Computer-Assisted Proof of the Main Theorem of 'The Classification of Branched Willmore Spheres in the 3-Sphere and the 4-Sphere'

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Abstract

We provide a computer-assisted proof of the holomorphy of the quartic and the octic meromorphic differentials arising in the main theorem 4.11 of our paper 'The Classification of Branched Willmore spheres in the 3-Sphere and the 4-Sphere' (arXiv:1706.01405), using the free mathematical software Sage.

Mathematical subject classification :
35J35, 35R01, 49Q10, 53A05, 53A10, 53A30, 53C42, 58E15.

Contents

Chapter 1

Introduction and organisation of the paper

1.1 Description of the results

For a broader picture on the motivations, we refer the reader to [4].

By a classical theorem of Robert Bryant (see [2]), there exists a holomorphic quartic differential $\mathscr{Q}_{\vec{\Phi}}$ associated to any Willmore immersion $\vec{\Phi} : \Sigma^2 \to S^3$ from a closed Riemann immersion Σ^2 with the following property : if $\mathscr{Q}_{\vec{\Phi}} = 0$, then there exists a stereographic projection $\pi : S^3 \setminus \{p\} \to \mathbb{R}^3$ from some $p \in S^3$ such that the mean curvature of the composition

$$\pi \circ \vec{\Phi} : \Sigma^2 \setminus \vec{\Phi}^{-1}(\{p\}) \to \mathbb{R}^3$$

vanishes identically. In particular, if Σ^2 is a topological sphere, we thereby deduce by Riemann-Roch theorem that $\mathscr{Q}_{\vec{\Phi}}$ must always vanish identically. At this point, this is not hard to show the following theorem.

Theorem (Bryant, [2])**.** *Let $\vec{\Phi} : S^2 \to S^3$ a conformal Willmore immersion. Then $\vec{\Phi}$ is the inverse stereographic projection of a complete minimal immersion $\vec{\Psi} : S^2 \setminus \{p_1, \cdots, p_m\} \to \mathbb{R}^3$ with embedded planar ends.*

Let us now consider branched Willmore immersion $\vec{\Phi} : \Sigma^2 \to S^3$. It was showed by Tobias Lamm and Huy The Nguyen ([3]) that the quartic differential $\mathscr{Q}_{\vec{\Phi}}$ is only meromorphic, and may have poles up to order equal to 2 at each branch point. Let $\{p_1, \cdots, p_m\} \subset \Sigma^2$ these branch points. That $\mathscr{Q}_{\vec{\Phi}}$ has poles of order at most 2 at $p_1, \cdots, p_m$ is equivalent to say that it is a *holomorphic* section of the holomorphic line bundle

$$\mathscr{L} = K_{\Sigma^2}^4 \otimes \mathscr{O}(2\,p_1 + \cdots + 2\,p_m),$$

where $K_{\Sigma^2} = T^*\Sigma^2$ is the canonical bundle of the compact connected Riemann surface Σ^2. If Σ^2 has genus g, then the degree (or the first Chern class) of $\mathscr{L}$ is given by

$$\begin{aligned}\deg(\mathscr{L}) &= 4\deg(\mathscr{L}) + 2\deg(\mathscr{O}(p_1 + \cdots + p_m)) \\ &= 4(2g-2) + 2m.\end{aligned}$$

In particular, if $g \geq 1$ or $m \geq 4$, then $\deg(\mathscr{L}) \geq 0$ and by Riemann-Roch theorem, the space $H^0(\Sigma^2, \mathscr{L})$ of holomorphic sections of $\mathscr{L}$ has positive dimension.

However, if Σ^2 has genus 0, and $m \leq 3$, we deduce that

$$\deg(\mathscr{L}) = -8 + 2m \leq -2 < 0,$$

so all holomorphic sections of $\mathscr{L}$ must vanish identically. In particular, we have $\mathscr{Q}_{\vec{\Phi}} = 0$, and this easily implies the following theorem.

Theorem (Lamm, Nguyen, [3]). *Let $\vec{\Phi} : S^2 \to S^3$ a branched Willmore immersion with less than 3 branch points. Then $\vec{\Phi}$ is the inverse stereographic projection of a complete minimal immersion $\vec{\Psi} : S^2 \setminus \{p_1, \cdots, p_m\} \to \mathbb{R}^3$ with finite total curvature.*

By performing a precise Taylor expansion of $\mathscr{Q}_{\vec{\Phi}}$ and obtaining informations on the first ([4]) and second residue ([5]) of *variational* branched Willmore immersions, we were able to show that $\mathscr{Q}_{\vec{\Phi}}$ has poles of order at most 1. Furthermore, for $\theta_0 \leq 3$, we showed in [4] that poles are completely removable. This shows that

$$\mathscr{Q}_{\vec{\Phi}} \in H^0(S^2, K_{S^2}^4 \otimes \mathscr{O}(p_1 + \cdots + p_m))$$

and we have

$$\deg\left(K_{S^2}^4 \otimes \mathscr{O}(p_1 + \cdots + p_m)\right) = -8 + m < 0$$

for $m \leq 7$. Therefore, we obtain in particular the following improvement of (1.1).

Theorem A ([4]). *Let $\vec{\Phi} : S^2 \to S^3$ a* variational *branched Willmore immersion with less than 7 branch points. Then $\vec{\Phi}$ is the inverse stereographic projection of a a complete minimal immersion $\vec{\Psi} : S^2 \setminus \{p_1, \cdots, p_n\} \to \mathbb{R}^3$ with finite total curvature and* zero flux.

However, as we shall see this is almost impossible to check by hand that $\mathscr{Q}_{\vec{\Phi}}$ has no poles at branch points of multiplicity $\theta_0 \geq 4$. The goal of this paper is to show that there are indeed no poles. This permits to obtain the following result, which is a combination of [4] and of the forthcoming computations.

Theorem B. *Let $n \geq 3$ and $\vec{\Phi} : D^2 \to S^n$ a branched Willmore sphere with a unique branch point at the origin of multiplicity $\theta_0 \geq 1$. Assume that $\vec{\Phi}$ is a* true *branched Willmore disk for $1 \leq \theta_0 \leq 3$, and provided $\theta_0 \geq 2$, the second residue $r(0) \in \{0, \cdots, \theta_0 - 1\}$ satisfies $r(0) \leq \theta_0 - 2$. If the quartic differential $\mathscr{Q}_{\vec{\Phi}}$ defined by*

$$\mathscr{Q}_{\vec{\Phi}} = g^{-1} \otimes \left(\partial^N \overline{\partial}^N \vec{h}_0 \,\dot{\otimes}\, \vec{h}_0 - \partial^N \vec{h}_0 \,\dot{\otimes}\, \overline{\partial}^N \vec{h}_0\right) + \frac{1}{4}\left(1 + |\vec{H}|^2\right) \vec{h}_0 \,\dot{\otimes}\, \vec{h}_0$$

is meromorphic. *Then $\mathscr{Q}_{\vec{\Phi}}$ is holomorphic, and the form of degree 8*

$$\begin{aligned}
\mathscr{O}_{\vec{\Phi}} &= g^{-2} \otimes \left\{\frac{1}{4}(\partial^N\overline{\partial}^N\vec{h}_0 \,\dot{\otimes}\, \partial^N\overline{\partial}^N\vec{h}_0) \otimes (\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0) + \frac{1}{4}(\partial^N\vec{h}_0 \,\dot{\otimes}\, \partial^N\vec{h}_0) \otimes (\overline{\partial}^N\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}^N\vec{h}_0)\right. \\
&\left.- \frac{1}{2}(\partial^N\overline{\partial}^N\vec{h}_0 \,\dot{\otimes}\, \partial^N\vec{h}_0) \otimes (\overline{\partial}^N\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0) - \frac{1}{2}(\partial^N\overline{\partial}^N\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}^N\vec{h}_0) \otimes (\partial^N\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0) + \frac{1}{2}(\partial^N\overline{\partial}^N\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0) \otimes (\partial^N\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}^N\vec{h}_0)\right\} \\
&+ \frac{1}{4}(1 + |\vec{H}|^2)\, g^{-1} \otimes \left\{\frac{1}{2}(\partial^N\overline{\partial}^N\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0) \otimes (\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0) - (\partial^N\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0) \otimes (\overline{\partial}^N\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0) + \frac{1}{2}(\partial^N\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}^N\vec{h}_0) \otimes (\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0)\right\} \\
&+ \frac{1}{64}\left(1 + |\vec{H}|^2\right)^2 \left(\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right)^2.
\end{aligned}$$

is bounded, i.e.

$$\mathscr{O}_{\vec{\Phi}} \in L^\infty(D^2). \tag{1.1.1}$$

In particular, $\mathscr{O}_{\vec{\Phi}}$ is holomorphic once it is meromorphic.

Therefore, we deduce the following special case.

Theorem C ([4]). Variational *branched Willmore sphere in S^3 are inverse stereographic projection of complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature and* zero flux.

Indeed, by [4], *variational* branched Willmore spheres are *true* and the condition on the second residue is satisfied by [5], [6].

Finally, as the analysis is codimension free, the computations involving only the meromorphy of the quartic form will imply also that the poles of order *a priori* equal to 4 of Montiel's octic form $\mathscr{O}_{\vec{\Phi}}$ are completely removable.

Theorem D. *Let* $\vec{\Phi}:S^2\to S^4$ *be* variational *branched Willmore surface. Then* $\vec{\Phi}$ *is either the inverse stereographic projection of a complete branched minimal surface in* $\mathbb{R}^4$ *with finite total curvature and zero flux or the image by the Penrose twistor fibration of a (singular) algebraic curve* $C\subset\mathbb{CP}^3$*.*

Furthermore, the two possibilities coincide if and only if the algebraic curve $C\subset\mathbb{CP}^3$ *lies in some hypersurface* $H\simeq\mathbb{CP}^2\subset\mathbb{CP}^3$*.*

1.2 Description of the argument

The notations are consistent with [4] and will not be reintroduced again here.

First, recall that the quartic form has the following structure for immersions $\vec{\Phi}:\Sigma^2\to\mathbb{R}^3$

$$\begin{aligned}\mathscr{Q}_{\vec{\Phi}}&=g^{-1}\otimes\left(\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\vec{h}_0-\partial^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0\right)+\frac{1}{4}|\vec{H}|^2\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\\&=g^{-1}\otimes\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0-\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)+\left(\frac{1}{4}|\vec{H}|^2+|\vec{h}_0|^2_{WP}\right)\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle^2.\end{aligned}$$

In particular, the expression makes sense for immersions $\vec{\Phi}:\Sigma^2\to\mathbb{R}^n$ for any $n\geq 3$, and if $n\geq 4$, then the quartic differential $\mathscr{Q}_{\vec{\Phi}}$ need not be holomorphic. Therefore, if we compute its Taylor expansion for arbitrary codimension immersions, there shall be some non-meromorphic components appearing, which must therefore vanish once we suppose that $\mathscr{Q}_{\vec{\Phi}}$ is meromorphic. Furthermore, one of the main achievements of this paper is to show that the relations furnished by these cancellations are non-trivial, and show the holomorphy of $\mathscr{Q}_{\vec{\Phi}}$, once we combine them with the conservation laws.

Indeed, we show that for any branched Willmore disk $\vec{\Phi}:D^2\to\mathbb{R}^n$ with a unique branch point of multiplicity $\theta_0\geq 3$ at 0, there exists $\vec{A}_1,\vec{C}_1\in\mathbb{C}^n$ such that

$$\begin{aligned}\mathscr{Q}_{\vec{\Phi}}&=(\theta_0-1)(\theta_0-2)\langle\vec{A}_1,\vec{C}_1\rangle\frac{dz^4}{z}-6(\theta_0-2)\left(|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle-\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_1,\vec{A}_1\rangle\right)\overline{z}\,dz^4\\&+\frac{3(\theta_0-2)}{2\theta_0}\left(|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle-\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\langle\vec{A}_1,\vec{C}_1\rangle\right)z^{\theta_0}\,\overline{z}^{2-\theta_0}\,dz^4+O(|z|^3).\end{aligned}$$

If we suppose that $\mathscr{Q}_{\vec{\Phi}}$ is meromorphic, then we obtain (see (2.8.13))

$$\begin{cases}|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle=\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_1,\vec{A}_1\rangle\\|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle=\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\langle\vec{A}_1,\vec{C}_1\rangle,\end{cases}\tag{1.2.1}$$

Remarking that is a linear system in $(\langle\vec{A}_1,\vec{C}_1\rangle,\langle\vec{A}_1,\vec{A}_1\rangle)$, we can recast (1.2.1) as

$$\begin{pmatrix}|\vec{A}_1|^2&-\langle\overline{\vec{A}_1},\vec{C}_1\rangle\\-\langle\vec{A}_1,\overline{\vec{C}_1}\rangle&|\vec{C}_1|^2\end{pmatrix}\begin{pmatrix}\langle\vec{A}_1,\vec{C}_1\rangle\\\langle\vec{A}_1,\vec{A}_1\rangle\end{pmatrix}=0.\tag{1.2.2}$$

Thanks of Cauchy-Schwarz inequality, we obtain

$$\det\begin{pmatrix}|\vec{A}_1|^2&-\langle\overline{\vec{A}_1},\vec{C}_1\rangle\\-\langle\vec{A}_1,\overline{\vec{C}_1}\rangle&|\vec{C}_1|^2\end{pmatrix}=|\vec{A}_1|^2|\vec{C}_1|^2-|\langle\vec{A}_1,\overline{\vec{C}_1}\rangle|^2\geq 0.\tag{1.2.3}$$

Therefore, if the determinant is positive, we obtain

$$\langle\vec{A}_1,\vec{C}_1\rangle=0,$$

and the holomorphy of the quartic form, and if the determinant vanishes,

$$\vec{A}_1\ \text{and}\ \vec{C}_1\ \text{are proportional.}\tag{1.2.4}$$

Furthermore, in the case $\theta_0 = 4$, we obtain

$$\mathscr{Q}_{\vec{\Phi}} = 6\langle \vec{A}_1, \vec{C}_1\rangle \frac{dz^4}{z} - 12\left(|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle - \langle \overline{\vec{A}_1}, \vec{C}_1\rangle\langle \vec{A}_1, \vec{A}_1\rangle\right)\overline{z}\,dz^4$$
$$+\frac{3}{4}\left(|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle - \langle \vec{A}_1, \overline{\vec{C}_1}\rangle\langle \vec{A}_1, \vec{C}_1\rangle\right) z^{\theta_0}\,\overline{z}^{2-\theta_0}\,dz^4 - \frac{3}{8}\langle \vec{A}_1, \vec{C}_1\rangle\overline{\langle \vec{C}_1, \vec{C}_1\rangle}\frac{\overline{z}^4}{z}\log|z| + O(|z|^4).$$

so we obtain the additional

$$\langle \vec{A}_1, \vec{C}_1\rangle\overline{\langle \vec{C}_1, \vec{C}_1\rangle} = 0, \tag{1.2.5}$$

which implies by the preceding argument, that

$$\langle \vec{A}_1, \vec{C}_1\rangle = 0,$$

along with the holomorphy of the quartic form. So we can suppose that $\theta_0 \geq 5$ in the following.

Now, the final argument is to combine the invariance under inversions of the Willmore energy, which translates into the conservation law

$$d\,\mathrm{Im}\left(\mathscr{I}_{\vec{\Phi}}\left(\partial\vec{H} + |\vec{H}|^2\partial\vec{\Phi} + 2\,g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}\right) - g^{-1}\otimes\left(\overline{\partial}|\vec{\Phi}|^2\otimes\vec{h}_0 - 2\,\langle\vec{\Phi},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}\right)\right) = 0$$

where for all $\vec{X}\in\mathbb{C}^n$, we have

$$\mathscr{I}_{\vec{\Phi}} = |\vec{\Phi}|^2\vec{X} - 2\langle\vec{\Phi},\vec{X}\rangle\vec{\Phi}$$

and gives as coefficient in $z^{\theta_0+3}dx_1\wedge dx_2$, by supposing that

$$|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle = \langle \overline{\vec{A}_1}, \vec{C}_1\rangle\langle \vec{A}_1, \vec{A}_1\rangle$$

the following quantity (see (4.6.35))

$$\frac{4(\theta_0-4)}{\theta_0^2(\theta_0-3)}|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle = 0. \tag{1.2.6}$$

Finally, as (1.2.6) trivially shows that

$$\langle \vec{A}_1, \vec{C}_1\rangle = 0$$

we also obtain the holomorphy of the quartic form $\mathscr{Q}_{\vec{\Phi}}$.

Finally, let us mention that there is a result on the removability of poles of a 8-differential for branched Willmore spheres with values in S^4. The removability of these poles does not necessitate additional analysis as one can see in chapter 5.

1.3 Heuristic argument

We can see easily that for a true Willmore sphere with branch points of multiplicity $\theta_0 \leq 3$ the quartic form $\mathscr{Q}_{\vec{\Phi}}$ is holomorphic, as the coefficient $c\in\mathbb{C}$ in

$$\mathscr{Q}_{\vec{\Phi}} = \frac{c\,dz^4}{z} + O(1)$$

is a function of the first residue $\vec{\gamma}_0$, which vanishes whenever the first residue does. For branch points of multiplicity $\theta_0 \geq 4$, we can even remove the hypothesis as $\langle \vec{A}_1, \vec{C}_1\rangle$ is independent of $\vec{\gamma}_0$. In this case, the coefficient has no geometric meaning, and this fact is general, as for any holomorphic function $h : D^2 \to \mathbb{C}$ such that $h(0)\neq 0$

$$\frac{h(z)\,dz^4}{z},$$

there exists a local coordinate w around 0 such that

$$\frac{h(z)\,dz^4}{z} = \frac{dw^4}{w}. \tag{1.3.1}$$

We refer to [1] for more details on this.

1.4 On notations

In order to obtain an extremely readable format for the code, a function

$$\vec{F}(z) = \lambda_1 \vec{A} z^a \overline{z}^b + \lambda_2 \vec{B} z^c \overline{z}^d \log^p |z|, \quad \lambda_1, \lambda_2 \in \mathbb{C}, \ \vec{A}, \vec{B} \in \mathbb{C}^n, \ a, b, c, d \in \mathbb{Z}, \ p \in \mathbb{N}.$$

will be written as the following matrix

$$\vec{F} = \begin{pmatrix} \lambda_1 & \vec{A} & a & b & 0 \\ \lambda_2 & \vec{B} & c & d & p \end{pmatrix}$$

and in the case $p = 0$, we will simply remove the last column of zeroes and write instead

$$\vec{F} = \begin{pmatrix} \lambda_1 & \vec{A} & a & b \\ \lambda_2 & \vec{B} & c & d \end{pmatrix}$$

All program shall deal with matrices of this type, and as logarithm terms only appear at branch points of multiplicity $\theta_0 \leq 4$, this will be very convenient to remove this last column to speed up computations. The error in $O(|z|^*)$ will be most of the time omitted for simplicity of notations.

We used this notations as in the Sage code, all variables are complex or real numbers, so to keep track of the order scalar product arise, this notations is quite convenient.

Finally, we will indicate with a red cross scalar products which vanish, while the scalar products which cancel will be indicated by a blue cross. For example, we have

$$\langle \vec{A}_0, \vec{A}_0 \rangle = 0, \quad \langle \vec{A}_1, \vec{A}_1 \rangle + 2\langle \vec{A}_0, \vec{A}_2 \rangle = 0,$$

so we will write in expressions involving several scalar products

$$\cancel{\langle \vec{A}_0, \vec{A}_0 \rangle}, \quad \cancel{\langle \vec{A}_1, \vec{A}_1 \rangle} + \cancel{2\langle \vec{A}_0, \vec{A}_2 \rangle}$$

to indicate to the reader which cancellation we have used.

Finally, remark that there are hyperlinks to any equation in this document.

1.5 The code

The Python code from Sage is available on the personal website of the first author at the following address:

https://people.math.ethz.ch/ alexism/publications

The link is available under the name *Sage source code*, which can be found after the link to this present paper.

The Sage version we used is the version 7.3 ([7]), and at the end of the project, ETH's Sage version moved to version 7.6, which did not create compatibility issues in the last computations, which were performed with this more recent version.

Remark A. The code was runned on ETH's supercomputers, and some programms needed around 3 hours to finish (on the model described below), especially for the last order development of the quartic form and the last order development of the conservation laws. The last order of the Weingarten tensor also took some time (less than one hour though), so we do not know how much time it would take (if it can terminate) on a personal computer. Nevertheless, the first two steps of the code (in the two files `Code_Classification_1.txt` and `Code_Classification_2.txt`) are not very resource demanding, and one shall be able to run it on a personal computer. As an indication, the strongest machine we had access to had the following characteristics:

$$\begin{cases} \text{Type : Superserver \ 1027TR-TF} \\ \text{CPU : } 48 \times \text{Intel(R) Xeon(R) CPU E5} - 2697 \text{ v2 @ 2.70GHz} \\ \text{RAM : 256GB} \end{cases}$$

For more informations on the hardware aspect, we refer the reader to the following link:

https://blogs.ethz.ch/isgdmath/central-clients/

Chapter 2

The direct verification of the proof

2.1 Step 1 verification : upper regularity and poles of order 2

First, recall that by the very first development of the four tensors is

$$\begin{cases} \partial_z \vec{\Phi} = \vec{A}_0 z^{\theta_0 - 1} + O(|z|^{\theta_0}) \\ g = |z|^{2\theta_0 - 2} + O(|z|^{2\theta_0 - 1}) \\ \vec{H} = \mathrm{Re}\left(\dfrac{\vec{C}_1}{z^{\theta_0 - 2}}\right) + O(|z|^{3-\theta_0}) \\ \vec{h}_0 = O(|z|^{\theta_0 - 1}). \end{cases} \tag{2.1.1}$$

One can check that each development made in [4] has an error in $O(|z|^{\alpha+1-\varepsilon})$, where α is the maximal order of products of z and $\overline{z}$. From now on, we will not write these errors to simplify notations. With Sage, we define

```
dzphi = matrix([1,a0,n-1,0])
g = matrix([1,n-1,n-1])
ginv = matrix([1,1-n,1-n])
H = matrix([[1/2,c1,1-n,0],[1/2,c1b,0,1-n]])
```

with obvious notations (here `n` corresponds to the multiplicity θ_0), and using the `latex()` function with the function

```
latex(dzphi),latex(g),latex(H)
```

yields

$$\begin{cases} \begin{pmatrix} 1 & A_0 & \theta_0 - 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} 1 & \theta_0 - 1 & \theta_0 - 1 \end{pmatrix}, \\ \begin{pmatrix} \frac{1}{2} & C_1 & -\theta_0 + 2 & 0 \\ \frac{1}{2} & \overline{C_1} & 0 & -\theta_0 + 2 \end{pmatrix} \end{cases}$$

which we interpret as

$$\begin{cases} \partial_z \vec{\Phi} = \begin{pmatrix} 1 & \vec{A}_0 & \theta_0 - 1 & 0 \end{pmatrix} \\ g = \begin{pmatrix} 1 & \theta_0 - 1 & \theta_0 - 1 \end{pmatrix} \\ \vec{H} = \begin{pmatrix} \frac{1}{2} & \vec{C}_1 & -\theta_0 + 2 & 0 \\ \frac{1}{2} & \overline{\vec{C}_1} & 0 & -\theta_0 + 2 \end{pmatrix} \end{cases}$$

In these notations, the first column is a numerical constant in $\mathbb{C}$, the second the vector coefficient in $\mathbb{C}^n$ (if the matrix has 4 columns), and the two last columns are the power in z and $\overline{z}$ respectively.

Remark A. From step 5, as logarithm will appear in the developments, the scalars will have 4 columns and the vectors 5. This shall not create confusions as we will consistently come back and force between this matrix notation and the more traditional one from [4].

We will systematically omit all $dz, dz^2, dz^3, dz^4, dz^8, |dz|^2$ factors in the following expressions. In particular, scalars have 3 columns, and vectors 4.

We will always write first the result given by Sage, then the transcription in correct mathematical notations, so that one can check consistently each step. We will also indicate to which equation one needs to compare with.

Assuming that the second residue satisfies $r(0) \leq \theta_0 - 2$, we start from the expansion obtained in [4]

$$\begin{cases}\partial_z\vec{\Phi} = \vec{A}_0 z^{\theta_0-1} + \vec{A}_1 z^{\theta_0} + \vec{A}_2 z^{\theta_0+1} + \dfrac{1}{4\theta_0}\vec{C}_1 z\,\overline{z}^{\theta_0} + \dfrac{1}{8}\overline{\vec{C}_1} z^{\theta_0-1}\overline{z}^2 + O(|z|^{\theta_0+2-\varepsilon}) \\ \vec{H} = \mathrm{Re}\left(\dfrac{\vec{C}_1}{z^{\theta_0-2}}\right) + O(|z|^{3-\theta_0-\varepsilon}),\end{cases} \tag{2.1.2}$$

obtained by direct integration of

$$\begin{aligned}\partial_{\overline{z}}\left(\partial_z\vec{\Phi}\right) &= \frac{1}{2}e^{2\lambda}\vec{H} = \frac{1}{2}|z|^{2\theta_0-2}\left(1+O(|z|)\right)\times\left(\mathrm{Re}\left(\frac{\vec{C}_1}{z^{\theta_0-2}}\right) + O(|z|^{3-\theta_0-\varepsilon})\right) \\ &= \frac{1}{2}\mathrm{Re}\left(\vec{C}_1 z\overline{z}^{\theta_0-1}\right) + O(|z|^{\theta_0+1-\varepsilon}) \\ &= \frac{1}{4}\vec{C}_1 z\overline{z}^{\theta_0-1} + \frac{1}{4}\overline{\vec{C}_1} z^{\theta_0-1}\overline{z} + O(|z|^{\theta_0+1-\varepsilon}),\end{aligned}$$

using the first order expansions of $e^{2\lambda}$ and $\vec{H}$ in (2.1.1) (and using Proposition 6.7. of [4] to integrate it).

