Optimal estimate for the gradient of Green's function on degenerating surfaces and applications

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Abstract

In this paper we prove a uniform estimate for the gradient of the Green function on a closed Riemann surface, independent of its conformal class, and we derive compactness results for immersions with L^2-bounded second fundamental form and for riemannian surfaces of uniformly bounded gaussian curvature entropy.

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Introduction

Let Σ a closed smooth surface of genus g. We can endow Σ with a metric h, then thanks to the uniformization theorem, see [13] or [20], there exists in the conformal class of h, i.e. the set of metric on Σ which can be written $e^{2u}h$ where u is a smooth function, a metric of constant curvature, equal to 1 if $g = 0$, 0 if $g = 1$ and -1 otherwise. The sphere case is very particular, since the conformal group is not compact but this case is not of great interest here since there is only one conformal class. In the hyperbolic case the metric is unique and in the torus it is also true up to normalized the area. In the following, we assume this normalization and we still denote by h the metric of constant curvature (when $g \geq 1$) and we associate to h its Laplace-Beltrami operator Δ_h. Then there exists, up to normalization, a unique nonnegative Green function G_h associated to Δ_h.
The main goal of this paper is to give estimates on G_h independently of the conformal class defined by the metric h. This is a very classical subject in the theory of Riemann surfaces strongly related with the behavior of the spectrum of the Laplace operator, see [3]. Let (Σ, h_k) a sequence of hyperbolic surface whose conformal class degenerate, that is to say that some geodesics are pinching. Let us assume that there is only one degenerating geodesic γ_k, let denote Σ_∞ its nodal limit, see section 1 for precise definition, then Ji, see [12], proved that If γ_k does not separate Σ_k then G_k is uniformly bounded on every compact of $\Sigma_\infty \times \Sigma_\infty$, else $\lim_{k\to+\infty} |G_k| = +\infty$ on $\Sigma_\infty \times \Sigma_\infty$.

Here we see that we have a very different behavior with respect to the manner the conformal class degenerates. Is a similar behavior is possible for derivatives? Indeed, formally we can write

$$G_k(x,y) = \sum_{i\geq 1} \frac{\varphi_i^k(x)\varphi_i^k(y)}{\lambda_i^k},$$

where λ_i^k and φ_i^k are respectively the ith (non vanishing) eigenvalue and the ith (non constant) eigenfunction of Δ_{h_k}, repeating indices according to multiplicity. Of course if the the nodal limit is disconnected then the first eigenvalue goes to zero while the first eigenfunction goes to a positive locally constant function depending on the genus of each connected component. Looking at derivatives instead, one can

expect a better behavior of the Green function, even in the collar region. However, the gradient of the Green function gets a simple pole on the diagonal, hence it is not in L^2. The main result of this paper establishes that it is however true in a slightly weaker norm.

Theorem 0.1. *Let Σ be a closed surface then there exits C a positive constant and an integer N depending only on the genus of Σ such that for any metric h on Σ with constant curvature equal to 1, 0 or -1 and with normalized volume*[1]*, and any Green function associated to h, there exists a finite atlas of N conformal charts (U_i, ψ_i), such that for any $y \in \Sigma$ we get*

$$\sup_{t>0} t^2 \left|\left\{x \in V_i \mid |d_x G_h^i(x,y)| \geq t\right\}\right| \leq C, \tag{1}$$

where $V_i = \psi_i(U_i)$ and $G_h^i(\,.\,,y) = (\psi_i)_*(G_h(\,.\,,y))$.

We can remark that on a fix Riemann surface (Σ, h), the Green function is always bounded for this weak L^2-norm. We can deduce it from the standard pointwise estimate, see [1],

$$|d_x(G_h^i(x,y))| \leq \frac{C_h}{d_h(x,y)},$$

but of course this estimate depends on the metric we take on Σ.

At the knowledge of authors, this result is the first control of the Green function independent of the conformal class. Moreover it looks quite optimal since the result is clearly false in L^2. In fact the atlas is very explicit, since, for intense considering the hyperbolic case, the surface divides in thick and thin part, on the thick part we can consider any disc with radius smaller than the injectivity radius. And in the thin part, using collar lemma, chart are given by degenerating annuli.

This result is optimal in the sense that we also prove that the weak L^2-norm of the Green function computed with respect to the intrinsic metric is not bounded when the singularity hold to a collapsing region, which is make clear in the torus case by the proposition 2.1.

Regarding the proof of the theorem, once we have rule out the trivial case of the sphere then we treat the case of a degenerating torus and of an hyperbolic surface quite differently. For the torus, our proof relies on an estimate of the coefficient of the Fourier decomposition on a long thin cylinder using the periodicity condition, and in the hyperbolic case, it relies on the coarea formula and the decomposition of the surface in thin and thick part.

In the last sections, we give some applications of theorem 0.1 in differential geometry both from extrinsic and intrinsic point of view. First, we prove that the gradient of the conformal factor of an immersion with L^2-bounded second fundamental form is uniformly bounded in $L^{2,\infty}$, up to chose a convent atlas given by theorem 0.1. Then we deduce a theorem of weak compactness for sequences of immersions with L^2-bounded second fundamental. This last result was partially already proved by Kuwert and Li and the second author, see [14] and [17]. Finally, in the last section we prove that considering a sequence of Riemann surface with bounded total curvature and entropy(see below for precise definition) then we can find a finite conformal atlas in which the conformal factor is uniformly bounded.

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[1] 4π in the sphere case and 1 in the torus case

1 Preliminaries

1.1 Lorentz spaces

Here we recall some classical facts about Lorentz spaces, [7] for details.

Definition 1.1. *Let D be a domain of $\mathbb{R}^k$, $p \in (1, +\infty)$ and $q \in [1, +\infty]$. The Lorentz space $L^{p,q}(D)$ is the set of measurable functions $f : D \to \mathbb{R}$ such that*

$$|f|_{p,q} = \left(\int_0^{+\infty} \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < +\infty \text{ if } q < +\infty$$

or

$$|f|_{p,\infty} = \sup \left(t^{\frac{1}{p}} f^*(t) \right) \text{ if } q = +\infty$$

where f^* the decreasing rearrangement of f.

$| \, |_{p,q}$ happens to be a quasi norm equivalent to a norm for which $L^{p,q}$ is a Banach space. Each $L^{p,q}$ may be seen as a deformation of L^p. For instance, we have the strict inclusions

$$L^{p,1} \subset L^{p,q'} \subset L^{p,q''} \subset L^{p,\infty},$$

if $1 < q' < q''$. Moreover,

$$L^{p,p} = L^p.$$

Furthermore, if $|D|$ is finite, we have that for all q and q',

$$p > p' \Rightarrow L^{p,q} \subset L^{p',q'}.$$

Using the fact that $f^*(t) = \inf\{s > 0 \text{ s.t. } d_f(s) \leq t\}$ where d_f is the distribution function of f, we see that the $L^{2,\infty}$ norm of f is finite if and only if $\sup_{t>0} t^2 \, |\{x \in D \mid |f(x, \, .)| \geq t\}|$ is finite.

Finally, for $p \in (1, +\infty)$ and $q \in [1, +\infty]$, $L^{\frac{p}{p-1}, \frac{q}{q-1}}$ is the dual of $L^{p,q}$.

1.2 Degenerating Riemann surfaces

Here we remind the Deligne-Mumford's description of the loss of compactness of the conformal class for a sequence of Riemann surfaces with fixed topology, see [11] for details.

