^2\ \vec{H} + \tilde{A}(\vec{H}) = 0 \qquad , \tag{IV.2}$$

where $\Delta_\perp$ is the negative covariant laplacian for the connection D in the normal bundle $N\vec{\Phi}(\Sigma)$ to $\vec{\Phi}(\Sigma)$ issued from the ambiant scalar product in $\mathbb{R}^m$: for every section σ of $N\vec{\Phi}(\Sigma)$ one has $D_{\vec{X}}\sigma := \pi_{\vec{n}}(\sigma_* \vec{X})$. In the particular case when $m = 3$, the mean curvature vector $\vec{H}$ is oriented along the unit normal to $\vec{\Phi}(\Sigma)$, $\vec{H} = H\,\vec{n}$, and (IV.2) is equivalent to the following equation satisfied by the mean curvature function H :

$$\Delta_g H + 2H\ (|H|^2 - K) = 0 \qquad , \tag{IV.3}$$

where Δ_g is the negative laplace operator for the induced metric g on $\vec{\Phi}(\Sigma)$ and K is the scalar curvature of (Σ, g).

The Geometric-Analysis questions and problems one can adress regarding Willmore immersions and equations (IV.2) can be listed as follows

i) Try to give an *explicit* description of the space of Willmore immersions for a given surface Σ (in the spirit of Weierstras type representation of minimal surfaces for instance).

ii) Analyse the compactness or the lack of compactness (weak or strong ?) of the space of Willmore immersions below a certain level of energy for a given Σ.

iii) Study the existence of a minimizer (and try to identify it) of W among all smooth immersions of a given surface Σ.

iv) Does there exists a notion of weak Willmore immersions ? if so, are they necessarily regular or can these *Weak Willmore Surfaces* be singular somewhere ?

These 4 problems are strongly related one with another. The first commentarii one can make at this stage about this list are the following. Problem 1 was maybe one of the first studied chronologically and succesfully solved by R.Bryant when Σ is the sphere S^2 and for $m = 3$. In [Bry] a description of the *Moduli Space of Willmore immersions* of S^2 using algebraic-geometry tools is given. Later on similar descriptions were obtain in this direction for instance in [Mon] by S. Montiel for $m = 4$ and still for $\Sigma = S^2$. A *Weierstrass-type* representation of Willmore torii using loop groups and infinite dimensional Lie Algebras in [Hel1], [Hel2]...etc This list of results in the direction of attacking problem i) is absolutely not exhaustiv but was just intended to illustrate it.

Because of the difficulty of reaching explicit descriptions of the *Moduli Spaces* of Willmore immersions for various Σ and m, one can look instead at more qualitative properties of Willmore immersions and try to solve questions related to problem ii). The first remark one can make regarding ii) is that, given the lack of compactness of the Möbius group of conformal transformations of $\mathbb{R}^m$, it is clear that the space of Willmore immersions of a given surface below a certain level of energy is not compact. One example for this lack of compactness is the following : take the Willmore torus in $\mathbb{R}^3$ which is given by the rotation about the $z-$axis of the vertical circle contained in the Oxz-plane, of radius 1 and center the point of coordinates $(\sqrt{2}, 0, 0)$. This torus is one of the simplest example of Willmore surface in $\mathbb{R}^3$, after the round sphere S^2. Apply to $\mathbb{R}^3$ a transformation given by the composition of an inversion of center a point A that one takes outside the torus ($M \to MA/|MA|^2$) and a dilation adjusted in order to keep the area of the torus fixed. Because of the conformality of this transformation, the resulting surface is still Willmore and it has the same energy as the Willmore torus. By taking a familly of centers A closer and closer to the Willmore torus one "observes" that the sequence of resulting surfaces, after applying these transformations, is made of torii whose "holes" are getting smaller and smaller and "tend" to disappear moreover this sequence of surfaces "looks" more and more like "converging" to a round sphere. This example indicates clearly that the moduli space of Willmore torii, below a certain level of energy, cannot be strongly compact and one has to take the quotient modulo the action of the Moebius group before to hope any kind of strong compactness.

Problem iii) is easy when Σ is the round sphere S^2. A classical elementary result in differential geometry (see [Wil2]) asserts that, for any closed oriented surface S in $\mathbb{R}^m$, the following inequality holds

$$\int_S |H|^2 \, dvol_g \geq 4\pi \quad ,$$

with equality if and only if S is a round sphere. When Σ is a 2-dimensional torus the question iii) was partly solved in [Si2] by L.Simon, where it is proved, using quite involved arguments, that there exists indeed a minimizer of the Willmore energy among all possible smooth immersions of the 2-torus T^2 in $\mathbb{R}^m$. Based on results established in [Si2], M.Bauer and E. Kuwert in [BaK] were able to establish the existence of a minimizer of the Willmore energy among all possible immersions of a given arbitrary oriented closed surface Σ. Hence comes then the problem of identifying the minimizer(s). In the case where Σ is the 2-torus T^2 this question is related to the so called *Willmore Conjecture*. This conjecture claims that the Willmore torus described above is the only minimizer, modulo the Möbius group action, of the Willmore energy among all possible immersions of T^2. Until now, no proof of Willmore conjecture has been recognised and published.

