HARMONIC MAPS FROM B^3 INTO S^2 HAVING A LINE OF SINGULARITIES

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I.Introduction

In this paper, we consider maps from a bounded domain $\Omega \subset \mathbb{R}^n$ taking values in the unit sphere S^{m-1}, of $\mathbb{R}^m$. If u is a map in $H^1(\Omega, S^{m-1})$ we denote by $E(u)$ the Dirichlet energy:

$$E(u) = \int_\Omega |\nabla u|^2 dx$$

Weakly harmonic maps from Ω into S^{m-1} are critical points of $E(u)$ for variations on range; more precisely u is harmonic from Ω into S^{m-1} if:

$$\forall \xi \in C_0^\infty(\Omega; \mathbb{R}^m) \qquad \frac{d}{dt} E(\frac{u+t\xi}{|u+t\xi|})(0) = 0$$

this turns out to be equivalent to the fact that u is a weak solution of the equation

$$\Delta u + |\nabla u|^2 u = 0 \qquad (1)$$

Two classes of the weak harmonic maps are particulary studied:

I *the minimizing harmonic maps*: those are the maps which minimize the energy $E(u)$ among the maps of $H^1(\Omega; \mathbb{R}^m)$ with value into the sphere S^{m-1}.

II *the stationary harmonic maps*: those are the weak harmonic maps which are also critical points of the energy $E(u)$ for the variations in the domain Ω: more precisely u is harmonic and

$$\forall \eta \in C_0^\infty(\Omega; \mathbb{R}^n) \qquad \frac{d}{dt} E(u(x + t\eta(x)))(0) = 0 \qquad (2)$$

There are several results concerning partial regularity of harmonic maps we recall briefly most of them in the following board. By $Sing$ we denote the singular set of the considered maps, $dim(Sing)$ the Hausdorff dimension of $Sing$ and $\mathcal{H}^k(Sing)$ the value of the k-th Hausdorff measure of $Sing$. In italic we precise the name of the authors of the proofs followed by the references of the proofs.

	Minimizers	Stationary maps	Weak harmonic maps
$n = 2$	$Sing = \emptyset$:	$Sing = \emptyset$:	$Sing = \emptyset$:
	Morrey [10]	*Gruter* [7] ,*Schoen* [11]	*Helein* [9]
$n \geq 3$	$dim(Sing) \leq n - 3$:	$\mathcal{H}^{n-2}(Sing) = 0$:	???
	Schoen-Uhlenbeck [12]	*Evans* [3]	

The preceeding results remain valid for maps from a Riemannian Manifold with boundary into a compact Riemannian Manifold without boundary, except the result of L.C.Evans, whose proof fundamentaly uses the symmetries of the sphere.
No general result concerning the regularity of weakly harmonic maps from a domain of $\mathbb{R}^n$ ($n \geq 3$) into a sphere has been found. We give here examples of harmonic maps from B^3 into S^2 whose singular set is exactly a segment in B^3 and thus with Hausdorff measure $\mathcal{H}^1$ not equal to zero; this shows that the regularity results, concerning the stationary maps, do not extend to arbitrary weakly harmonic maps: indeed, stationarity and more precisely monoticity formulas (see the preceding references) are an essential tool in the regularity theory for these maps.
We may even conjecture that there exists harmonic maps for $n = 3$ singular everywhere.

Our starting point is a prescribed singularity result of R.Hardt, F.H.Lin and C.Poon [8] concerning axially symmetric maps from B^3 into S^2 : a map u from B^3 into S^2 is axially symmetric if it can be written in cylindrical coordinates in the following way:

$$u(r;\theta;z) = (cos\theta\, sin\phi(r,z); sin\theta\, sin\phi(r,z); cos\phi(r,z))$$

$\phi(r,z)$ is usually called the angle function. Now let g be a map from ∂B^3 into S^2 verifying the following condition:

$$(C) \begin{cases} g \text{ is } C^\infty(\partial B^3; S^2) \\ g \text{ is axially symmetric} \\ g \text{ has an angle function in } [-\pi; +\pi] \\ g((0,0,1)) = g((0,0,-1)) \end{cases}$$

The result of Hardt, Lin and Poon asserts that for any set of consecutive points of B^3 on the z axis $\{a_1^+, a_1^-, ..., a_j^+, a_j^-\}$ there exists an axially symmetric harmonic map $u \in C^\infty(B^3/\{a_1^+, a_1^-, ..., a_j^+, a_j^-\})$ having singularities of degree ± 1 if $g((0,0,1)) = (0,0,1)$ or ∓ 1 if $g((0,0,1)) = (0,0,-1)$ at the a_i and such that $u/_{\partial B^3} = g$.