As $\langle\partial_z\vec{\Phi}, \partial_z\vec{\Phi}\rangle = 0$, we find directly

$$\langle\vec{A}_0, \vec{A}_0\rangle = \langle\vec{A}_0, \vec{A}_1\rangle = 0, \qquad \langle\vec{A}_0, \vec{C}_1\rangle = \langle\overline{\vec{A}_0}, \vec{C}_1\rangle = 0. \tag{2.1.3}$$

Now define

$$\begin{cases}\alpha_0 = 2\langle\overline{\vec{A}_0}, \vec{A}_1\rangle, \\ \alpha_1 = 2\langle\overline{\vec{A}_0}, \vec{A}_2\rangle\end{cases}$$

As $|\vec{A}_0|^2 = \dfrac{1}{2}$, (2.1.2) implies that

$$e^{2\lambda} = 2|\partial_z\vec{\Phi}|^2 = |z|^{2\theta_0-2}\left(1 + 2\,\mathrm{Re}\left(\alpha_0 z + \alpha_1 z^2\right) + 2|\vec{A}_1|^2|z|^2 + O(|z|^{3-\varepsilon})\right)$$

which is coded as

$$e^{2\lambda} = \begin{pmatrix} 1 & \theta_0-1 & \theta_0-1 \\ 2\,|A_1|^2 & \theta_0 & \theta_0 \\ \alpha_0 & \theta_0 & \theta_0-1 \end{pmatrix}\begin{pmatrix} \alpha_1 & \theta_0+1 & \theta_0-1 \\ \overline{\alpha_0} & \theta_0-1 & \theta_0 \\ \overline{\alpha_1} & \theta_0-1 & \theta_0+1 \end{pmatrix} \tag{2.1.4}$$

as expected. We now compute

$$\vec{h}_0 = \begin{pmatrix} -\dfrac{\theta_0-2}{2\,\theta_0} & C_1 & 0 & \theta_0 \\ 4 & A_2 & \theta_0 & 0 \\ -2\,\alpha_0 & A_1 & \theta_0 & 0 \\ 2\,\alpha_0^2-4\,\alpha_1 & A_0 & \theta_0 & 0 \\ 2 & A_1 & \theta_0-1 & 0 \\ -2\,\alpha_0 & A_0 & \theta_0-1 & 0 \\ -4\,|A_1|^2+2\,\alpha_0\overline{\alpha_0} & A_0 & \theta_0-1 & 1 \end{pmatrix} + O(|z|^{\theta_0+1-\varepsilon}) \tag{2.1.5}$$

to be compared with

$$\vec{h}_0 = 2\left(\vec{A}_1-\alpha_0\vec{A}_0+\left(2|\vec{A}_1|^2-|\alpha_0|^2\right)\vec{A}_0\overline{z}\right)z^{\theta_0-1}+2\left(2\vec{A}_2-\alpha_0\vec{A}_1-(2\alpha_1-\alpha_0^2)\vec{A}_0\right)z^{\theta_0}$$
$$-\frac{(\theta_0-2)}{2\theta_0}\vec{C}_1\overline{z}^{\theta_0}+O(|z|^{\theta_0+1-\varepsilon})$$

so we see that both developments coincide to this point. We also check that

$$\mathscr{Q}_{\vec{\Phi}} = (\theta_0-1)(\theta_0-2)\langle\vec{A}_1,\vec{C}_1\rangle\frac{dz^4}{z}+O(1)$$

as in the code, we have

$$\mathscr{Q}_{\vec{\Phi}} = \begin{pmatrix} -\left(\left(\alpha_0\theta_0^2-3\,\alpha_0\theta_0+2\,\alpha_0\right)A_0-\left(\theta_0^2-3\,\theta_0+2\right)A_1\right)C_1 & -1 & 0 \end{pmatrix}+O(1)$$

and as $\langle\vec{A}_0,\vec{C}_1\rangle=0$, and $(\theta_0-1)(\theta_0-2)=\theta_0^2-3\theta_0+2$, we indeed recover the same formula.

2.2 Step 2 verification : higher development for $\theta_0\geq 4$

From now we suppose that $\theta_0\geq 4$.

Let us sum up the different developments we obtained so far. From now, we can be much faster and check directly the important steps, that is the development of $\vec{Q}$, $\vec{H}$, $\partial_z\vec{\Phi}$, $e^{2\lambda}$ and $\vec{h}_0$.

As previously, $\vec{Q}$ is such that

$$\partial\vec{Q} = -|\vec{H}|^2\partial\vec{\Phi}-2\,g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}$$

and $\vec{Q}$ has no anti-holomorphic components of high singularity. Now, as $\vec{H}=O(|z|^{2-\theta_0})$ and $\partial_z\vec{\Phi}=O(|z|^{\theta_0-1})$, we have

$$|\vec{H}|^2\partial\vec{\Phi}=O(|z|^{3-\theta_0}) \tag{2.2.1}$$

and (see `del1H3`)

$$g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi} = \begin{pmatrix} \vec{A}_1\cdot\overline{\vec{C}_1} & \overline{\vec{A}_0} & -\theta_0+2 & 0 \\ \vec{A}_1\cdot\overline{\vec{C}_1} & \overline{\vec{A}_0} & 0 & -\theta_0+2 \end{pmatrix}+O(|z|^{3-\theta_0}) \tag{2.2.2}$$

which checks.

While it is not used in this step, we will need the development of $|\vec{H}|^2\partial\vec{\Phi}$ in the next one, so we check that

$$|\vec{H}|^2\partial\vec{\Phi} = \begin{pmatrix} \frac{1}{4}\,C_1^2 & A_0 & -\theta_0+3 & 0 \\ \frac{1}{2}\,C_1\overline{C_1} & A_0 & 1 & -\theta_0+2 \\ \frac{1}{4}\,\overline{C_1}^2 & A_0 & \theta_0-1 & -2\,\theta_0+4. \end{pmatrix} \tag{2.2.3}$$

In particular, thanks of (2.2.1) and (2.2.2), we see that $|\vec{H}|^2\partial\vec{\Phi}$ is an error, and we obtain

$$\vec{Q} = \begin{pmatrix} \dfrac{2\left((\alpha_0(\theta_0-1)-\alpha_0\theta_0)A_0C_1+A_1C_1\right)}{\theta_0-3} & \overline{A_0} & -\theta_0+3 & 0 \\ -2\left(\alpha_0(\theta_0-1)-\alpha_0\theta_0\right)A_0\overline{C_1}-2\,A_1\overline{C_1} & \overline{A_0} & 1 & -\theta_0+2 \end{pmatrix}$$

and as $\langle\vec{A}_0,\vec{C}_1\rangle = \langle\overline{\vec{A}_0},\vec{C}_1\rangle = 0$, this reduces to

$$\vec{Q} = \begin{pmatrix} \frac{2\,\vec{A}_1\cdot\vec{C}_1}{\theta_0-3} & \overline{\vec{A}_0} & -\theta_0+3 & 0 \\ -2\,\vec{A}_1\cdot\overline{\vec{C}_1} & \overline{\vec{A}_0} & 1 & -\theta_0+2 \end{pmatrix}$$

and as

$$\partial\left(\vec{H}-2i\vec{L}+\vec{\gamma}_0\log|z|-\vec{Q}\right)=0$$

there exists $\vec{C}_2\in\mathbb{C}^n$ such that

$$\begin{aligned}\vec{H}-2i\vec{L}+\vec{\gamma}_0\log|z| &= \frac{\overline{\vec{C}_1}}{\overline{z}^{\theta_0-2}}+\frac{\overline{\vec{C}_2}}{\overline{z}^{\theta_0-3}}+\vec{Q}\\ &= \frac{\overline{\vec{C}_1}}{\overline{z}^{\theta_0-2}}+\frac{\overline{\vec{C}_2}}{\overline{z}^{\theta_0-3}}+\begin{pmatrix} \dfrac{2\left((\alpha_0(\theta_0-1)-\alpha_0\theta_0)A_0C_1+A_1C_1\right)}{\theta_0-3} & \overline{A_0} & -\theta_0+3 & 0 \\ -2\left(\alpha_0(\theta_0-1)-\alpha_0\theta_0\right)A_0\overline{C_1}-2\,A_1\overline{C_1} & \overline{A_0} & 1 & -\theta_0+2 \end{pmatrix}\end{aligned}$$

so taking the real part we get

$$\vec{H} = \mathrm{Re}\left(\frac{\vec{C}_1}{z^{\theta_0-2}}\right)+\begin{pmatrix} \dfrac{(\alpha_0(\theta_0-1)-\alpha_0\theta_0)A_0C_1+A_1C_1}{\theta_0-3} & \overline{A_0} & -\theta_0+3 & 0 \\ \dfrac{((\theta_0-1)\overline{\alpha_0}-\theta_0\overline{\alpha_0})\overline{A_0C_1}+\overline{A_1C_1}}{\theta_0-3} & A_0 & 0 & -\theta_0+3 \\ -(\alpha_0(\theta_0-1)-\alpha_0\theta_0)A_0\overline{C_1}-A_1\overline{C_1} & \overline{A_0} & 1 & -\theta_0+2 \\ -((\theta_0-1)\overline{\alpha_0}-\theta_0\overline{\alpha_0})C_1\overline{A_0}-C_1\overline{A_1} & A_0 & -\theta_0+2 & 1 \end{pmatrix}$$

so using again $\langle\vec{A}_0,\vec{C}_1\rangle = \langle\overline{\vec{A}_0},\vec{C}_1\rangle = 0$, we finally obtain

$$\vec{H} = \begin{pmatrix} \frac{1}{2} & \overline{\vec{C}_1} & 0 & \theta_0-2 \\ \frac{1}{2} & \vec{C}_1 & \theta_0-2 & 0 \\ \frac{1}{2} & \overline{\vec{D}_2} & 0 & -\theta_0+3 \\ \frac{1}{2} & \vec{D}_2 & -\theta_0+3 & 0 \end{pmatrix}\quad\begin{pmatrix} \dfrac{\vec{A}_1\cdot\vec{C}_1}{\theta_0-3} & \overline{\vec{A}_0} & -\theta_0+3 & 0 \\ \overline{\dfrac{\vec{A}_1\cdot\vec{C}_1}{\theta_0-3}} & A_0 & 0 & -\theta_0+3 \\ -\vec{A}_1\cdot\overline{\vec{C}_1} & \overline{\vec{A}_0} & 1 & -\theta_0+2 \\ -\vec{C}_1\cdot\overline{\vec{A}_1} & \vec{A}_0 & -\theta_0+2 & 1 \end{pmatrix}$$

so for

$$\begin{cases} \vec{C}_2 = \vec{D}_2+\dfrac{2}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_0}, \\ \vec{B}_1 = -2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0 \end{cases} \tag{2.2.4}$$

we obtain

$$\vec{H} = \mathrm{Re}\left(\frac{\vec{C}_1}{z^{\theta_0-2}} + \frac{\vec{C}_2}{z^{\theta_0-3}} + \vec{B}_1\frac{\overline{z}}{z^{\theta_0-2}}\right) + O(|z|^{4-\theta_0-\varepsilon}). \tag{2.2.5}$$

2.3 Upper development for $\theta_0 \geq 5$

From now on, all computations are exclusively computer generated.

Reinserting the new development of $\vec{H}$ allows one to obtain for some $\vec{D}_4 \in \mathbb{C}^n$

$$\vec{H} = \mathrm{Re}\left(\frac{\vec{C}_1}{z^{\theta_0-2}} + \frac{\vec{C}_2}{z^{\theta_0-3}} + \frac{\vec{D}_3}{z^{\theta_0-4}} + \frac{\vec{D}_4}{z^{\theta_0-5}}\right) \tag{2.3.1}$$

$$+\left(\begin{array}{cccc}
-\frac{\left(\overline{A_0}\,\overline{\alpha_0}-\overline{A_1}\right)\overline{C_1}}{\theta_0-3} & A_0 & 0 & -\theta_0+3 \\
\frac{2\left(\left(\overline{\alpha_0}^2-\overline{\alpha_1}\right)\overline{A_0}-\overline{A_1}\,\overline{\alpha_0}+\overline{A_2}\right)\overline{C_1}-\left(\overline{A_0}\,\overline{\alpha_0}-\overline{A_1}\right)\overline{C_2}}{\theta_0-4} & A_0 & 0 & -\theta_0+4 \\
\frac{\overline{C_1}^2}{8\,(\theta_0-4)} & \overline{A_0} & 0 & -\theta_0+4 \\
(A_0\alpha_0-A_1)\overline{C_1} & \overline{A_0} & 1 & -\theta_0+2 \\
-\frac{\left(\overline{A_0}\,\overline{\alpha_0}-\overline{A_1}\right)\overline{B_1}+\left(2\,|A_1|^2\overline{A_0}-2\,\alpha_0\overline{A_0}\,\overline{\alpha_0}+\alpha_0\overline{A_1}\right)\overline{C_1}}{\theta_0-3} & A_0 & 1 & -\theta_0+3 \\
-\frac{\left(\overline{A_0}\,\overline{\alpha_0}-\overline{A_1}\right)\overline{C_1}}{\theta_0-3} & A_1 & 1 & -\theta_0+3 \\
(A_0\alpha_0-A_1)\overline{C_1} & \overline{A_1} & 1 & -\theta_0+3 \\
\left(2\,A_0|A_1|^2-2\,A_0\alpha_0\overline{\alpha_0}+A_1\overline{\alpha_0}\right)\overline{C_1}+(A_0\alpha_0-A_1)\overline{C_2} & \overline{A_0} & 1 & -\theta_0+3 \\
-\frac{C_1(\theta_0+2)\overline{C_1}}{8\,\theta_0} & A_0 & 2 & -\theta_0+2 \\
\frac{1}{2}\,(A_0\alpha_0-A_1)\overline{B_1}-\left(\left(\alpha_0^2-\alpha_1\right)A_0-A_1\alpha_0+A_2\right)\overline{C_1} & \overline{A_0} & 2 & -\theta_0+2 \\
-\frac{C_1^2}{4\,\theta_0} & \overline{A_0} & -2\,\theta_0+4 & \theta_0 \\
\left(\overline{A_0}\,\overline{\alpha_0}-\overline{A_1}\right)C_1 & A_0 & -\theta_0+2 & 1 \\
-\frac{C_1(\theta_0+2)\overline{C_1}}{8\,\theta_0} & \overline{A_0} & -\theta_0+2 & 2 \\
\frac{1}{2}\left(\overline{A_0}\,\overline{\alpha_0}-\overline{A_1}\right)B_1-\left(\left(\overline{\alpha_0}^2-\overline{\alpha_1}\right)\overline{A_0}-\overline{A_1}\,\overline{\alpha_0}+\overline{A_2}\right)C_1 & A_0 & -\theta_0+2 & 2 \\
-\frac{(A_0\alpha_0-A_1)C_1}{\theta_0-3} & \overline{A_0} & -\theta_0+3 & 0 \\
-\frac{(A_0\alpha_0-A_1)B_1+\left(2\,A_0|A_1|^2-2\,A_0\alpha_0\overline{\alpha_0}+A_1\overline{\alpha_0}\right)C_1}{\theta_0-3} & \overline{A_0} & -\theta_0+3 & 1 \\
-\frac{(A_0\alpha_0-A_1)C_1}{\theta_0-3} & \overline{A_1} & -\theta_0+3 & 1 \\
\left(\overline{A_0}\,\overline{\alpha_0}-\overline{A_1}\right)C_1 & A_1 & -\theta_0+3 & 1 \\
\left(2\,|A_1|^2\overline{A_0}-2\,\alpha_0\overline{A_0}\,\overline{\alpha_0}+\alpha_0\overline{A_1}\right)C_1+\left(\overline{A_0}\,\overline{\alpha_0}-\overline{A_1}\right)C_2 & A_0 & -\theta_0+3 & 1 \\
\frac{2\left(\left(\alpha_0^2-\alpha_1\right)A_0-A_1\alpha_0+A_2\right)C_1-(A_0\alpha_0-A_1)C_2}{\theta_0-4} & \overline{A_0} & -\theta_0+4 & 0 \\
\frac{C_1^2}{8\,(\theta_0-4)} & A_0 & -\theta_0+4 & 0 \\
-\frac{\overline{C_1}^2}{4\,\theta_0} & A_0 & \theta_0 & -2\,\theta_0+4
\end{array}\right) \tag{2.3.2}$$

so for some $\vec{C}_3, \vec{B}_2, \vec{B}_3, \vec{E}_1 \in \mathbb{C}^n$ we have

$$\vec{H} = \mathrm{Re}\left(\frac{\vec{C}_1}{z^{\theta_0-2}} + \frac{\vec{C}_2}{z^{\theta_0-3}} + \frac{\vec{C}_3}{z^{\theta_0-4}} + \vec{B}_1\frac{\overline{z}}{z^{\theta_0-2}} + \vec{B}_2\frac{\overline{z}^2}{z^{\theta_0-2}} + \vec{B}_3\frac{\overline{z}}{z^{\theta_0-3}}\right) + O(|z|^{5-\theta_0}).$$

or on the code

$$\vec{H} = \begin{pmatrix} \frac{1}{2} & C_1 & -\theta_0+2 & 0 \\ \frac{1}{2} & C_2 & -\theta_0+3 & 0 \\ \frac{1}{2} & C_3 & -\theta_0+4 & 0 \\ \frac{1}{2} & B_1 & -\theta_0+2 & 1 \\ \frac{1}{2} & B_2 & -\theta_0+2 & 2 \\ \frac{1}{2} & B_3 & -\theta_0+3 & 1 \\ \frac{1}{2} & E_1 & -2\,\theta_0+4 & \theta_0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \overline{C_1} & 0 & -\theta_0+2 \\ \frac{1}{2} & \overline{C_2} & 0 & -\theta_0+3 \\ \frac{1}{2} & \overline{C_3} & 0 & -\theta_0+4 \\ \frac{1}{2} & \overline{B_1} & 1 & -\theta_0+2 \\ \frac{1}{2} & \overline{B_2} & 2 & -\theta_0+2 \\ \frac{1}{2} & \overline{B_3} & 1 & -\theta_0+3 \\ \frac{1}{2} & \overline{E_1} & \theta_0 & -2\,\theta_0+4 \end{pmatrix} \tag{2.3.3}$$

We will only need the precise expressions of $\vec{B}_3$ and $\vec{E}_1$. First, recall that (2.2.4) we have

$$\vec{B}_1 = -2\langle\overline{\vec{A}_1}, \vec{C}_1\rangle\vec{A}_0$$

while by (2.1.3), we have

$$\langle\vec{A}_0, \vec{A}_0\rangle = \langle\vec{A}_0, \vec{A}_1\rangle = \langle\vec{A}_0, \vec{C}_1\rangle = \langle\vec{A}_0, \overline{\vec{C}_1}\rangle = 0.$$

Therefore, comparing (2.3.3) and (2.3.1), we obtain the expression

$$\begin{aligned} \frac{1}{2}\vec{B}_3 &= \begin{pmatrix} -\frac{(A_0\alpha_0 - A_1)B_1 + (2A_0|A_1|^2 - 2A_0\alpha_0\overline{\alpha_0} + A_1\overline{\alpha_0})C_1}{\theta_0-3} & \overline{A_0} & -\theta_0+3 & 1 \\ -\frac{(A_0\alpha_0 - A_1)C_1}{\theta_0-3} & \overline{A_1} & -\theta_0+3 & 1 \\ \left(\overline{A_0}\overline{\alpha_0} - \overline{A_1}\right)C_1 & A_1 & -\theta_0+3 & 1 \\ (2|A_1|^2\overline{A_0} - 2\alpha_0\overline{A_0}\overline{\alpha_0} + \alpha_0\overline{A_1})C_1 + \left(\overline{A_0}\overline{\alpha_0} - \overline{A_1}\right)C_2 & A_0 & -\theta_0+3 & 1 \end{pmatrix} \\ &= \frac{\overline{\alpha_0}}{\theta_0-3}\langle\vec{A}_1, \vec{C}_1\rangle\overline{\vec{A}_0} + \frac{1}{\theta_0-3}\langle\vec{A}_1, \vec{C}_1\rangle\overline{\vec{A}_1} - \langle\overline{\vec{A}_1}, \vec{C}_1\rangle\vec{A}_1 + \left(\alpha_0\langle\overline{\vec{A}_1}, \vec{C}_1\rangle - \langle\overline{\vec{A}_1}, \vec{C}_2\rangle\right)\vec{A}_0 \end{aligned}$$

while we trivially have

$$\begin{aligned} \frac{1}{2}\vec{E}_1 &= \begin{pmatrix} -\frac{C_1^2}{4\,\theta_0} & \overline{A_0} & -2\,\theta_0+4 & \theta_0 \end{pmatrix} \\ &= -\frac{1}{4\theta_0}\langle\vec{C}_1, \vec{C}_1\rangle\overline{\vec{A}_0} \end{aligned}$$

Now, we have as $\langle\overline{\vec{A}_0}, \vec{B}_1\rangle = -\langle\overline{\vec{A}_1}, \vec{C}_1\rangle$

$$\frac{1}{2}\vec{B}_2 = \begin{pmatrix} -\frac{C_1(\theta_0+2)\overline{C_1}}{8\,\theta_0} & \overline{A_0} & -\theta_0+2 & 2 \\ \frac{1}{2}\left(\overline{A_0}\overline{\alpha_0} - \overline{A_1}\right)B_1 - \left(\left(\overline{\alpha_0}^2 - \overline{\alpha_1}\right)\overline{A_0} - \overline{A_1}\overline{\alpha_0} + \overline{A_2}\right)C_1 & A_0 & -\theta_0+2 & 2 \end{pmatrix}$$

$$
\begin{aligned}
&= -\frac{(\theta_0+2)}{8\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0} + \left(-\frac{1}{2}\overline{\alpha_0}\langle\overline{\vec{A}_1},\vec{C}_1\rangle + \overline{\alpha_0}\langle\overline{\vec{A}_1},\vec{C}_1\rangle - \langle\overline{\vec{A}_2},\vec{C}_1\rangle\right)\vec{A}_0 \\
&= -\frac{(\theta_0+2)}{8\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0} + \left(\frac{1}{2}\overline{\alpha_0}\langle\overline{\vec{A}_1},\vec{C}_1\rangle - \langle\overline{\vec{A}_2},\vec{C}_1\rangle\right)\vec{A}_0
\end{aligned}
$$

so we obtain

$$
\begin{cases}
\vec{B}_2 = -\dfrac{(\theta_0+2)}{4\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0} + \left(\overline{\alpha_0}\langle\overline{\vec{A}_1},\vec{C}_1\rangle - 2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\right)\vec{A}_0 \\
\vec{B}_3 = -2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_1 + \dfrac{2}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_1} + \dfrac{2\overline{\alpha_0}}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_0} + 2\left(\alpha_0\langle\overline{\vec{A}_1},\vec{C}_1\rangle - \langle\overline{\vec{A}_1},\vec{C}_2\rangle\right)\vec{A}_0 \\
\vec{E}_1 = -\dfrac{1}{2\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_0}.
\end{cases}
\tag{2.3.4}
$$