Finally one could adress problem iv). The interrest of such a question should be clear to an analyst in calculus of variations. The existence of a weak notion to Willmore surfaces would give enough flexibility for instance in order to attack questions related to the compactness like ii) or to provide a new simpler proof to the question iii). This assertion is nothing less than saying that, for instance, the concept of Distributional type solutions to PDEs in such or such Sobolev space is offering a flexible setting for the understanding the PDE (and it's strong solutions too !).

IV.2 An Elliptic-Divergence Form Formulation of Willmore Equation.

Despite their elegant aspects equations (IV.2) and (IV.3) offer challenging mathematical difficulties. First of all one has to observe that the highest order term $\Delta_\perp \vec{H}$ for (IV.2) or $\Delta_g H$ for (IV.3) is non-linear since the metric g defining the Laplace operator depends on the variable immersion $\vec{\Phi}$. Another difficulty comes from the fact that the Euler-Lagrange equations (IV.2), (IV.3) are a-priori non compatible with the Lagrangian (IV.1) in the following sense. Making the minimal regularity assumption which ensures that the Lagrangian (I.1) is finite - the second fundamental form $\vec{B}$ is L^2 on $\vec{\Phi}(\Sigma)$ - is not enough in order for the non-linearities in the equations (I.2) or (I.3) to have a distributional meaning : the expression $|\vec{H}|^2\,\vec{H}$ requires at least that $\vec{H}$ is in L^3 and not only in L^2...etc.

In section 3 of the present paper we showed that any Euler Lagrange equation of any 2-dimensional conformally invariant lagrangian with quadratic growth (such as the harmonic map equations into riemannian submanifolds or such as the precribed mean curvature equation) can be written in divergence-elliptic form, see (III.14). This divergence elliptic form has numerous consequences for the analysis of this equation. We saw how the regularity of $W^{1,2}$ solutions could be deduced from it. It permits also , in particular, to extend the set of solutions to subspaces of distribution with very low regularity requirement (lower than $W^{1,2}$). Going back now to the Willmore problem, it seems that the analysis developped in [Ri1] can be extended to other conformally invariant equations such as the harmonic map equations into Lorentzian manifolds. Granting this observation together with the correspondance established by Bryant [Bry] between Willmore surfaces in $\mathbb{R}^3$ and harmonic maps into the Minkowski sphere $S^{3,1}$ in $\mathbb{R}^{4,1}$, the author found not really the technic but at least a strong encouragement for looking for an elliptic-divergence form to the Willmore Euler-Lagrange equation (IV.1).

Before to go to this elliptic-divergence form we have to make a distinction between a *divergence form* of an elliptic PDE and an *elliptic-divergence form* of such a PDE. Take for instance the harmonic map equation into the sphere S^n which is, as we saw it in the previous sections,

$$-\Delta u = u|\nabla u|^2 \quad .$$

As we mentioned it above it has been proved in [Sha] that, for $W^{1,2}$ maps into S^n, the above harmonic map equation is equivalent to

$$\forall i,j = 1\cdots n+1 \qquad div(u^i\nabla u^j - u^j\nabla u^i) = 0 \quad . \tag{IV.4}$$

This form is clearly a *divergence form* but it is not of an *elliptic-divergence form* and properties such as the regularity of the solution were not a-priori accessible from (IV.4) until F. Hélein, some years later, discovered the following *elliptic-divergence form* of the same equation discussed in the previous section

$$-\Delta u = \nabla^\perp B\cdot\nabla u \quad .$$

Hence the *divergence form* does not provides what we are looking for as long as it is not an *elliptic-divergence* form. Looking now at Willmore equation, assuming our surface realizes a graph f over a 2 dimensional domain D , the Willmore equation is obtained by looking at the condition for f to be critical of W, which is of the form

$$\int_D F(\nabla f,\nabla^2 f)\ dx$$

It is then clear that the equation could be written in *divergence form.* The difficulty however is more to identify each term in this equation, to check it's ellipticity, the nature of the non-linear part...etc.

The author established then the following result :

Theorem IV.2 *[Ri2] Willmore Euler-Lagrange Equation (I.2) is equivalent to*

$$d\left(*_g\, d\vec{H} - 3\,*_g\ \pi_{\vec{n}}(d\vec{H})\right) - d\star\left(d\vec{n}\wedge\vec{H}\right) = 0 \tag{IV.5}$$

where $*_g$ is the Hodge operator on Σ associated to the induced metric g and where $\star$ is the Hodge operator on $p-$vectors in $\mathbb{R}^m$: $\star 1$ is the unit positively oriented $m-$vector in $\mathbb{R}^m$ and in general for every pair (α,β) of $p-$vectors in $\mathbb{R}^m$ one has

$$\alpha\wedge\star\beta = <\alpha,\beta>\ \star 1$$

where $<\alpha,\beta>$ is the scalar product between α and β for the canonical metric in $\mathbb{R}^m$ (with these notations $\star(\vec{H}\wedge d\vec{n})$ is then a 1-form on Σ with values into $\mathbb{R}^m$).