Let (Σ, c_k) a sequence of closed Riemann surface of fixed genus g. If $g = 0$ then the conformal class is fixed since there is only one conformal class on the sphere. If $g = 1$ then, we know that, (Σ, c_k) is conformally equivalent to $\mathbb{R}^2 / \left(\frac{1}{\sqrt{\Im(v_k)}} \mathbb{Z} \times \frac{v_l}{\sqrt{\Im(v_k)}} \mathbb{Z} \right)$ where v_k lies in the fundamental domain $\{z \in \mathbb{C} \text{ s.t. } |\Re(z)| \leq 1 \text{ and } |z| \geq 1\}$ of $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$, and we say that c_k degenerates if $|v_k| \to +\infty$. If $g \geq 1$, let h_k the hyperbolic metric associated with c_k, then (Σ, c_k) degenerates if there exits a closed geodesic whose length goes to zero. In that case, up to a subsequence, there exists

1. an integer $N \in \{1, \ldots, 3g - 3\}$,
2. a sequence $\mathcal{L}_k = \{\Gamma_k^i \, ; \, i = 1 \ldots N\}$ of finitely many pairwise disjoint simple closed geodesics of (Σ, h_k) with length converging to zero,
3. a closed Riemann surfaces $(\overline{\Sigma}, \overline{c})$,

4. a complete hyperbolic surface $(\widetilde{\Sigma},\widetilde{h})$ with $2N$ cups $\{(q_1^i,q_2^i)\,;\,i=1\dots N\}$ such that $\widetilde{\Sigma}$ has been obtain topologically after removing the geodesic of $\mathcal{L}_k$ to Σ and after closing each component of the boundary of $\Sigma\setminus\mathcal{L}_k$ by adding a puncture q_l^i at each of these component. Moreover $\overline{\Sigma}$ is topologically equal to $\widetilde{\Sigma}$ and the complex structure defined by $\widetilde{h}$ on $\widetilde{\Sigma}\setminus\{q_l^i\}$ extends uniquely to $\overline{c}$. We can also equipped $\overline{\Sigma}$ with a metric $\overline{h}$ with constant curvature, but not necessarily hyperbolic since the genus of $\overline{\Sigma}$ can be lower than the one of Σ.

$(\widetilde{\Sigma},\widetilde{h})$ is called the nodal surface of the covering sequence and $(\overline{\Sigma},\overline{c})$ is its renormalization. These objects are related, in the sense that, there exists a diffeomorphism $\psi_k:\widetilde{\Sigma}\setminus\{q_l^i\}\to\Sigma\setminus\mathcal{L}_k$ such that $\widetilde{h}_k=\psi_k^*h_k$ converge in C^∞_{loc} topology to $\widetilde{h}$.

2 Proof of theorem 0.1

Before starting the proof, we present a "baby case" illustrating the difficulty for getting some $L^{2,\infty}$-estimate for functions whose laplacian is in L^1 on a long thin cylinder. On a fix domain, such an estimate is a classical result, see theorem 3.3.6 of [9]. Let us now consider the cylinder $C_l=\frac{1}{\sqrt{2\pi l}}\left(S^1\times\left[-\frac{l}{2},\frac{l}{2}\right]\right)$ which is identified with $S^1\times\left[-\frac{l}{2},\frac{l}{2}\right]$ endowed with the conformal metric $g=\frac{1}{2\pi l}(d\theta^2+dt^2)$. We set $u_l(t,\theta)=\frac{t^2}{4\pi l}$ which solves $\Delta_g u_l=1$. Then $|du_l|_g=\frac{t}{\sqrt{2\pi l}}$ and we easily check that $\|du_l\|_{L^{2,\infty}_g}\sim l$. While, considering the conformal chart $\psi_l:A_l\to C_l$ with $A_l=\mathbb{D}\setminus B\left(0,e^{-l}\right)$ and

$$\psi_l(\theta,r)=\left(\cos(\theta),\sin(\theta),\ln(r)+\frac{l}{2}\right).$$

Then, $\overline{u}_l=u_l\circ\psi_l=\frac{\left(\ln(r)+\frac{l}{2}\right)^2}{4\pi l}$ is uniformly bounded in $L^{2,\infty}$ **with respect to the euclidean metric**. Indeed,

$$|\nabla\overline{u}_l|=\left|\frac{\ln(r)+\frac{l}{2}}{2\pi l r}\right|\leq\frac{1}{r}.$$

This fact also illustrate that despite its closeness to the L^2-norm, the $L^{2,\infty}$-one is not conformally invariant[2]. This is one of the reason why we need to construct a specific conformal atlas.

All along the proof, for any given chart,unless otherwise stated, all the norms are computed with respect to the euclidean metric.

Proof of theorem 0.1 :

The sphere case :
Any sphere with constant curvature is conformal to the standard one, then the chart are given by south and north stereographic projection composed by the conformal diffeomorphism. The Green function (up to a constant) is the one of the standard sphere, its gradient is clearly bounded in $L^{2,\infty}$, hence there is nothing to prove for theorem 0.1.

The torus Case :

Let (Σ_l,g_l) be a sequence of flat tori of volume 1. Thanks to uniformization theorem, see [13], we know that, (Σ_l,g_l) is isometric to $\mathbb{R}^2/\left(\frac{1}{\sqrt{\Im(v_l)}}\mathbb{Z}\times\frac{v_l}{\sqrt{\Im(v_l)}}\mathbb{Z}\right)$ where v_l lies in a fundamental domain

[2] Although, it is invariant by dilation.

of $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$. Of course in the following we assume that the sequence degenerate, i.e. $|v_l| \to +\infty$, else the metric strongly converge and also the Green function.

We are going to treat first the rectangular torus (the case of $v_l \in i\mathbb{R}$) and we will explain how to deduce from it the general case. Up to some normalizations, our torus is isometric to long and thin cylinder : $C_l = \frac{1}{\sqrt{2\pi l}}\left(S^1 \times \left[-\frac{l}{2}, \frac{l}{2}\right]\right)$ with the standard identification of its boundary components. Then this cylinder is conformal to the annular $A_l = \mathbb{D} \setminus B\left(0, e^{-l}\right)$ through the following diffeomorphism

$$\psi_l(\theta, r) = \left(\frac{\cos(\theta)}{\sqrt{2\pi l}}, \frac{sin(\theta)}{\sqrt{2\pi l}}, \frac{\ln(r) + \frac{l}{2}}{\sqrt{2\pi l}}\right).$$

Let G_l be the pull back of a Green function on A_l. It satisfies[3]

$$\Delta_z G_l(\,.\,, w) = \delta_w - \frac{1}{2\pi l r^2} \text{ on } A_l,$$

and

$$G_l\left((\theta, e^{-l}), w\right) = G_l\left((\theta, 1), w\right) \text{ and } e^{-l}\partial_r G_l\left((\theta, e^{-l}), w\right) = \partial_r G_l\left((\theta, 1), w\right) \text{ for all } \theta.$$

Then we split G_l in three parts: a singular part s_l, a diffusion part u_l and an harmonic part g_l, as follows $G_l = s_l + u_l + g_l$ where

$$s_l\left((\theta, r), w\right) = \begin{cases} \frac{1}{2\pi}\ln\left|re^{i\theta} - w\right| + \frac{1}{2\pi}\ln\left|re^{i\theta} - e^l w\right| & \text{if } |w| \leq \frac{1}{2} \\ \frac{1}{2\pi}\ln\left|re^{i\theta} - w\right| + \frac{1}{2\pi}\ln\left|re^{i\theta} - e^{-l} w\right| & \text{if } |w| > \frac{1}{2} \end{cases}$$

and

$$u_l\left((\theta, r), w\right) = -\frac{1}{4\pi l}\left(\ln(r)\right)^2.$$

We easily check that, on the one hand $\Delta s_l = \delta_w$ on A_l and $\|\nabla s_l\|_{2,\infty} = O(1)$, independently of l and w, and on the other hand $\Delta u_l = -\frac{1}{2\pi l r^2}$ on A_l and $\|\nabla u_l\|_{2,\infty} = O(1)$, in fact we get even the more precise estimate $|\nabla u_l| = O\left(\frac{1}{r}\right)$. Finally we estimate g_l, in that aim we assume that $w \leq \frac{1}{2}$, the other case can be done in a similar way. Then g_l satisfies

$$\Delta g_l = 0,$$

$$g_l\left((\theta, 1), w\right) - g_l\left((\theta, e^{-l}), w\right) = -\frac{3l}{4\pi} + \frac{1}{2\pi}\ln\left|\frac{e^{-l}e^{i\theta} - e^l w}{e^{i\theta} - w}\right| = -\frac{3l}{4\pi} + F_l(\theta), \tag{2}$$

and

$$\begin{aligned} \partial_r g_l\left((\theta, 1), w\right) - e^{-l}\partial_r g_l\left((\theta, e^{-l}), w\right) &= \frac{1}{2\pi}\left(-\frac{\langle e^{i\theta}, e^{i\theta} - w\rangle}{|e^{i\theta} - w|^2} + \frac{\langle e^{-l}e^{i\theta}, e^{-l}e^{i\theta} - e^l w\rangle}{|e^{-l}e^{i\theta} - e^l w|^2} + 1\right) \\ &= H_l(\theta). \end{aligned} \tag{3}$$

Using Fourier analysis, we can decompose g_l as follows

$$g_l((\theta, r), w) = c_0 + a_0\ln(r) + \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}(a_n r^n + b_n r^{-n})e^{in\theta}.$$

Thanks to (2) we easily check that,

$$a_0 = O(1).$$

[3]This equation must be understood in a weak sense and be tested against smooth function of A_l whose composition with ψ_l^{-1} extends to a smooth function on Σ_l.