Integrating the equation

$$
\partial_{\overline{z}}\left(\partial_z\vec{\Phi}\right) = \frac{e^{2\lambda}}{2}\vec{H},
$$

one obtains for some $\vec{A}_3,\vec{A}_4\in\mathbb{C}^n$ the development

$$
\partial_z\vec{\Phi} = \begin{pmatrix}
1 & A_0 & \theta_0-1 & 0 \\
1 & A_1 & \theta_0 & 0 \\
1 & A_2 & \theta_0+1 & 0 \\
1 & A_3 & \theta_0+2 & 0 \\
1 & A_4 & \theta_0+3 & 0 \\
\dfrac{1}{4\,\theta_0} & C_1 & 1 & \theta_0 \\
\dfrac{1}{4\,(\theta_0+1)} & B_1 & 1 & \theta_0+1 \\
\dfrac{\overline{\alpha_0}}{4\,(\theta_0+1)} & C_1 & 1 & \theta_0+1 \\
\dfrac{1}{4\,(\theta_0+2)} & B_2 & 1 & \theta_0+2 \\
\dfrac{\overline{\alpha_1}}{4\,(\theta_0+2)} & C_1 & 1 & \theta_0+2 \\
\dfrac{\overline{\alpha_0}}{4\,(\theta_0+2)} & B_1 & 1 & \theta_0+2 \\
\dfrac{1}{4\,\theta_0} & C_2 & 2 & \theta_0 \\
\dfrac{\alpha_0}{4\,\theta_0} & C_1 & 2 & \theta_0 \\
\dfrac{1}{4\,(\theta_0+1)} & B_3 & 2 & \theta_0+1 \\
\dfrac{\overline{\alpha_0}}{4\,(\theta_0+1)} & C_2 & 2 & \theta_0+1 \\
\dfrac{\alpha_0}{4\,(\theta_0+1)} & B_1 & 2 & \theta_0+1 \\
\dfrac{|A_1|^2}{2\,(\theta_0+1)} & C_1 & 2 & \theta_0+1
\end{pmatrix}
\begin{pmatrix}
\dfrac{1}{4\,\theta_0} & C_3 & 3 & \theta_0 \\
\dfrac{\alpha_1}{4\,\theta_0} & C_1 & 3 & \theta_0 \\
\dfrac{\alpha_0}{4\,\theta_0} & C_2 & 3 & \theta_0 \\
\dfrac{1}{8\,\theta_0} & E_1 & -\theta_0+3 & 2\,\theta_0 \\
\dfrac{1}{8} & \overline{B_1} & \theta_0 & 2 \\
\dfrac{1}{8}\,\alpha_0 & \overline{C_1} & \theta_0 & 2 \\
\dfrac{1}{12} & \overline{B_3} & \theta_0 & 3 \\
\dfrac{1}{12}\,\overline{\alpha_0} & \overline{B_1} & \theta_0 & 3 \\
\dfrac{1}{12}\,\alpha_0 & \overline{C_2} & \theta_0 & 3 \\
\dfrac{1}{6}\,|A_1|^2 & \overline{C_1} & \theta_0 & 3 \\
\dfrac{1}{8} & \overline{C_1} & \theta_0-1 & 2 \\
\dfrac{1}{12} & \overline{C_2} & \theta_0-1 & 3 \\
\dfrac{1}{12}\,\overline{\alpha_0} & \overline{C_1} & \theta_0-1 & 3 \\
\dfrac{1}{16} & \overline{C_3} & \theta_0-1 & 4 \\
\dfrac{1}{16}\,\overline{\alpha_1} & \overline{C_1} & \theta_0-1 & 4 \\
\dfrac{1}{16}\,\overline{\alpha_0} & \overline{C_2} & \theta_0-1 & 4 \\
\dfrac{1}{8} & \overline{B_2} & \theta_0+1 & 2 \\
\dfrac{1}{8}\,\alpha_1 & \overline{C_1} & \theta_0+1 & 2 \\
\dfrac{1}{8}\,\alpha_0 & \overline{B_1} & \theta_0+1 & 2 \\
-\dfrac{1}{4\,(\theta_0-4)} & \overline{E_1} & 2\,\theta_0-1 & -\theta_0+4
\end{pmatrix}
+ O(|z|^{\theta_0+4-\varepsilon})
\tag{2.3.5}
$$

Throwing all terms of degree more than equal to θ_0+2 yields

$$\partial_z\vec{\Phi}=\begin{pmatrix} 1 & A_0 & \theta_0-1 & 0\\ 1 & A_1 & \theta_0 & 0\\ 1 & A_2 & \theta_0+1 & 0\\ \frac{1}{4\,\theta_0} & C_1 & 1 & \theta_0\\ \frac{1}{8} & \overline{C_1} & \theta_0-1 & 2 \end{pmatrix}$$

so we recover (2.1.2). The next step is to look at relations given by the conformality of $\vec{\Phi}$. We have

$$0=\langle\partial_z\vec{\Phi},\partial_z\vec{\Phi}\rangle=\begin{pmatrix}
A_0^2 & 2\,\theta_0-2 & 0\\
2\,A_0A_1 & 2\,\theta_0-1 & 0\\
A_1^2+2\,A_0A_2 & 2\,\theta_0 & 0\\
2\,A_1A_2+2\,A_0A_3 & 2\,\theta_0+1 & 0\\
A_2^2+2\,A_1A_3+2\,A_0A_4 & 2\,\theta_0+2 & 0\\
\frac{A_0C_1}{2\,\theta_0} & \theta_0 & \theta_0\\
\frac{A_0C_1\overline{\alpha_0}+A_0B_1}{2\,(\theta_0+1)} & \theta_0 & \theta_0+1\\
\frac{8\,A_0B_1\theta_0\overline{\alpha_0}+8\,A_0C_1\theta_0\overline{\alpha_1}+8\,A_0B_2\theta_0+C_1(\theta_0+2)\overline{C_1}}{16\,(\theta_0^2+2\,\theta_0)} & \theta_0 & \theta_0+2\\
\frac{(A_0\alpha_0+A_1)C_1+A_0C_2}{2\,\theta_0} & \theta_0+1 & \theta_0\\
\frac{A_0C_2\overline{\alpha_0}+(A_0\alpha_0+A_1)B_1+A_0B_3+\left(2\,A_0|A_1|^2+A_1\overline{\alpha_0}\right)C_1}{2\,(\theta_0+1)} & \theta_0+1 & \theta_0+1\\
\frac{(A_1\alpha_0+A_0\alpha_1+A_2)C_1+(A_0\alpha_0+A_1)C_2+A_0C_3}{2\,\theta_0} & \theta_0+2 & \theta_0\\
\frac{4\,A_0E_1\theta_0+C_1^2}{16\,\theta_0^2} & 2 & 2\,\theta_0\\
\frac{1}{4}\,A_0\overline{B_1}+\frac{1}{4}\,(A_0\alpha_0+A_1)\overline{C_1} & 2\,\theta_0-1 & 2\\
\frac{1}{6}\,A_0\overline{B_1}\overline{\alpha_0}+\frac{1}{6}\,A_0\overline{B_3}+\frac{1}{6}\left(2\,A_0|A_1|^2+A_1\overline{\alpha_0}\right)\overline{C_1}+\frac{1}{6}\,(A_0\alpha_0+A_1)\overline{C_2} & 2\,\theta_0-1 & 3\\
\frac{1}{4}\,A_0\overline{C_1} & 2\,\theta_0-2 & 2\\
\frac{1}{6}\,A_0\overline{C_1}\overline{\alpha_0}+\frac{1}{6}\,A_0\overline{C_2} & 2\,\theta_0-2 & 3\\
\frac{1}{8}\,A_0\overline{C_2}\overline{\alpha_0}+\frac{1}{8}\,A_0\overline{C_1}\overline{\alpha_1}+\frac{1}{64}\,\overline{C_1}^2+\frac{1}{8}\,A_0\overline{C_3} & 2\,\theta_0-2 & 4\\
\frac{1}{4}\,(A_0\alpha_0+A_1)\overline{B_1}+\frac{1}{4}\,A_0\overline{B_2}+\frac{1}{4}\,(A_1\alpha_0+A_0\alpha_1+A_2)\overline{C_1} & 2\,\theta_0 & 2\\
-\frac{A_0\overline{E_1}}{2\,(\theta_0-4)} & 3\,\theta_0-2 & -\theta_0+4
\end{pmatrix}$$

Most of these relations are trivial or uninteresting, and we will only need besides the previous ones (as

$\langle \vec{A}_0, \vec{C}_1 \rangle = \langle \vec{A}_0, \overline{\vec{C}_1} \rangle = 0)$

$$\langle \vec{A}_1, \vec{A}_1 \rangle + 2\langle \vec{A}_0, \vec{A}_2 \rangle = 0, \quad \langle \vec{A}_1, \vec{C}_1 \rangle + \langle \vec{A}_0, \vec{C}_2 \rangle = 0, \quad \langle \vec{A}_0, \overline{\vec{C}_2} \rangle = 0, \tag{2.3.6}$$

furnished by the lines $(2\theta_0, 0)$, $(\theta_0 + 1, \theta_0)$, and $(2\theta_0 - 2, 3)$. Therefore, we deduce that

$$0 = \frac{1}{8}\, A_0 \overline{C_2} \overline{\alpha_0} + \frac{1}{8}\, A_0 \overline{C_1} \overline{\alpha_1} + \frac{1}{64}\, \overline{C_1}^2 + \frac{1}{8}\, A_0 \overline{C_3} = \frac{1}{64} \overline{\langle \vec{C}_1, \vec{C}_1 \rangle} + \frac{1}{8} \langle \vec{A}_0, \overline{\vec{C}_3} \rangle$$

so

$$\langle \vec{C}_1, \vec{C}_1 \rangle + 8\langle \overline{\vec{A}_0}, \vec{C}_3 \rangle = 0.$$

We now stress out all relations we shall need in the sequel

$$\begin{cases} \langle \vec{A}_0, \vec{A}_0 \rangle = \langle \vec{A}_0, \vec{A}_1 \rangle = \langle \vec{A}_0, \vec{C}_1 \rangle = \langle \vec{A}_0, \overline{\vec{C}_1} \rangle = \langle \vec{A}_0, \overline{\vec{C}_2} \rangle = 0 \\ \langle \vec{A}_1, \vec{A}_1 \rangle + 2\langle \vec{A}_0, \vec{A}_2 \rangle = 0, \quad \langle \vec{A}_1, \vec{C}_1 \rangle + \langle \vec{A}_0, \vec{C}_2 \rangle = 0, \quad \langle \vec{C}_1, \vec{C}_1 \rangle + 8\langle \overline{\vec{A}_0}, \vec{C}_3 \rangle = 0. \end{cases} \tag{2.3.7}$$

As $e^{2\lambda} = 2\langle \partial_z \vec{\Phi}, \partial_{\bar{z}} \vec{\Phi} \rangle$, we compute

$$
e^{2\lambda} = \begin{pmatrix}
2\,A_0\overline{A_0} & \theta_0 - 1 & \theta_0 - 1 \\
2\,A_0\overline{A_1} & \theta_0 - 1 & \theta_0 \\
2\,A_0\overline{A_2} + \frac{1}{4}\,\overline{A_0C_1} & \theta_0 - 1 & \theta_0 + 1 \\
2\,A_0\overline{A_3} + \frac{1}{12}\,\left(2\,\overline{A_0}\overline{\alpha_0} + 3\,\overline{A_1}\right)\overline{C_1} + \frac{1}{6}\,\overline{A_0C_2} & \theta_0 - 1 & \theta_0 + 2 \\
2\,A_0\overline{A_4} + \frac{1}{24}\,\left(4\,\overline{A_1}\overline{\alpha_0} + 3\,\overline{A_0}\overline{\alpha_1} + 6\,\overline{A_2}\right)\overline{C_1} + \frac{1}{24}\,\left(3\,\overline{A_0}\overline{\alpha_0} + 4\,\overline{A_1}\right)\overline{C_2} + \frac{1}{8}\,\overline{A_0C_3} & \theta_0 - 1 & \theta_0 + 3 \\
\frac{\cancel{A_0\overline{C_1}}}{2\,\theta_0} & 2\,\theta_0 - 1 & 1 \\
\frac{A_0\theta_0\overline{B_1} + (A_0\alpha_0\theta_0 + A_1(\theta_0 + 1))\overline{C_1}}{2\,(\theta_0^2 + \theta_0)} & 2\,\theta_0 & 1 \\
\mu_1 & 2\,\theta_0 + 1 & 1 \\
\frac{\cancel{A_0\overline{C_1}\overline{\alpha_0}} + \cancel{A_0\overline{C_2}}}{2\,\theta_0} & 2\,\theta_0 - 1 & 2 \\
\frac{A_0\theta_0\overline{B_1}\overline{\alpha_0} + A_0\theta_0\overline{B_3} + \left(2\,A_0\theta_0|A_1|^2 + (\theta_0\overline{\alpha_0} + \overline{\alpha_0})A_1\right)\overline{C_1} + (A_0\alpha_0\theta_0 + A_1(\theta_0 + 1))\overline{C_2}}{2\,(\theta_0^2 + \theta_0)} & 2\,\theta_0 & 2 \\
\frac{8\,(\theta_0\overline{\alpha_1} - 4\,\overline{\alpha_1})A_0\overline{C_1} + (\theta_0 - 4)\overline{C_1}^2 + 8\,(\theta_0\overline{\alpha_0} - 4\,\overline{\alpha_0})A_0\overline{C_2} + 8\,A_0(\theta_0 - 4)\overline{C_3} - 8\,\theta_0\overline{A_0E_1}}{16\,(\theta_0^2 - 4\,\theta_0)} & 2\,\theta_0 - 1 & 3 \\
\frac{\cancel{A_0\overline{E_1}}}{4\,\theta_0} & 3\,\theta_0 - 1 & -\theta_0 + 3 \\
\frac{1}{4}\,A_0C_1\overline{\alpha_0} + \frac{1}{4}\,A_0B_1 + 2\,A_2\overline{A_1} & \theta_0 + 1 & \theta_0 \\
\frac{1}{6}\,A_0C_2\overline{\alpha_0} + \frac{1}{12}\,(2\,A_0\alpha_0 + 3\,A_1)B_1 + \frac{1}{6}\,A_0B_3 + \frac{1}{12}\,\left(4\,A_0|A_1|^2 + 3\,A_1\overline{\alpha_0}\right)C_1 + 2\,A_3\overline{A_1} & \theta_0 + 2 & \theta_0 \\
\frac{1}{4}\,A_0C_1 + 2\,A_2\overline{A_0} & \theta_0 + 1 & \theta_0 - 1 \\
\frac{1}{12}\,(2\,A_0\alpha_0 + 3\,A_1)C_1 + \frac{1}{6}\,A_0C_2 + 2\,A_3\overline{A_0} & \theta_0 + 2 & \theta_0 - 1 \\
\frac{1}{24}\,(4\,A_1\alpha_0 + 3\,A_0\alpha_1 + 6\,A_2)C_1 + \frac{1}{24}\,(3\,A_0\alpha_0 + 4\,A_1)C_2 + \frac{1}{8}\,A_0C_3 + 2\,A_4\overline{A_0} & \theta_0 + 3 & \theta_0 - 1 \\
\mu_2 & \theta_0 + 1 & \theta_0 + 1 \\
\frac{C_1^2(\theta_0 - 4) - 8\,A_0E_1\theta_0 + 8\,(\alpha_1\theta_0 - 4\,\alpha_1)C_1\overline{A_0} + 8\,(\alpha_0\theta_0 - 4\,\alpha_0)C_2\overline{A_0} + 8\,C_3(\theta_0 - 4)\overline{A_0}}{16\,(\theta_0^2 - 4\,\theta_0)} & 3 & 2\,\theta_0 - 1 \\
2\,A_1\overline{A_0} & \theta_0 & \theta_0 - 1 \\
2\,A_1\overline{A_1} & \theta_0 & \theta_0 \\
\frac{1}{4}\,\alpha_0\overline{A_0C_1} + 2\,A_1\overline{A_2} + \frac{1}{4}\,\overline{A_0B_1} & \theta_0 & \theta_0 + 1 \\
\frac{1}{6}\,\alpha_0\overline{A_0C_2} + 2\,A_1\overline{A_3} + \frac{1}{12}\,\left(2\,\overline{A_0}\overline{\alpha_0} + 3\,\overline{A_1}\right)\overline{B_1} + \frac{1}{6}\,\overline{A_0B_3} + \frac{1}{12}\,\left(4\,|A_1|^2\overline{A_0} + 3\,\alpha_0\overline{A_1}\right)\overline{C_1} & \theta_0 & \theta_0 + 2 \\
\frac{\cancel{C_1\overline{A_0}}}{2\,\theta_0} & 1 & 2\,\theta_0 - 1 \\
\frac{B_1\theta_0\overline{A_0} + \left(\theta_0\overline{A_0}\overline{\alpha_0} + (\theta_0 + 1)\overline{A_1}\right)C_1}{2\,(\theta_0^2 + \theta_0)} & 1 & 2\,\theta_0 \\
\mu_3 & 1 & 2\,\theta_0 + 1 \\
\frac{\cancel{C_1\alpha_0\overline{A_0}} + \cancel{C_2\overline{A_0}}}{2\,\theta_0} & 2 & 2\,\theta_0 - 1 \\
\frac{B_1\alpha_0\theta_0\overline{A_0} + B_3\theta_0\overline{A_0} + \left(2\,\theta_0|A_1|^2\overline{A_0} + (\alpha_0\theta_0 + \alpha_0)\overline{A_1}\right)C_1 + \left(\theta_0\overline{A_0}\overline{\alpha_0} + (\theta_0 + 1)\overline{A_1}\right)C_2}{2\,(\theta_0^2 + \theta_0)} & 2 & 2\,\theta_0 \\
\frac{\cancel{E_1\overline{A_0}}}{4\,\theta_0} & -\theta_0 + 3 & 3\,\theta_0 - 1
\end{pmatrix}
$$

where

$$\mu_1 = \frac{1}{2\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)}\Big\{\left(\theta_0^2 + \theta_0\right)A_0\overline{B_2} + \left(\left(\alpha_0\theta_0^2 + \alpha_0\theta_0\right)A_0 + \left(\theta_0^2 + 2\,\theta_0\right)A_1\right)\overline{B_1}$$

$$+ \left(\left(\alpha_1\theta_0^2 + \alpha_1\theta_0\right)A_0 + \left(\alpha_0\theta_0^2 + 2\,\alpha_0\theta_0\right)A_1 + \left(\theta_0^2 + 3\,\theta_0 + 2\right)A_2\right)\overline{C_1}\Big\}$$

$$\mu_2 = \frac{1}{32\,\theta_0^2}\Big\{8\,\alpha_0\theta_0^2\overline{A_0B_1} + 8\,A_0B_1\theta_0^2\overline{\alpha_0} + 8\,A_0C_1\theta_0^2\overline{\alpha_1} + 8\,A_0B_2\theta_0^2 + 64\,A_2\theta_0^2\overline{A_2} + 8\,\theta_0^2\overline{A_0B_2}$$

$$+ \left(8\,\alpha_1\theta_0^2\overline{A_0} + \left(\theta_0^2 + 4\right)C_1\right)\overline{C_1}\Big\}$$

$$\mu_3 = \frac{1}{2\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)}\Big\{\left(\theta_0^2 + \theta_0\right)B_2\overline{A_0} + \left(\left(\theta_0^2\overline{\alpha_0} + \theta_0\overline{\alpha_0}\right)\overline{A_0} + \left(\theta_0^2 + 2\,\theta_0\right)\overline{A_1}\right)B_1$$

$$+ \left(\left(\theta_0^2\overline{\alpha_1} + \theta_0\overline{\alpha_1}\right)\overline{A_0} + \left(\theta_0^2\overline{\alpha_0} + 2\,\theta_0\overline{\alpha_0}\right)\overline{A_1} + \left(\theta_0^2 + 3\,\theta_0 + 2\right)\overline{A_2}\right)C_1\Big\}.$$

As $\vec{E}_1 \in \mathrm{Span}(\overline{\vec{A}_0})$ thanks of (2.3.4), we have

$$\langle \vec{E}_1, \overline{\vec{A}_0}\rangle = 0$$

so there is no coefficient in $\mathrm{Re}\left(*\dfrac{\overline{z}^{3\theta_0-1}}{z^{\theta_0-3}}\right)$ in the Taylor expansion of $e^{2\lambda}$. Also, thanks of (2.3.7), we know that

$$\langle \overline{\vec{A}_0}, \vec{C}_1\rangle = 0$$

the coefficient in $\mathrm{Re}\left(*z\overline{z}^{2\theta_0-1}\right)$ also vanish. Also, by (2.3.7), we have

$$\langle \overline{\vec{A}_0}, \vec{C}_2\rangle = 0$$

so the coefficient in $\mathrm{Re}\left(*z^2\overline{z}^{2\theta_0-2}\right)$ also vanishes.

Finally, we have

$$
e^{2\lambda} = \begin{pmatrix}
2\,A_0\overline{A_0} & \theta_0 - 1 & \theta_0 - 1 \\
2\,A_0\overline{A_1} & \theta_0 - 1 & \theta_0 \\
2\,A_0\overline{A_2} + \frac{1}{4}\,\overline{A_0C_1} & \theta_0 - 1 & \theta_0 + 1 \\
2\,A_0\overline{A_3} + \frac{1}{12}\,\left(2\,\overline{A_0\alpha_0} + 3\,\overline{A_1}\right)\overline{C_1} + \frac{1}{6}\,\overline{A_0C_2} & \theta_0 - 1 & \theta_0 + 2 \\
2\,A_0\overline{A_4} + \frac{1}{24}\,\left(4\,\overline{A_1\alpha_0} + 3\,\overline{A_0\alpha_1} + 6\,\overline{A_2}\right)\overline{C_1} + \frac{1}{24}\,\left(3\,\overline{A_0\alpha_0} + 4\,\overline{A_1}\right)\overline{C_2} + \frac{1}{8}\,\overline{A_0C_3} & \theta_0 - 1 & \theta_0 + 3 \\
\frac{A_0\theta_0\overline{B_1} + (A_0\alpha_0\theta_0 + A_1(\theta_0+1))\overline{C_1}}{2\,(\theta_0^2 + \theta_0)} & 2\,\theta_0 & 1 \\
\mu_1 & 2\,\theta_0 + 1 & 1 \\
\frac{A_0\theta_0\overline{B_1}\,\overline{\alpha_0} + A_0\theta_0\overline{B_3} + \left(2\,A_0\theta_0|A_1|^2 + (\theta_0\overline{\alpha_0} + \overline{\alpha_0})A_1\right)\overline{C_1} + (A_0\alpha_0\theta_0 + A_1(\theta_0+1))\overline{C_2}}{2\,(\theta_0^2 + \theta_0)} & 2\,\theta_0 & 2 \\
\frac{8\,(\theta_0\overline{\alpha_1} - 4\,\overline{\alpha_1})A_0\overline{C_1} + (\theta_0 - 4)\overline{C_1}^2 + 8\,(\theta_0\overline{\alpha_0} - 4\,\overline{\alpha_0})A_0\overline{C_2} + 8\,A_0(\theta_0 - 4)\overline{C_3} - 8\,\theta_0\overline{A_0E_1}}{16\,(\theta_0^2 - 4\,\theta_0)} & 2\,\theta_0 - 1 & 3 \\
\frac{1}{4}\,A_0C_1\overline{\alpha_0} + \frac{1}{4}\,A_0B_1 + 2\,A_2\overline{A_1} & \theta_0 + 1 & \theta_0 \\
\frac{1}{6}\,A_0C_2\overline{\alpha_0} + \frac{1}{12}\,(2\,A_0\alpha_0 + 3\,A_1)B_1 + \frac{1}{6}\,A_0B_3 + \frac{1}{12}\,\left(4\,A_0|A_1|^2 + 3\,A_1\overline{\alpha_0}\right)C_1 + 2\,A_3\overline{A_1} & \theta_0 + 2 & \theta_0 \\
\frac{1}{4}\,A_0C_1 + 2\,A_2\overline{A_0} & \theta_0 + 1 & \theta_0 - 1 \\
\frac{1}{12}\,(2\,A_0\alpha_0 + 3\,A_1)C_1 + \frac{1}{6}\,A_0C_2 + 2\,A_3\overline{A_0} & \theta_0 + 2 & \theta_0 - 1 \\
\frac{1}{24}\,(4\,A_1\alpha_0 + 3\,A_0\alpha_1 + 6\,A_2)C_1 + \frac{1}{24}\,(3\,A_0\alpha_0 + 4\,A_1)C_2 + \frac{1}{8}\,A_0C_3 + 2\,A_4\overline{A_0} & \theta_0 + 3 & \theta_0 - 1 \\
\mu_2 & \theta_0 + 1 & \theta_0 + 1 \\
\frac{C_1^2(\theta_0 - 4) - 8\,A_0E_1\theta_0 + 8\,(\alpha_1\theta_0 - 4\,\alpha_1)C_1\overline{A_0} + 8\,(\alpha_0\theta_0 - 4\,\alpha_0)C_2\overline{A_0} + 8\,C_3(\theta_0 - 4)\overline{A_0}}{16\,(\theta_0^2 - 4\,\theta_0)} & 3 & 2\,\theta_0 - 1 \\
2\,A_1\overline{A_0} & \theta_0 & \theta_0 - 1 \\
2\,A_1\overline{A_1} & \theta_0 & \theta_0 \\
\frac{1}{4}\,\alpha_0\overline{A_0C_1} + 2\,A_1\overline{A_2} + \frac{1}{4}\,\overline{A_0B_1} & \theta_0 & \theta_0 + 1 \\
\frac{1}{6}\,\alpha_0\overline{A_0C_2} + 2\,A_1\overline{A_3} + \frac{1}{12}\,\left(2\,\overline{A_0\alpha_0} + 3\,\overline{A_1}\right)\overline{B_1} + \frac{1}{6}\,\overline{A_0B_3} + \frac{1}{12}\,\left(4\,|A_1|^2\overline{A_0} + 3\,\alpha_0\overline{A_1}\right)\overline{C_1} & \theta_0 & \theta_0 + 2 \\
\frac{B_1\theta_0\overline{A_0} + \left(\theta_0\overline{A_0}\,\overline{\alpha_0} + (\theta_0 + 1)\overline{A_1}\right)C_1}{2\,(\theta_0^2 + \theta_0)} & 1 & 2\,\theta_0 \\
\mu_3 & 1 & 2\,\theta_0 + 1 \\
\frac{B_1\alpha_0\theta_0\overline{A_0} + B_3\theta_0\overline{A_0} + \left(2\,\theta_0|A_1|^2\overline{A_0} + (\alpha_0\theta_0 + \alpha_0)\overline{A_1}\right)C_1 + \left(\theta_0\overline{A_0}\,\overline{\alpha_0} + (\theta_0 + 1)\overline{A_1}\right)C_2}{2\,(\theta_0^2 + \theta_0)} & 2 & 2\,\theta_0
\end{pmatrix}
$$

for some uninteresting $\mu_3 \in \vec{C}$. First, recall that by (2.3.4)

$$\vec{E}_1 = -\frac{1}{2\theta_0}\langle \vec{C}_1, \vec{C}_1\rangle \overline{\vec{A}_0}$$

while by (2.3.7)

$$\langle \vec{C}_1, \vec{C}_1\rangle + 8\langle \overline{\vec{A}_0}, \vec{C}_3\rangle = 0,$$

so we deduce that

$$\frac{C_1^2(\theta_0 - 4) - 8\,A_0E_1\theta_0 + 8\,(\alpha_1\theta_0 - 4\,\alpha_1)C_1\overline{A_0} + 8\,(\alpha_0\theta_0 - 4\,\alpha_0)C_2\overline{A_0} + 8\,C_3(\theta_0 - 4)\overline{A_0}}{16\,(\theta_0^2 - 4\,\theta_0)} = -\frac{\theta_0}{2\theta_0(\theta_0 - 4)}\langle \vec{A}_0, \vec{E}_1\rangle$$

$$= -\frac{1}{2(\theta_0-4)}\left\langle \vec{A}_0, -\frac{1}{2\theta_0}\langle \vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_0}\right\rangle = \frac{1}{8\theta_0(\theta_0-4)}\langle\vec{C}_1,\vec{C}_1\rangle. \tag{2.3.8}$$

as $|\vec{A}_0|^2 = \dfrac{1}{2}$.