Assuming the immersion $\vec{\Phi}$ is conformal from the flat disc $D^2=\Sigma$ into $\mathbb{R}^m$ then $\vec{\Phi}$ is Willmore if and only if

$$\Delta\vec{H} - 3\ div(\pi_{\vec{n}}(\nabla\vec{H})) + div\star\left(\nabla^\perp\vec{n}\wedge\vec{H}\right) = 0 \tag{IV.6}$$

where the operators Δ, div and ∇ are taken with respect to the flat metric on D^2 ($\Delta = \partial^2_{x_1}+\partial^2_{x_2}$, $divX = tr\circ\nabla$ and $\nabla = (\partial_{x_1},\partial_{x_2})$). The operator $\nabla^\perp$ denotes the rotation by $\pi/2$ of ∇ : $\nabla^\perp := (-\partial_{x_2},\partial_{x_1})$. □

The proof of the equivalence between Willmore equation (IV.2) and the new equation (IV.6) in conformal coordinates is quite short and we present it now in 3 dimensions.

The following equation is a general result in differential geometry of surfaces and holds for any smooth conformal parametrization $\vec{\Phi}$.

$$-2H\ \nabla\vec{\Phi} = \nabla\vec{n} + \vec{n}\wedge\nabla^\perp\vec{n} \quad . \tag{IV.7}$$

Taking the divergence of this equation yields

$$-2H\ \Delta\vec{\Phi} - 2\nabla H\cdot\nabla\vec{\Phi} = div\left(\nabla\vec{n} + \vec{n}\wedge\nabla^\perp\vec{n}\right) \quad . \tag{IV.8}$$

Multiplying the later equation by H, using (IV.7) and the fact that in conformal parametrization the mean curvature vector $\vec{H}$ is given by $\vec{H} = e^{-2\lambda}\ \Delta\vec{\Phi}$, where $e^\lambda = |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$ is the conformal factor, gives

$$-4\ H^2\ e^{2\lambda}\ \vec{H} + \nabla H\cdot\left(\nabla\vec{n} + \vec{n}\wedge\nabla^\perp\vec{n}\right) = H\ div\left(\nabla\vec{n} + \vec{n}\wedge\nabla^\perp\vec{n}\right) \quad . \tag{IV.9}$$

The Willmore equation in it's original form (IV.3) in conformal coordinates writes

$$\Delta H\ \vec{n} = -2e^{2\lambda}\ H\ (H^2 - K)\ \vec{n} \quad , \tag{IV.10}$$

and the definition of the Gauss curvature K is

$$K\ \vec{n} = \frac{e^{-2\lambda}}{2}\ \nabla\vec{n}\wedge\nabla^\perp\vec{n} \quad . \tag{IV.11}$$

Combining the three last equations we obtain

$$div\left(H\ \nabla\vec{n} - 2\nabla H\ \vec{n} - H\ \vec{n}\wedge\nabla^\perp\vec{n}\right) = 0 \tag{IV.12}$$

which is (IV.6) in 3 dimension (in this dimension one has $\vec{H} = H\ \vec{n}$ and therefore $\nabla\vec{H} = \nabla H\ \vec{n} + H\ \nabla\vec{n}$ and $\pi_{\vec{n}}(\nabla\vec{H}) = \nabla H\ \vec{n}$).

We can now make the following important information about (IV.6)

$$\pi_{\vec{n}}(\vec{v}) := \vec{n}\llcorner(\vec{n}\llcorner\vec{v})$$

where $\llcorner$ is the interior multiplication between $q-$ and $p-$vectors $q \geq p$ producing $q-p-$vectors in $\mathbb{R}^m$ obtained from the usual interior multiplication between $q-$vectors and $p-$forms (see [Fe] 1.5.1 combined with 1.7.5) and the duality issued from the canonical scalar product in $\mathbb{R}^m$ (denoted by γ_p in [Fe]) : for every choice of $q-$, $p-$ and $m-q+p-$forms, respectively α, β and γ the following holds

$$\langle\alpha\llcorner\beta,\gamma\rangle = \langle\alpha,\beta\wedge\gamma\rangle \quad .$$

With this notations one has then

$$div(\pi_{\vec{n}}(\nabla\vec{w})) = \Delta\left[\vec{n}\llcorner(\vec{n}\llcorner\vec{w}(x))\right] - \vec{n}\llcorner(\nabla\vec{n}\llcorner\vec{w}(x)) - \nabla\vec{n}\llcorner(\vec{n}\llcorner\vec{w})(x) \tag{IV.13}$$

Thus, assuming now that the unit $m-2-$vector $\vec{n}$ is in $W^{1,2}$, the distribution

$$\mathcal{L}_{\vec{n}}\vec{w} := \Delta\vec{w} - 3\ div(\pi_{\vec{n}}(\nabla\vec{w})) + div\star(\nabla^\perp\vec{n}\wedge\vec{w}) \quad . \tag{IV.14}$$

is well defined for an arbitrary choice of $\vec{w}$ in $L^2(D^2)$. This shows that the Euler-Lagrange equation in the form (IV.5) or (IV.6) is compatible with the Lagrangian (IV.1). Indeed the equation has a distributional sense under the least possible regularity requirement for the immersion $\vec{\Phi}(\Sigma)$: This minimal requirement is for the Gauss map to be in $W^{1,2}$ on Σ with respect to the induced metric. This leads to the definition of *Weak Willmore Immersions*.