On the one hand, thanks to (2) and (3), we get

$$a_n(1-e^{-nl})+b_n(1-e^{nl})=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}F_l(\theta)e^{-in\theta}\,d\theta \tag{4}$$

and

$$a_n(1-e^{-nl})-b_n(1-e^{nl})=\frac{1}{n\sqrt{2\pi}}\int_0^{2\pi}H_l(\theta)e^{-in\theta}\,d\theta. \tag{5}$$

On the other hand,

$$\begin{aligned}\|\nabla\left(g_l-a_0\ln(r)\right)\|_2^2&=O\left(\sum_{n\in\mathbb{Z}}n^2a_n^2\int_{e^{-l}}^1 r^{2n-1}\,dr+n^2b_n^2\int_{e^{-l}}^1 r^{-2n-1}\,dt\right)\\&=O\left(\sum_{n\in\mathbb{Z}}na_n^2\left(1-e^{-2nl}\right)+nb_n^2\left(1-e^{2nl}\right)\right)\end{aligned}$$

But thanks to (4) and (5) and the fact that F_l and H_l converge in $C^2(S^1)$, as l goes to infinity, we have

$$|a_n|=O\left(\frac{1}{n^2(1-e^{-nl})}\right)\text{ and }|b_n|=O\left(\frac{1}{n^2(1-e^{nl})}\right)$$

uniformly with respect to l. Hence

$$\sum_{n\in\mathbb{Z}}na_n^2\left(1-e^{-2nl}\right)+nb_n^2\left(1-e^{2nl}\right)=O(1)$$

uniformly with respect to l. Which prove that

$$\|\nabla\left(g_l-a_0\ln(r)\right)\|_2^2=O(1)$$

Finally we conclude that

$$\|\nabla G_l\|_{2,\infty}=O(1),$$

This achieves the proof of theorem 0.1 in the case $v_l\in i\mathbb{R}$. In the general case, the torus is isometric to (A_{l_k},g_k) where $l_k=2\pi\Im(v_k)$ and $g_k=\frac{1}{2\pi l_k r^2}\left(r^2d\theta^2+\frac{dr^2}{\cos(\alpha_k)}\right)$ with $\alpha_k=\frac{\pi}{2}-\arg(v_k)\to 0$. Hence G_{l_k} split as follows $G_{l_k}=s_k+u_k+g_k$ where

$$s_k\left((\theta,r),w\right)=\begin{cases}\frac{1}{2\pi\cos(\alpha_k)}\ln\left|re^{i\theta}-w\right|_{g_k}+\frac{1}{2\pi\cos(\alpha_k)}\ln\left|re^{i\theta}-e^{l}w\right|_{h_k}&\text{if }|w|\le\frac{1}{2}\\\frac{1}{2\pi\cos(\alpha_k)}\ln\left|re^{i\theta}-w\right|_{g_k}+\frac{1}{2\pi\cos(\alpha_k)}\ln\left|re^{i\theta}-e^{-l}w\right|_{g_k}&\text{if }|w|>\frac{1}{2}\end{cases}$$

and

$$u_k\left((\theta,r),w\right)=-\frac{1}{4\pi l_k\cos(\alpha_k)}\left(\ln(r)\right)^2,$$

and then the analysis of g_k is the same.

The fact that we cannot bound the weak-L^2 norm with respect to the intrinsic metric of the torus is illustrated by the following proposition.

Proposition 2.1. *There exists a sequence of metric h_k on $\mathbb{T}^2$ with constant curvature and volume equal to 1 which is unbounded in the moduli space and such that, for any $y \in \Sigma$, we get*

$$\sup_{t>0} t^2 \left|\{x \in \Sigma \mid |dG_{h_k}(x,y)| \geq t\}\right| \to +\infty. \tag{6}$$

Proof of proposition 2.1:

In order to prove this proposition ,we go back to the case of degenerating rectangular tori viewed as a long thin cylinder: $C_l = \frac{1}{\sqrt{2\pi l}}\left(S^1 \times \left[-\frac{l}{2}, \frac{l}{2}\right]\right)$ with the two ends being identified in an obvious way. Let $g_\phi(\theta,t) = G((\theta,t),(\phi,0))$, then if this function is uniformly bounded in $L^{2,\infty}$ it would be the case for $g(\theta,t) = \frac{1}{2\pi}\int_0^{2\pi} g_\phi(\theta,t)\, d\phi$, thanks to the invariant by rotation. Then we easily check that $g(\theta,t) = \frac{|t|}{4\pi}$, hence $|dg| = \frac{\sqrt{2\pi l}}{4\pi}$, here the norm is computed with respect to the metric $h = \frac{1}{2\pi l}(d\theta^2 + dt^2)$. Finally we easily check that $\|dg\|_{L^{2,\infty}_h} \sim l$, which is a contradiction and prove proposition 2.1. □

The case of genus ≥ 2 :

Let (Σ, c_k) be a sequence of Riemann surfaces of fixed genus $g \geq 2$. Thanks to the uniformization theorem, see [13], we know that, we can endow Σ with a conformal metric h_k such that (Σ, h_k) is isometric to $\mathbb{H}/\Gamma_k$ where Γ_k is a discrete group of $\mathrm{PSL}_2(\mathbb{R})$. Then, the Green functions associated to h_k satisfy

$$(\Delta_{h_k})_x G_k(\,.\,,y) = \delta_y - \frac{1}{v_k},$$

where v_k is the volume of (Σ, h_k) which depends only on the genus thanks to Gauss-Bonnet theorem. In order to study the behavior of the Green's function and following the classic description of hyperbolic surfaces, see [11] or [10], we set $\delta < arcsinh(1)$ and then we split (Σ, h_k) in two parts: a thick part $E_k^\delta = \{s \in \Sigma \,|\, \mathrm{injrad}((\Sigma,h_k),s) \geq \delta\}$ and a thin part $F_k^\delta = \{s \in \Sigma \,|\, \mathrm{injrad}((\Sigma,h_k),s) < \delta\}$. Thanks to the decomposition theorem of Deligne-Mumford, we know that the sequence of metrics converges strongly on the thick part and develops collar in the thin part. We are going to split our proof depending on the case whether y lies in the thick or int thin part. But before we prove a general estimate for Green functions on a closed surfaces.

Step 1: Coarea formula for Green's functions.

In this step, G_k is any Green function associated to h_k, there is no normalization. Volumes and lengths are taken with respect to h_k here.

Let $t > 0$, integrating by part on a level set, we get

$$\int_{G_k(x,y)=t} \frac{\partial G_k(x,y)}{\partial \nu_k}\, d\sigma_k(x) = \int_{G_k(x,y)\geq t} (\Delta_{h_k})_x G_k(x,y)\, dv_k(x) = 1 - \frac{vol(\{G_k(x,y)\geq t\})}{v_k},$$

where ν_k is the exterior normal of the open set $\{x \in \Sigma \mid G_k(x,y) > t\}$. Then, we get

$$\int_{G_k(x,y)=t} |dG_k(x,y)|^2_{h_k}\, d\sigma_k(x) \leq 2.$$

Let $1 < p < 2$ and $a > 0$, then thanks to coarea-formula, see [22] or [4], we get that

$$\int_{G_k(x,y)\geq a} |dG_k^{1-p/2}(x,y)|^2_{h_k}\, dv_k(x) = \int_a^{+\infty}\left(\int_{G_k(x,y)=t} \frac{|dG_k(x,y)|_{h_k}}{G_k^p(x,y)}\, d\sigma_k(x)\right) dt \leq 2\int_a^{+\infty} \frac{1}{t^p}\, dt \leq C_{p,a},$$

where $C_{p,a}$ is a positive constant depending only on p and a. Moreover, we can also prove, considering negative level set, that

$$\int_{G_k(x,y)\leq -a} |dG_k^{1-\frac{p}{2}}(x,y)|^2_{h_k}\, dv_k(x) \leq C_{p,a},$$

where $C_{p,a}$ is a positive constant depending only on p and a. Finally we get,

$$\int_{|G_k(x,y)|\geq a} |dG_k^{1-\frac{p}{2}}(x,y)|^2_{h_k}\, dv_k(x) \leq C_{p,a}, \tag{7}$$

where $C_{p,a}$ is a positive constant depending only on p and a.