In particular, recalling that $|\vec{A}_0|^2 = \dfrac{1}{2}$, we see that there exists some constants

$$\beta\in\mathbb{R},\quad \alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8,\alpha_9\in\mathbb{C}$$

such that

$$e^{2\lambda} = |z|^{2\theta_0-2} + 2|\vec{A}_1|^2|z|^{2\theta_0} + \beta|z|^{2\theta_0+2} + 2\,\mathrm{Re}\Big(\alpha_0 z^{\theta_0}\overline{z}^{\theta_0-1} + \alpha_1 z^{\theta_0+1}\overline{z}^{\theta_0-1} + \alpha_2 z\overline{z}^{2\theta_0} + \alpha_3 z^{\theta_0+2}\overline{z}^{\theta_0-1} + \alpha_4 z^{\theta_0+3}\overline{z}^{\theta_0-1}$$
$$+ \alpha_5 z^{\theta_0+1}\overline{z}^{\theta_0} + \alpha_6 z^{\theta_0+2}\overline{z}^{\theta_0} + \alpha_7 z^3\overline{z}^{2\theta_0-1} + \alpha_8 z^2\overline{z}^{2\theta_0} + \alpha_9 z\overline{z}^{2\theta_0+1}\Big) + O(|z|^{2\theta_0+3-\varepsilon})$$

so that

$$e^{2\lambda} = \begin{pmatrix} 1 & \theta_0-1 & \theta_0-1 \\ 2\,|A_1|^2 & \theta_0 & \theta_0 \\ \beta & \theta_0+1 & \theta_0+1 \\ \alpha_0 & \theta_0 & \theta_0-1 \\ \alpha_1 & \theta_0+1 & \theta_0-1 \\ \alpha_2 & 1 & 2\,\theta_0 \\ \alpha_3 & \theta_0+2 & \theta_0-1 \\ \alpha_4 & \theta_0+3 & \theta_0-1 \\ \alpha_5 & \theta_0+1 & \theta_0 \\ \alpha_6 & \theta_0+2 & \theta_0 \\ \alpha_7 & 3 & 2\,\theta_0-1 \\ \alpha_8 & 2 & 2\,\theta_0 \\ \alpha_9 & 1 & 2\,\theta_0+1 \end{pmatrix} \begin{pmatrix} \overline{\alpha_0} & \theta_0-1 & \theta_0 \\ \overline{\alpha_1} & \theta_0-1 & \theta_0+1 \\ \overline{\alpha_2} & 2\,\theta_0 & 1 \\ \overline{\alpha_3} & \theta_0-1 & \theta_0+2 \\ \overline{\alpha_4} & \theta_0-1 & \theta_0+3 \\ \overline{\alpha_5} & \theta_0 & \theta_0+1 \\ \overline{\alpha_6} & \theta_0 & \theta_0+2 \\ \overline{\alpha_7} & 2\,\theta_0-1 & 3 \\ \overline{\alpha_8} & 2\,\theta_0 & 2 \\ \overline{\alpha_9} & 2\,\theta_0+1 & 1 \end{pmatrix}$$

In these coefficients, the only interesting ones are $|\vec{A}_1|^2$, α_0 and α_2. As we need the exact formula for α_2, we first recall that

$$\vec{B}_1 = -2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0,\quad |\vec{A}_0|^2 = \frac{1}{2},\quad \langle\overline{\vec{A}_0},\vec{C}_1\rangle = 0.$$

We deduce that

$$\langle\overline{\vec{A}_0},\vec{B}_1\rangle = \langle\overline{\vec{A}_0}, -2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0\rangle = -\langle\overline{\vec{A}_1},\vec{C}_1\rangle$$

and finally

$$\alpha_2 = \frac{B_1\theta_0\overline{A_0} + \big(\cancel{\theta_0\overline{A_0}\alpha_0} + (\theta_0+1)\overline{A_1}\big)C_1}{2\,(\theta_0^2+\theta_0)} = \frac{1}{2\theta_0(\theta_0+1)}\Big(-\theta_0\langle\overline{\vec{A}_1},\vec{C}_1\rangle + (\theta_0+1)\langle\overline{\vec{A}_1},\vec{C}_1\rangle\Big)$$

$$= \frac{1}{2\theta_0(\theta_0+1)}\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \tag{2.3.9}$$

while by (2.3.8), we have

$$\alpha_7 = \frac{1}{8\theta_0(\theta_0-4)}\langle \vec{C}_1, \vec{C}_1\rangle. \tag{2.3.10}$$

Now, we have

$$\begin{aligned}
\alpha_3 &= \frac{1}{12}\,(\cancel{2\,A_0\alpha_0} + 3\,A_1)C_1 + \frac{1}{6}\,A_0C_2 + 2\,A_3\overline{A_0} \\
&= \frac{1}{4}\langle \vec{A}_1, \vec{C}_1\rangle + \frac{1}{6}\langle \vec{A}_0, \vec{C}_2\rangle + 2\langle \overline{\vec{A}_0}, \vec{A}_3\rangle = \frac{1}{12}\langle \vec{A}_1, \vec{C}_1\rangle + 2\langle \overline{\vec{A}_0}, \vec{A}_3\rangle
\end{aligned}$$

Finally, we have (using the result not proved yet that $\alpha_0 = 0$, but α_6 does not enter into play in this proof at the reader may check)

$$\begin{aligned}
\alpha_6 &= \frac{1}{6}\,A_0C_2\cancel{\overline{\alpha_0}} + \frac{1}{12}\,(\cancel{2\,A_0\alpha_0} + \cancel{3\,A_1})B_1 + \cancel{\frac{1}{6}\,A_0B_3} + \frac{1}{12}\,\left(\cancel{4\,A_0|A_1|^2} + 3\,A_1\cancel{\overline{\alpha_0}}\right)C_1 + 2\,A_3\overline{A_1} \\
&= 2\langle \overline{\vec{A}_1}, \vec{A}_3\rangle.
\end{aligned}$$

Now, we also have as $\langle \vec{A}_1, \vec{C}_1\rangle + \langle \vec{A}_0, \vec{C}_2\rangle = 0$ by conformality

$$0 = (\cancel{A_1\alpha_0} + \cancel{A_0\alpha_1} + A_2)C_1 + (\cancel{A_0\alpha_0} + A_1)C_2 + A_0C_3 = \langle \vec{A}_2, \vec{C}_1\rangle + \langle \vec{A}_1, \vec{C}_2\rangle + \langle \vec{A}_0, \vec{C}_3\rangle$$

and the development

$$\begin{aligned}
\alpha_4 &= \frac{1}{24}\,(4\,A_1\alpha_0 + \cancel{3\,A_0\alpha_1} + 6\,A_2)C_1 + \frac{1}{24}\,(3\,A_0\alpha_0 + 4\,A_1)C_2 + \frac{1}{8}\,A_0C_3 + 2\,A_4\overline{A_0} \\
&= \frac{1}{6}\alpha_0\langle \vec{A}_1, \vec{C}_1\rangle + \frac{1}{4}\langle \vec{A}_2, \vec{C}_1\rangle - \frac{1}{8}\alpha_0\langle \vec{A}_1, \vec{C}_1\rangle + \frac{1}{6}\langle \vec{A}_1, \vec{C}_2\rangle - \frac{1}{8}\left(\langle \vec{A}_2, \vec{C}_1\rangle + \langle \vec{A}_1, \vec{C}_2\rangle\right) + 2\langle \overline{\vec{A}_0}, \vec{A}_4\rangle \\
&= \frac{1}{24}\alpha_0\langle \vec{A}_1, \vec{C}_1\rangle + \frac{1}{24}\left(3\langle \vec{A}_2, \vec{C}_1\rangle + \langle \vec{A}_1, \vec{C}_2\rangle\right) + 2\langle \overline{\vec{A}_0}, \vec{A}_4\rangle.
\end{aligned}$$

$$\begin{cases}
\alpha_1 = 2\langle \overline{\vec{A}_0}, \vec{A}_2\rangle \\
\alpha_2 = \dfrac{1}{2\theta_0(\theta_0+1)}\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \\
\alpha_3 = \dfrac{1}{12}\langle \vec{A}_1, \vec{C}_1\rangle + 2\langle \overline{\vec{A}_0}, \vec{A}_3\rangle \\
\alpha_4 = \dfrac{1}{24}\alpha_0\langle \vec{A}_1, \vec{C}_1\rangle + \dfrac{1}{24}\left(3\langle \vec{A}_2, \vec{C}_1\rangle + \langle \vec{A}_1, \vec{C}_2\rangle\right) + 2\langle \overline{\vec{A}_0}, \vec{A}_4\rangle \\
\alpha_5 = 2\langle \overline{\vec{A}_1}, \vec{A}_2\rangle \\
\alpha_6 = 2\langle \overline{\vec{A}_1}, \vec{A}_3\rangle \\
\alpha_7 = \dfrac{1}{8\theta_0(\theta_0-4)}\langle \vec{C}_1, \vec{C}_1\rangle
\end{cases} \tag{2.3.11}$$

We now obtain

$$\vec{h}_0 = \begin{pmatrix}
2 & A_1 & \theta_0 - 1 & 0 \\
4 & A_2 & \theta_0 & 0 \\
6 & A_3 & \theta_0 + 1 & 0 \\
8 & A_4 & \theta_0 + 2 & 0 \\
-\frac{\theta_0 - 2}{2\,\theta_0} & C_1 & 0 & \theta_0 \\
-\frac{\theta_0 - 2}{2\,(\theta_0 + 1)} & B_1 & 0 & \theta_0 + 1 \\
-\frac{\theta_0 \overline{\alpha_0} - 2\,\overline{\alpha_0}}{2\,(\theta_0 + 1)} & C_1 & 0 & \theta_0 + 1 \\
-\frac{\theta_0 - 2}{2\,(\theta_0 + 2)} & B_2 & 0 & \theta_0 + 2 \\
-\frac{\theta_0 \overline{\alpha_1} - 2\,\overline{\alpha_1}}{2\,(\theta_0 + 2)} & C_1 & 0 & \theta_0 + 2 \\
-\frac{\theta_0 \overline{\alpha_0} - 2\,\overline{\alpha_0}}{2\,(\theta_0 + 2)} & B_1 & 0 & \theta_0 + 2 \\
-\frac{\theta_0 - 3}{2\,\theta_0} & C_2 & 1 & \theta_0 \\
-\frac{\alpha_0 \theta_0 - 2\,\alpha_0}{2\,\theta_0} & C_1 & 1 & \theta_0 \\
-\frac{\theta_0 - 3}{2\,(\theta_0 + 1)} & B_3 & 1 & \theta_0 + 1 \\
-\frac{\theta_0 \overline{\alpha_0} - 3\,\overline{\alpha_0}}{2\,(\theta_0 + 1)} & C_2 & 1 & \theta_0 + 1 \\
-\frac{\alpha_0 \theta_0 - 2\,\alpha_0}{2\,(\theta_0 + 1)} & B_1 & 1 & \theta_0 + 1 \\
-\frac{2\left(\theta_0^2 - 2\,\theta_0 + 1\right) |A_1|^2 - \alpha_0 \overline{\alpha_0}}{2\,(\theta_0^2 + \theta_0)} & C_1 & 1 & \theta_0 + 1 \\
-\frac{\theta_0 - 4}{2\,\theta_0} & C_3 & 2 & \theta_0 \\
-\frac{\alpha_1 \theta_0 - 2\,\alpha_1}{2\,\theta_0} & C_1 & 2 & \theta_0 \\
-\frac{\alpha_0 \theta_0 - 3\,\alpha_0}{2\,\theta_0} & C_2 & 2 & \theta_0 \\
-\frac{\theta_0 - 2}{2\,\theta_0} & E_1 & -\theta_0 + 2 & 2\,\theta_0 \\
\frac{1}{4} & \overline{B_1} & \theta_0 - 1 & 2 \\
\frac{1}{6} & \overline{B_3} & \theta_0 - 1 & 3 \\
\frac{1}{6}\,\overline{\alpha_0} & \overline{B_1} & \theta_0 - 1 & 3 \\
-\frac{1}{6}\,|A_1|^2 + \frac{1}{12}\,\alpha_0 \overline{\alpha_0} & \overline{C_1} & \theta_0 - 1 & 3 \\
\frac{1}{2} & \overline{B_2} & \theta_0 & 2 \\
\frac{1}{4}\,\alpha_0 & \overline{B_1} & \theta_0 & 2
\end{pmatrix}$$

$$
\begin{pmatrix}
-\frac{\theta_0}{2\,(\theta_0-4)} & \overline{E_1} & 2\,\theta_0-2 & -\theta_0+4 \\
-4\,|A_1|^2+2\,\alpha_0\overline{\alpha_0} & A_0 & \theta_0-1 & 1 \\
-4\,|A_1|^2+2\,\alpha_0\overline{\alpha_0} & A_1 & \theta_0 & 1 \\
-4\,|A_1|^2+2\,\alpha_0\overline{\alpha_0} & A_2 & \theta_0+1 & 1 \\
\mu_4 & A_0 & \theta_0 & 2 \\
-2\,\alpha_0 & A_0 & \theta_0-1 & 0 \\
-2\,\alpha_0 & A_1 & \theta_0 & 0 \\
-2\,\alpha_0 & A_2 & \theta_0+1 & 0 \\
-2\,\alpha_0 & A_3 & \theta_0+2 & 0 \\
2\,\alpha_0^2-4\,\alpha_1 & A_0 & \theta_0 & 0 \\
2\,\alpha_0^2-4\,\alpha_1 & A_1 & \theta_0+1 & 0 \\
2\,\alpha_0^2-4\,\alpha_1 & A_2 & \theta_0+2 & 0 \\
2\,\alpha_2\theta_0-4\,\alpha_2 & A_0 & 0 & \theta_0+1 \\
2\,\alpha_2\theta_0-4\,\alpha_2 & A_1 & 1 & \theta_0+1 \\
-4\,\alpha_0^3\theta_0+2\,\alpha_0^3+6\,\alpha_0\alpha_1-6\,\alpha_3 & A_0 & \theta_0+1 & 0 \\
-4\,\alpha_0^3\theta_0+2\,\alpha_0^3+6\,\alpha_0\alpha_1-6\,\alpha_3 & A_1 & \theta_0+2 & 0 \\
2\,\alpha_0^4+4\,\alpha_0^2\alpha_1+4\,\alpha_1^2+8\,\alpha_0\alpha_3-4\left(\alpha_0^4+3\,\alpha_0^2\alpha_1\right)\theta_0-8\,\alpha_4 & A_0 & \theta_0+2 & 0 \\
-12\,\alpha_0^2\theta_0\overline{\alpha_0}+8\,\alpha_0|A_1|^2+8\,\alpha_0^2\overline{\alpha_0}+4\,\alpha_1\overline{\alpha_0}-4\,\alpha_5 & A_0 & \theta_0 & 1 \\
-12\,\alpha_0^2\theta_0\overline{\alpha_0}+8\,\alpha_0|A_1|^2+8\,\alpha_0^2\overline{\alpha_0}+4\,\alpha_1\overline{\alpha_0}-4\,\alpha_5 & A_1 & \theta_0+1 & 1 \\
\mu_5 & A_0 & \theta_0+1 & 1 \\
2\,\alpha_7\theta_0-8\,\alpha_7 & A_0 & 2 & \theta_0 \\
6\,\alpha_0\alpha_2-2\,(\alpha_0\alpha_2-\alpha_8)\theta_0-6\,\alpha_8 & A_0 & 1 & \theta_0+1 \\
-2\,(\alpha_2\overline{\alpha_0}-\alpha_9)\theta_0+4\,\alpha_2\overline{\alpha_0}-4\,\alpha_9 & A_0 & 0 & \theta_0+2 \\
-2\,\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2} & A_0 & 2\,\theta_0-1 & -\theta_0+2 \\
-2\,\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2} & A_1 & 2\,\theta_0 & -\theta_0+2 \\
-4\,\theta_0\overline{\alpha_0}^3+4\,\overline{\alpha_0}^3 & A_0 & \theta_0-2 & 3 \\
-4\,\theta_0\overline{\alpha_0}^3+4\,\overline{\alpha_0}^3 & A_1 & \theta_0-1 & 3 \\
4\,\overline{\alpha_0}^4+12\,\overline{\alpha_0}^2\overline{\alpha_1}-4\left(\overline{\alpha_0}^4+3\,\overline{\alpha_0}^2\overline{\alpha_1}\right)\theta_0 & A_0 & \theta_0-2 & 4
\end{pmatrix}
$$

$$\begin{pmatrix} -12\,\alpha_0\theta_0\overline{\alpha_0}^2 + 4\,|A_1|^2\overline{\alpha_0} + 10\,\alpha_0\overline{\alpha_0}^2 + 2\,\alpha_0\overline{\alpha_1} - 2\,\overline{\alpha_5} & A_0 & \theta_0 - 1 & 2 \\ -12\,\alpha_0\theta_0\overline{\alpha_0}^2 + 4\,|A_1|^2\overline{\alpha_0} + 10\,\alpha_0\overline{\alpha_0}^2 + 2\,\alpha_0\overline{\alpha_1} - 2\,\overline{\alpha_5} & A_1 & \theta_0 & 2 \\ \mu_6 & A_0 & \theta_0 - 1 & 3 \\ -2\,\theta_0\overline{\alpha_7} & A_0 & 2\,\theta_0 - 2 & -\theta_0 + 4 \\ 2\,(\overline{\alpha_0\alpha_2} - \overline{\alpha_8})\theta_0 + 2\,\overline{\alpha_0\alpha_2} - 2\,\overline{\alpha_8} & A_0 & 2\,\theta_0 - 1 & -\theta_0 + 3 \\ 2\,(\alpha_0\overline{\alpha_2} - \overline{\alpha_9})\theta_0 + 4\,\alpha_0\overline{\alpha_2} - 4\,\overline{\alpha_9} & A_0 & 2\,\theta_0 & -\theta_0 + 2 \end{pmatrix}$$

where

$$\mu_4 = 8\,|A_1|^4 + 18\,\alpha_0^2\overline{\alpha_0}^2 - 16\,(3\,\alpha_0\theta_0\overline{\alpha_0} - 2\,\alpha_0\overline{\alpha_0})|A_1|^2 + 8\,\alpha_1\overline{\alpha_0}^2 - 12\left(2\,\alpha_0^2\overline{\alpha_0}^2 + \alpha_1\overline{\alpha_0}^2 + \alpha_0^2\overline{\alpha_1}\right)\theta_0$$
$$+ 4\,\alpha_5\overline{\alpha_0} + 4\left(2\,\alpha_0^2 + \alpha_1\right)\overline{\alpha_1} + 4\,\alpha_0\overline{\alpha_5} - 4\,\beta$$
$$\mu_5 = 10\,\alpha_0^3\overline{\alpha_0} - 12\left(2\,\alpha_0^2\theta_0 - \alpha_0^2 - \alpha_1\right)|A_1|^2 + 12\,\alpha_0\alpha_1\overline{\alpha_0} + 6\,\alpha_0\alpha_5 - 8\left(2\,\alpha_0^3\overline{\alpha_0} + 3\,\alpha_0\alpha_1\overline{\alpha_0}\right)\theta_0 + 6\,\alpha_3\overline{\alpha_0} - 6\,\alpha_6$$
$$\mu_6 = 14\,\alpha_0\overline{\alpha_0}^3 - 4\left(6\,\theta_0\overline{\alpha_0}^2 - 5\,\overline{\alpha_0}^2 - \overline{\alpha_1}\right)|A_1|^2 + 20\,\alpha_0\overline{\alpha_0\alpha_1} - 8\left(2\,\alpha_0\overline{\alpha_0}^3 + 3\,\alpha_0\overline{\alpha_0\alpha_1}\right)\theta_0 + 2\,\alpha_0\overline{\alpha_3} + 2\,\overline{\alpha_0\alpha_5} - 2\,\overline{\alpha_6}$$

Throwing all terms of order larger or equal that $\theta_0 + 1$, we obtain

$$\vec{h}_0 = \begin{pmatrix} 2 & A_1 & \theta_0 - 1 & 0 \\ 4 & A_2 & \theta_0 & 0 \\ -\frac{\theta_0 - 2}{2\,\theta_0} & C_1 & 0 & \theta_0 \\ -4\,|A_1|^2 + 2\,\alpha_0\overline{\alpha_0} & A_0 & \theta_0 - 1 & 1 \\ -2\,\alpha_0 & A_0 & \theta_0 - 1 & 0 \\ -2\,\alpha_0 & A_1 & \theta_0 & 0 \\ 2\,\alpha_0^2 - 4\,\alpha_1 & A_0 & \theta_0 & 0 \end{pmatrix}$$

which agrees with the previous development (2.1.5).