This theorem is in fact a regularity result for minimizers of the generalised relaxed energy functional $F_v(u)$ introduced by F.Bethuel, H.Brezis and J.-M.Coron in [1], where $F_v(u)$ is given by

$$F_v(u) = E(u) + 8\pi L(u,v) \quad \text{with} \quad E(u) = \int_{B^3} |\nabla u|^2 dx,$$

for a fixed $v \in H^1(B^3, S^2)$

$L(u,v)$ is given by $L(u,v) = \frac{1}{4\pi} sup_{\xi: B^3 \to \mathbb{R}, \|\nabla\xi\|_\infty \leq 1} \{ \int_{B^3} (D(u) - D(v)).\nabla\xi \, dx \}$

and $D(u)$ by $D(u) = (u.u_y \wedge u_z; u.u_z \wedge u_x; u.u_x \wedge u_y)$, vector field introduced in [2]

It is easy to verify that $L(u, v_1) = L(u, v_2)$ if v_1 and v_2 have the same singularities . R.Hardt, F.H.Lin and C.Poon have proved that, for a fixed v having a finite number of singularities, any minimizer of $F_v(u)$ (such a minimizer is weakly harmonic see [1]) among axially symmetric map having an angle function in $[-\pi; +\pi]$ and a fixed boundary condition g verifying (C), has exactly the same singularities as v, with same degrees. (This implies in particular that $L(u,v) = 0$). Indeed, when u and v have only a finite number of singularities $L(u,v)$ can be interpreted as the minimal connexion, see [2], between all the singularities of u and v after having inversed the sign of the degree of the singularities of v. Note also that minimizers of $F_v(u)$ do not depend only on v but only on its singularities along the z axis.

The idea of our proof is a construction of a H^1-convergent sequence v_n of axially symmetric maps from B^3 into S^2 having more and more singularities along the z axis and we observe the behavior of the minimizers of F_{v_n} by passing to the limit. Our aim is to find a special configuration of the added singularities for which there exists a strong H^1 convergence of a special sequence of minimizers. This strong H^1 convergence will indeed preserve, at the limit, all the singularities we have added.

II. Prescribed singularities for axially symmetric maps.

We recall here without proof the result of R.Hardt, F.H.Lin and C.Poon [8] in the form we will use later.For this we introduce the following notations:
for $g : \partial B^3 \to S^2$ verifying (C) let

$$H^1_g(B^3; S^2) = \{u \in H^1(B^3; S^2); u_{\backslash \partial B^3} = g\}$$

$$\mathcal{R}^\pi_{AS} = \left\{ \begin{array}{l} u \in H^1(B^3; S^2); u \text{ is axially symetric,with angle function in } [-\pi, +\pi], \\ u \text{ has only a finite number of singularities which alternate} \\ \text{with degrees } \pm 1 \text{ along the } z \text{ axis} \end{array} \right\}$$

and $\mathcal{A}^\pi_{AS} = \overline{\mathcal{R}^\pi_{AS}}^{H^1}$

Remark: in [8], the notation $\mathcal{R}^{\pi}_{AS}$ denotes a slightly different and smaller set, more precisely, they impose at every singular point of any map in $\mathcal{R}^{\pi}_{AS}$ a special asymptotic configuration, nevertheless this set has the same closure in $H^1_g(B^3, S^2)$.

THEOREM 1 [8] . *Let g be a map from ∂B^3 into S^2 verifying the condition (C) and v in $H^1_g(B^3,S^2) \cap \mathcal{R}^{\pi}_{AS}$ then any minimizer of F_v among $H^1_g(B^3,S^2) \cap \mathcal{A}^{\pi}_{AS}$ has the same singularities with the same degree as v.*

III. The construction of maps having a line of singularities

We prove in this part the following result:

THEOREM 2 . *Let g be a map from ∂B^3 into S^2 verifying the condition (C), let a, b be arbitrary real numbers such that $-1 < a < b < 1$, there exists an axially symmetric harmonic map u in $\mathcal{A}^{\pi}_{AS} \cap C^{\infty}(\overline{B}^3/\{(0,0,z); a \le z \le b\}) \cap H^1_g(B^3,S^2)$ whose singular set is exactly the segment $[a,b]$ on the z axis.*

Remark: we can make a similar construction for an union of disjoint segments on $B^3\cap$ z axis.