Step 2: Estimate in the thick part.

In order to obtain the estimate on the whole thick part, we will cover it by a finite number of balls with radius $\frac{\delta}{2}$, where have been chosen such that $0 < \delta < acrsinh(1)$. Since we consider a general sequence of Green functions $G_k(\,.\,, y_k)$, we have to pay attention to the location of these balls with respect to the singularity y_k. In fact if y_k is in the thick part then we will center one of the ball of the covering at y_k and then the others won't have to deal with this singularity.

Let $x_k \in E_k^\delta$ and we first assume that $y_k \notin B_{h_k}\left(x_k, \frac{\delta}{2}\right)$. Then $B_{h_k}\left(x_k, \frac{\delta}{2}\right)$ is isometric to $B\left(0, \tanh\left(\frac{\delta}{4}\right)\right)$ in the Poincaré disc. In the following, we make all computations in the conformal chart $B(0,3r)$ with $r = \frac{\tanh\left(\frac{\delta}{4}\right)}{3}$ and the metric $h_p = \frac{4dx^2}{(1-|x|^2)^2}$. But the hyperbolic metric is equivalent to the euclidean one on this ball.

On $B(0,3r)$ we decompose G_k as follows

$$G_k(\,.\,, y_k) = u_k + g_k,$$

$$G_k(\,.\,, y_k) = u_k + g_k,$$

where $u_k(x) = \frac{1}{v_k} \ln\left(\frac{1}{1-|z|^2}\right)$ and g_k be a smooth harmonic function. Hence we can apply (7), with $p = \frac{3}{2}$ and $a = \frac{2}{v_k} \ln\left(\frac{1}{1-(3r)^2}\right)$ to $\widetilde{G}_k = G_k - g_k(0)$, which gives that

$$\int_{\{|\widetilde{G}_k(x,y_k)|\geq a\}\cap B(0,3r)} |d\widetilde{G}_k^{\frac{1}{4}}(x,y_k)|^2_{h_p}\, dv_{h_p}(x) \leq C,$$

where C is a positive constant depending only on the genus and δ. Then, by the mean value property, there exists $\rho \in [2r, 3r]$ such that

$$\int_{\{|\widetilde{G}_k(x_k,y)|\geq a\}\cap \partial B(0,\rho)} |d\widetilde{G}_k^{\frac{1}{4}}(x,y_k)|_{h_p}\, d\sigma_{h_p}(x) \leq C,$$

where C is positive constant depending only on the genus and δ. Then, using the fact harmonic functions satisfy the mean value property, we get that the mean value of $g_k - g_k(0)$ is 0 on $\partial B(0,\rho)$ and we easily deduce that

$$|G_k(x,y_k) - g_k(0)| \leq C, \text{ for all } x \in \partial B(0,\rho),$$

where C is positive constant depending only on the genus and δ. Then using classical elliptic estimate, we have

$$\|dG_k(\,.\,, y_k)\|_{L^\infty_{h_p}(B(0,r))} \leq C, \tag{8}$$

where C is positive constant depending only on the genus and δ. Then, in the case $x_k = y_k$, equivalently the ball we consider is centered at the singularity of $G(x, .)$, we obtain the same estimate decomposing G_k as follows

$$G_k(.\,, y_k) = s_k + u_k + g_k,$$

where $s_k(x) = \frac{1}{2\pi}\ln(|x|)$. But, of course, in that case, due to the presence of s_k, the estimate is in $L^{2,\infty}$. Finally covering the thick part with a uniformly bounded number of balls we get the desired estimate on $G(.\,, y_k)$ on the thick part. Indeed, either y_k is not in the thick part and the result will follows directly from (8), or we start by taking a ball centered at y_k and then we cover the rest of the thick part by balls which does not contain y_k.

Step 3: Estimate in the thin part.

Let $x_k \in F_k^\delta$ and $y_k \in \Sigma$ two converging sequences in $\overline{\Sigma}$. First, thanks to the collar lemma, see [10], we know that each connected component of the thin part (i.e. at most 3g-3), contains a simple closed geodesic γ_k of length $\varepsilon_k = l(\gamma_k) < 2arcsinh(1)$, and is isometric to

$$B_k = \left\{ z = re^{i\varphi} \in \mathbb{H} : 1 \leq r \leq e^{\varepsilon_k}, arctan\left(sinh\left(\frac{\varepsilon_k}{2}\right)\right) < \varphi < \pi - arctan\left(sinh\left(\frac{\varepsilon_k}{2}\right)\right) \right\},$$

where the geodesic corresponds to $\left\{ re^{i\frac{\pi}{2}} \in \mathbb{H} : 1 \leq r \leq e^{\varepsilon_k} \right\}$ and the line $\{r = 1\}$ and $\{r = e^{\varepsilon_k}\}$ are identified via $z \mapsto e^{\varepsilon_k} z$. It is often easier to consider the following cylindrical parametrization. Let $\varphi_k = arctan\left(sinh\left(\frac{\varepsilon_k}{2}\right)\right)$ and we set

$$C_k = \left\{ (\cos(\theta), sin(\theta), t) \,|\, 0 \leq \theta < 2\pi, \frac{2\pi}{\varepsilon_k}\varphi_k < t < \frac{2\pi}{\varepsilon_k}(\pi - \varphi_k) \right\}$$

equipped with the metric

$$h_c = \left(\frac{\varepsilon_k}{2\pi sin\left(\frac{\varepsilon_k t}{2\pi}\right)} \right)^2 (d\theta^2 + dt^2),$$

where the geodesic correspond to $\left\{ t = \frac{\pi^2}{\varepsilon_k} \right\}$.

We are going to make the proof assuming that y_k lies in the thin part. When this is not the case the proof carries over after the simply operation consisting of withdrawing the singular part s_k. We can also assume that $y_k \notin \left(\left[\frac{2\pi}{\varepsilon_k}\varphi_k, \frac{2\pi}{\varepsilon_k}\varphi_k + \frac{\delta}{10} \right] \cup \left[\frac{2\pi}{\varepsilon_k}(\pi - \varphi_k) - \frac{\delta}{10}, \frac{2\pi}{\varepsilon_k}(\pi - \varphi_k) \right] \right) \times S^1$, replacing δ by $\frac{\delta}{2}$ if necessary.

Then, as for the torus case, we choose an annulus as conformal chart. Precisely, let $A_k = \mathbb{D} \setminus B(0, e^{-l_k})$ and $\psi_k : A_k \to C_k$ defined as follows

$$\psi_k(\theta, r) = \left(\cos(\theta), sin(\theta), \ln(r) + \frac{2\pi}{\varepsilon_k}(\pi - \varphi_k) \right),$$

where $l_k = \frac{2\pi}{\varepsilon_k}(\pi - 2\varphi_k)$. Then, the pull back of a Green function on A_k, that we keep denoting G_k, satisfies

$$\Delta_z G_k(.\,, w_k) = \delta_{w_k} - \left(\frac{\varepsilon_k}{r2\pi \sin\left(\frac{\varepsilon_k}{2\pi}\left(\ln(r) + \frac{2\pi}{\varepsilon_k}(\pi - \varphi_k)\right)\right)} \right)^2 \text{ on } A_k,$$

where $\psi_k(w_k) = y_k$ with $w_k \in B(0, e^{-\frac{\delta}{10}}) \setminus B(0, e^{-l_k+\frac{\delta}{10}})$.