IV.3 Weak Willmore Immersions.

Before to give a weak formulation to Willmore immersions we define first the notion of *Weak immersion with L^2-bounded second fundamental form.*

Definition IV.1 [Weak immersions with L^2-bounded second fundamental form] *Let $\vec{\Phi}$ be a $W^{1,2}$ map from a 2-dimensional manifold Σ into $\mathbb{R}^m$. $\vec{\Phi}$ is called a weak immersion with locally L^2-bounded second fundamental form if the following holds : For every $x \in \Sigma$ there exists an open disk D in Σ, a constant $C > 0$ and a sequence of smooth embeddings $\vec{\Phi}^k$ from D into $\mathbb{R}^m$ such that*

i)

$$\mathcal{H}^2(\vec{\Phi}(D)) \neq 0$$

ii)

$$\mathcal{H}^2(\vec{\Phi}^k(D)) \leq C < +\infty$$

iii)

$$\int_D |B^k|^2 \, dvol_{g^k} \leq \frac{8\pi}{3}$$

iv)

$$\vec{\Phi}^k \rightharpoonup \vec{\Phi} \quad \textit{weakly in } W^{1,2}$$

where $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure, B^k is the second fundamental form associated to the embedding $\vec{\Phi}^k$ and g^k denotes the metric obtained by the pull-back by $\vec{\Phi}^k$ of the induced metric on $\vec{\Phi}^k(\Sigma)$. □

For example $W^{2,2}$ graphs in $\mathbb{R}^3$ of maps from $\mathbb{R}^2$ into $\mathbb{R}$ are weak immersions with L^2-bounded second fundamental form.

The following result was established by F. Hélein (théorème 5.1.1 of [Hel]) as a generalization of results due to T. Toro [To1], [To2] and S. Müller and V. Sverak [MS].

Theorem IV.3 *[Hel]* **[Existence of local conformal coordinates for weak immersions]** *Let $\vec{\Phi}$ be a weak immersion from a 2-dimensional manifold Σ into $\mathbb{R}^m$ with L^2-bounded second fundamental form. Then for every x in Σ there exists an open disk D in Σ containing x and an homeomorphism Ψ of D such that $\vec{\Phi} \circ \Psi$ is a conformal bilipschitz immersion and the induced metric g on D from the standard metric of $\mathbb{R}^m$ is continuous in this parametrization. Moreover the Gauss map $\vec{n}$ of this immersion is in $W^{1,2}(D, Gr_{m-2}(\mathbb{R}^m)$ for the induced metric g.* □

We can now give the definition of a *Weak Willmore immersions with L^2-bounded second fundamental form.*

Definition IV.2 [Weak Willmore immersions with L^2-bounded second fundamental form]. *A weak immersion $\vec{\Phi}$ from a 2-dimensional manifold Σ into $\mathbb{R}^m$ with L^2- bounded second fundamental form is Willmore when, about every point x in Σ, in a conformal parametrization given by theorem IV.3 from the 2 dimensional disk D^2 the following equation holds*

$$\Delta\vec{H} - 3\ div(\pi_{\vec{n}}(\nabla\vec{H})) + div \star \left(\nabla^\perp\vec{n}\wedge\vec{H}\right) = 0 \quad in\ \mathcal{D}'(D^2). \tag{IV.15}$$

where Δ, ∇, $\nabla^\perp$ □

This definition makes sense since, as observed above, the expression

$$\nabla\vec{H} - 3\pi_{\vec{n}}(\nabla\vec{H}) + \star(\nabla^\perp\vec{n}\wedge\vec{H})$$

has a distributional meaning as soon as the Gauss map $\vec{n}$ is in $W^{1,2}$.

For example, in 3 dimension, the graph of a $W^{2,2}$ function f over the disc D^2 defines a weak immersions in $\mathbb{R}^3$ with L^2-bounded second fundamental form. According to [To1], [MS] it admits locally a bilipschitz conformal parametrization and hence definition IV.2 applies to that situation.

We have hence defined the notion of $W^{2,2}$ Willmore graph in $\mathbb{R}^3$ and $W^{2,2}$ is clearly the minimal requirement for having an L^2 bounded second fundamental form. In this sense our weak notion of Willmore immersion is clearly optimal.

IV.4 Integrability by compensation and the regularity of Weak Willmore immersions with L^2-bounded second fundamental form.

It is not difficult to see that for any $\vec{n}\in W^{1,2}(D^2, Gr_{m-2}(\mathbb{R}^m)$ the *Willmore operator* $\mathcal{L}_{\vec{n}}$ defined by (IV.14) is elliptic and continuous from L^p into $W^{-2,p}$ for any $p>2$. One checks also that $\mathcal{L}_{\vec{n}}$ is formally selfadjoint. Moreover the following invertibility property holds again for any $p>2$.