III.1) Presentation and reduction of the problem

Without loss of generality we may assume that $g((0,0,1)) = (0,0,1)$. we will often make use of the formalism of the cartesian currents introduced in this context of maps in $H^1_g(B^3,S^2)$ by M.Giaquinta, G.Modica and J.Soucek in [5], [6] and adapted to the axially symmetric case in [8] (lemma 3.1):

LEMMA 1 . *For any u in $H^1_g(B^3,S^2) \cap \mathcal{A}^{\pi}_{AS}$ ($deg\, g = 0$) there exists J a Lebesgue mesurable subset of z axis $\cap B^3$ such that*

$$\partial[graph\, u]\lfloor(B^3 \times S^2) = -\partial[J] \times [S^2] \qquad (3)$$

(We replace the sign - by + if $g((0,0,1)) = (0,0,-1)$)

For the convenience of the reader, we illustrate the meaning of (3) on a simple example: assume that u has a finite number of singularities $a_i^{\pm}$ of degree ± 1 which necessarily alternate along the z axis (this is imposed by topological reasons in the axially symmetric case: see [8]) the graph of u has the following boundary:

$$\partial[graph\, u]\lfloor(B^3 \times S^2) = -\partial[\bigcup_{i=1}^{j}[a_i^-, a_i^+]] \times [S^2] = -(\sum_{i=1}^{j}[\{a_i^+\}] - [\{a_i^-\}]) \times [S^2]$$

Using this formalism we can give the following interpretation of $L(u,v)$: for u and v in $H^1_g(B^3,S^2)\cap \mathcal{A}^{\pi}_{AS}$ there exists a one dimensional current $I_{u,v}$ with support on the z axis such that

$$-\partial I_{u,v}\times[S^2]=(\partial[graph\, u]-\partial[graph v])\lfloor(B^3\times S^2),$$
$$\text{and } M(I_{u,v})=L(u,v)$$

see [5]; in the particular case where one of the two maps is in $\mathcal{R}^{\pi}_{AS}$ this current is rectifiable, with multiplicity 1 and of the form $+[J_1]-[J_2]$ where J_1 and J_2 are Lebesgue measurable on $B^3\cap$ Z axis.

In order to prove theorem 2, we are going to construct u and v verifying conditions (C1) (C2) and (C3) below; we have the following:

THEOREM 3 . *There exists u and v in $\mathcal{A}^{\pi}_{AS}$ verifying the three following conditions:*

$(C1)\,\partial[graph\, v]\lfloor(B^3\times S^2)=-\partial[J]\times[S^2]$ *where J is a Lebesgue measurable subset of (a,b) verifying $0<\mathcal{H}^1(J\cap(\alpha,\beta))<\beta-\alpha$ for any $a<\alpha<\beta<b$*

$v\in C^{\infty}(\overline{B^3}\setminus[a,b];S^2)$

$(C2)\,u$ *is a minimizer of F_v among $H^1_g(B^3,S^2)\cap\mathcal{A}^{\pi}_{AS}$*

$(C3)\quad L(u,v)=0$

Theorem 2, then is a consequence of theorem 3 and the following:

LEMMA 2 . *Let u and v in $H^1_g(B^3,S^2)\cap\mathcal{A}^{\pi}_{AS}$ verifying (C1) (C2) and (C3) then u is an axially symmetric harmonic map in $C^{\infty}(\overline{B^3}/\{(0,0,z);a\le z\le b\})$ whose singular set is exactly the segment $[a,b]$ on the Z axis.*

Proof of lemma 2:The regularity of u away from the z axis comes directly from the axial symmetry of u and the fact that u is harmonic: indeed the angle function of u verifies the following elliptic equation for $r>0$:

$$\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r})+\frac{\partial}{\partial z}(r\frac{\partial\phi}{\partial z})+\frac{sin2\phi}{2r}=0$$

In any ball $B_\rho(x)$ centered on the z axis which does not intersect $[a,b]$ we observe, as in theorem 9 [8] that u minimize the classical relaxed energy $E(u)+8\pi L(u)$ among $\mathcal{A}^{\pi}_{AS}$, so, by theorem 7.2 [8] u is C^{∞} away from $[a,b]$. Let us now consider a small ball $B_\rho(x)$ centered on the z axis between a and b then

$$L(u,v)=0\ \Rightarrow\ \partial[graph\, u]\lfloor B_\rho(x)=\partial[graph\, v]\lfloor B_\rho(x)$$