2.4 The conservation law associated to the invariance by inversions

Finally, we have thanks of the third conservation law

$$d\,\mathrm{Im}\left(|\vec{\Phi}|^2\vec{\alpha} - 2\langle\vec{\Phi},\vec{\alpha}\rangle\vec{\Phi} - g^{-1}\otimes\left(\vec{h}_0\otimes\overline{\partial}|\vec{\Phi}|^2 - 2\langle\vec{h}_0,\vec{\Phi}\rangle\otimes\overline{\partial}\vec{\Phi}\right)\right) = 0$$

if

$$\vec{\alpha} = \partial\vec{H} + |\vec{H}|^2\partial\vec{\Phi} + 2\,g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}. \tag{2.4.1}$$

As the form

$$\vec{\beta} = |\vec{\Phi}|^2\vec{\alpha} - 2\langle\vec{\Phi},\vec{\alpha}\rangle\vec{\Phi} - g^{-1}\otimes\left(\vec{h}_0\otimes\overline{\partial}|\vec{\Phi}|^2 - 2\langle\vec{h}_0,\vec{\Phi}\rangle\otimes\overline{\partial}\vec{\Phi}\right)$$

is a $\mathbb{C}^n$-valued 1 differential of type $(1,0)$ (in $\mathrm{Span}_{\mathbb{C}^n}(dz)$), we write for some $\vec{F}: D^2 \to \mathbb{C}^n$,

$$\vec{\beta} = \vec{F}(z)dz, \tag{2.4.2}$$

and we have $\partial\vec{\beta} = 0$. Therefore, we have

$$d\vec{\beta} = \partial\vec{\beta} + \overline{\partial}\vec{\beta} = \overline{\partial}\vec{\beta}$$

and by (2.4.2), we deduce by linearity of imaginary part that

$$d\,\mathrm{Im}\left(\vec{\beta}\right) = \mathrm{Im}\left(\overline{\partial}\vec{\beta}\right) = \mathrm{Im}\left(\partial_{\overline{z}}\vec{F}(z)\,d\overline{z}\wedge dz\right) = -2\,\mathrm{Re}\left(\partial_{\overline{z}}\vec{F}(z)\right)dx_1\wedge dx_2 = 0$$

as

$$d\overline{z}\wedge dz = (dx_1 - idx_2)\wedge(dx_1 + idx_2) = 2i\,dx_1\wedge dx_2, \quad \text{and } \mathrm{Im}\,(i\,\cdot\,) = -\mathrm{Re}\,(\,\cdot\,)$$

In particular, we have $d\,\mathrm{Im}\left(\vec{\beta}\right) = 0$ if and only if

$$\mathrm{Re}\left(\partial_{\overline{z}}\vec{F}(z)\right) = 0. \tag{2.4.3}$$

We first compute

$$\vec{\alpha} = \begin{pmatrix}
-\frac{1}{2}\,\theta_0 + 1 & C_1 & -\theta_0 + 1 & 0 \\
-\frac{1}{2}\,\theta_0 + \frac{3}{2} & C_2 & -\theta_0 + 2 & 0 \\
-\frac{1}{2}\,\theta_0 + 2 & C_3 & -\theta_0 + 3 & 0 \\
-\frac{1}{2}\,\theta_0 + 1 & B_1 & -\theta_0 + 1 & 1 \\
-\frac{1}{2}\,\theta_0 + 1 & B_2 & -\theta_0 + 1 & 2 \\
-\frac{1}{2}\,\theta_0 + \frac{3}{2} & B_3 & -\theta_0 + 2 & 1 \\
-\theta_0 + 2 & E_1 & -2\,\theta_0 + 3 & \theta_0 \\
\frac{1}{2} & \overline{B_1} & 0 & -\theta_0 + 2 \\
1 & \overline{B_2} & 1 & -\theta_0 + 2 \\
\frac{1}{2} & \overline{B_3} & 0 & -\theta_0 + 3 \\
\frac{1}{2}\,\theta_0 & \overline{E_1} & \theta_0 - 1 & -2\,\theta_0 + 4 \\
\frac{1}{4}\,C_1^2 & A_0 & -\theta_0 + 3 & 0 \\
\frac{1}{2}\,C_1\overline{C_1} & A_0 & 1 & -\theta_0 + 2 \\
\frac{1}{4}\,\overline{C_1}^2 & A_0 & \theta_0 - 1 & -2\,\theta_0 + 4 \\
-2\,A_0C_1\alpha_0 + 2\,A_1C_1 & \overline{A_0} & -\theta_0 + 2 & 0 \\
-2\,A_0C_1\alpha_0 + 2\,A_1C_1 & \overline{A_1} & -\theta_0 + 2 & 1 \\
2\,A_0C_1\alpha_0^2 + 2\left(\alpha_0^2 - 2\,\alpha_1\right)A_0C_1 - 4\,A_1C_1\alpha_0 - 2\,A_0C_2\alpha_0 + 4\,A_2C_1 + 2\,A_1C_2 & \overline{A_0} & -\theta_0 + 3 & 0 \\
2\,A_0C_1\alpha_0\overline{\alpha_0} - 2\left(2\,|A_1|^2 - \alpha_0\overline{\alpha_0}\right)A_0C_1 - 2\,A_0B_1\alpha_0 - 2\,A_1C_1\overline{\alpha_0} + 2\,A_1B_1 & \overline{A_0} & -\theta_0 + 2 & 1 \\
-\frac{C_1^2(\theta_0 - 2)}{2\,\theta_0} & \overline{A_0} & -2\,\theta_0 + 3 & \theta_0 \\
-2\,A_0\alpha_0\overline{C_1} + 2\,A_1\overline{C_1} & \overline{A_0} & 0 & -\theta_0 + 2 \\
-2\,A_0\alpha_0\overline{C_1} + 2\,A_1\overline{C_1} & \overline{A_1} & 0 & -\theta_0 + 3 \\
2\,A_0\alpha_0^2\overline{C_1} - 2\,A_0\alpha_0\overline{B_1} + 2\left(\alpha_0^2 - 2\,\alpha_1\right)A_0\overline{C_1} - 4\,A_1\alpha_0\overline{C_1} + 2\,A_1\overline{B_1} + 4\,A_2\overline{C_1} & \overline{A_0} & 1 & -\theta_0 + 2 \\
2\,A_0\alpha_0\overline{C_1}\overline{\alpha_0} - 2\left(2\,|A_1|^2 - \alpha_0\overline{\alpha_0}\right)A_0\overline{C_1} - 2\,A_0\alpha_0\overline{C_2} - 2\,A_1\overline{C_1}\overline{\alpha_0} + 2\,A_1\overline{C_2} & \overline{A_0} & 0 & -\theta_0 + 3 \\
-\frac{C_1(\theta_0 - 2)\overline{C_1}}{2\,\theta_0} & \overline{A_0} & -\theta_0 + 1 & 2
\end{pmatrix}$$

As $\vec{\Phi}$ is Willmore we have with the command

```
latex(matrix_sort(matrix_full_simplify(real_part(diffzb(alpha)))))
```

the identity

$$0 = \mathrm{Re}\,(\partial_{\overline{z}}\vec{\alpha}) = \begin{pmatrix}
-\frac{1}{2}\,\theta_0 + 1 & \overline{B_1} & 0 & -\theta_0 + 1 \\
((\alpha_0\theta_0 - 2\,\alpha_0)A_0 - A_1(\theta_0 - 2))\overline{C_1} & \overline{A_0} & 0 & -\theta_0 + 1 \\
-\frac{1}{2}\,\theta_0 + \frac{3}{2} & \overline{B_3} & 0 & -\theta_0 + 2 \\
-\left(\overline{A_0}\overline{\alpha_0} - \overline{A_1}\right)\overline{C_1} & A_1 & 0 & -\theta_0 + 2 \\
((\alpha_0\theta_0 - 3\,\alpha_0)A_0 - A_1(\theta_0 - 3))\overline{C_1} & \overline{A_1} & 0 & -\theta_0 + 2 \\
\nu_1 & \overline{A_0} & 0 & -\theta_0 + 2 \\
-\left(\overline{A_0}\overline{\alpha_0} - \overline{A_1}\right)\overline{B_1} - \left(2\,|A_1|^2\overline{A_0} - 2\,\alpha_0\overline{A_0}\overline{\alpha_0} + \alpha_0\overline{A_1}\right)\overline{C_1} & A_0 & 0 & -\theta_0 + 2 \\
-\theta_0 + 2 & \overline{B_2} & 1 & -\theta_0 + 1 \\
-\frac{(\theta_0^2 - 4)C_1\overline{C_1}}{4\,\theta_0} & A_0 & 1 & -\theta_0 + 1 \\
\nu_2 & \overline{A_0} & 1 & -\theta_0 + 1 \\
-\theta_0^2 + 2\,\theta_0 & E_1 & -2\,\theta_0 + 3 & \theta_0 - 1 \\
-\frac{1}{2}\,C_1^2(\theta_0 - 2) & \overline{A_0} & -2\,\theta_0 + 3 & \theta_0 - 1 \\
-\frac{1}{2}\,\theta_0 + 1 & B_1 & -\theta_0 + 1 & 0 \\
\left((\theta_0\overline{\alpha_0} - 2\,\overline{\alpha_0})\overline{A_0} - (\theta_0 - 2)\overline{A_1}\right)C_1 & A_0 & -\theta_0 + 1 & 0 \\
-\theta_0 + 2 & B_2 & -\theta_0 + 1 & 1 \\
-\frac{(\theta_0^2 - 4)C_1\overline{C_1}}{4\,\theta_0} & \overline{A_0} & -\theta_0 + 1 & 1 \\
\nu_3 & A_0 & -\theta_0 + 1 & 1 \\
-\frac{1}{2}\,\theta_0 + \frac{3}{2} & B_3 & -\theta_0 + 2 & 0 \\
\left((\theta_0\overline{\alpha_0} - 3\,\overline{\alpha_0})\overline{A_0} - (\theta_0 - 3)\overline{A_1}\right)C_1 & A_1 & -\theta_0 + 2 & 0 \\
-(A_0\alpha_0 - A_1)C_1 & \overline{A_1} & -\theta_0 + 2 & 0 \\
\nu_4 & A_0 & -\theta_0 + 2 & 0 \\
-(A_0\alpha_0 - A_1)B_1 - \left(2\,A_0|A_1|^2 - 2\,A_0\alpha_0\overline{\alpha_0} + A_1\overline{\alpha_0}\right)C_1 & \overline{A_0} & -\theta_0 + 2 & 0 \\
-\theta_0^2 + 2\,\theta_0 & \overline{E_1} & \theta_0 - 1 & -2\,\theta_0 + 3 \\
-\frac{1}{2}\,(\theta_0 - 2)\overline{C_1}^2 & A_0 & \theta_0 - 1 & -2\,\theta_0 + 3
\end{pmatrix}$$

where

$$\nu_1 = \left(2\,A_0(\theta_0 - 3)|A_1|^2 - 2\,(\alpha_0\theta_0\overline{\alpha_0} - 3\,\alpha_0\overline{\alpha_0})A_0 + (\theta_0\overline{\alpha_0} - 3\,\overline{\alpha_0})A_1\right)\overline{C_1} + ((\alpha_0\theta_0 - 3\,\alpha_0)A_0 - A_1(\theta_0 - 3))\overline{C_2}$$

$$\nu_2 = ((\alpha_0\theta_0 - 2\,\alpha_0)A_0 - A_1(\theta_0 - 2))\overline{B_1} + 2\left(\left(2\,\alpha_0^2 - \left(\alpha_0^2 - \alpha_1\right)\theta_0 - 2\,\alpha_1\right)A_0 + (\alpha_0\theta_0 - 2\,\alpha_0)A_1 - A_2(\theta_0 - 2)\right)\overline{C_1}$$

$$\nu_3 = \left((\theta_0\overline{\alpha_0} - 2\,\overline{\alpha_0})\overline{A_0} - (\theta_0 - 2)\overline{A_1}\right)B_1 - 2\left(\left(\left(\overline{\alpha_0}^2 - \overline{\alpha_1}\right)\theta_0 - 2\,\overline{\alpha_0}^2 + 2\,\overline{\alpha_1}\right)\overline{A_0} - (\theta_0\overline{\alpha_0} - 2\,\overline{\alpha_0})\overline{A_1} + (\theta_0 - 2)\overline{A_2}\right)C_1$$

$$\nu_4 = \Big(2\,(\theta_0-3)|A_1|^2\overline{A_0} - 2\,(\alpha_0\theta_0\overline{\alpha_0} - 3\,\alpha_0\overline{\alpha_0})\overline{A_0} + (\alpha_0\theta_0 - 3\,\alpha_0)\overline{A_1}\Big)C_1 + \Big((\theta_0\overline{\alpha_0} - 3\,\overline{\alpha_0})\overline{A_0} - (\theta_0-3)\overline{A_1}\Big)C_2.$$

Taking one order less in the expansion, we get

$$\vec{\alpha} = \begin{pmatrix} -\frac{1}{2}\,\theta_0+1 & C_1 & -\theta_0+1 & 0 \\ -\frac{1}{2}\,\theta_0+\frac{3}{2} & C_2 & -\theta_0+2 & 0 \\ -\frac{1}{2}\,\theta_0+1 & B_1 & -\theta_0+1 & 1 \\ \frac{1}{2} & \overline{B_1} & 0 & -\theta_0+2 \\ -2\,A_0C_1\alpha_0+2\,A_1C_1 & \overline{A_0} & -\theta_0+2 & 0 \\ -2\,A_0\alpha_0\overline{C_1}+2\,A_1\overline{C_1} & \overline{A_0} & 0 & -\theta_0+2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\,\theta_0+1 & C_1 & -\theta_0+1 & 0 \\ -\frac{1}{2}\,\theta_0+\frac{3}{2} & C_2 & -\theta_0+2 & 0 \\ -\frac{1}{2}\,\theta_0+1 & B_1 & -\theta_0+1 & 1 \\ \frac{1}{2} & \overline{B_1} & 0 & -\theta_0+2 \\ 2\,A_1C_1 & \overline{A_0} & -\theta_0+2 & 0 \\ 2\,A_1\overline{C_1} & \overline{A_0} & 0 & -\theta_0+2 \end{pmatrix}$$

as $\langle\vec{A}_0,\vec{C}_1\rangle = \langle\overline{\vec{A}_0},\vec{C}_1\rangle = 0$.

We will prove thanks of the invariance by inversions that

$$\alpha_0 = 2\langle\overline{\vec{A}_0},\vec{A}_1\rangle = 0$$

What follows is valid for all $\theta_0 \geq 4$, as we do not use the complete developments of tensors. We compute

$$0 = \mathrm{Re}\left(\partial_{\overline{z}}\vec{F}(z)\right) = \begin{pmatrix} \frac{4\left(\theta_0^2\overline{\alpha_0}^3+2\,\theta_0\overline{\alpha_0}^3-3\,\overline{\alpha_0}^3\right)\overline{A_0}^2}{\theta_0} & A_0 & -1 & \theta_0+2 \\ -\frac{4\left(\theta_0^2\overline{\alpha_0}^3+2\,\theta_0\overline{\alpha_0}^3-3\,\overline{\alpha_0}^3\right)A_0\overline{A_0}}{\theta_0} & \overline{A_0} & -1 & \theta_0+2 \\ 2\,\overline{A_0}^2\overline{\alpha_0}-4\,\overline{A_0A_1} & A_1 & 0 & \theta_0 \\ -2\,A_0\alpha_0\overline{A_0}+2\,A_1\overline{A_0} & \overline{A_1} & 0 & \theta_0 \\ 4\,|A_1|^2\overline{A_0}^2-4\,\alpha_0\overline{A_0}^2\overline{\alpha_0}+4\,\alpha_0\overline{A_0A_1} & A_0 & 0 & \theta_0 \\ -4\,A_0|A_1|^2\overline{A_0}+4\,A_0\alpha_0\overline{A_0}\overline{\alpha_0}-2\,A_1\overline{A_0}\overline{\alpha_0}-2\,(A_0\alpha_0-A_1)\overline{A_1} & \overline{A_0} & 0 & \theta_0 \\ -2\,\overline{A_0}^2 & A_1 & 0 & \theta_0-1 \\ 2\,\alpha_0\overline{A_0}^2 & A_0 & 0 & \theta_0-1 \\ -2\,A_0\alpha_0\overline{A_0}+2\,A_1\overline{A_0} & \overline{A_0} & 0 & \theta_0-1 \\ \lambda_1 & \overline{A_0} & 0 & \theta_0+1 \\ \lambda_2 & A_0 & 0 & \theta_0+1 \\ -\frac{\left(\theta_0^2+2\,\theta_0-4\right)\overline{A_0}^2}{4\,\theta_0^2} & \overline{B_1} & 0 & \theta_0+1 \\ -2\left(\overline{\alpha_0}^2-\overline{\alpha_1}\right)\overline{A_0}^2+4\,\overline{A_0A_1}\overline{\alpha_0}-2\,\overline{A_1}^2-4\,\overline{A_0A_2} & A_1 & 0 & \theta_0+1 \\ -2\,A_0\alpha_0\overline{A_0}+2\,A_1\overline{A_0} & \overline{A_2} & 0 & \theta_0+1 \\ -4\,A_0|A_1|^2\overline{A_0}+4\,A_0\alpha_0\overline{A_0}\overline{\alpha_0}-2\,A_1\overline{A_0}\overline{\alpha_0}-2\,(A_0\alpha_0-A_1)\overline{A_1} & \overline{A_1} & 0 & \theta_0+1 \end{pmatrix} \tag{2.4.4}$$

$$
\begin{pmatrix}
\lambda_3 & \overline{A_0} & 1 & \theta_0 \\
\lambda_4 & A_0 & 1 & \theta_0 \\
-\frac{A_0\overline{A_0}\,\overline{\alpha_0} - A_0\overline{A_1}}{2\,\theta_0} & C_1 & 1 & \theta_0 \\
\frac{4\left(\alpha_0^2 - \alpha_1\right)A_0\theta_0\overline{A_0} - 4\,A_1\alpha_0\theta_0\overline{A_0} + 4\,A_2\theta_0\overline{A_0} - A_0C_1}{\theta_0} & \overline{A_1} & 1 & \theta_0 \\
-\frac{A_0(\theta_0 - 2)\overline{A_0}}{\theta_0^2} & B_1 & 1 & \theta_0 \\
4\,\overline{A_0}^2\overline{\alpha_0} - 8\,\overline{A_0A_1} & A_2 & 1 & \theta_0 \\
8\,|A_1|^2\overline{A_0}^2 - 8\,\alpha_0\overline{A_0}^2\overline{\alpha_0} + 8\,\alpha_0\overline{A_0A_1} & A_1 & 1 & \theta_0 \\
-\frac{8\left(\alpha_0^2 - \alpha_1\right)\theta_0\overline{A_0}^2 - C_1(\theta_0 - 2)\overline{A_0}}{2\,\theta_0} & A_0 & 1 & \theta_0 - 1 \\
-4\,\overline{A_0}^2 & A_2 & 1 & \theta_0 - 1 \\
4\,\alpha_0\overline{A_0}^2 & A_1 & 1 & \theta_0 - 1 \\
4\left(\alpha_0^2 - \alpha_1\right)A_0\overline{A_0} - 4\,A_1\alpha_0\overline{A_0} + 4\,A_2\overline{A_0} & \overline{A_0} & 1 & \theta_0 - 1 \\
\lambda_5 & A_0 & 2 & \theta_0 - 1 \\
\lambda_6 & \overline{A_0} & 2 & \theta_0 - 1 \\
-\frac{3\left(4\left(\alpha_0^2 + \left(\alpha_0^2 - \alpha_1\right)\theta_0 - \alpha_1\right)\overline{A_0}^2 + C_1\overline{A_0}\right)}{2\,(\theta_0 + 1)} & A_1 & 2 & \theta_0 - 1 \\
-6\,\overline{A_0}^2 & A_3 & 2 & \theta_0 - 1 \\
6\,\alpha_0\overline{A_0}^2 & A_2 & 2 & \theta_0 - 1 \\
-\frac{1}{4}\,A_0\alpha_0\overline{A_0} + \frac{1}{4}\,A_1\overline{A_0} & C_1 & 2 & \theta_0 - 1 \\
\frac{(\theta_0 - 2)\overline{A_0}^2}{2\,\theta_0} & C_1 & -\theta_0 + 1 & 2\,\theta_0 - 1 \\
-\frac{2\left(\alpha_2\theta_0^3 - 2\,\alpha_2\theta_0^2\right)\overline{A_0}^2 - \left(\left(\theta_0\overline{\alpha_0} - 2\,\overline{\alpha_0}\right)\overline{A_0}^3 - (\theta_0 - 2)\overline{A_0}^2\overline{A_1}\right)C_1}{\theta_0^2} & A_0 & -\theta_0 + 1 & 2\,\theta_0 \\
-\frac{\left(\theta_0\overline{\alpha_0} - 2\,\overline{\alpha_0}\right)\overline{A_0}^2 - 2\left(\theta_0^2 - \theta_0 - 2\right)\overline{A_0A_1}}{2\,(\theta_0^2 + \theta_0)} & C_1 & -\theta_0 + 1 & 2\,\theta_0 \\
\lambda_7 & \overline{A_0} & -\theta_0 + 1 & 2\,\theta_0 \\
\frac{\left(\theta_0^3 - 3\,\theta_0^2 + \theta_0 + 2\right)\overline{A_0}^2}{2\,(\theta_0^3 + \theta_0^2)} & B_1 & -\theta_0 + 1 & 2\,\theta_0 \\
-\frac{C_1(\theta_0 - 2)\overline{A_0}}{2\,(\theta_0^2 + \theta_0)} & \overline{A_1} & -\theta_0 + 1 & 2\,\theta_0 \\
\frac{2\left(A_0\alpha_0\overline{A_0}^2 - A_1\overline{A_0}^2\right)C_1}{\theta_0} & \overline{A_0} & -\theta_0 + 2 & 2\,\theta_0 - 1 \\
\frac{(\theta_0 - 3)\overline{A_0}^2}{2\,\theta_0} & C_2 & -\theta_0 + 2 & 2\,\theta_0 - 1 \\
-2\,A_0\overline{A_0}\,\overline{\alpha_0} + 2\,A_0\overline{A_1} & A_1 & \theta_0 & 0 \\
2\,A_0^2\alpha_0 - 4\,A_0A_1 & \overline{A_1} & \theta_0 & 0
\end{pmatrix}
$$

$$\begin{pmatrix}
-4\,A_0|A_1|^2\overline{A_0}+4\,A_0\alpha_0\overline{A_0}\overline{\alpha_0}-2\,A_1\overline{A_0}\overline{\alpha_0}-2\,(A_0\alpha_0-A_1)\overline{A_1} & A_0 & \theta_0 & 0 \\
4\,A_0^2|A_1|^2-4\,A_0^2\alpha_0\overline{\alpha_0}+4\,A_0A_1\overline{\alpha_0} & \overline{A_0} & \theta_0 & 0 \\
\lambda_8 & A_0 & \theta_0 & 1 \\
\lambda_9 & \overline{A_0} & \theta_0 & 1 \\
-\frac{A_0\alpha_0\overline{A_0}-A_1\overline{A_0}}{2\,\theta_0} & \overline{C_1} & \theta_0 & 1 \\
\frac{4\left(\overline{\alpha_0}^2-\overline{\alpha_1}\right)A_0\theta_0\overline{A_0}-4\,A_0\theta_0\overline{A_1}\overline{\alpha_0}+4\,A_0\theta_0\overline{A_2}-\overline{A_0C_1}}{\theta_0} & A_1 & \theta_0 & 1 \\
-\frac{A_0(\theta_0-2)\overline{A_0}}{\theta_0^2} & \overline{B_1} & \theta_0 & 1 \\
4\,A_0^2\alpha_0-8\,A_0A_1 & \overline{A_2} & \theta_0 & 1 \\
8\,A_0^2|A_1|^2-8\,A_0^2\alpha_0\overline{\alpha_0}+8\,A_0A_1\overline{\alpha_0} & \overline{A_1} & \theta_0 & 1 \\
-2\,A_0^2 & \overline{A_1} & \theta_0-1 & 0 \\
-2\,A_0\overline{A_0}\overline{\alpha_0}+2\,A_0\overline{A_1} & A_0 & \theta_0-1 & 0 \\
2\,A_0^2\overline{\alpha_0} & \overline{A_0} & \theta_0-1 & 0 \\
-\frac{8\left(\overline{\alpha_0}^2-\overline{\alpha_1}\right)A_0^2\theta_0-A_0(\theta_0-2)\overline{C_1}}{2\,\theta_0} & \overline{A_0} & \theta_0-1 & 1 \\
-4\,A_0^2 & \overline{A_2} & \theta_0-1 & 1 \\
4\,A_0^2\overline{\alpha_0} & \overline{A_1} & \theta_0-1 & 1 \\
4\left(\overline{\alpha_0}^2-\overline{\alpha_1}\right)A_0\overline{A_0}-4\,A_0\overline{A_1}\overline{\alpha_0}+4\,A_0\overline{A_2} & A_0 & \theta_0-1 & 1 \\
\lambda_{10} & \overline{A_0} & \theta_0-1 & 2 \\
\lambda_{11} & A_0 & \theta_0-1 & 2 \\
-\frac{3\left(4\left(\left(\overline{\alpha_0}^2-\overline{\alpha_1}\right)\theta_0+\overline{\alpha_0}^2-\overline{\alpha_1}\right)A_0^2+A_0\overline{C_1}\right)}{2\,(\theta_0+1)} & \overline{A_1} & \theta_0-1 & 2 \\
-6\,A_0^2 & \overline{A_3} & \theta_0-1 & 2 \\
-\frac{1}{4}\,A_0\overline{A_0}\overline{\alpha_0}+\frac{1}{4}\,A_0\overline{A_1} & \overline{C_1} & \theta_0-1 & 2 \\
6\,A_0^2\overline{\alpha_0} & \overline{A_2} & \theta_0-1 & 2 \\
\lambda_{12} & A_0 & \theta_0+1 & 0 \\
\lambda_{13} & \overline{A_0} & \theta_0+1 & 0 \\
-\frac{\left(\theta_0^2+2\,\theta_0-4\right)A_0^2}{4\,\theta_0^2} & B_1 & \theta_0+1 & 0
\end{pmatrix}$$