First of all, thanks to our previous step, see (8), we remark that

$$|\nabla G_k(\,.\,, w_k)| \leq \frac{C}{r} \text{ on } B(0, e^{-l_k+\frac{\delta}{10}}) \setminus B(0, e^{-l_k}), \tag{9}$$

and

$$|\nabla G_k(\,.\,, w_k)| \leq C \text{ on } B(0,1) \setminus B(0, e^{-\frac{\delta}{10}}), \tag{10}$$

where C is a positive constant depending only on the genus and δ. Then we split G_k as follows

$$G_k((\theta, r), w_k) = s_k(\theta, r) + u_k(\theta, r) + g_k(\theta, r),$$

where

$$u_k(\theta, r) = \frac{\ln\left(\sin\left(\frac{\varepsilon_k}{2\pi}\left(\ln(r) + \frac{2\pi}{\varepsilon_k}\right)\right)\right)}{v_k},$$

and

$$s_k(\theta, r) = \frac{1}{2\pi}\ln\left(\left|re^{i\theta} - w_k\right|\right).$$

We easily check that

$$\Delta u_k = -\left(\frac{\varepsilon_k}{r2\pi\sin\left(\frac{\varepsilon_k}{2\pi}\left(\ln(r) + \frac{2\pi}{\varepsilon_k}(\pi - \varphi_k)\right)\right)}\right)^2$$

$$\begin{aligned}
\|\nabla u_k\|_2^2 &= \frac{1}{2\pi v_k^2}\int_{e^{-l_k}}^1 \left(\frac{\cos\left(\frac{\varepsilon_k}{2\pi}\left(\ln(r) + \frac{2\pi}{\varepsilon_k}(\pi - \varphi_k)\right)\right)}{\sin\left(\frac{\varepsilon_k}{2\pi}\left(\ln(r) + \frac{2\pi}{\varepsilon_k}(\pi - \varphi_k)\right)\right)}\right)^2 \frac{\varepsilon_k^2}{r} dr \\
&= \frac{1}{2\pi v_k^2}\int_{e^{-l_k}}^1 \left(\frac{-\varepsilon_k^2}{r} + \frac{1}{\left(\sin\left(\frac{\varepsilon_k}{2\pi}\left(\ln(r) + \frac{2\pi}{\varepsilon_k}(\pi - \varphi_k)\right)\right)\right)^2}\frac{\varepsilon_k^2}{r}\right) dr \qquad (11) \\
&= \frac{1}{v_k^2}\left[-\frac{\varepsilon_k^2}{2\pi}\ln(r) - \varepsilon_k \frac{\cos\left(\frac{\varepsilon_k}{2\pi}\left(\ln(r) + \frac{2\pi}{\varepsilon_k}(\pi - \varphi_k)\right)\right)}{\sin\left(\frac{\varepsilon_k}{2\pi}\left(\ln(r) + \frac{2\pi}{\varepsilon_k}(\pi - \varphi_k)\right)\right)}\right]_{e^{-l_k}}^1 \leq C,
\end{aligned}$$

$$\Delta s_k = \delta_{w_k},$$

and

$$\|\nabla s_k\|_{2,\infty} \leq C, \tag{12}$$

where C is a positive function depending only on the genus and δ.

Then g_k, which has been obtained from G_k after subtracting s_k and u_k, is a smooth harmonic function. Let $\overline{g}_k(r)$ be the mean value of g_k on the circle of radius r centered at 0. It is also harmonic and radial, hence $\overline{g}_k(t) = a_k \ln(r) + b_k$. Moreover, thanks to (9) and (10), we get that

$$|\nabla g_k| \leq \frac{C}{r} \text{ on } B(0, e^{-l_k+\frac{\delta}{20}}) \setminus B(0, e^{-l_k}), \tag{13}$$

and

$$|\nabla g_k| \leq C \text{ on } B(0,1) \setminus B(0, e^{-\frac{\delta}{20}}). \tag{14}$$

In particular, a_k is uniformly bounded and we get,

$$\|\nabla \overline{g}_k\|_{L^{2,\infty}\left(B(0,1)\setminus B(0,e^{-l_k})\right)} \leq C, \tag{15}$$

Then using the fact the mean value of $g_k - \overline{g}_k$ is zero and the previous estimate, we get that

$$\|g_k - \overline{g}_k\|_{L^\infty\left(\left(B(0,e^{-l_k+\frac{\delta}{20}})\setminus B(0,e^{-l_k})\right)\cup\left(B(0,1)\setminus B(0,e^{-\frac{\delta}{20}})\right)\right)} \leq C, \tag{16}$$

where C is positive constant depending only on the genus and δ. Then, since $g_k - \overline{g}_k$ is harmonic and with radial mean value equal to zero,

$$\|\nabla(g_k - \overline{g}_k)\|_{L^2\left(B(0,e^{-\frac{\delta}{10}})\setminus B(0,e^{-l_k+\frac{\delta}{10}})\right)} \leq C, \tag{17}$$

where C is positive constant depending only on the genus and δ. The last inequality can be proved using the furrier decomposition and remarking that $g_k - \overline{g}_k$ has no logarithmic part. Finally, thanks to (11), (12), (15) and (17), we get the desired estimate, which concludes the proof of the theorem 0.1. ■

3 Weak compactness result for immersions with second fundamental form bounded in L^2

The first application of theorem 0.1 regards the control of the conformal factor for immersions with L^2-bounded second fundamental form. Before to state the main result, we shall first remind the notion of weak immersions introduced by the second author in [17] .

Let Σ a smooth compact surface equipped with a reference smooth metric g_0. One define the Sobolev spaces $W^{k,p}(\Sigma, \mathbb{R}^m)$ of measurable maps from Σ into $\mathbb{R}^m$ into the following way

$$W^{k,p}(\Sigma, \mathbb{R}^m) = \left\{ f : \Sigma \to \mathbb{R}^m \text{measurables.t.} \sum_{l=0}^{k} \int_\Sigma |\nabla^l f|^p_{g_0} \, dv_{g_0} < +\infty \right\}.$$

Since Σ is compact it is not difficult to see that this space is independent of the choice we have made of g_0.

Let $f \in W^{1,\infty}(\Sigma, \mathbb{R}^m)$, we define g_f to be the following symmetric bilinear form

$$g_f(X,Y) = \langle df(X), df(Y) \rangle,$$

and we shall assume that there exists $C_f > 1$ such that

$$C_f^{-1} g_0(X,X) \leq g(X,X) \leq C_f g_0(X,X). \tag{18}$$

For such a map, we can define the Gauss map as being the following measurable map in $L^\infty(\Sigma)$ taking values int the Grassmannian of oriented $m-2$-planes of $\mathbb{R}^m$,

$$\vec{n}_f = \star \frac{\frac{\partial f}{\partial x} \wedge \frac{\partial f}{\partial x}}{\left| \frac{\partial f}{\partial x} \wedge \frac{\partial f}{\partial x} \right|}.$$

We then introduce the space $\mathcal{E}_\Sigma$ of weak immersions of Σ with bounded second fundamental form as follow:

$$\mathcal{E}_\Sigma = \left\{ \begin{array}{c} \Phi \in W^{1,\infty}(\Sigma) \text{ which satisfies (18) for some } C_\Phi > 0 \\ \\ \text{and } \int_\Sigma |d\vec{n}_\Phi|_g^2 \, dv_g < +\infty \end{array} \right\},$$

where $g = \Phi^* \xi$.

It is proved in [19] that any weak immersion defines a smooth conformal structure on Σ. Let $\Phi \in \mathcal{E}_\Sigma$, we denote by $\pi_{\vec{n}_\Phi}$ the orthonormal projection of vector in $\mathbb{R}^m$ onto the $m-2$-plane given by $\vec{n}_\Phi$. With these notations the second fundamental form of the immersion at p is given by

$$\forall X, Y \in T_p\Sigma \;\; \vec{\mathbb{I}}_p(X,Y) = \pi_{\vec{n}_\Phi} d^2\Phi(X,Y),$$

and the mean curvature vector of the immersion at p is given by

$$\vec{H} = \frac{1}{2} \mathrm{tr}_g(\vec{\mathbb{I}}).$$

A natural quantity while considering such immersions is the Lagrangian given by the L^2-norm of the second fundamental form :

$$E(\phi) = \int_\Sigma |\vec{\mathbb{I}}|_g^2 \, dv_g.$$

An elementary computation, using Gauss-Bonnet formula, gives

$$E(\phi) = \int_\Sigma |\vec{\mathbb{I}}|_g^2 \, dv_g = \int_\Sigma |d\vec{n}_\Phi|_g^2 \, dv_g = 4W(\phi) - 4\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic and

$$W(\Phi) = \int_\Sigma |\vec{H}|_g^2 \, dv_g,$$

is the so called Willmore energy.