- $\exists\,\epsilon_0>0$ s.t. if $\int_{D^2}|\nabla\vec{n}|^2<\epsilon_0$ then

$$\forall\vec{\xi}\in C_0^\infty(D^2,\mathbb{R}^3) \qquad \|\vec{\xi}\|_{L^p}\le C\|\mathcal{L}_{\vec{n}}\vec{\xi}\|_{W^{-2,p}} \quad . \tag{IV.16}$$

These last 2 facts, continuity and local invertibility of $\mathcal{L}_{\vec{n}}$, do not work anymore for $p=2$!

This makes the relevant case for Willmore, the case $p=2$, critical in the usual non-linear PDE sense and some "$\varepsilon-$gain" of integrability has to be found in the non-linear terms in the equation $\mathcal{L}_{\vec{n}}\vec{H}=0$ in order to establish some regularity of $\vec{H}$. This $\varepsilon-$gain of regularity will come from the application of the *Integrability by compensation theory* to the following *conservation laws* for *Weak Willmore Immersions.*

Theorem IV.4 [Conservation laws for weak Willmore immersions] *[Ri2] Let $\vec{\Phi}$ be a weak Willmore immersion from a disk D^2 into $\mathbb{R}^m$ with L^2-bounded second fundamental form. Assume $\vec{\Phi}$ is conformal and denote $\vec{L}$ the map from D^2 into $\mathbb{R}^m$ satisfying*

$$\nabla^\perp\vec{L} := \nabla\vec{H} - 3\pi_{\vec{n}}(\nabla\vec{H}) + \star(\nabla^\perp\vec{n}\wedge\vec{H}) \quad , \tag{IV.17}$$

then the following identities hold

$$\nabla\vec{\Phi}\cdot\nabla^{\perp}\vec{L} = 0 \quad , \tag{IV.18}$$

and

$$\nabla\vec{\Phi}\wedge\nabla^{\perp}\vec{L} = 2\ (-1)^m\ \nabla\left[\star(\vec{n}\llcorner\vec{H})\right]\llcorner\nabla^{\perp}\vec{\Phi} \tag{IV.19}$$

Moreover, denote respectively S and $\vec{R}$ the scalar and $2-$vector valued functions on D^2 given by $\nabla S := \vec{L}\cdot\nabla\vec{\Phi}$ and $\nabla\vec{R} := \nabla\vec{\Phi}\wedge\vec{L} + 2\ (-1)^m\left[\star(\vec{n}\llcorner\vec{H})\right]\llcorner\nabla\vec{\Phi}$, then the following identity holds

$$\nabla^{\perp}\vec{R} = (-1)^{m-1}\ \star\left(\vec{n}\bullet\nabla\vec{R}\right) + \star\vec{n}\ \nabla S \tag{IV.20}$$

Where $\bullet$ is the first order contraction between multivectors given by $\alpha\bullet\beta = \alpha\llcorner\beta$ when β is a $1-$vector and $\alpha\bullet(\beta\wedge\gamma) = (\alpha\bullet\beta)\wedge\gamma + (-1)^{pq}(\alpha\bullet\gamma)\wedge\beta$ where β and γ are arbitrary $p-$ and $q-$vectors. The equation (IV.20) implies in particular that on D^2 the following equation holds

$$\Delta\vec{R} = (-1)^{m-1}\star\nabla^{\perp}\vec{n}\bullet\nabla\vec{R} + \nabla^{\perp}(\star\vec{n})\cdot\nabla S \tag{IV.21}$$

□

Observe that the operators $\star$, $\llcorner$ and $\bullet$ commute with derivatives in D^2. Therefore the identities (IV.18) and (IV.19) express the vanishing of linear combinations of certain jacobians (i.e. are divergence free quantities in particular) and because of this very special structures they pass to the limit under weak convergence of Willmore surfaces having uniformly bounded Willmore energies. This observations justify then the name *conservation laws* for the identities (IV.18) and (IV.19). The last equation (IV.21) tells us that the laplacian of $\vec{R}$ is given by a linear combination of jacobians. In order to prove the regularity of *Weak Willmore Immersions*, this last fact will be interpreted by the mean of the Wente's estimates described in section 1 of the present paper in the following way.

The proof of the continuous differentiability of Weak Willmore Immersions using the integrability by compensation theory.

Let $\vec{\Phi}$ be a *Weak Willmore Immersion* in conformal parametrization given by theorem IV.3. Then, from (IV.17), the gradient of $\vec{L}$ is in the space $H^{-1}+L^1$. Therefore $\vec{L}$ is in the Lorentz space $L^{2,\infty}$, this implies that both ∇S and $\nabla\vec{R}$ are in $L^{2,\infty}$. Since $\nabla\vec{n}$ is in L^2 we are in the position to apply theorem I.2 to equation (IV.21) and to deduce that in fact $\nabla\vec{R}$ is in L^2. Taking the scalar product between $\vec{n}$ and equation (IV.20) we obtain from the fact that $\nabla\vec{R}$ is in L^2 the fact that ∇S is also in L^2. Going back now to equation (IV.21), these latest informations permit to deduce that $\vec{R}$ is in fact in $W^{2,1}$ by invoquing the *integrability by compensation* result theorem I.1. By taking the scalar product again between $\vec{n}$ and (IV.20) we obtain also that S is in $W^{2,1}$. Denote by $(\vec{e}_1,\vec{e}_2)$ the orthonormal basis $(\vec{e}_1,\vec{e}_2) = e^{-\lambda}(\partial_x\vec{\Phi},\partial_y\vec{\Phi})$ (where we recall that e^{λ} is the conformal factor $e^{\lambda} = |\partial_x\vec{\Phi}| = |\partial_y\vec{\Phi}|$). An elementary computation gives