$$
\begin{pmatrix}
-2\,A_0\overline{A_0}\overline{\alpha_0}+2\,A_0\overline{A_1} & A_2 & \theta_0+1 & 0 \\
-2\left(\alpha_0^2-\alpha_1\right)A_0^2+4\,A_0A_1\alpha_0-2\,A_1^2-4\,A_0A_2 & \overline{A_1} & \theta_0+1 & 0 \\
-4\,A_0|A_1|^2\overline{A_0}+4\,A_0\alpha_0\overline{A_0}\overline{\alpha_0}-2\,A_1\overline{A_0}\overline{\alpha_0}-2\left(A_0\alpha_0-A_1\right)\overline{A_1} & A_1 & \theta_0+1 & 0 \\
-\frac{4\left(\alpha_0^3\theta_0^2+2\,\alpha_0^3\theta_0-3\,\alpha_0^3\right)A_0\overline{A_0}}{\theta_0} & A_0 & \theta_0+2 & -1 \\
\frac{4\left(\alpha_0^3\theta_0^2+2\,\alpha_0^3\theta_0-3\,\alpha_0^3\right)A_0^2}{\theta_0} & \overline{A_0} & \theta_0+2 & -1 \\
\frac{A_0^2(\theta_0-2)}{2\,\theta_0} & \overline{C_1} & 2\,\theta_0-1 & -\theta_0+1 \\
\frac{2\left(A_0^2\overline{A_0}\overline{\alpha_0}-A_0^2\overline{A_1}\right)\overline{C_1}}{\theta_0} & A_0 & 2\,\theta_0-1 & -\theta_0+2 \\
\frac{A_0^2(\theta_0-3)}{2\,\theta_0} & \overline{C_2} & 2\,\theta_0-1 & -\theta_0+2 \\
-\frac{2\left(\theta_0^3\overline{\alpha_2}-2\,\theta_0^2\overline{\alpha_2}\right)A_0^2-\left((\alpha_0\theta_0-2\,\alpha_0)A_0^3-A_0^2A_1(\theta_0-2)\right)\overline{C_1}}{\theta_0^2} & \overline{A_0} & 2\,\theta_0 & -\theta_0+1 \\
-\frac{(\alpha_0\theta_0-2\,\alpha_0)A_0^2-2\left(\theta_0^2-\theta_0-2\right)A_0A_1}{2\,(\theta_0^2+\theta_0)} & \overline{C_1} & 2\,\theta_0 & -\theta_0+1 \\
\lambda_{14} & A_0 & 2\,\theta_0 & -\theta_0+1 \\
-\frac{A_0(\theta_0-2)\overline{C_1}}{2\,(\theta_0^2+\theta_0)} & A_1 & 2\,\theta_0 & -\theta_0+1 \\
\frac{\left(\theta_0^3-3\,\theta_0^2+\theta_0+2\right)A_0^2}{2\,(\theta_0^3+\theta_0^2)} & \overline{B_1} & 2\,\theta_0 & -\theta_0+1
\end{pmatrix}
\qquad (2.4.5)
$$

where

$$
\begin{aligned}
\lambda_1 = \frac{1}{4\,\theta_0^2}\Big\{ & 8\left(\overline{\alpha_0}^2-\overline{\alpha_1}\right)A_1\theta_0^2\overline{A_0}+16\left(2\,A_0\theta_0^2\overline{A_0}\overline{\alpha_0}-A_0\theta_0^2\overline{A_1}\right)|A_1|^2 \\
& -8\left(6\,\alpha_0\theta_0^3\overline{\alpha_0}^2-6\,\alpha_0\theta_0\overline{\alpha_0}^2+\left(3\,\alpha_0\overline{\alpha_0}^2-2\,\alpha_0\overline{\alpha_1}+\overline{\alpha_5}\right)\theta_0^2\right)A_0\overline{A_0}+\left(\theta_0^2+2\,\theta_0-8\right)\overline{A_0B_1} \\
& +8\left(2\,A_0\alpha_0\theta_0^2\overline{\alpha_0}-A_1\theta_0^2\overline{\alpha_0}\right)\overline{A_1}-8\left(A_0\alpha_0\theta_0^2-A_1\theta_0^2\right)\overline{A_2}+4\left((\alpha_0\theta_0+2\,\alpha_0)A_0\overline{A_0}^2-A_1(\theta_0+2)\overline{A_0}^2\right)\overline{C_1}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_2 = -\frac{2}{\theta_0}\Big\{ & 4\,\alpha_0\theta_0\overline{A_0A_1}\overline{\alpha_0}-\alpha_0\theta_0\overline{A_1}^2-2\,\alpha_0\theta_0\overline{A_0A_2}+4\left(\theta_0\overline{A_0}^2\overline{\alpha_0}-\theta_0\overline{A_0A_1}\right)|A_1|^2 \\
& -\left(6\,\alpha_0\theta_0^2\overline{\alpha_0}^2-6\,\alpha_0\overline{\alpha_0}^2+\left(3\,\alpha_0\overline{\alpha_0}^2-2\,\alpha_0\overline{\alpha_1}+\overline{\alpha_5}\right)\theta_0\right)\overline{A_0}^2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_3 = \frac{1}{2\,\theta_0^2}\Big\{ & 16\,A_1\alpha_0\theta_0^2\overline{A_0}\overline{\alpha_0}+4\left(\alpha_2\theta_0^3+\alpha_2\theta_0^2\right)A_0^2+A_0B_1(\theta_0-4)+16\left(2\,A_0\alpha_0\theta_0^2\overline{A_0}-A_1\theta_0^2\overline{A_0}\right)|A_1|^2 \\
& -8\left(3\,\alpha_0^2\theta_0^3\overline{\alpha_0}+3\,\alpha_0^2\theta_0\overline{\alpha_0}-\left(3\,\alpha_0^2\overline{\alpha_0}+2\,\alpha_1\overline{\alpha_0}-\alpha_5\right)\theta_0^2\right)A_0\overline{A_0}-8\left(\theta_0^2\overline{A_0}\overline{\alpha_0}-\theta_0^2\overline{A_1}\right)A_2 \\
& +2\left(4\,A_0^2\overline{A_0}\overline{\alpha_0}-4\,A_0^2\overline{A_1}+A_0\theta_0\overline{\alpha_0}\right)C_1+8\left(\left(\alpha_0^2-\alpha_1\right)A_0\theta_0^2-A_1\alpha_0\theta_0^2\right)\overline{A_1}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_4 = -\frac{1}{2\,\theta_0^2}\Big\{ & 32\,\alpha_0\theta_0^2|A_1|^2\overline{A_0}^2+16\left(\alpha_0^2-\alpha_1\right)\theta_0^2\overline{A_0A_1}+4\left(\alpha_2\theta_0^3+\alpha_2\theta_0^2\right)A_0\overline{A_0}-\left(\theta_0^2-4\right)B_1\overline{A_0} \\
& -8\left(3\,\alpha_0^2\theta_0^3\overline{\alpha_0}+3\,\alpha_0^2\theta_0\overline{\alpha_0}-\left(3\,\alpha_0^2\overline{\alpha_0}+2\,\alpha_1\overline{\alpha_0}-\alpha_5\right)\theta_0^2\right)\overline{A_0}^2+\left(\theta_0\overline{A_0}\overline{\alpha_0}-\left(\theta_0^2-\theta_0\right)\overline{A_1}\right)C_1\Big\}
\end{aligned}
$$

$$
\lambda_5 = \frac{C_2(\theta_0-3)\overline{A_0}+4\left(2\,\alpha_0^3\theta_0^2+6\,\alpha_0^3-\left(5\,\alpha_0^3+6\,\alpha_0\alpha_1-3\,\alpha_3\right)\theta_0\right)\overline{A_0}^2+\left(4\,A_0\alpha_0\overline{A_0}^2+\alpha_0\theta_0\overline{A_0}-4\,A_1\overline{A_0}^2\right)C_1}{2\,\theta_0}
$$

$$
\lambda_6 = \frac{1}{4\,\theta_0}\Big\{24\left(\alpha_0^2-\alpha_1\right)A_1\theta_0\overline{A_0}-24\,A_2\alpha_0\theta_0\overline{A_0}-8\left(2\,\alpha_0^3\theta_0^2+6\,\alpha_0^3-\left(5\,\alpha_0^3+6\,\alpha_0\alpha_1-3\,\alpha_3\right)\theta_0\right)A_0\overline{A_0}+24\,A_3\theta_0\overline{A_0}
$$

$$
\left. - (A_0\alpha_0\theta_0 - A_1\theta_0)C_1 \right\}
$$

$$
\lambda_7 = \frac{1}{2\,(\theta_0^3+\theta_0^2)}\left\{4\left(\alpha_2\theta_0^4 - \alpha_2\theta_0^3 - 2\,\alpha_2\theta_0^2\right)A_0\overline{A_0} + \left(3\,\theta_0^2 - 4\,\theta_0 - 4\right)B_1\overline{A_0}\right.
$$

$$
\left. - \left(4\left(\theta_0^2\overline{\alpha_0} - \theta_0\overline{\alpha_0} - 2\,\overline{\alpha_0}\right)A_0\overline{A_0}^2 - 4\left(\theta_0^2 - \theta_0 - 2\right)A_0\overline{A_0A_1} - \left(\theta_0^2\overline{\alpha_0} - 2\,\theta_0\overline{\alpha_0}\right)\overline{A_0}\right)C_1\right\}
$$

$$
\lambda_8 = \frac{1}{2\,\theta_0^2}\left\{8\left(\overline{\alpha_0}^2 - \overline{\alpha_1}\right)A_1\theta_0^2\overline{A_0} + 16\left(2\,A_0\theta_0^2\overline{A_0}\overline{\alpha_0} - A_0\theta_0^2\overline{A_1}\right)|A_1|^2\right.
$$

$$
- 8\left(3\,\alpha_0\theta_0^3\overline{\alpha_0}^2 + 3\,\alpha_0\theta_0\overline{\alpha_0}^2 - \left(3\,\alpha_0\overline{\alpha_0}^2 + 2\,\alpha_0\overline{\alpha_1} - \overline{\alpha_5}\right)\theta_0^2\right)A_0\overline{A_0} + 4\left(\theta_0^3\overline{\alpha_2} + \theta_0^2\overline{\alpha_2}\right)\overline{A_0}^2 + (\theta_0 - 4)\overline{A_0B_1}
$$

$$
\left. + 8\left(2\,A_0\alpha_0\theta_0^2\overline{\alpha_0} - A_1\theta_0^2\overline{\alpha_0}\right)\overline{A_1} - 8\left(A_0\alpha_0\theta_0^2 - A_1\theta_0^2\right)\overline{A_2} + 2\left(4\,A_0\alpha_0\overline{A_0}^2 + \alpha_0\theta_0\overline{A_0} - 4\,A_1\overline{A_0}^2\right)\overline{C_1}\right\}
$$

$$
\lambda_9 = -\frac{1}{2\,\theta_0^2}\left\{32\,A_0^2\theta_0^2|A_1|^2\overline{\alpha_0} + 16\left(\overline{\alpha_0}^2 - \overline{\alpha_1}\right)A_0A_1\theta_0^2 - 8\left(3\,\alpha_0\theta_0^3\overline{\alpha_0}^2 + 3\,\alpha_0\theta_0\overline{\alpha_0}^2 - \left(3\,\alpha_0\overline{\alpha_0}^2 + 2\,\alpha_0\overline{\alpha_1} - \overline{\alpha_5}\right)\theta_0^2\right)A_0^2\right.
$$

$$
\left. + 4\left(\theta_0^3\overline{\alpha_2} + \theta_0^2\overline{\alpha_2}\right)A_0\overline{A_0} - \left(\theta_0^2 - 4\right)A_0\overline{B_1} + \left(A_0\alpha_0\theta_0 - \left(\theta_0^2 - \theta_0\right)A_1\right)\overline{C_1}\right\}
$$

$$
\lambda_{10} = \frac{4\left(2\,\theta_0^2\overline{\alpha_0}^3 + 6\,\overline{\alpha_0}^3 - \left(5\,\overline{\alpha_0}^3 + 6\,\overline{\alpha_0\alpha_1} - 3\,\overline{\alpha_3}\right)\theta_0\right)A_0^2 + A_0(\theta_0 - 3)\overline{C_2} + \left(4\,A_0^2\overline{A_0}\overline{\alpha_0} - 4\,A_0^2\overline{A_1} + A_0\theta_0\overline{\alpha_0}\right)\overline{C_1}}{2\,\theta_0}
$$

$$
\lambda_{11} = \frac{1}{4\,\theta_0}\left\{24\left(\overline{\alpha_0}^2 - \overline{\alpha_1}\right)A_0\theta_0\overline{A_1} - 24\,A_0\theta_0\overline{A_2}\overline{\alpha_0} - 8\left(2\,\theta_0^2\overline{\alpha_0}^3 + 6\,\overline{\alpha_0}^3 - \left(5\,\overline{\alpha_0}^3 + 6\,\overline{\alpha_0\alpha_1} - 3\,\overline{\alpha_3}\right)\theta_0\right)A_0\overline{A_0}\right.
$$

$$
\left. + 24\,A_0\theta_0\overline{A_3} - \left(\theta_0\overline{A_0}\overline{\alpha_0} - \theta_0\overline{A_1}\right)\overline{C_1}\right\}
$$

$$
\lambda_{12} = \frac{1}{4\,\theta_0^2}\left\{16\,A_1\alpha_0\theta_0^2\overline{A_0}\overline{\alpha_0} + \left(\theta_0^2 + 2\,\theta_0 - 8\right)A_0B_1 + 16\left(2\,A_0\alpha_0\theta_0^2\overline{A_0} - A_1\theta_0^2\overline{A_0}\right)|A_1|^2\right.
$$

$$
- 8\left(6\,\alpha_0^2\theta_0^3\overline{\alpha_0} - 6\,\alpha_0^2\theta_0\overline{\alpha_0} + \left(3\,\alpha_0^2\overline{\alpha_0} - 2\,\alpha_1\overline{\alpha_0} + \alpha_5\right)\theta_0^2\right)A_0\overline{A_0} - 8\left(\theta_0^2\overline{A_0}\overline{\alpha_0} - \theta_0^2\overline{A_1}\right)A_2
$$

$$
\left. + 4\left((\theta_0\overline{\alpha_0} + 2\,\overline{\alpha_0})A_0^2\overline{A_0} - A_0^2(\theta_0 + 2)\overline{A_1}\right)C_1 + 8\left(\left(\alpha_0^2 - \alpha_1\right)A_0\theta_0^2 - A_1\alpha_0\theta_0^2\right)\overline{A_1}\right\}
$$

$$
\lambda_{13} = -\frac{2}{\theta_0}\left\{4\,A_0A_1\alpha_0\theta_0\overline{\alpha_0} - A_1^2\theta_0\overline{\alpha_0} - 2\,A_0A_2\theta_0\overline{\alpha_0} - \left(6\,\alpha_0^2\theta_0^2\overline{\alpha_0} - 6\,\alpha_0^2\overline{\alpha_0} + \left(3\,\alpha_0^2\overline{\alpha_0} - 2\,\alpha_1\overline{\alpha_0} + \alpha_5\right)\theta_0\right)A_0^2\right.
$$

$$
\left. + 4\left(A_0^2\alpha_0\theta_0 - A_0A_1\theta_0\right)|A_1|^2\right\}
$$

$$
\lambda_{14} = \frac{1}{2\,(\theta_0^3+\theta_0^2)}\left\{4\left(\theta_0^4\overline{\alpha_2} - \theta_0^3\overline{\alpha_2} - 2\,\theta_0^2\overline{\alpha_2}\right)A_0\overline{A_0} + \left(3\,\theta_0^2 - 4\,\theta_0 - 4\right)A_0\overline{B_1}\right.
$$

$$
\left. - \left(4\left(\alpha_0\theta_0^2 - \alpha_0\theta_0 - 2\,\alpha_0\right)A_0^2\overline{A_0} - 4\left(\theta_0^2 - \theta_0 - 2\right)A_0A_1\overline{A_0} - \left(\alpha_0\theta_0^2 - 2\,\alpha_0\theta_0\right)A_0\right)\overline{C_1}\right\}
$$

The only interesting coefficient is

$$
\frac{\overline{z}^{\theta_0+2}}{z}
$$

which is as $\langle \vec{A}_0, \vec{A}_0\rangle = 0$ and $|\vec{A}_0|^2 = \dfrac{1}{2}$

$$
\begin{aligned}
\Omega_0 &= \begin{pmatrix} \dfrac{4\left(\theta_0^2\overline{\alpha_0}^3 + 2\,\theta_0\overline{\alpha_0}^3 - 3\,\overline{\alpha_0}^3\right)\cancel{\overline{A_0}^2}}{\theta_0} & A_0 & -1 & \theta_0 + 2 \\ -\dfrac{4\left(\theta_0^2\overline{\alpha_0}^3 + 2\,\theta_0\overline{\alpha_0}^3 - 3\,\overline{\alpha_0}^3\right)A_0\overline{A_0}}{\theta_0} & \overline{A_0} & -1 & \theta_0 + 2 \end{pmatrix} \\
&= -\frac{2}{\theta_0}\left(\theta_0^2 + 2\theta_0 - 3\right)\overline{\alpha_0}^3\overline{\vec{A}_0} \\
&= -\frac{2}{\theta_0}(\theta_0 + 3)(\theta_0 - 1)\overline{\alpha_0}^3\overline{\vec{A}_0}
\end{aligned}
$$

$$= 0 \tag{2.4.6}$$

as $\theta_0^2 + 2\theta_0 - 1 = (\theta_0+3)(\theta_0-1)$. As $\theta_0 \geq 4$, and $\vec{A}_0 \neq 0$ by the very definition of a branch point of multiplicity $\theta_0 \geq 1$, we obtain

$$\alpha_0 = 2\langle \overline{\vec{A}_0}, \vec{A}_1 \rangle = 0. \tag{2.4.7}$$

However, the rest of these expressions all vanish identically by previous results. Therefore, we have

$$\begin{cases} \vec{B}_2 = -\dfrac{(\theta_0+2)}{4\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0} + \left(\cancel{\dfrac{1}{2}\overline{\alpha_0}\langle \overline{\vec{A}_1}, \vec{C}_1\rangle} - 2\langle \overline{\vec{A}_2}, \vec{C}_1\rangle\right)\vec{A}_0 \\ \vec{B}_3 = -2\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \vec{A}_1 + \dfrac{2}{\theta_0 - 3}\langle \vec{A}_1, \vec{C}_1\rangle \overline{\vec{A}_1} + \cancel{\dfrac{2\overline{\alpha_0}}{\theta_0-3}\langle \vec{A}_1, \vec{C}_1\rangle\overline{\vec{A}_0}} + 2\left(\cancel{\alpha_0\langle \overline{\vec{A}_1}, \vec{C}_1\rangle} - \langle \overline{\vec{A}_1}, \vec{C}_2\rangle\right)\vec{A}_0 \\ \vec{E}_1 = -\dfrac{1}{2\theta_0}\langle \vec{C}_1, \vec{C}_1\rangle \overline{\vec{A}_0}. \end{cases}$$

so

$$\begin{cases} \vec{B}_1 = -2\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \vec{A}_0 \\ \vec{B}_2 = -\dfrac{(\theta_0+2)}{4\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0} - 2\langle \overline{\vec{A}_2}, \vec{C}_1\rangle \vec{A}_0 \\ \vec{B}_3 = -2\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \vec{A}_1 + \dfrac{2}{\theta_0-3}\langle \vec{A}_1, \vec{C}_1\rangle \overline{\vec{A}_1} - 2\langle \overline{\vec{A}_1}, \vec{C}_2\rangle \vec{A}_0 \\ \vec{E}_1 = -\dfrac{1}{2\theta_0}\langle \vec{C}_1, \vec{C}_1\rangle \overline{\vec{A}_0}. \end{cases} \tag{2.4.8}$$

2.5 Relations given by the meromorphy

We have

$$\mathscr{Q}_{\vec{\Phi}} = \begin{pmatrix} 16\,A_0A_2|A_1|^2 - 8\,A_0A_1\alpha_5 - 8\left(2\,A_0^2\alpha_1 - A_1^2\right)|A_1|^2 & \theta_0 - 1 & -\theta_0 + 1 \\ 48\,A_1A_2|A_1|^2 + 48\,A_0A_3|A_1|^2 - 24\,A_0A_1\alpha_6 - 48\left(A_0A_1\alpha_1 + A_0^2\alpha_3\right)|A_1|^2 & \theta_0 & -\theta_0+1 \\ \omega_0 & 0 & 1 \\ \omega_1 & \theta_0 - 1 & -\theta_0 + 2 \\ \omega_2 & 2\,\theta_0 - 1 & -2\,\theta_0 + 2 \\ \left(\theta_0^2 - 5\,\theta_0 + 6\right)A_1C_2 - 2\left(\left(\alpha_1\theta_0^2 - 2\,\alpha_1\theta_0\right)A_0 - \left(\theta_0^2 - 2\,\theta_0\right)A_2\right)C_1 & 0 & 0 \\ \omega_3 & 1 & 0 \\ 2\left(2\,\theta_0^2 - 7\,\theta_0 + 6\right)A_1E_1 & -\theta_0 + 1 & \theta_0 \\ \omega_4 & -1 & 2 \\ 4\left(\theta_0^3\overline{\alpha_7} - 5\,\theta_0^2\overline{\alpha_7} + 4\,\theta_0\overline{\alpha_7}\right)A_0A_1 + \left(\theta_0^2 - \theta_0\right)A_1\overline{E_1} & 2\,\theta_0 - 3 & -2\,\theta_0 + 4 \\ \omega_5 & -1 & 1 \\ -8\left(\theta_0^3\overline{\alpha_2} - \theta_0\overline{\alpha_2}\right)A_0^2|A_1|^2 + 4\left(\theta_0^3\overline{\alpha_8} - 2\,\theta_0^2\overline{\alpha_8} - 3\,\theta_0\overline{\alpha_8}\right)A_0A_1 & 2\,\theta_0 - 2 & -2\,\theta_0 + 3 \\ 4\left(\theta_0^3\overline{\alpha_2} - \theta_0^2\overline{\alpha_2} - 2\,\theta_0\overline{\alpha_2}\right)A_0A_1 & 2\,\theta_0 - 2 & -2\,\theta_0 + 2 \\ \left(\theta_0^2 - 3\,\theta_0 + 2\right)A_1C_1 & -1 & 0 \end{pmatrix}$$

where

$$\omega_0 = -\frac{1}{\theta_0}\Big\{2\left(\theta_0^3 - 6\,\theta_0^2 + 11\,\theta_0 - 6\right)A_0C_2|A_1|^2 - 8\left(\alpha_1\alpha_2\theta_0^4 - \alpha_1\alpha_2\theta_0^3 - 2\,\alpha_1\alpha_2\theta_0^2\right)A_0^2$$
$$+ 4\left(\alpha_8\theta_0^4 - 4\,\alpha_8\theta_0^3 + \alpha_8\theta_0^2 + 6\,\alpha_8\theta_0\right)A_0A_1 + 4\left(\alpha_2\theta_0^4 - \alpha_2\theta_0^3 - 2\,\alpha_2\theta_0^2\right)A_1^2 + 8\left(\alpha_2\theta_0^4 - \alpha_2\theta_0^3 - 2\,\alpha_2\theta_0^2\right)A_0A_2$$
$$- \left(\theta_0^3 - 5\,\theta_0^2 + 6\,\theta_0\right)A_1B_3 + 2\left(\left(\alpha_1\theta_0^3 - 2\,\alpha_1\theta_0^2\right)A_0 - \left(\theta_0^3 - 2\,\theta_0^2\right)A_2\right)B_1$$
$$+ 2\Big(\left(\theta_0^3 - 2\,\theta_0^2 + 3\,\theta_0 - 6\right)A_1|A_1|^2 + \left(\alpha_5\theta_0^3 - 3\,\alpha_5\theta_0^2 + 2\,\alpha_5\theta_0\right)A_0\Big)C_1\Big\}$$

$$\omega_1 = -\frac{2}{\theta_0}\Big\{16\,A_0A_1\theta_0|A_1|^4 + 8\,A_0^2\alpha_1\theta_0\overline{\alpha_5} - 8\left(\alpha_1\overline{\alpha_1} - \beta\right)A_0A_1\theta_0 - 4\,A_1^2\theta_0\overline{\alpha_5} - 8\,A_0A_2\theta_0\overline{\alpha_5}$$
$$+ \left(2\,\theta_0^4\overline{\alpha_2} - 5\,\theta_0^3\overline{\alpha_2} + 5\,\theta_0\overline{\alpha_2} - 2\,\overline{\alpha_2}\right)A_0C_1 - A_1\theta_0\overline{B_2} - \left(A_0\alpha_1\theta_0 - A_2\theta_0\right)\overline{B_1}\Big\}$$

$$\omega_2 = -8\left(\alpha_1\theta_0^3\overline{\alpha_2} - 2\,\alpha_1\theta_0^2\overline{\alpha_2} - \alpha_1\theta_0\overline{\alpha_2} + 2\,\alpha_1\overline{\alpha_2}\right)A_0^2 + 4\left(\theta_0^3\overline{\alpha_9} + \theta_0^2\overline{\alpha_9} - 4\,\theta_0\overline{\alpha_9} - 4\,\overline{\alpha_9}\right)A_0A_1$$
$$+ 4\left(\theta_0^3\overline{\alpha_2} - 2\,\theta_0^2\overline{\alpha_2} - \theta_0\overline{\alpha_2} + 2\,\overline{\alpha_2}\right)A_1^2 + 8\left(\theta_0^3\overline{\alpha_2} - 2\,\theta_0^2\overline{\alpha_2} - \theta_0\overline{\alpha_2} + 2\,\overline{\alpha_2}\right)A_0A_2$$

$$\omega_3 = -4\left(\alpha_7\theta_0^3 - 7\,\alpha_7\theta_0^2 + 12\,\alpha_7\theta_0\right)A_0A_1 + \left(\theta_0^2 - 7\,\theta_0 + 12\right)A_1C_3$$
$$- \left(3\left(\alpha_3\theta_0^2 - \alpha_3\theta_0 - 2\,\alpha_3\right)A_0 + 2\left(\alpha_1\theta_0^2 - 4\,\alpha_1\right)A_1 - 3\left(\theta_0^2 - \theta_0 - 2\right)A_3\right)C_1$$
$$- 2\left(\left(\alpha_1\theta_0^2 - 4\,\alpha_1\theta_0 + 3\,\alpha_1\right)A_0 - \left(\theta_0^2 - 4\,\theta_0 + 3\right)A_2\right)C_2$$

$$\omega_4 = \frac{1}{8\left(\theta_0^2 + \theta_0\right)}\Big\{64\left(\alpha_2\theta_0^5 - 2\,\alpha_2\theta_0^4 - \alpha_2\theta_0^3 + 2\,\alpha_2\theta_0^2\right)A_0^2|A_1|^2 - 16\left(\theta_0^4 - 3\,\theta_0^3 + 2\,\theta_0^2\right)A_0B_1|A_1|^2$$
$$- 32\left(\alpha_9\theta_0^5 - 5\,\alpha_9\theta_0^3 + 4\,\alpha_9\theta_0\right)A_0A_1 + 8\left(\theta_0^4 - 2\,\theta_0^3 - \theta_0^2 + 2\,\theta_0\right)A_1B_2$$
$$- \left(8\left(\theta_0^4\overline{\alpha_5} - 4\,\theta_0^3\overline{\alpha_5} + 3\,\theta_0^2\overline{\alpha_5} + 4\,\theta_0\overline{\alpha_5} - 4\,\overline{\alpha_5}\right)A_0 - \left(\theta_0^4 - 4\,\theta_0^3 + 3\,\theta_0^2 + 4\,\theta_0 - 4\right)\overline{B_1}\right)C_1\Big\}$$

$$\omega_5 = -\frac{2}{\theta_0}\Big\{\left(\theta_0^3 - 4\,\theta_0^2 + 5\,\theta_0 - 2\right)A_0C_1|A_1|^2 + 4\left(\alpha_2\theta_0^4 - 2\,\alpha_2\theta_0^3 - \alpha_2\theta_0^2 + 2\,\alpha_2\theta_0\right)A_0A_1 - \left(\theta_0^3 - 3\,\theta_0^2 + 2\,\theta_0\right)A_1B_1\Big\}$$