Theorem 3.1. *Let (Σ, c_k) be a sequence of closed Riemann surface of fixed genus greater than one. Let denote h_k the metric with constant curvature (and volume equal to one in the torus case) in c_k and Φ_k a sequence of weak conformal immersion of Σ into $\mathbb{R}^m$, i.e.*

$$\Phi_k^* \xi = e^{2u_k} h_k,$$

where $u_k \in L^\infty(\Sigma)$. Then there exists a finite conformal atlas (U_i, ψ_i) independent of k and a positive constant C depending only on the genus of Σ, such that

$$\|dv_k^i\|_{L^{2,\infty}(V_i)} \leq CW(\Phi_k),$$

where v_k^i is the conformal factor of $\Phi^k \circ \psi_i^{-1}$ in $V_i = \psi_i(U_i)$, i.e. $v_k^i = \frac{1}{2} \ln \left| \frac{\partial \Phi^k \circ \psi_i^{-1}}{\partial x} \right| = \frac{1}{2} \ln \left| \frac{\partial \Phi^k \circ \psi_i^{-1}}{\partial y} \right|$.

Proof of theorem 3.1:

Let $K_g = 0$ or -1 if the genus g of Σ is 1 or greater than 1 and be $K_{\Phi_k^* \xi}$ the Gauss curvature associated to $\Phi_k^* \xi$. It is classical that u_k satisfies the following Liouville equation

$$-\Delta_{h_k} u_k = K_{\Phi_k^* \xi} e^{2u_k} - K_g. \tag{19}$$

Let G_k be the nonnegative Green function of (Σ, h_k), then using the representation formula, we get that

$$u_k = G_k \star \left(K_{\Phi_k^*\xi} e^{2u_k} - K_g\right). \tag{20}$$

We have the following straightforward estimate,

$$\int_\Sigma |K_{\Phi_k^*\xi} e^{2u_k}|\, dv_{h_k} = \int_\Sigma |K_{\Phi_k^*\xi}|\, dv_{\Phi_k^*\xi} \leq \frac{1}{2}\int_\Sigma |\vec{\mathbb{I}}_{\Phi_k^*\xi}|\, dv_{\Phi_k^*(\xi)} \leq W(\Phi_k), \tag{21}$$

this proves that the right hand side of (19) is uniformly bounded in L^1-norm by $W(\Phi_k)$ with respect to the metric h_k. Then let (U_i, ψ_i) the conformal atlas given by theorem 0.1 and let $\phi_k^i = \Phi_k \circ \psi_i^{-1} : V_i \to \mathbb{R}^m$ and $v_k^i : V_i \to \mathbb{R}$ such that $(\phi_k^i)^*(\xi) = e^{2v_k^i} dz^2$. First we observe that $v_k^i = u_k \circ \psi_i^{-1} + w_k^i$ where $w_k^i : V_i \to \mathbb{R}$ such that $(\psi_i)^*(h_k) = e^{2w_k^i} dz^2$. Moreover, we can easily check that ∇w_k^i is uniformly bounded in $L^{2,\infty}$ since in the torus case a chart is given by an annulus and $w_k^i = -\ln(r) + c_k$ and in the hyperbolic case the chart is either a disc with radius strictly less than 1 and $w_k^i = -\ln(1-r^2) + c_k^i$ or an annulus $A_k = \mathbb{D} \setminus B(0, e^{-l_k})$ where $l_k = \frac{2\pi}{\varepsilon_k}(\pi - 2\varphi_k)$ with $w_k^i = -\ln\left(r \sin\left(\frac{\varepsilon_k}{2\pi}\left(\ln(r) + \frac{2\pi}{\varepsilon_k}(\pi - \varphi_k)\right)\right)\right) + c_k^i$.

Then it suffices to check that $\nabla(u_k \circ \psi_i^{-1})$ is uniformly bounded in $L^{2,\infty}(V_i)$ in order to prove the theorem. Thanks to (20), we have

$$u_k(y) = \int_\Sigma G_k(x,y) F_k(x)\, dv_{h_k} + \frac{1}{\mathrm{vol}(\Sigma, h_k)} \int_\Sigma u_k\, dv_{h_k}, \tag{22}$$

where $F_k = K_{\Phi_k^*\xi} e^{2u_k} - K_g$. Hence,

$$\nabla_y u_k \circ \psi_i^{-1}(y) = \int_\Sigma \nabla_y G_k(x, \psi_i^{-1}(y)) F_k(x)\, dv_{h_k}, \tag{23}$$

then using the fact $\sup_{x\in\Sigma} \|\nabla_y G_k(x, \psi_i^{-1}(y))\|_{L^{2,\infty}}$ is uniformly bounded, thanks to theorem 0.1, thanks to (21) and standard inequality on convolution we get that $\|\nabla(u_k \circ \psi_i^{-1})\|_{L^{2,\infty}(V_i)}$ is uniformly bounded which concludes the proof of theorem 3.1. ■

Then our second application concerns the weak compactness of the conformal immersion with L^2-bounded second fundamental form. The following result was proved first in [17] when the conformal classes of the surfaces do not degenerate and has been extended to the general case of degenerating riemmann surfaces in [14]. We shall present a different approach for proving this result as being a consequence of our main theorem 0.1

Theorem 3.2. *Let Σ a closed surface of genus strictly greater than 1 and $\Phi_k \in \mathcal{E}_\Sigma$ a sequence of weak immersion into $\mathbb{R}^m$ with L^2-bounded second fundamental form. Then, up to a subsequence, for any connected component σ of $\widetilde{\Sigma}$, the nodal surface of the converging sequence $(\Sigma, \Phi_k^*\xi)$, there exists a Möbius transformation Ξ_k of $\mathbb{R}^m$ such that*

$$\Xi_k \circ \Phi_k(\Sigma) \subset B(0,R)$$

where R depends only on m and there exists at most finitely many point $\{a_1, \ldots, a_L\}$ of σ such that if we denote $\Psi_k = \Xi_k \circ \Phi_k \circ \phi_k$, then

$$\Psi_k \rightharpoonup \Psi \text{ weakly in } W^{2,2}_{loc}\left(\sigma \setminus \{a_1, \ldots, a_L, q_1, \ldots q_K\}, \tilde{h}\right),$$

where Ψ is a weak conformal(possibly branched) immersion of $(\sigma, \tilde{h})$ into $\mathbb{R}^m$ and the q_i are the punctures of $(\sigma, \tilde{h})$ and $\phi_k : \widetilde{\Sigma} \to \Sigma$ such $\phi_k^*(h_k) \to \tilde{h}$ in $C^\infty_{loc}(\widetilde{\Sigma})$.

Moreover, for any compact $K \subset \sigma \setminus \{a_1, \dots, a_L, q_1, \dots q_K\}$ there exists $C_K > 0$ such that

$$\sup_{k\in\mathbb{N}} \|Log|d\Psi_k|_{\phi_k^* h_k}\|_{L^\infty(K)} \leq C_K,$$

where C_K depends only on m, K and the L^2-bound on the second fundamental form of Φ_k.

Here we consider the hyperbolic case, since in the sphere case the existence of a non compact conformal group is the additional difficulty already treated in [16] and in the torus when it degenerate, the injectivity radius uniformly blow down.

Proof of theorem 3.2:

By assumption there exists Λ, a positive constant , such that

$$\sup_{k\in\mathbb{N}} W(\Phi_k) \leq \Lambda. \tag{24}$$

We denote by u_k the conformal factor of this weak immersion with respect to the hyperbolic metric h_k in the conformal class of $\Phi_k^*\xi$. That is to say

$$\Phi_k^*\xi = e^{2u_k} h_k,$$

where $K_{h_k} \equiv -1$.