Moreover, since $0 < \mathcal{H}^1(J \cap (x-\rho, x+\rho)) < 2\rho$, $\partial[J]\lfloor B_\rho(x) \neq 0$, $\partial[graph\, v]\lfloor B_\rho(x) \neq 0$ thus $\partial[graph\, u]\lfloor B_\rho(x) \neq 0$ and u is not regular on $B_\rho(x)$ so $sing(u) = [a, b]$. This completes the proof of lemma 2.$\triangle$

Remark: We may conjecture (C1) and (C2) imply (C3). Actually if we replace (C1) by "v have a finite number of singularity" this is true: that is the Theorem1. By constructing two such maps u and v which verify (C1) (C2) and (C3) we avoid a more general eventuel result.

The rest of the paper (section III.2) is devoted to the proof of theorem 3; that is the construction of u and v verifying (C1) (C2) and (C3). They will be obtained as strong limits of sequences u_n and v_n in $\mathcal{R}^\pi_{AS}$.

III.2) Proof of theorem 3

we will construct u_n and v_n two sequences of $\mathcal{R}^\pi_{AS} \cap H^1_g(B^3, S^2)$ which verifie (D1) and (D2):

$(D1)$ v_n converges strongly in H^1 to v which verifies $(C1)$

$(D2)$ there exists a sequence of minimizers u_n of F_{v_n} among $H^1_g(B^3, S^2) \cap \mathcal{A}^\pi_{AS}$ which strongly converges in H^1

The strong limits of those two sequences verifie (C1) (C2) and (C3): (C1) is contained in (D1), (C2) comes easily from the sequentially lower semicontinuity for the weak H^1 topology of F_v (exactly like in lemma 5), finally theorem 1 implies that $L(u_n, v_n) = 0$ and the two strong convergences enable us to pass to the limit in L, since L is continuous for the strong H^1 topology (see[1]) then (C3) is verified.

Remark: (D2) is of course the most difficult condition to obtain and the strong convergence is necessary: a simple weak convergence could efface all the singularities we have added.

III.2) a) Construction of a familly of sequences v_n verifying (D1)

As we have announced in the introduction v_n will be constructed by adding more and more singularities, and because of v_n must strongly converge, we have to do this by spending little energy. So we need the following lemma:

Lemma 3 . *Let u in $H^1_g(B^3,S^2)\cap\mathcal{A}^{\pi}_{AS}$ and $-1<\alpha<\beta<1$ two reals such that u be regular in a neighborhood V of the segment $[\alpha;\beta]$ on the Z axis. Then for any $C>8\pi$ there exists $\tilde u$ in $H^1_g(B^3,S^2)\cap\mathcal{A}^{\pi}_{AS}$ such that*

(*a*) $(0,0,\alpha)$ *and* $(0,0,\beta)$ *are the only one singularities of* $\tilde u$ *in* V *and with opposite degrees* ± 1 *or* ∓ 1

b) $\tilde u = u$ *in* B^3/V

c) $\int_{B^3}|\nabla(u-\tilde u)|^2dx\le C|\beta-\alpha|$

Proof of lemma 3: This lemma is a direct application of the "construction of the dipole" in the axially axially symmetric case made in [8](lemma 6.1) and we shortly recall it here for the convenience of the reader.

We modify u into two balls of radii $\varepsilon|\beta-\alpha|$, centered in $(0,0,\alpha)$ and in $(0,0,\beta)$ and we also modify u into the cylinder joining them; precisely into

$$\begin{aligned}\Omega_\varepsilon = \{(r,\theta,z)/\alpha\le z\le\beta \text{ and } r\le\varepsilon|\beta-\alpha|\}\\ \cup\{(r,\theta,z)/r^2+(z-\alpha)^2\le\varepsilon^2|\beta-\alpha|^2\}\\ \cup\{(r,\theta,z)/r^2+(z-\beta)^2\le\varepsilon^2|\beta-\alpha|^2\}\end{aligned}$$