The only non-trivial component is as $\langle\vec{A}_1,\vec{A}_1\rangle + 2\langle\vec{A}_0,\vec{A}_2\rangle = 0$

$$\omega_0 = -\frac{1}{\theta_0}\Big\{2\left(\theta_0^3 - 6\,\theta_0^2 + 11\,\theta_0 - 6\right)A_0C_2|A_1|^2 - 8\left(\alpha_1\alpha_2\theta_0^4 - \alpha_1\alpha_2\theta_0^3 - 2\,\alpha_1\alpha_2\theta_0^2\right)A_0^2$$
$$+ 4\left(\alpha_8\theta_0^4 - 4\,\alpha_8\theta_0^3 + \alpha_8\theta_0^2 + 6\,\alpha_8\theta_0\right)A_0A_1 + 4\left(\alpha_2\theta_0^4 - \alpha_2\theta_0^3 - 2\,\alpha_2\theta_0^2\right)A_1^2 + 8\left(\alpha_2\theta_0^4 - \alpha_2\theta_0^3 - 2\,\alpha_2\theta_0^2\right)A_0A_2$$
$$- \left(\theta_0^3 - 5\,\theta_0^2 + 6\,\theta_0\right)A_1B_3 + 2\Big(\left(\alpha_1\theta_0^3 - 2\,\alpha_1\theta_0^2\right)A_0 - \left(\theta_0^3 - 2\,\theta_0^2\right)A_2\Big)B_1$$
$$+ 2\Big(\left(\theta_0^3 - 2\,\theta_0^2 + 3\,\theta_0 - 6\right)A_1|A_1|^2 + \left(\alpha_5\theta_0^3 - 3\,\alpha_5\theta_0^2 + 2\,\alpha_5\theta_0\right)A_0\Big)C_1\Big\}$$
$$= -\frac{1}{\theta_0}\Big\{-2\left(\theta_0^3 - 6\theta_0^2 + 11\theta_0 - 6\right)A_1C_1|A_1|^2 - \left(\theta_0^3 - 5\theta_0^2 + 6\theta_0\right)A_1B_3 - 2(\theta_0^3 - 2\theta_0^2)A_2B_1$$
$$- 2\left(\theta_0^3 - 2\theta_0^2 + 3\theta_0 - 6\right)A_1C_1|A_1|^2\Big\}.$$

Recall that

$$\begin{cases}\vec{B}_1 = -2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0\\ \vec{B}_2 = -\dfrac{(\theta_0+2)}{4\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0} - 2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\vec{A}_0\\ \vec{B}_3 = -2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_1 + \dfrac{2}{\theta_0-3}\overline{\langle\vec{A}_1,\vec{C}_1\rangle}\overline{\vec{A}_1} - 2\langle\overline{\vec{A}_1},\vec{C}_2\rangle\vec{A}_0\\ \vec{E}_1 = -\dfrac{1}{2\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_0}.\end{cases}$$

We first have

$$\langle\vec{A}_1,\vec{B}_3\rangle = -2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_1,\vec{A}_1\rangle + \frac{2}{\theta_0-3}|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle$$

$$\langle \vec{A}_2, \vec{B}_1 \rangle = -2\langle \overline{\vec{A}_1}, \vec{C}_1 \rangle \langle \vec{A}_0, \vec{A}_2 \rangle = \langle \overline{\vec{A}_1}, \vec{C}_1 \rangle \langle \vec{A}_1, \vec{A}_1 \rangle$$

and finally

$$\begin{aligned}
\omega_0 &= -\frac{1}{\theta_0}\Big\{ -2\left(\theta_0^3 - 6\theta_0^2 + 11\theta_0 - 6\right) A_1 C_1 |A_1|^2 - \left(\theta_0^3 - 5\theta_0^2 + 6\theta_0\right) A_1 B_3 - 2(\theta_0^3 - 2\theta_0^2) A_2 B_1 \\
&\quad - 2\left(\theta_0^3 - 2\theta_0^2 + 3\theta_0 - 6\right) A_1 C_1 |A_1|^2 \Big\} \\
&= -\frac{1}{\theta_0}\Big\{ \left(-2\left(\theta_0^3 - 6\theta_0^2 + 11\theta_0 - 6\right) - \left(\theta_0^3 - 5\theta_0^2 + 6\theta_0\right)\frac{2}{\theta_0 - 3} + 2\left(\theta_0^3 - 2\theta_0^2 + 3\theta_0 - 6\right)\right) |\vec{A}_1|^2 \langle \vec{A}_1, \vec{C}_1 \rangle \\
&\quad + 2\left(-(\theta_0^3 - 5\theta_0^2 + 6\theta_0) + (\theta_0^3 - 2\theta_0)^2\right) \langle \overline{\vec{A}_1}, \vec{C}_1 \rangle \langle \vec{A}_1, \vec{A}_1 \rangle \Big\} \\
&= -\frac{1}{\theta_0}\left\{ 6\theta_0(\theta_0 - 2)|\vec{A}_1|^2 \langle \vec{A}_1, \vec{C}_1 \rangle - 6\theta(\theta_0 - 2)\langle \overline{\vec{A}_1}, \vec{C}_1 \rangle \right\} \\
&= -6(\theta_0 - 2)\left(|\vec{A}_1|^2 \langle \vec{A}_1, \vec{C}_1 \rangle - \langle \overline{\vec{A}_1}, \vec{C}_1 \rangle \langle \vec{A}_1, \vec{A}_1 \rangle \right) \\
&= 0
\end{aligned}$$

so

$$|\vec{A}_1|^2 \langle \vec{A}_1, \vec{C}_1 \rangle = \langle \overline{\vec{A}_1}, \vec{C}_1 \rangle \langle \vec{A}_1, \vec{A}_1 \rangle \tag{2.5.1}$$

In particular, if $\vec{A}_1 \neq 0$, we have by Cauchy-Schwarz inequality $|\langle \vec{A}_1, \vec{A}_1 \rangle| \leq |\vec{A}_1|^2$, so we obtain

$$\left| \langle \vec{A}_1, \vec{C}_1 \rangle \right| = \left| \langle \overline{\vec{A}_1}, \vec{C}_1 \rangle \frac{\langle \vec{A}_1, \vec{A}_1 \rangle}{|\vec{A}_1|^2} \right| = |\langle \overline{\vec{A}_1}, \vec{C}_1 \rangle| \frac{|\langle \vec{A}_1, \vec{A}_1 \rangle|}{|\vec{A}_1|^2} \leq |\langle \overline{\vec{A}_1}, \vec{C}_1 \rangle|. \tag{2.5.2}$$

As $\vec{A}_1 = 0$ would finish the proof, we can safely assume that (2.5.2) holds.

2.6 Another conservation law

We check that the scaling invariance is trivial, as it is algebraically trivial. Here, if

$$2i\,\partial\vec{L}=\vec{\alpha}=\partial\vec{H}+|\vec{H}|^2\partial\vec{\Phi}+2\,g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle+\partial\left(\vec{\gamma_0}\log|z|\right),$$

it corresponds to

$$0=\langle\partial_z\vec{L},\partial_{\overline{z}}\vec{\Phi}\rangle=\begin{pmatrix}
\left(\overline{A_0B_1}\right)\left(\frac{1}{2}\right) & 0 & 1\\
\left(\overline{A_0A_0}\right)\left(\left(2\,A_1\overline{C_1}\right)\right) & 0 & 1\\
\left(\overline{A_0B_3}\right)\left(\frac{1}{2}\right) & 0 & 2\\
\left(\overline{A_1B_1}\right)\left(\frac{1}{2}\right) & 0 & 2\\
\left(\overline{A_0A_1}\right)\left(\left(2\,A_1\overline{C_1}\right)\right) & 0 & 2\\
\left(\overline{A_0A_1}\right)\left(\left(2\,A_1\overline{C_1}\right)\right) & 0 & 2\\
\left(\overline{A_0A_0}\right)\left(\left(-4\,A_0|A_1|^2\overline{C_1}+2\,A_1\overline{C_2}\right)\right) & 0 & 2\\
\left(\overline{A_0B_2}\right)() & 1 & 1\\
\left(A_0\overline{A_0}\right)\left(\left(\frac{1}{2}\,C_1\overline{C_1}\right)\right) & 1 & 1\\
\left(\overline{A_0A_0}\right)\left(\left(-4\,A_0\alpha_1\overline{C_1}+2\,A_1\overline{B_1}+4\,A_2\overline{C_1}\right)\right) & 1 & 1\\
\left(\left(-\frac{1}{2}\,\theta_0+1\right)\left(\frac{1}{4\,\theta_0}\right)\right)\left(C_1\overline{C_1}\right) & 1 & 1\\
\left(E_1\overline{A_0}\right)\left(\left(-\theta_0+2\right)\right) & -2\,\theta_0+3 & 2\,\theta_0-1\\
\left(\overline{A_0A_0}\right)\left(\left(-\frac{C_1^2(\theta_0-2)}{2\,\theta_0}\right)\right) & -2\,\theta_0+3 & 2\,\theta_0-1\\
\left(C_1\overline{A_1}\right)\left(\left(-\frac{1}{2}\,\theta_0+1\right)\right) & -\theta_0+1 & \theta_0\\
\left(B_1\overline{A_0}\right)\left(\left(-\frac{1}{2}\,\theta_0+1\right)\right) & -\theta_0+1 & \theta_0\\
\left(C_1\overline{A_0}\right)\left(\left(-\frac{1}{2}\,\theta_0+1\right)\right) & -\theta_0+1 & \theta_0-1\\
\left(C_1\overline{A_2}\right)\left(\left(-\frac{1}{2}\,\theta_0+1\right)\right) & -\theta_0+1 & \theta_0+1\\
\left(B_1\overline{A_1}\right)\left(\left(-\frac{1}{2}\,\theta_0+1\right)\right) & -\theta_0+1 & \theta_0+1\\
\left(B_2\overline{A_0}\right)\left(\left(-\frac{1}{2}\,\theta_0+1\right)\right) & -\theta_0+1 & \theta_0+1\\
\left(\overline{A_0A_0}\right)\left(\left(-\frac{C_1(\theta_0-2)\overline{C_1}}{2\,\theta_0}\right)\right) & -\theta_0+1 & \theta_0+1
\end{pmatrix}$$

$$
\begin{pmatrix}
(C_2\overline{A_1})\left(\left(-\frac{1}{2}\,\theta_0+\frac{3}{2}\right)\right) & -\theta_0+2 & \theta_0 \\
(B_3\overline{A_0})\left(\left(-\frac{1}{2}\,\theta_0+\frac{3}{2}\right)\right) & -\theta_0+2 & \theta_0 \\
(\overline{A_0A_1})\,((2\,A_1C_1)) & -\theta_0+2 & \theta_0 \\
(\overline{A_0A_1})\,((2\,A_1C_1)) & -\theta_0+2 & \theta_0 \\
(\overline{A_0A_0})\left(\left(-4\,A_0C_1|A_1|^2+2\,A_1B_1\right)\right) & -\theta_0+2 & \theta_0 \\
(C_2\overline{A_0})\left(\left(-\frac{1}{2}\,\theta_0+\frac{3}{2}\right)\right) & -\theta_0+2 & \theta_0-1 \\
(\overline{A_0A_0})\,((2\,A_1C_1)) & -\theta_0+2 & \theta_0-1 \\
(C_1C_1)\left(\frac{1}{8}\left(-\frac{1}{2}\,\theta_0+1\right)\right) & -\theta_0+3 & \theta_0-1 \\
(C_3\overline{A_0})\left(\left(-\frac{1}{2}\,\theta_0+2\right)\right) & -\theta_0+3 & \theta_0-1 \\
(A_0\overline{A_0})\left(\left(\frac{1}{4}\,C_1^2\right)\right) & -\theta_0+3 & \theta_0-1 \\
(\overline{A_0A_0})\,((-4\,A_0C_1\alpha_1+4\,A_2C_1+2\,A_1C_2)) & -\theta_0+3 & \theta_0-1 \\
(\overline{A_0E_1})\left(\left(\frac{1}{2}\,\theta_0\right)\right) & \theta_0-1 & -\theta_0+3 \\
(A_0\overline{A_0})\left(\left(\frac{1}{4}\,\overline{C_1}^2\right)\right) & \theta_0-1 & -\theta_0+3
\end{pmatrix}
$$

We fist look at

$$
\Gamma_1=\begin{pmatrix}
(\overline{A_0B_3})\left(\frac{1}{2}\right) & 0 & 2 \\
(\overline{A_1B_1})\left(\frac{1}{2}\right) & 0 & 2 \\
(\overline{A_0A_1})\left(\left(2\,A_1\overline{C_1}\right)\right) & 0 & 2 \\
(\overline{A_0A_1})\left(\left(2\,A_1\overline{C_1}\right)\right) & 0 & 2 \\
(\overline{A_0A_0})\left(\left(-4\,A_0|A_1|^2\overline{C_1}+2\,A_1\overline{C_2}\right)\right) & 0 & 2
\end{pmatrix}=0
$$

As $\langle\vec{A}_0,\vec{A}_0\rangle=0$ and $\langle\vec{A}_0,\vec{A}_1\rangle=\langle\overline{\vec{A}_0},\vec{A}_1\rangle=0$, and $\vec{B}_1\in\mathrm{Span}(\vec{A}_0)$, we have

$$
\Gamma_1=\frac{1}{2}\overline{\langle\vec{A}_0,\vec{B}_3\rangle}=0
$$

but

$$
\vec{B}_3=-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_1+\frac{2}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_1}-2\langle\overline{\vec{A}_1},\vec{C}_2\rangle\vec{A}_0
$$

so

$$
\langle\vec{A}_0,\vec{B}_3\rangle=0
$$

Now, we have

$$
\langle\overline{\vec{A}_0},\vec{B}_3\rangle=-\langle\overline{\vec{A}_1},\vec{C}_2\rangle.
$$

so everything is in fact trivial.

2.7 Next order developments of tensors

Up to harmonic factors, we have

$$
\vec{H} = \begin{pmatrix}
\frac{\overline{A_1 C_1}}{\theta_0 - 3} & A_0 & 0 & -\theta_0 + 3 \\
-\frac{2\left(\overline{A_0}\overline{\alpha_1} - \overline{A_2}\right)\overline{C_1} - \overline{A_1 C_2}}{\theta_0 - 4} & A_0 & 0 & -\theta_0 + 4 \\
\frac{\overline{C_1}^2}{8\left(\theta_0 - 4\right)} & \overline{A_0} & 0 & -\theta_0 + 4 \\
-\frac{3\left(\overline{A_1}\overline{\alpha_1} + \overline{A_0}\overline{\alpha_3} - \overline{A_3}\right)\overline{C_1} + 2\left(\overline{A_0}\overline{\alpha_1} - \overline{A_2}\right)\overline{C_2} - \overline{A_1 C_3}}{\theta_0 - 5} & A_0 & 0 & -\theta_0 + 5 \\
\frac{\overline{C_1 C_2}}{4\left(\theta_0 - 5\right)} & \overline{A_0} & 0 & -\theta_0 + 5 \\
\frac{\overline{C_1}^2}{8\left(\theta_0 - 5\right)} & \overline{A_1} & 0 & -\theta_0 + 5 \\
\frac{\overline{A_1 C_1}}{8\left(\theta_0 - 5\right)} & \overline{C_1} & 0 & -\theta_0 + 5 \\
-A_1\overline{C_1} & \overline{A_0} & 1 & -\theta_0 + 2 \\
-\frac{2\,|A_1|^2\overline{A_0 C_1} - \overline{A_1 B_1}}{\theta_0 - 3} & A_0 & 1 & -\theta_0 + 3 \\
\frac{\overline{A_1 C_1}}{\theta_0 - 3} & A_1 & 1 & -\theta_0 + 3 \\
-A_1\overline{C_1} & \overline{A_1} & 1 & -\theta_0 + 3 \\
2\,A_0|A_1|^2\overline{C_1} - A_1\overline{C_2} & \overline{A_0} & 1 & -\theta_0 + 3 \\
-\frac{2\left(\overline{A_0}\overline{\alpha_1} - \overline{A_2}\right)\overline{C_1} - \overline{A_1 C_2}}{\theta_0 - 4} & A_1 & 1 & -\theta_0 + 4 \\
-\frac{2\,|A_1|^2\overline{A_0 C_2} + 2\left(\overline{A_0}\overline{\alpha_1} - \overline{A_2}\right)\overline{B_1} - \overline{A_1 B_3} + 2\left(2\,|A_1|^2\overline{A_1} + \overline{A_0}\overline{\alpha_5}\right)\overline{C_1}}{\theta_0 - 4} & A_0 & 1 & -\theta_0 + 4 \\
\frac{16\,A_0(\theta_0 - 4)|A_1|^2\overline{C_2} - 8\,A_1(\theta_0 - 4)\overline{C_3} + \left(8\left(\theta_0\overline{\alpha_5} - 4\,\overline{\alpha_5}\right)A_0 + 8\left(\theta_0\overline{\alpha_1} - 4\,\overline{\alpha_1}\right)A_1 - (\theta_0 - 6)\overline{B_1}\right)\overline{C_1}}{8\left(\theta_0 - 4\right)} & \overline{A_0} & 1 & -\theta_0 + 4 \\
-A_1\overline{C_1} & \overline{A_2} & 1 & -\theta_0 + 4 \\
2\,A_0|A_1|^2\overline{C_1} - A_1\overline{C_2} & \overline{A_1} & 1 & -\theta_0 + 4 \\
-\frac{C_1(\theta_0 + 2)\overline{C_1}}{8\,\theta_0} & A_0 & 2 & -\theta_0 + 2 \\
-\frac{1}{2}\,A_1\overline{B_1} + (A_0\alpha_1 - A_2)\overline{C_1} & \overline{A_0} & 2 & -\theta_0 + 2 \\
-\frac{2\,|A_1|^2\overline{A_0 C_1} - \overline{A_1 B_1}}{\theta_0 - 3} & A_1 & 2 & -\theta_0 + 3 \\
-\frac{16\,\theta_0|A_1|^2\overline{A_0 B_1} - 8\,\theta_0\overline{A_1 B_2} + \left(\theta_0^2 - \theta_0 - 6\right)C_1\overline{C_2} + \left(8\,\alpha_5\theta_0\overline{A_0} + 8\,\alpha_1\theta_0\overline{A_1} + \left(\theta_0^2 - 2\,\theta_0 - 4\right)B_1\right)\overline{C_1}}{8\left(\theta_0^2 - 3\,\theta_0\right)} & A_0 & 2 & -\theta_0 + 3 \\
\frac{\overline{A_1 C_1}}{\theta_0 - 3} & A_2 & 2 & -\theta_0 + 3 \\
-\frac{1}{2}\,A_1\overline{B_1} + (A_0\alpha_1 - A_2)\overline{C_1} & \overline{A_1} & 2 & -\theta_0 + 3 \\
A_0|A_1|^2\overline{B_1} - \frac{1}{2}\,A_1\overline{B_3} + \left(2\,A_1|A_1|^2 + A_0\alpha_5\right)\overline{C_1} + (A_0\alpha_1 - A_2)\overline{C_2} & \overline{A_0} & 2 & -\theta_0 + 3
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{\left(\theta_0^2+4\,\theta_0+3\right)C_2\overline{C_1}-\left(12\left(\theta_0^2\overline{\alpha_2}+\theta_0\overline{\alpha_2}\right)\overline{A_0}-\left(\theta_0^2+4\,\theta_0\right)\overline{B_1}\right)C_1}{12\left(\theta_0^2+\theta_0\right)} & A_0 & 3 & -\theta_0+2 \\
-\frac{A_1C_1}{12\,\theta_0} & \overline{C_1} & 3 & -\theta_0+2 \\
-\frac{C_1(\theta_0+3)\overline{C_1}}{12\,\theta_0} & A_1 & 3 & -\theta_0+2 \\
-\frac{1}{24}\,A_1\overline{C_1} & C_1 & 3 & -\theta_0+2 \\
\frac{1}{3}\left(\theta_0\overline{\alpha_2}+\overline{\alpha_2}\right)A_0C_1+\frac{2}{3}\left(A_0\alpha_1-A_2\right)\overline{B_1}-\frac{1}{3}\,A_1\overline{B_2}+\left(A_1\alpha_1+A_0\alpha_3-A_3\right)\overline{C_1} & \overline{A_0} & 3 & -\theta_0+2 \\
-\frac{C_1^2}{4\,\theta_0} & \overline{A_0} & -2\,\theta_0+4 & \theta_0 \\
\frac{(\alpha_2\theta_0+\alpha_2)C_1\overline{A_0}-E_1\overline{A_1}}{\theta_0+1} & A_0 & -2\,\theta_0+4 & \theta_0+1 \\
-\frac{C_1\overline{A_1}}{4\left(\theta_0^2+\theta_0\right)} & C_1 & -2\,\theta_0+4 & \theta_0+1 \\
\frac{\left(4\left(\alpha_2\theta_0^2+\alpha_2\theta_0\right)A_0-B_1(4\,\theta_0+1)\right)C_1}{8\left(\theta_0^2+\theta_0\right)} & \overline{A_0} & -2\,\theta_0+4 & \theta_0+1 \\
-\frac{C_1^2(2\,\theta_0+1)}{8\left(\theta_0^2+\theta_0\right)} & \overline{A_1} & -2\,\theta_0+4 & \theta_0+1 \\
-\frac{C_1C_2(2\,\theta_0-5)-2\,A_1E_1\theta_0}{2\left(2\,\theta_0^2-5\,\theta_0\right)} & \overline{A_0} & -2\,\theta_0+5 & \theta_0 \\
-C_1\overline{A_1} & A_0 & -\theta_0+2 & 1 \\
-\frac{C_1(\theta_0+2)\overline{C_1}}{8\,\theta_0} & \overline{A_0} & -\theta_0+2 & 2 \\
\left(\overline{A_0}\,\overline{\alpha_1}-\overline{A_2}\right)C_1-\frac{1}{2}\,B_1\overline{A_1} & A_0 & -\theta_0+2 & 2 \\
-\frac{\left(\theta_0^2+4\,\theta_0+3\right)C_1\overline{C_2}-\left(12\left(\alpha_2\theta_0^2+\alpha_2\theta_0\right)A_0-\left(\theta_0^2+4\,\theta_0\right)B_1\right)\overline{C_1}}{12\left(\theta_0^2+\theta_0\right)} & \overline{A_0} & -\theta_0+2 & 3 \\
-\frac{\overline{A_1C_1}}{12\,\theta_0} & C_1 & -\theta_0+2 & 3 \\
-\frac{C_1(\theta_0+3)\overline{C_1}}{12\,\theta_0} & \overline{A_1} & -\theta_0+2 & 3 \\
-\frac{1}{24}\,C_1\overline{A_1} & \overline{C_1} & -\theta_0+2 & 3 \\
\frac{1}{3}\left(\alpha_2\theta_0+\alpha_2\right)\overline{A_0C_1}+\frac{2}{3}\left(\overline{A_0}\,\overline{\alpha_1}-\overline{A_2}\right)B_1+\left(\overline{A_1}\,\overline{\alpha_1}+\overline{A_0}\,\overline{\alpha_3}-\overline{A_3}\right)C_1-\frac{1}{3}\,B_2\overline{A_1} & A_0 & -\theta_0+2 & 3 \\
\frac{A_1C_1}{\theta_0-3} & \overline{A_0} & -\theta_0+3 & 0 \\
-\frac{2\,A_0C_1|A_1|^2-A_1B_1}{\theta_0-3} & \overline{A_0} & -\theta_0+3 & 1 \\
\frac{A_1C_1}{\theta_0-3} & \overline{A_1} & -\theta_0+3 & 1 \\
-C_1\overline{A_1} & A_1 & -\theta_0+3 & 1 \\
2\,C_1|A_1|^2\overline{A_0}-C_2\overline{A_1} & A_0 & -\theta_0+3 & 1 \\
-\frac{2\,A_0C_1|A_1|^2-A_1B_1}{\theta_0-3} & \overline{A_1} & -\theta_0+3 & 2 \\
-\frac{16\,A_0B_1\theta_0|A_1|^2-8\,A_1B_2\theta_0+\left(\theta_0^2-\theta_0-6\right)C_2\overline{C_1}+\left(8\,A_1\theta_0\overline{\alpha_1}+8\,A_0\theta_0\overline{\alpha_5}+\left(\theta_0^2-2\,\theta_0-4\right)\overline{B_1}\right)C_1}{8\left(\theta_0^2-3\,\theta_0\right)} & \overline{A_0} & -\theta_0+3 & 2 \\
\frac{A_1C_1}{\theta_0-3} & \overline{A_2} & -\theta_0+3 & 2 \\
\left(\overline{A_0}\,\overline{\alpha_1}-\overline{A_2}\right)C_1-\frac{1}{2}\,B_1\overline{A_1} & A_1 & -\theta_0+3 & 2 \\
B_1|A_1|^2\overline{A_0}+\left(2\,|A_1|^2\overline{A_1}+\overline{A_0}\,\overline{\alpha_5}\right)C_1+\left(\overline{A_0}\,\overline{\alpha_1}-\overline{A_2}\right)C_2-\frac{1}{2}\,B_3\overline{A_1} & A_0 & -\theta_0+3 & 2
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{2\,(A_0\alpha_1 - A_2)C_1 - A_1C_2}{\theta_0 - 4} & \overline{A_0} & -\theta_0 + 4 & 0 \\
\frac{C_1^2}{8\,(\theta_0 - 4)} & A_0 & -\theta_0 + 4 & 0 \\
-\frac{2\,(A_0\alpha_1 - A_2)C_1 - A_1C_2}{\theta_0 - 4} & \overline{A_1} & -\theta_0 + 4 & 1 \\
-\frac{2\,A_0C_2|A_1|^2 + 2\,(A_0\alpha_1 - A_2)B_1 - A_1B_3 + 2\,\left(2\,A_1|A_1|^2 + A_0\alpha_5\right)C_1}{\theta_0 - 4} & \overline{A_0} & -\theta_0 + 4 & 1 \\
\frac{16\,C_2(\theta_0 - 4)|A_1|^2\overline{A_0} - 8\,C_3(\theta_0 - 4)\overline{A_1} - \left(B_1(\theta_0 - 6) - 8\,(\alpha_5\theta_0 - 4\,\alpha_5)\overline{A_0} - 8\,(\alpha_1\theta_0 - 4\,\alpha_1)\overline{A_1}\right)C_1}{8\,(\theta_0 - 4)} & A_0 & -\theta_0 + 4 & 1 \\
-C_1\overline{A_1} & A_2 & -\theta_0 + 4 & 1 \\
2\,C_1|A_1|^2\overline{A_0} - C_2\overline{A_1} & A_1 & -\theta_0 + 4 & 1 \\
-\frac{3\,(A_1\alpha_1 + A_0\alpha_3 - A_3)C_1 + 2\,(A_0\alpha_1 - A_2)C_2 - A_1C_3}{\theta_0 - 5} & \overline{A_0} & -\theta_0 + 5 & 0 \\
\frac{C_1C_2}{4\,(\theta_0 - 5)} & A_0 & -\theta_0 + 5 & 0 \\
\frac{C_1^2}{8\,(\theta_0 - 5)} & A_1 & -\theta_0 + 5 & 0 \\
\frac{A_1C_1}{8\,(\theta_0 - 5)} & C_1 & -\theta_0 + 5 & 0 \\
-\frac{\overline{C_1}^2}{4\,\theta_0} & A_0 & \theta_0 & -2\,\theta_0 + 4 \\
-\frac{(2\,\theta_0 - 5)\overline{C_1C_2} - 2\,\theta_0\overline{A_1E_1}}{2\,(2\,\theta_0^2 - 5\,\theta_0)} & A_0 & \theta_0 & -2\,\theta_0 + 5 \\
\frac{(\theta_0\overline{\alpha_2} + \overline{\alpha_2})A_0\overline{C_1} - A_1\overline{E_1}}{\theta_0 + 1} & \overline{A_0} & \theta_0 + 1 & -2\,\theta_0 + 4 \\
-\frac{A_1\overline{C_1}}{4\,(\theta_0^2 + \theta_0)} & \overline{C_1} & \theta_0 + 1 & -2\,\theta_0 + 4 \\
\frac{\left(4\,\left(\theta_0^2\overline{\alpha_2} + \theta_0\overline{\alpha_2}\right)\overline{A_0} - (4\,\theta_0 + 1)\overline{B_1}\right)\overline{C_1}}{8\,(\theta_0^2 + \theta_0)} & A_0 & \theta_0 + 1 & -2\,\theta_0 + 4 \\
-\frac{(2\,\theta_0 + 1)\overline{C_1}^2}{8\,(\theta_0^2 + \theta_0)} & A_1 & \theta_0 + 1 & -2\,\theta_0 + 4
\end{pmatrix}
$$