$$-2e^{\lambda}\ \vec{H} = \frac{\partial\vec{R}}{\partial x}\llcorner\vec{e}_2 + (\vec{L}\cdot\vec{e}_2)\ \frac{\partial\vec{\Phi}}{\partial x} = \frac{\partial\vec{R}}{\partial x}\llcorner\vec{e}_2 + \frac{\partial S}{\partial y}\vec{e}_1 \quad . \tag{IV.22}$$

Since, from theorem IV.3, $\vec{\Phi} \in W^{2,2}$ and since $\lambda \in L^\infty \cap W^{1,2}$, we deduce from $\vec{R},\ S \in W^{2,1}$ that $\vec{H} \in W^{1,1}$. This implies that $\vec{H}$ is in the Lorentz space $L^{2,1}$. Hence, using the equation giving the mean curvature vector in conformal parametrization

$$\Delta\vec{\Phi} = 2\, e^{2\lambda}\ \vec{H} \quad . \tag{IV.23}$$

we obtain that $\nabla^2\vec{\Phi}$ is in $L^{2,1}$ also and, by Lorentz-Sobolev embedding that $\nabla\vec{\Phi}$ and $\vec{n}$ are continuous. Then, locally, the immersion realizes a C^1 Willmore graph everywhere. □

It is explained in [Ri2] section III.2 how to pass from the fact that the mean curvature vector is in $W^{1,1}$ to the full regularity of the Weak Willmore Immersions. Precisely we have.

Theorem IV.5 [Regularity for weak Willmore immersions] *Let $\vec{\Phi}$ be a weak Willmore immersion from a 2-dimensional manifold Σ into $\mathbb{R}^m$ with L^2-bounded second fundamental form, then $\vec{\Phi}(\Sigma)$ is the image of a real analytic immersion.* □

The deduction of the smoothness of weak Willmore immersions from the fact that the mean curvature vector is in $W^{1,1}$, that we previously established above, comes from the following $\epsilon-$regularity result. *Integrability by compensation theory* is used again for proving that result see [Ri2] section III.2. The $\epsilon-$regularity result says the following.

Theorem IV.6 [$\epsilon-$regularity for weak Willmore Immersions] *Let $\vec{\Phi}$ be a weak Willmore immersion from the unit 2 dimensional disk D^2 into $\mathbb{R}^m$ with L^2-bounded second fundamental form. There exists $\varepsilon > 0$, independent of $\vec{\Phi}$ such that the following holds. Let $\vec{n}$ be the Gauss map associated to the weak immersion $\vec{\Phi}$, assuming $\vec{\Phi}$ is the bilipschitz parametrization given by theorem IV.3, if*

$$\int_{D^2} |\nabla\vec{n}|^2\ dvol_g \le \varepsilon \quad , \tag{IV.24}$$

then for every $k \in \mathbb{N}$ there is a positive constant C_k depending only on k such that

$$\|\nabla^k\vec{n}\|^2_{L^\infty(D^2_{1/2})} \le C_k\ \int_{D^2} |\nabla\vec{n}|^2\ dvol_g \tag{IV.25}$$

where $D^2_{1/2}$ is the disk of radius $1/2$ for the flat metric on D^2. □

This $\epsilon-$regularity result was already established using a different approch in [KS1] under the assumption that the Willmore surface is smooth. Hence this original $\epsilon-$regularity result, with the regularity assumption, could not be applied in our situation. Our goal was instead to prove that these regularity assumptions are correct.

IV.5 Integrability by compensation and compactness questions related to Willmore surfaces.

In this part we adress problem ii) above namely “Analyse the compactness or the lack of compactness (weak or strong ?) of the space of Willmore immersions below a certain level of energy for a given Σ”. And we explain how the existing answers provided to such kind of questions are related to our resolution of problem iv) in the previous subsection and are then related to *Integrability by Compensation* phenomena.

IV.5.1 The small energy case.

From the $\epsilon-$regularity result it is clear that a sequence of Willmore immersions of a disk having small enough energy, converges in any norm, to a limiting Willmore Immersion on any strictly included subset of the Disk. An interresting question regarding compactness and Willmore energy under small energy assumption is the following.

Problem 1 : Given a sequence of conformal immersions $\vec{\Phi}_k$ of the 2-Disk D^2 into $\mathbb{R}^m$ satisfying $\int_{D^2} |\nabla\vec{n}_k|^2 \le \varepsilon$ for some $\epsilon > 0$ and assuming that

$$\mathcal{L}_{\vec{n}_k}\vec{H}_k \longrightarrow 0 \text{ strongly in } H^{-2} \quad , \tag{IV.26}$$

is it true that, modulo extraction of a subsequence, $\vec{\Phi}_k$ converges to a Willmore Immersion $\vec{\Phi}$?