and for ε sufficiently small such that $\Omega_\varepsilon\subset V$. Let $\Omega^+_\varepsilon=\Omega_\varepsilon\cap\{z\ge\frac{\beta+\alpha}{2}\}$ and $\Omega^-_\varepsilon=\Omega_\varepsilon\cap\{z\le\frac{\beta+\alpha}{2}\}$. $\partial\Omega^+_\varepsilon\cap\partial\Omega^-_\varepsilon$ is the horizontal disk $B^2_{\varepsilon|\beta-\alpha|}((0,0,\frac{\beta+\alpha}{2}))$. Let $\tilde u$ coincide, on this disk, with the unique conformal axially homeomorphism that maps onto the large spherical region $\{X\in S^2:|X-(0,0,1)|>tan(\phi(\varepsilon|\beta-\alpha|;\frac{\beta+\alpha}{2}))\}$ Let $\tilde u$ coincide with u in B^3/Ω_ε. The rest of $\tilde u$ is, in Ω^-_ε the natural radial extension centered in $(0,0,\alpha)$ of its value on $\partial\Omega^-_\varepsilon$ and in Ω^+_ε, $\tilde u$ is the natural radial extention centered in $(0,0,\beta)$ of its value on $\partial\Omega^+_\varepsilon$. We remark that such modifications preserve the axial symmetry of the map and maintain its angle function in $[-\pi;+\pi]$. The map that we obtain is only lipschitz in $V/\{(0,0,\alpha);(0,0,\beta)\}$ but it does not spend a lot of H^1 energy to smooth the angle function in this domain.
So we obtain (see [8]):

$$\int_{\Omega_\varepsilon}|\nabla\tilde u|^2dx\le 8\pi|\beta-\alpha|+O(\varepsilon)+2\pi\int_0^{\varepsilon|\beta-\alpha|}\int_{-\frac{r}{\varepsilon}}^{\frac{r}{\varepsilon}}\varepsilon^2|\beta-\alpha|^2\frac{z^2}{r^3}|\frac{\partial\phi}{\partial z}|^2dz\,dr$$

then $\int_{\Omega_\varepsilon}|\nabla\tilde u|^2dx\le 8\pi|\beta-\alpha|+O(\varepsilon)+C||\frac{\partial\phi}{\partial z}||^2_{L^\infty(\Omega^\varepsilon)}$

But $\frac{\partial \phi}{\partial z} = 0$ on the z axis in V then $||\frac{\partial \phi}{\partial z}||^2_{L^\infty(\Omega^\varepsilon)} \to 0$ as $\varepsilon \to 0$. Since $\tilde{u} = u$ in B^3/Ω_ε and since $\int_{\Omega_\varepsilon} |\nabla u|^2 dx \to 0$ as $\varepsilon \to 0$ we obtain the desired result.$\triangle$

Thus the singularities will be added dipole by dipole. Each singularity will be an end of a segment of the sequence of sets on the Z axis that we introduce now.
Let μ_n and δ_n be two sequences of positive reals verifying the following condition:

$$(\Delta) \begin{cases} 0 < \delta_0 < \dfrac{b-a}{2} \\ 0 < \delta_n < \mu_n^2 & \forall n \in \mathbb{N} \\ 0 < \mu_{n+1} < \delta_n^2 \end{cases}$$

We denote by $E = (E_n)$ the associated sequence of subset of $[a,b]$ constructed by induction as follows:

$$E_0 = [a,b], \quad E_{n+\frac{1}{2}} = E_n \setminus \left\{ \begin{array}{r} \text{open segments of length } \delta_n < \mu_n^2 \\ \text{each centred at the middles of the segments of } E_n \end{array} \right\}$$

$$E_{n+1} = E_{n+\frac{1}{2}} \bigcup \left\{ \begin{array}{c} \text{closed segments of length } \mu_{n+1} < \delta_n^2 \text{ each centered at} \\ \text{the middles of the segments of } [a,b] \setminus E_{n+\frac{1}{2}} \text{ in } [a,b] \end{array} \right\}$$

Let $\mathcal{E}_{a,b}$ be the set of such sequences (E_n); we have the following lemma:

Lemma 4 . *For any E in $\mathcal{E}_{a,b}$ there exists v_n in $\mathcal{R}^\pi_{AS} \cap H^1_g(B^3, S^2)$ such that*

$$\begin{array}{ll} a) & \partial[graph\, v_n] \lfloor (B^3 \times S^2) = -\partial[E_n] \times [S^2] \\ b) & v_n \text{ verifie (D1)} \end{array}$$

Remark: Since v_n is in $\mathcal{R}^\pi_{AS}$ the cartesian currents equality of the lemma signifies that the singularities of v_n are exactly the ends of the segments of E_n and for any given segment of E_n the degree of the singularity at the superior end is +1 and the degree of the singularity at the inferior end is -1