Now let $(\widetilde{\Sigma}, \tilde{h})$ be the nodal surface of the converging sequence (Σ, h_k), $\{q_i\}$ the set of punctures, $(\overline{\Sigma}, \overline{h})$ its renormalisation and $\phi_k : \widetilde{\Sigma} \to \Sigma$ the continuous map given by to Deligne-Mumford compactification recalled in section 1. Let σ be any connected component of $\widetilde{\Sigma}$.

Then $\widetilde{\Phi}_k = \Phi_k \circ \phi_k$ is a conformal weak immersion of $(\sigma, \tilde{h}_k)$ where $\tilde{h}_k = \phi_k^* h_k$. Hence, we get

$$(\Phi_k \circ \phi_k)^*\xi = e^{2u_k}\tilde{h}_k.$$

Let $\delta > 0$ and $K_\delta = \{x \in \sigma \text{ s.t. } d_{\overline{h}}(x, q_i) \geq \delta \text{ for all } i\}$, thanks to the local convergence of ϕ_k, then

$$\|\nabla u_k\|_{L^{2,\infty}_{\tilde{h}}(\Sigma)} \text{ is uniformly bounded on } K_\delta. \tag{25}$$

Here we use the fact that on the thick part the euclidean metric and the hyperbolic one are equivalent then theorem 3.1 can be consider intrinsically on the thick part.

Then, in order to find the correct Möbius transformation, we follow the procedure introduced by the second author in [17]. For each $x \in K_\delta$ there exists $\rho_x^k > 0$ such that

$$\int_{B_{\tilde{h}_k}(x,\rho_x^k)} |d\vec{n}_{\widetilde{\Phi}_k}|^2_{\tilde{h}_k} dv_{\tilde{h}_k} = \min\left(\frac{8\pi}{3}, \int_{K_\delta} |d\vec{n}_{\widetilde{\Phi}_k}|^2_{\tilde{h}_k} dv_{\tilde{h}_k}\right)$$

where $B_{\tilde{h}}(x, \rho_x^k)$ is the geodesic ball in $(\sigma, \tilde{h})$ of center x and radius ρ_x^k. Then, using the Besicovitch covering lemma, we can extract a finite covering of $K_\delta \subset \cup_{i\in I_k} B_{\tilde{h}}\left(x_i^k, \frac{\rho_{x_i^k}}{2}\right)$, such that each point is covered at most N time, where N is independent of k. Then, thanks to (24), we can extract a finite covering, $I \subset \cup_k I_k$, which is independent of k where x_i^k converges to x_i^∞ and $\rho_{x_i^k}$ converges to $\rho_{x_i^\infty}$. Then we set $I_0 = \{i \in I \text{ s.t. } \rho_{x_i^\infty} = 0\}$ and $I_1 = I \setminus I_0$.

Claim :For each $i \in I \setminus I_0$ there exist $\overline{v}_k^i \in \mathbb{R}$ such that

$$\|v_k - \overline{v}_k^i\|_{L^\infty\left(B_{\tilde{h}}\left(x_i^k, \frac{\rho_{x_i^k}}{2}\right)\right)} \leq C,$$

where C is a constant which depends only on Λ.

Proof of the Claim :

Let fix $i \in I \setminus I_0$ and identify, up to uniformly bounded conformal diffeomorphism, $B_{\tilde{h}_k}(x, \rho_x^k)$ with $\mathbb{D}$. Then, thanks to lemma 5.1.4 of [9] there exists a moving frame $(\vec{e}_1^k, \vec{e}_2^k) \in W^{1,2}(\mathbb{D}, S^{m-1})$ such that

$$\int_{\mathbb{D}} (|\nabla \vec{e}_1^k|^2 + |\nabla \vec{e}_2^k|^2)\, dz \leq \int_{\mathbb{D}} |d\vec{n}_{\tilde{\Phi}_k}|^2 dz \leq \frac{8\pi}{3},$$

and moreover

$$\star \vec{n}_{\tilde{\Phi}_k} = \vec{e}_1^k \wedge \vec{e}_2^k \text{ and } \Delta u_k = \left(\nabla^\perp \vec{e}_1^k, \vec{e}_2^k\right).$$

Let v_k be the solution of

$$\begin{cases} \Delta v_k = \left(\nabla^\perp \vec{e}_1^k, \vec{e}_2^k\right) \text{ on } \mathbb{D} \\ v_k = 0 \text{ on } \partial\mathbb{D} \end{cases}.$$

Then, thanks to Wente inequality, see section 3 of [9], we get

$$\|v_k\|_\infty + \|\nabla v_k\|_2 \leq \frac{1}{2\pi} \|\nabla \vec{e}_1^k\| \|\nabla \vec{e}_2^k\|. \tag{26}$$

Finally using the fact $u_k - v_k$ is harmonies with $\|\nabla(u_k - v_k)\|_{2,\infty}$ uniformly bounded, we proved that there exist $c_k \in \mathbb{R}$ and C a positive constant independent of k such that

$$\|u_k - v_k - c_k\|_{L^\infty\left(B\left(0, \frac{1}{2}\right)\right)} \leq C, \tag{27}$$

Finally, putting (26) and (27) together concludes the proof of the claim. □

Then using the fact that each point is covered by a universally number of ball, we easily get that there exists $\overline{v}_k \in \mathbb{R}$ such that

$$\|v_k - \overline{v}_k\|_{L^\infty\left(K_\delta \setminus \cup_{i \in I_0} B_{\tilde{h}}\left(x_i^\infty, \frac{\delta}{2}\right)\right)} \leq C. \tag{28}$$

We also remark that he constant $\overline{v}_k$ is independent of δ. Let $x_0 \in \sigma$ then we set

$$\widehat{\Phi}_k = e^{-\overline{v}_k} \left(\tilde{\Phi}_k - \tilde{\Phi}_k(x_0)\right).$$

Then, using Simon monotonicity formula, see [22], as in [17] we proved that there exists $y_0 \in B(0,1) \subset \mathbb{R}^m$ and $t > 0$ such that

$$\widehat{\Phi}_k \left(K_\delta \setminus \cup_{i \in I_0} B_{\overline{h}}\left(x_i^\infty, \frac{\delta}{2}\right)\right) \cap B(x_0, t) = \emptyset. \tag{29}$$

Finally, we set

$$\Xi_k = I_{x_0,t}\left(e^{-\overline{v}_k}\left(\,.\, - \tilde{\Phi}_k(x_0)\right)\right)$$

where $I_{x_0,t}$ is the inversion of $\mathbb{R}^m$ centered at x_0 and with ratio t. Hence Ξ_k is a Möbius transformation such that, if we set $\Psi_k = \Xi_k \circ \Phi_k$, we get thanks to (28) that

$$\sup_{k \in \mathbb{N}} \|Log|d\Psi_k|_{\tilde{h}_k}\|_{L^\infty(K)} \leq C_K,$$

and thanks to (29), that there exist $R > 0$ such that,

$$\Psi_k \left(K_\delta \setminus \cup_{i \in I_0} B_{\overline{h}} \left(x_i, \frac{\delta}{2} \right) \right) \subset B(0, R).$$

Finally, Using a classical argument of functional analysis , see for instance [19] beginning of section VI.7.1, we easily deduce that Ψ_k converge to Ψ in $W^{2,2}_{loc}\left(\sigma \setminus \{x_1^\infty, \dots, x_L^\infty, q_1, \dots q_K\}, \tilde{h}\right)$, moreover Ψ is a weak immersion away from $\{x_1^\infty, \dots, x_L^\infty, q_1, \dots q_K\}$ satisfying

$$W(\Psi) < +\infty$$

Finally Lemma A.5 of [18] permits us to extend Ψ as a conformal, possibly branched, immersion of σ.■

4 Weak compactness of Riemannian surfaces with bounded Gaussian curvature entropy

The last application of our main result is a compactness result in the spirit of Cheeger and Gromov [5], Trojanov [24] and most recently Shioya [21] for Riemannian surfaces. Indeed, we prove a general compactness result for sequence of metrics on a given closed surface assuming only that the area and the total curvature are uniformly bounded and that the entropy of the Gaussian curvature is also bounded. The first assumptions are the weaker we can assume in order to the problem makes sense. And the second is made necessarily if one consider a long thin cylinder closed by a two spherical cap, see [24] and reference therein for more examples of degenerating metrics with bounded curvature and area.