In other word the question we are raising here is to know whether Palais Smale sequences of Willmore Functional, under the small energy assumption, converge necessarily to Willmore immersions. This question is presently still open. The resolution of that question would lead for instance to a new proof of L.Simon's result (problem iii)) of the existence of a minimizer to Willmore energy among all possible immersions of torii.

Inspired by the results presented in the previous subsection one can observe that the conservation laws (IV.18), (IV.19) will be automatically satisfied. No comes the following question :

Problem 2 : Let $\vec{\Phi}$ be a bilipschitz $W^{2,2}$ conformal immersion of the 2-dimensional disk D^2 into $\mathbb{R}^m$. Assuming that there exists a map $\vec{L}$ in $L^{2,\infty}(D^2,\mathbb{R}^m)$ satisfying

$$\nabla\vec{\Phi}\cdot\nabla^\perp\vec{L} = 0 \quad , \tag{IV.27}$$

and

$$\nabla\vec{\Phi}\wedge\nabla^\perp\vec{L} = 2\,(-1)^m\,\nabla\left[\star(\vec{n}\llcorner\vec{H})\right]\llcorner\nabla^\perp\vec{\Phi} \tag{IV.28}$$

where $\vec{n}$ is the unit normal ($\vec{n} = e^{-2\lambda}\partial_x\vec{\Phi}\wedge\partial_y\vec{\Phi}$) and where $\vec{H}$ is the mean curvature vector given by equation (IV.23). Does it imply that $\vec{\Phi}$ is Willmore ?

If the answer to the previous question would be positive then the above problem 1 about Palais-Smale sequences for Willmore under the small energy assumption would be solved.

IV.5.2 The medium energy case. The Li-Yau 8π condition.

The following Energy condition is one of the beautiful achievement of the paper by P.Li and S.T.Yau [LY].

Theorem IV.7 *[LY] Let Σ be a closed 2-manifold. Let $\vec{\Phi}\ :\ \Sigma \longrightarrow \mathbb{R}^m$ be an immersion. Assume there exists $p \in \mathbb{R}^m$ such that*

$$\vec{\Phi}^{-1}(\{p\}) = \{x_1,\cdots,x_n\} \quad ,$$

x_i *distinct. Then*

$$\int_\Sigma |\vec{H}|^2 \, dvol_g \geq 4\pi \, n \quad .$$

As consequence we have that, if an immersion satisfies

$$\int_\Sigma |\vec{H}|^2 < 8\pi \quad , \tag{IV.29}$$

then it is an embedding. We will call the condition given by (IV.29), the *Li-Yau* 8π *condition.*

The aim now is to prove that *weak limits of Willmore surfaces under the Li-Yau condition are still Willmore.*

Consider a familly of Willmore immersions S_n with fixed topology and bounded Willmore energy. Modulo the action of the Moebius group of conformal transformations of $\mathbb{R}^m$, which preseves Willmore Lagrangian, and therefore Willmore equation (IV.5), we can always fix the area of each S_n to be equal to 1. Now using Federer Fleming argument we can extract a subsequence to that sequence such that the current of integration on S_n converges for the Flat topology to some limiting integral current of integration S (see [Fe] for the terminology of integral currents). Since S_n has a uniformly bounded Willmore energy and a fixed topology, the L^2 norm for the induced metric of it's second fundamental form, and hence the $W^{1,2}-$norm on the surface of the Gauss map, are bounded. Applying then theorem IV.6 and a classical argument of concentration compactness we then deduce that S_n converges, in a suitable parametrization, in the C^k topology to S outside finitely many points $\{p_1, \cdots, p_k\}$. This strong convergence implies that S is a smooth Willmore surface a-priori outside these points. The question to know whether these singular points are so called "removable" or not is then fundamental. In the case where p_k is a point of density less than 2 the regularity of S is given by following result which extends to arbitrary codimensions the main result of [KS3].

Theorem IV.8 [Point removability for Willmore immersions.] *Let $\vec{\Phi}$ be a continuous map from D^2 into $\mathbb{R}^m$ with $\vec{\Phi}(0) = x_0$. Assume that $\vec{\Phi}$ realizes a finite area Willmore immersion over $D^2 \setminus \{0\}$ and that the $W^{1,2}$ energy of the Gauss map on $D^2 \setminus \{0\}$ is bounded. Let μ be the measure given by the product of the restriction to $\vec{\Phi}(D^2)$ of the 2-dimensional Hausdorff measure $\mathcal{H}^2$ in $\mathbb{R}^m$ with the multiplicity function from $\vec{\Phi}(D^2)$ into $\mathbb{N}$ which to each point in $\vec{\Phi}(D^2)$ assigns it's number of preimage by $\vec{\Phi}$. Assume further that*