Proof of lemma 4: We construct v_n by induction: let v_0 be any map in $H^1_g(B^3, S^2) \cap \mathcal{R}^\pi_{AS}$ whose singularities are exactly a and b. From v_n we construct $v_{n+1/2} \in \mathcal{R}^\pi_{AS}$ whose graph has the boundary $-\partial[E_{n+1/2}] \times [S^2]$ by inserting dipoles of length $\delta_n < \mu_n^2$ each centered at the center of the segments of E_n as it is made in lemma 2 for a given constant $C > 8\pi$ independent of n. then, similary, from $v_{n+1/2}$ we construct $v_{n+1} \in \mathcal{R}^\pi_{AS}$ whose graph has the boundary $-\partial[E_{n+1}] \times [S^2]$ by inserting dipoles of length $\mu_{n+1} < \delta_n^2$ each

centered at the center of the segments of $[a,b]/E_{n+1/2}$ in $[a,b]$; those dipoles of course will have an opposite orientation than those that we have added between n and $n+1/2$.
We now verify that v_n strongly converges in H^1. N_n the number of segment of E_n is

$$N_n = 2/3 * 4^n + 1/3$$

$$\text{so} \int_{B^3} |\nabla(v_n - v_{n+1/2})|^2 dx \leq C\,(4^n(2/3)+1/3)\,\delta_n \leq C'\,4^n * (\mu_1^2)^{4^{n-1}}$$

$$\text{and} \int_{B^3} |\nabla(v_{n+1/2} - v_{n+1})|^2 dx \leq C\,(2\,(4^n(2/3)+1/3)-1)\,\mu_{n+1} \leq C'\,4^n * (\mu_1^2)^{4^n}$$

$$\text{then} \int_{B^3} |\nabla(v_{n+1} - v_n)|^2 dx \leq C"\,4^n * (\mu_1^2)^{4^n}$$

it is clear that v_n is a Cauchy sequence in H^1, let v be its limit there exists a Lebesgue measurable subset J of $B^3 \cap$z axis such that $\partial[graph\, v] \lfloor B^3 \times S^2 = -\partial[J] \times [S^2]$. $[E_n]-[J]$ is a mass equibounded sequence of 1 dimension currents, we can extract a weakly convergent sequence of currents always denoted $[E_n]-[J]$; let L be the limit

$$[E_n] - [J] \rightharpoonup L \qquad \text{and} \qquad spt(L) \subset [a,b].$$

Since $v_n \longrightarrow v$ strongly in H^1 we have

$$\partial[graph\, v_n] \rightharpoonup \partial[graph\, v]$$

$$\text{then} \qquad \partial[E_n] - \partial[J] \rightharpoonup 0 \qquad \text{so} \qquad \partial[L] = 0$$

The constancy theorem (see [4]) implies $L=0$ and $[E_n] \rightharpoonup [J]$. Thus, χ_n, the characteristic function of E_n weakly converges in $L^{\infty *}$ to χ the characteristic function of J.

Let us show now that

$$\forall(\alpha,\beta) \in \mathbb{R}^2 \quad a < \alpha < \beta < b \quad \text{then} \quad 0 < \mathcal{H}^1(J \cap (\alpha,\beta)) < \beta - \alpha.$$

For p sufficiently large there exists a segment S of E_p such that $S \subset [\alpha,\beta]$ thus it is sufficient to show that $0 < \int_S \chi\, dx < |S|$. Since $\chi_n \rightharpoonup \chi$ in $L^{\infty *}$, $\int_S \chi_n\, dx \longrightarrow \int_S \chi\, dx$; moreover

$$\int_S \chi_{n+1+p}\, dx = \int_S \chi_{n+p}\, dx - (2*4^n+1)\,\frac{\delta_{n+p}}{3} + (4^{n+1}-1)\,\frac{\mu_{n+p+1}}{3}$$

$$= |S| - \delta_p - \sum_{k=1}^{n} \frac{\delta_{k+p}}{6}(4^{k+1}+2) + \sum_{k=1}^{n+1} \frac{\mu_{k+p}}{3}(4^k-1)$$

$$\text{thus} \quad \int_S \chi\, dx = |S| - \delta_p - \sum_{k=1}^{+\infty} \frac{\delta_{k+p}}{6}(4^{k+1}+2) + \sum_{k=1}^{+\infty} \frac{\mu_{k+p}}{3}(4^k-1)$$

using the condition (C) verified by δ_n and μ_n the result easily follow.$\triangle$