The entropy of the Gaussian curvature of a given metric is defined as follows, let Σ be a closed surface and g a Riemannian metric with Gaussian curvature equal to K_g, then we set

$$E(g) = \int_\Sigma K_g^+ \ln(K_g^+)\, dv_g,$$

where $K_g^+ = \max(0, K_g)$ and we set $K_g^+ \ln(K_g^+) = 0$ when $K_g^+ = 0$. This was introduce by Hamilton in the context of Ricci flow on surfaces. He notably proved that it is monotonically increasing along the Ricci flow on spheres with positive curvature, see [8] and [6]. In order to apply directly our preceding result, we introduce a slightly stronger notion of entropy. Let Σ a closed surface with a reference metric g_0, then we set

$$\tilde{E}_0(g) = \int_\Sigma \ln(e + |K_g dv_g|_{g_0})|K_g|\, dv_g.$$

Then considering this notion of entropy, we get the following compactness result.

Theorem 4.1. *For any closed Riemannian surface (Σ, g_0) and any sequence of smooth metric g_k such that*

$$\int_\Sigma |K_g|\, dv_g + \tilde{E}_0(g_k) = O(1),$$

then for each component σ of the thick part of (Σ, h_k), then, up to a dilatation of the metric by a factor e^{-C_k}, h_k converges weakly in $(L^\infty_{loc}(\sigma))^*$.

More precisely, up to a subsequence, one of the following occurs

i) $\text{genus}(\Sigma) = 0$*, then there exists C_k such that if $e^{-C_k} g_k = e^{2u_k} g_0$, where g_0 is the metric of the standard sphere, and u_k is uniformly bounded,*

ii) genus$(\Sigma) = 1$, *then up to a first dilation,* (Σ, g_k) *is isometric to* $\mathbb{C}/(\mathbb{Z} \times v_k\mathbb{Z})$ *where* v_k *lies in a fundamental domain of* $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$*, then there exists* C_k *such that if* $e^{-C_k} g_k = e^{2u_k} dz^2$ *then* u_k *is bounded in* $L^\infty_{loc}(\mathbb{C}/(\mathbb{Z} \times v_k\mathbb{Z}))$.

iii) genus$(\Sigma) \geq 1$, *then let* σ *a connected component* σ *of the nodal surface of* (Σ, h_k)*, then there exists* C_k *such that if* $e^{-C_k} g_k = e^{2u_k} h_k$ *and* u_k *is bounded in* $L^\infty_{loc}(\sigma)$.

Here, for a sake of simplicity we consider the standard sphere and a cylinder of fixed radius as thick part.

Proof of theorem 4.1:

We choose the atlas given by theorem 0.1, and let $g_k = e^{2u_k} h_k$ where u_k is the conformal factor with respect to a normalized metric of constant curvature. Let K be a compact set of σ a connected component of the thick part and U an open set of σ such that $\overline{U}$ is compact and $K \subset U$. Then on U the conformal factor satisfies

$$\Delta_{h_k} u_k = K_{g_k} e^{2u_k} - K_{h_k},$$

and h_k converges strongly to a smooth metric. On the one hand, since the total curvature is bounded, as in theorem 3.1, we get that ∇u_k is uniformly bounded in $L^{2,\infty}$. Then let $v_k \in H^1_0(U)$ such that

$$\Delta_{h_k} v_k = K_{g_k} e^{2u_k} - K_{h_k} \text{ on } U.$$

On the other hand, thanks to the theory of singular integral, see [23] exercise II 6.2.(b), we get that ∇v_k is uniformly bounded in L^2 and that v_k is uniformly bounded in L^∞. Then, since $u_k - v_k$ is harmonic on U whose gradient is bounded in $L^{2,\infty}$, then thanks to Harnack inequality there exists a constant C_k such that $u_k - v_k - C_k$ is uniformly bounded on K. Then, after checking that the constant is independent of K, we get that on each connected component of the thick part there exists a sequence of constant C_k such that $e^{-C_k} g_k = e^{\tilde{u}_k} h_k$ with $\tilde{u}_k$ uniformly bounded in $L^\infty_{loc}(\sigma)$, which prove the theorem.□

Remark : An interesting problem is to try to replace $\tilde{E}_0$ by E in the preceding theorem. In order to do this, on need to analyze the way $K_g dv_g$ concentrates as already done, in the particular case where the Gauss curvature converges uniformly, by [2] see also [15].

References

[1] Aubin, Thierry Some nonlinear problems in Riemannian geometry. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[2] Brezis, Haïm; Merle, Frank Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)\, e^u$ in two dimensions. Comm. Partial Differential Equations 16 (1991), no. 8-9, 12231253.

[3] Buser, Peter Geometry and spectra of compact Riemann surfaces. Reprint of the 1992 edition. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010.

[4] Chavel, Isaac Riemannian geometry. A modern introduction. Second edition. Cambridge Studies in Advanced Mathematics, 98. Cambridge University Press, Cambridge, 2006.

[5] Cheeger, Jeff; Gromov, Mikhael Collapsing Riemannian manifolds while keeping their curvature bounded. I. J. Differential Geom. 23 (1986), no. 3, 309346.

[6] Chow, Bennett The Ricci flow on the 2-sphere. J. Differential Geom. 33 (1991), no. 2, 325334.

[7] Grafakos, Loukas Classical Fourier analysis. Second edition. Graduate Texts in Mathematics, 249. Springer, New York, 2008.

[8] Hamilton, Richard S. The Ricci flow on surfaces. Mathematics and general relativity (Santa Cruz, CA, 1986), 237262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

[9] Hélein, Frédéric Harmonic maps, conservation laws and moving frames. Cambridge Tracts in Mathematics, 150. Cambridge University Press, Cambridge, 2002.

[10] Hubbard, John Hamal Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1. Teichmüller theory. Matrix Editions, Ithaca, NY, 2006.

[11] Hummel, Christoph Gromov's compactness theorem for pseudo-holomorphic curves. Progress in Mathematics, 151. Birkhuser Verlag, Basel, 1997.

[12] Ji, Lizhen The asymptotic behavior of Green's functions for degenerating hyperbolic surfaces. Math. Z. 212 (1993), no. 3, 375394.

[13] Jost, Jürgen Compact Riemann surfaces. An introduction to contemporary mathematics. Third edition. Universitext. Springer-Verlag, Berlin, 2006.

[14] Kuwert, Ernst; Li, Yuxiang $W^{2,2}$-conformal immersions of a closed Riemann surface into Rn. Comm. Anal. Geom. 20 (2012), no. 2, 313340.

[15] Li, Yan Yan; Shafrir, Itai Blow-up analysis for solutions of $-\Delta u = V(x)\, e^u$ in dimension two. Indiana Univ. Math. J. 43 (1994), no. 4, 12551270.

[16] Mondino Andrea, Rivière Tristan Immersed spheres of finite total curvature into manifolds. Adv. Calc. Var. (Ahead of print) 2013.

[17] Rivière, Tristan Variational Principles for immersed surfaces with L2-bounded second fundamental forms. J. reine angew. Math. Crelles *ahead of print* 2013.

[18] Rivière, Tristan Lipschitz conformal immersions from degenerating Riemann surfaces with L2-bounded second fundamental forms. Adv. Calc. Var. 6 (2013), no. 1, 131.

[19] Rivière, Tristan Weak immersions with L2-bounded second fundamental forms. PCMI - IAS lecture notes, Park City 2013.

[20] de Saint-Gervais, Henri Paul Uniformisation des surfaces de Riemann. Retour sur un théorème centenaire. ENS Éditions, Lyon, 2010.

[21] Shioya, Takashi The limit spaces of two-dimensional manifolds with uniformly bounded integral curvature. (English summary) Trans. Amer. Math. Soc. 351 (1999), no. 5, 17651801.

[22] Simon, Leon Lectures on geometric measure theory. Proceedings of the Centre for Mathematical Analysis, Australian National University, 3. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[23] Stein, Elias M. Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.

[24] Troyanov, M. Un principe de concentration-compacit pour les suites de surfaces riemanniennes. (French. English summary) [A concentration-compactness principle for sequences of Riemann surfaces] Ann. Inst. H. Poincar Anal. Non Linaire 8 (1991), no. 5, 419441.