$$\liminf_{r\rightarrow 0} \frac{\mu(B_r^m(x_0))}{\pi r^2} < 2 \tag{IV.30}$$

Then $\vec{\Phi}(D^2)$ is a $C^{1,\alpha}$ submanifold of $\mathbb{R}^m$ for every $\alpha < 1$. Moreover, if $\vec{H}$ denotes the mean curvature vector of this submanifold, there exists a constant vector $\vec{H}_0$ such that $\vec{H}(x) - \vec{H}_0 \log|x - x_0|$ is a $C^{0,\alpha}$ function on $\vec{\Phi}(D^2)$ where $|x - x_0|$ denotes the distance in $\vec{\Phi}(D^2)$ between x and x_0. If $\vec{H}_0 = 0$ then $\vec{\Phi}$ is an analytic Willmore immersion on the whole D^2. $\square$

Since the work of Robert Bryant [Bry] counter-examples to the above point removability results are known when instead of (IV.30) one has

$$\liminf_{r\to 0} \frac{\mu(B_r^m(x_0))}{\pi r^2} = 2 \quad . \tag{IV.31}$$

It goes as follows. Consider the cathenoid in $\mathbb{R}^3$ with center 0, axis the z axis and containing the horizontal unit circle in the $Oxy-$plane. It is known that this cathenoid is a minimal surface and hence Willmore. Apply to this surface the inversion (conformal) $x \to x/|x|^2$. We obtain a surface that we can compactify by adding the origin. Let S be this compact surface . It is of course Willmore outside the origin and condition (IV.31) is satisfied for $x_0 = 0$. this surface is self-tangent at the origin : it intersects itself at a point where the 2-tangent planes coincide and hence it is not an immersed Willmore surface and the conclusions of theorem IV.8 does not apply in that case.

The proof of theorem IV.8 for $m = 3$ given by Kuwert and Schätzle is using the original form (IV.3) of Willmore equation outside the point to be removed. It is quite involved and it goes through a technical and deep lemma, the "power decay lemma", which shows roughly that solutions to scalar equations of the form (IV.3) on the unit disk minus it's origin, under some decay assumption on $|H|$ near the origin, behave "nicely" (in the spirit of a Liouville-type theorem). This approach could have been very painful to extend in higher dimension since one has to extend the "power decay lemma" to systems. Our approach instead is based on the new formulation of Willmore equation (IV.6). The strength of it is that, because of it's divergence nature, the equation can be written on the whole disk D^2 without to have to remove the origin a-priori. A simple regularity consideration combined with a classical result on the distributions supported at one point leads to the following identity

$$\mathcal{L}_{\vec{n}}\vec{H} = \vec{c}_0\ \delta_0 \quad \text{ in } \mathcal{D}'(D^2). \tag{IV.32}$$

The goal is then to show that the vectorial residue $\vec{c}_0$ is zero. This is done in [Ri2] again by using an argument based on *Integrability by Compensation Theory.*

Granting theorem IV.8 for $m = 3$ Kuwert and Schätzle were able to establish the fact that the limit S of Willmore surfaces S_n is again a smooth Willmore submanifold in $\mathbb{R}^3$ under the assumption that the Willmore energy of the S_n is less than $8\pi - \delta$ for any fixed $\delta > 0$. This last fact ensures that S will be a graph about each p_i, $i = 1\cdots k$ and that the residues $\vec{H}_0$ will be equal to 0 at each p_i. The arguments, in order to prove that, under the *strict Li-Yau assumption* $W(S_n) < 8\pi - \delta$, S is a graph about each p_i and that the residues $\vec{H}_0$ at each p_i are equal to 0, can be found page 344 of [KS3] and are not specific to the codimension 1. Therefore, combining them with our point removability result, theorem IV.8, with these arguments we can now state our last main result

Theorem IV.9 [Weak compactness of Willmore Surfaces under the strict Li-Yau $8\pi-$condition] *Let m be an arbitrary integer larger than 2. Let $\delta > 0$. Consider $S_n \subset \mathbb{R}^m$ to be a sequence of smooth closed Willmore embeddings with uniformly bounded topology, area equal to one and Willmore energy $W(S_n)$ bounded by $8\pi - \delta$. Assume that S_n converges weakly as varifolds* *to some limit S which realizes a non zero current. Then S is a smooth Willmore embedding.* □

Finally, combining again arguments in [KS3] (pages 350-351) together with our point removability result and a theorem by Montiel in [Mon], which in particular implies that any non-umbillic Willmore 2-spheres in $\mathbb{R}^4$ has Willmore energy larger than 8π (this was known in $\mathbb{R}^3$ since the work of Bryant [Bry]), we obtain the following.

Theorem IV.10 [Strong compactness of Willmore torii under the strict Li-Yau $8\pi-$condition] *Let $m = 3$ or $m = 4$. Let $\delta > 0$ arbitrary. The space of Willmore embedded torii in $\mathbb{R}^m$ having Willmore energy less that $8\pi - \delta$ is compact up to Möbius transformations under smooth convergence of compactly contained surfaces in $\mathbb{R}^m$.* □

This extends to $m = 4$ theorem 5.3 of [KS3] where the above statement was proved for $m = 3$.

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