III.2) b) Choice of E_n; strong convergence of a sequence of minimizers:(D2)

α)A strong convergence lemma

The following result which is very useful is a simple consequence of theorem 1

LEMMA 5 . *Let w_n be in $H^1_g(B^3,S^2)\cap\mathcal{R}^\pi_{AS}$ which strongly converges in H^1 to w and let u_n be a sequence of minimizers of F_{w_n} among $H^1_g(B^3,S^2)\cap\mathcal{A}^\pi_{AS}$ which weakly converges to u in H^1; if w is in $\mathcal{R}^\pi_{AS}$ then*

a) *u minimizes F_w in $H^1_g(B^3,S^2)\cap\mathcal{A}^\pi_{AS}$*

b) *the convergence of u_n to u is strong*

Proof of lemma 5: First we show that $\lim_{n\to+\infty}F_{w_n}(u_n) = F_w(u)$. Let ξ be in $C^\infty(B^3;\mathbb{R})$ with $||\nabla\xi||\leq 1$, the sequentially lower semicontinuity for the weak H^1 topology of the functional

$$u\rightarrow\int_{B^3}|\nabla u|^2dx+2\int_{B^3}D(u).\nabla\xi dx \qquad \text{(see[1])}$$

implies that

$$\liminf_{n\to+\infty}\int_{B^3}|\nabla u_n|^2\,dx+2\int_{B^3}D(u_n).\nabla\xi\,dx\geq\int_{B^3}|\nabla u|^2\,dx+2\int_{B^3}D(u).\nabla\xi\,dx$$

Since $w_n\to w$ strongly in H^1: $\int_{B^3}D(w_n).\nabla\xi\,dx\to\int_{B^3}D(w).\nabla\xi\,dx$ then

$$\liminf_{n\to\infty}F_{w_n}(u_n)\geq F_w(u)$$

moreover , using the fact that $\lim_{n\to\infty}F_{w_n}(u)=F_w(u)$ and the minimality of u_n for F_{w_n} then $F_w(u)\geq\limsup_{n\to\infty}F_{w_n}(u_n)$ thus

$$\lim_{n\to\infty}F_{w_n}(u_n)=F_w(u)$$

Let $v\in H^1_g(B^3,S^2)\cap\mathcal{A}^\pi_{AS}$ $\quad F_{w_n}(v)\geq F_{w_n}(u_n)\Rightarrow F_w(v)\geq F_w(u)$ thus u minimizes F_w among $H^1_g(B^3,S^2)\cap\mathcal{A}^\pi_{AS}$. Suppose now that w is in $\mathcal{R}^\pi_{AS}$, we apply theorem 1 and we have $L(u,w)=0$ and $u\in\mathcal{R}^\pi_{AS}$; since

$$\lim_{n\to+\infty}E(u_n)=\lim_{n\to+\infty}F_{w_n}(u_n)=F_w(u)=E(u)$$

$,u_n$ strongly converge to u. This completes the proof of the lemma.$\triangle$

β)**the choice of** E_n

As we have announced in the introduction for v_n in $\mathcal{R}^\pi_{AS}$ the minimizing problem F_{v_n} among $\mathcal{A}^\pi_{AS} \cap H^1_g(B^3, S^2)$ only depends on the singularities of v_n and not intrinsically on v_n, this is a direct consequence of theorem1. Thus the set of the minimizers is exactly the following

$$I_n = \left\{ \begin{array}{r} u \in H^1_g(B^3,S^2) \cap \mathcal{R}^\pi_{AS} \text{ s.t. } u \text{ minimize } E(u) \text{ among } H^1_g(B^3,S^2) \cap \mathcal{A}^\pi_{AS} \\ \text{with the constraint} \\ \partial[graph\, u] \lfloor (B^3 \times S^2) = -\partial[E_n] \times [S^2] \end{array} \right\}$$

we consider now the following lemma which is an application of lemma 4

Lemma 6 . *For any sequence (ε_n) of positive real numbers tending to zero there exists (E_n) in $\mathcal{E}_{a,b}$ such that*

$$\forall n \geq 1 \quad \forall u \in I_n \qquad inf_{v \in I_{n-1}} \int_{B^3} |\nabla(u-v)|^2 dx < \varepsilon_n$$

Proof of lemma 6:We construct E_n by induction; $E_0 = [a,b]$; suppose E_n is constructed then $(\delta_p)_{p \leq n-1}$ and $(\mu_p)_{p \leq n}$ are fixed, the associated to E_n maps v_n of the lemma 4 are also fixed. we consider two sequences of positive reals $(\delta^k_n)_{k \in \mathbb{N}}$ and $(\mu^k_{n+1})_{k \in \mathbb{N}}$ tending to zero such that $\forall k \in \mathbb{N}$ $\mu^k_{n+1} < (\delta^k_n)^2$ and $\delta^k_n < (\mu^k_n)^2$. Let E^k_{n+1} be the sequence of subsets of $[a,b]$ constructed by adding dipoles of size δ^k_n and inversed dipoles of size μ^k_{n+1} exactly as in III 2¡) a) and let $v^k_{n+1} \in H^1_g(B^3,S^2) \cap \mathcal{R}^\pi_{AS}$ be the perturbation of v_n for each k after having added all the dipoles and the inversed dipoles of rank k, we have $\partial[graph\, v^k_{n+1}] \lfloor (B^3 \times S^2) = -\partial[E^k_{n+1}] \times [S^2]$. Clearly $v^k_{n+1} \to v_n$ strongly in H^1 as $k \to +\infty$. Let

$$I^k_{n+1} = \left\{ \begin{array}{r} u \in H^1_g(B^3,S^2) \cap \mathcal{R}^\pi_{AS} \text{ s.t. } u \text{ minimize } E(u) \text{ among } H^1_g(B^3,S^2) \cap \mathcal{A}^\pi_{AS} \\ \text{with the constraint} \\ \partial[graph\, u] \lfloor B^3 \times S^2 = -\partial[E^k_{n+1}] \times [S^2] \end{array} \right\}$$

suppose that

$$\begin{array}{l} \forall K \in \mathbb{N} \quad \exists k > K \quad \text{and} \quad u_k \in I^k_{n+1} \\ \text{such that} \quad inf_{v \in I_n} \displaystyle\int_{B^3} |\nabla(u_k - v)|^2 dx > \varepsilon_{n+1} \end{array} \qquad (4)$$

. u_k minimizes $F_{v^k_{n+1}}$. Since $E(u_k)$ is unifomly bounded we can extract a subsequence of u_k (allways denoted u_k) which weakly converge but, since $v^k_{n+1} \to v_n \in \mathcal{R}^{\pi}_{AS}$ strongly in H^1 from lemma 5 we know that this subsequence strongly converges to an element of I_n; we then contradict the inequality (4). The lemma is proved.$\triangle$

Let $\varepsilon_n = 1/2^n$, we consider as from now one the sequence $(E_n) \in \mathcal{E}_{a,b}$ associated to ε_n in lemma 5.

γ) construction of a strongly H^1 convergent sequence of minimizers

Let $p \geq 0$ we construct $(u^n_p)_{n \leq p}$ such that

$$u^n_p \in I_n \text{ and } \int_{B^3} |\nabla(u^n_p - u^{n-1}_p)|^2 dx \leq 1/2^n \qquad (5)$$

let v be any element of I_p, we note $u^p_p = v$; we know that

$$inf_{u \in I_{p-1}} \int_{B^3} |\nabla(u - u^p_p)|^2 dx < 1/2^p$$

let u^{p-1}_p be an element of I_{p-1} such that $\int_{B^3} |\nabla(u^{p-1}_p - u^p_p)|^2 dx < 1/2^p$...

We construct now $u_n \in I_n$ such that u_n strictly converges in H^1. From $(u^0_k)_{k \in \mathbb{N}}$ we extract $(u^0_{\phi(k)})_{k \in \mathbb{N}}$ which weakly converge in H^1 but those $u^0_{\phi(k)}$ are minimizers of F_{v_0}, from lemma 5 this convergence is strong and u_0, the limit,is a minimizer of F_{v_0}. From $u^1_{\phi(k)}$ we extract $u^1_{\phi'(\phi(k))}$ which weakly converge in H^1 but as before this convergence is strong and u_1, the limit, is a minimizer of F_{v_1}. Because of the strong convergence of $u^0_{\phi'(\phi(k))}$ and $u^1_{\phi'(\phi(k))}$ and the inequality (5) for n=1

$$\int_{B^3} |\nabla(u_0 - u_1)|^2 dx < 1/2$$

Then by induction we construct u_n in I_n such that

$$\int_{B^3} |\nabla(u_n - u_{n-1})|^2 dx < 1/2^n$$

This is a Cauchy sequence in H^1. u_n and v_n verify (D1) and (D2), this proves theorem 3. $\triangle$

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