which we translate for some $\vec{C}_4, \vec{B}_4, \vec{B}_5, \vec{B}_6, \vec{E}_2, \vec{E}_3 \in \mathbb{C}^n$ as

$$\vec{H} = \begin{pmatrix} \frac{1}{2} & C_1 & -\theta_0+2 & 0 \\ \frac{1}{2} & C_2 & -\theta_0+3 & 0 \\ \frac{1}{2} & C_3 & -\theta_0+4 & 0 \\ \frac{1}{2} & C_4 & -\theta_0+5 & 0 \\ \frac{1}{2} & B_1 & -\theta_0+2 & 1 \\ \frac{1}{2} & B_2 & -\theta_0+2 & 2 \\ \frac{1}{2} & B_3 & -\theta_0+3 & 1 \\ \frac{1}{2} & B_4 & -\theta_0+2 & 3 \\ \frac{1}{2} & B_5 & -\theta_0+3 & 2 \\ \frac{1}{2} & B_6 & -\theta_0+4 & 1 \\ \frac{1}{2} & E_1 & -2\,\theta_0+4 & \theta_0 \\ \frac{1}{2} & E_2 & -2\,\theta_0+4 & \theta_0+1 \\ \frac{1}{2} & E_3 & -2\,\theta_0+5 & \theta_0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \overline{C_1} & 0 & -\theta_0+2 \\ \frac{1}{2} & \overline{C_2} & 0 & -\theta_0+3 \\ \frac{1}{2} & \overline{C_3} & 0 & -\theta_0+4 \\ \frac{1}{2} & \overline{C_4} & 0 & -\theta_0+5 \\ \frac{1}{2} & \overline{B_1} & 1 & -\theta_0+2 \\ \frac{1}{2} & \overline{B_2} & 2 & -\theta_0+2 \\ \frac{1}{2} & \overline{B_3} & 1 & -\theta_0+3 \\ \frac{1}{2} & \overline{B_4} & 3 & -\theta_0+2 \\ \frac{1}{2} & \overline{B_5} & 2 & -\theta_0+3 \\ \frac{1}{2} & \overline{B_6} & 1 & -\theta_0+4 \\ \frac{1}{2} & \overline{E_1} & \theta_0 & -2\,\theta_0+4 \\ \frac{1}{2} & \overline{E_2} & \theta_0+1 & -2\,\theta_0+4 \\ \frac{1}{2} & \overline{E_3} & \theta_0 & -2\,\theta_0+5 \end{pmatrix}$$

We have as

$$\alpha_2 = \frac{1}{2\theta_0(\theta_0+1)}\langle \overline{\vec{A}_1}, \vec{C}_1\rangle, \quad \alpha_7 = \frac{1}{8\theta_0(\theta_0-4)}\langle \vec{C}_1, \vec{C}_1\rangle$$

word

$$\begin{aligned} \frac{1}{2}\vec{E}_2 &= \begin{pmatrix} \frac{(\alpha_2\theta_0+\alpha_2)C_1\overline{A_0} - E_1\overline{A_1}}{\theta_0+1} & A_0 & -2\,\theta_0+4 & \theta_0+1 \\ -\frac{C_1\overline{A_1}}{4\,(\theta_0^2+\theta_0)} & C_1 & -2\,\theta_0+4 & \theta_0+1 \\ \frac{\left(4\left(\alpha_2\theta_0^2+\alpha_2\theta_0\right)A_0 - B_1(4\,\theta_0+1)\right)C_1}{8\,(\theta_0^2+\theta_0)} & \overline{A_0} & -2\,\theta_0+4 & \theta_0+1 \\ -\frac{C_1^2(2\,\theta_0+1)}{8\,(\theta_0^2+\theta_0)} & \overline{A_1} & -2\,\theta_0+4 & \theta_0+1 \end{pmatrix} \\ &= -\frac{\alpha_2}{2}\vec{C}_1 - \frac{(2\theta_0+1)(\theta_0-4)}{\theta_0+1}\alpha_7\overline{\vec{A}_1} \end{aligned}$$

so

$$\vec{E}_2 = -\alpha_2\vec{C}_1 - \frac{2(2\theta_0+1)(\theta_0-4)}{\theta_0+1}\alpha_7\overline{\vec{A}_1} \tag{2.7.1}$$

Finally,

$$\begin{aligned} \frac{1}{2}\vec{E}_3 &= \begin{pmatrix} -\frac{C_1C_2(2\,\theta_0-5) - 2\,A_1E_1\theta_0}{2\,(2\,\theta_0^2-5\,\theta_0)} & \overline{A_0} & -2\,\theta_0+5 & \theta_0 \end{pmatrix} \\ &= -\frac{1}{2\theta_0}\langle \vec{C}_1, \vec{C}_2\rangle\overline{\vec{A}_0} \end{aligned}$$

So

$$\begin{cases} \vec{E}_2 = -\alpha_2\,\vec{C}_1 - \dfrac{2(2\theta_0+1)(\theta_0-4)}{\theta_0+1}\alpha_7\overline{\vec{A}_1} \\ \vec{E}_3 = -\dfrac{1}{\theta_0}\langle\vec{C}_1,\vec{C}_2\rangle\overline{\vec{A}_0} \end{cases}$$

Now, we have as $\vec{B}_1 \in \mathrm{Span}(\vec{A}_0)$, and $\vec{B}_2 \in \mathrm{Span}(\vec{A}_0, \overline{\vec{A}_0})$

$$\frac{1}{2}\vec{B}_4 = \begin{pmatrix} -\dfrac{(\theta_0^2+4\,\theta_0+3)C_1\overline{C_2} - \left(\cancel{12\,(\alpha_2\theta_0^2+\alpha_2\theta_0)A_0} - \cancel{(\theta_0^2+4\,\theta_0)B_1}\right)\overline{C_1}}{12\,(\theta_0^2+\theta_0)} & \overline{A_0} & -\theta_0+2 & 3 \\ -\dfrac{\overline{A_1C_1}}{12\,\theta_0} & C_1 & -\theta_0+2 & 3 \\ -\dfrac{C_1(\theta_0+3)\overline{C_1}}{12\,\theta_0} & \overline{A_1} & -\theta_0+2 & 3 \\ -\dfrac{1}{24}\,C_1\overline{A_1} & \overline{C_1} & -\theta_0+2 & 3 \\ \dfrac{1}{3}\,\cancel{(\alpha_2\theta_0+\alpha_2)\overline{A_0C_1}} + \dfrac{2}{3}\,(\overline{A_0}\overline{\alpha_1}-\overline{A_2})B_1 + \left(\overline{A_1}\overline{\alpha_1}+\cancel{\overline{A_0}\overline{\alpha_3}}-\overline{A_3}\right)C_1 - \cancel{\dfrac{1}{3}\,B_2\overline{A_1}} & A_0 & -\theta_0+2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{(\theta_0+3)C_1\overline{C_2}}{12\,\theta_0} & \overline{A_0} & -\theta_0+2 & 3 \\ -\dfrac{\overline{\zeta_2}}{12\,\theta_0} & C_1 & -\theta_0+2 & 3 \\ -\dfrac{(\theta_0+3)|C_1|^2}{12\,\theta_0} & \overline{A_1} & -\theta_0+2 & 3 \\ -\dfrac{\theta_0(\theta_0+1)}{12}\alpha_2 & \overline{C_1} & -\theta_0+2 & 3 \\ \dfrac{2}{3}\,(\overline{A_0}\overline{\alpha_1}-\overline{A_2})B_1 + (\overline{A_1}\overline{\alpha_1}-\overline{A_3})C_1 & A_0 & -\theta_0+2 & 3 \end{pmatrix}$$

as

$$\begin{cases} \alpha_2 = \dfrac{1}{2\theta_0(\theta_0+1)}\langle\overline{\vec{A}_1},\vec{C}_1\rangle \\ \zeta_2 = \langle\vec{A}_1,\vec{C}_1\rangle. \end{cases}$$

As $\vec{B}_1 = -2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0$, and

$$\langle\overline{\alpha_1}\overline{\vec{A}_0} - \overline{\vec{A}_2}, \vec{A}_0\rangle = \frac{1}{2}\left(\overline{\alpha_1} - \langle\vec{A}_0,\overline{\vec{A}_2}\rangle\right) = 0$$
$$\langle\overline{\alpha_1}\overline{\vec{A}_0} - \overline{\vec{A}_2}, \vec{B}_1\rangle = 0$$

so finally

$$\vec{B}_4 = -\frac{(\theta_0+3)}{6\theta_0}\langle\vec{C}_1,\overline{\vec{C}_2}\rangle\overline{\vec{A}_0} - \frac{\overline{\zeta_2}}{6\theta_0}\vec{C}_1 - \frac{(\theta_0+3)}{6\theta_0}|\vec{C}_1|^2\overline{\vec{A}_1} - \frac{\theta_0(\theta_0+1)}{6}\alpha_2\overline{\vec{C}_1} + 2\left(\overline{\alpha_1}\langle\overline{\vec{A}_1},\vec{C}_1\rangle - \langle\overline{\vec{A}_3},\vec{C}_1\rangle\right)\vec{A}_0.$$

Then, we have

$$\frac{1}{2}\vec{B}_5 =$$

$$\begin{pmatrix} -\dfrac{2\,\cancel{A_0C_1|A_1|^2} - \cancel{A_1B_1}}{\theta_0-3} & \overline{A_1} & -\theta_0+3 & 2 \\ -\dfrac{\cancel{16\,A_0B_1\theta_0|A_1|^2} - \cancel{8\,A_1B_2\theta_0} + \left(\theta_0^2-\theta_0-6\right)C_2\overline{C_1} + \left(8\,A_1\theta_0\overline{\alpha_1} + \cancel{8\,A_0\theta_0\overline{\alpha_5}} + \cancel{\left(\theta_0^2-2\,\theta_0-4\right)\overline{B_1}}\right)C_1}{8\,(\theta_0^2-3\,\theta_0)} & \overline{A_0} & -\theta_0+3 & 2 \\ \dfrac{A_1C_1}{\theta_0-3} & \overline{A_2} & -\theta_0+3 & 2 \\ \left(\cancel{\overline{A_0}\overline{\alpha_1}} - \overline{A_2}\right)C_1 - \cancel{\dfrac{1}{2}\,B_1\overline{A_1}} & A_1 & -\theta_0+3 & 2 \\ B_1|A_1|^2\overline{A_0} + \left(2\,|A_1|^2\overline{A_1} + \cancel{\overline{A_0}\overline{\alpha_5}}\right)C_1 + \left(\cancel{\overline{A_0}\overline{\alpha_1}} - \overline{A_2}\right)C_2 - \dfrac{1}{2}\,B_3\overline{A_1} & A_0 & -\theta_0+3 & 2 \end{pmatrix}$$

$$
= \begin{pmatrix} -\frac{\left(\theta_0^2-\theta_0-6\right)C_2\overline{C_1}+8\,\theta_0\overline{\alpha_1}\zeta_2}{8\,(\theta_0^2-3\,\theta_0)} & \overline{A_0} & -\theta_0+3 & 2 \\ \frac{\zeta_2}{\theta_0-3} & \overline{A_2} & -\theta_0+3 & 2 \\ -\overline{A_2}C_1 & A_1 & -\theta_0+3 & 2 \\ B_1|A_1|^2\overline{A_0}+2\,|A_1|^2\overline{A_1}C_1-\overline{A_2}C_2-\frac{1}{2}\,B_3\overline{A_1} & A_0 & -\theta_0+3 & 2 \end{pmatrix}
$$

As

$$
\begin{cases} \vec{B}_1 = -2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0 \\ \vec{B}_2 = -\frac{(\theta_0+2)}{4\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0}-2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\vec{A}_0 \\ \vec{B}_3 = -2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_1+\frac{2}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_1}-2\langle\overline{\vec{A}_1},\vec{C}_2\rangle\vec{A}_0 \\ \vec{E}_1 = -\frac{1}{2\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_0}. \end{cases}
$$

we have

$$
\begin{aligned}
&\langle\overline{\vec{A}_0},\vec{B}_1\rangle = -\langle\overline{\vec{A}_1},\vec{C}_1\rangle \\
&\langle\overline{\vec{A}_1},\vec{B}_3\rangle = -2|\vec{A}_1|^2\langle\overline{\vec{A}_1},\vec{C}_1\rangle+\frac{2}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\langle\vec{A}_1,\vec{A}_1\rangle} \\
&|\vec{A}_1|^2\langle\overline{\vec{A}_0},\vec{B}_1\rangle-\frac{1}{2}\langle\overline{\vec{A}_1},\vec{B}_3\rangle = -|\vec{A}_1|^2\langle\overline{\vec{A}_1},\vec{C}_1\rangle-\frac{1}{2}\left(-2|\vec{A}_1|^2\langle\overline{\vec{A}_1},\vec{C}_1\rangle+\frac{2}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\langle\vec{A}_1,\vec{A}_1\rangle}\right) \\
&= -\frac{1}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\langle\vec{A}_1,\vec{A}_1\rangle} \\
&= -\frac{1}{\theta_0-3}\overline{\zeta_0}\zeta_2
\end{aligned}
$$

where

$$
\zeta_0 = \langle\vec{A}_1,\vec{A}_1\rangle
$$

so we finally have as

$$
\alpha_2 = \frac{1}{2\theta_0(\theta_0+1)}\langle\overline{\vec{A}_1},\vec{C}_1\rangle
$$

the identity

$$
\frac{1}{2}\vec{B}_5 = \begin{pmatrix} -\frac{\left(\theta_0^2-\theta_0-6\right)C_2\overline{C_1}+8\,\theta_0\overline{\alpha_1}\zeta_2}{8\,(\theta_0^2-3\,\theta_0)} & \overline{A_0} & -\theta_0+3 & 2 \\ \frac{\zeta_2}{\theta_0-3} & \overline{A_2} & -\theta_0+3 & 2 \\ -\overline{A_2}C_1 & A_1 & -\theta_0+3 & 2 \\ 4\theta_0(\theta_0+1)\,|A_1|^2\alpha_2-\overline{A_2}C_2-\frac{1}{\theta_0-3}\overline{\zeta_0}\zeta_2 & A_0 & -\theta_0+3 & 2 \end{pmatrix}
$$

and finally

$$
\vec{B}_5 = -\left(\frac{(\theta_0+2)}{4}\langle\overline{\vec{C}_1},\vec{C}_2\rangle+\frac{2}{\theta_0-3}\overline{\alpha_1}\zeta_2\right)\overline{\vec{A}_0}+\frac{2\,\zeta_2}{\theta_0-3}\overline{\vec{A}_2}-2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\vec{A}_1
$$

$$+\left(8\theta_0(\theta_0+1)|\vec{A}_1|^2\alpha_2 - 2\langle\overline{\vec{A}_2},\vec{C}_2\rangle - \frac{2}{\theta_0-3}\overline{\zeta_0}\zeta_2\right)\vec{A}_0.$$

Finally, we have

$$\frac{1}{2}\vec{B}_6 =$$

$$\begin{pmatrix}
-\frac{2\,(A_0\alpha_1 - A_2)C_1 - A_1C_2}{\theta_0-4} & \overline{A_1} & -\theta_0+4 & 1\\
-\frac{2\,A_0C_2|A_1|^2 + 2\,(A_0\alpha_1 - A_2)B_1 - A_1B_3 + 2\left(2\,A_1|A_1|^2 + A_0\alpha_5\right)C_1}{\theta_0-4} & \overline{A_0} & -\theta_0+4 & 1\\
\frac{16\,C_2(\theta_0-4)|A_1|^2\overline{A_0} - 8\,C_3(\theta_0-4)\overline{A_1} - \left(B_1(\theta_0-6) - 8\,(\alpha_5\theta_0 - 4\,\alpha_5)\overline{A_0} - 8\,(\alpha_1\theta_0 - 4\,\alpha_1)\overline{A_1}\right)C_1}{8\,(\theta_0-4)} & A_0 & -\theta_0+4 & 1\\
-C_1\overline{A_1} & A_2 & -\theta_0+4 & 1\\
2\,C_1|A_1|^2\overline{A_0} - C_2\overline{A_1} & A_1 & -\theta_0+4 & 1
\end{pmatrix}$$

$$= \begin{pmatrix}
\frac{2A_2C_1 + A_1C_2}{\theta_0-4} & \overline{A_1} & -\theta_0+4 & 1\\
-\frac{-2A_2B_1 - A_1B_3 + 2\,|A_1|^2A_1C_1}{\theta_0-4} & \overline{A_0} & -\theta_0+4 & 1\\
-\overline{A_1}C_3 + \alpha_1\overline{A_1}C_1 & A_0 & -\theta_0+4 & 1\\
-2\theta_0(\theta_0+1)\alpha_2 & A_2 & -\theta_0+4 & 1\\
-C_2\overline{A_1} & A_1 & -\theta_0+4 & 1
\end{pmatrix}$$

We have

$$\begin{aligned}
2\langle\vec{A}_2,\vec{B}_1\rangle + \langle\vec{A}_1,\vec{B}_3\rangle &= -4\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_0,\vec{A}_2\rangle - 2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_1,\vec{A}_1\rangle + \frac{2}{\theta_0-3}|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle\\
&= \frac{2}{\theta_0-3}|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle
\end{aligned}$$

so we finally have

$$-\frac{-2A_2B_1 - A_1B_3 + 2\,|A_1|^2A_1C_1}{\theta_0-4} = -\frac{1}{\theta_0-4}\left(2 - \frac{2}{\theta_0-3}\right)|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle \tag{2.7.2}$$

$$= -\frac{2}{\theta_0-3}|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle \tag{2.7.3}$$

and

$$\vec{B}_6 = \frac{2\,\zeta_5}{\theta_0-4}\overline{\vec{A}_1} - \frac{4}{\theta_0-3}|\vec{A}_1|^2\zeta_2\overline{\vec{A}_0} + \left(-2\langle\overline{\vec{A}_1},\vec{C}_3\rangle + 4\theta_0(\theta_0+1)\alpha_1\alpha_2\right)\vec{A}_0 - 4\theta_0(\theta_0+1)\alpha_2\vec{A}_2 - 2\langle\overline{\vec{A}_1},C_2\rangle\vec{A}_1.$$

Finally,

$$\begin{cases}
\vec{B}_4 = -\frac{(\theta_0+3)}{6\theta_0}\langle\vec{C}_1,\overline{\vec{C}_2}\rangle\overline{\vec{A}_0} - \frac{\overline{\zeta_2}}{6\theta_0}\vec{C}_1 - \frac{(\theta_0+3)}{6\theta_0}|\vec{C}_1|^2\overline{\vec{A}_1} - \frac{\theta_0(\theta_0+1)}{6}\alpha_2\overline{\vec{C}_1} + 2\left(\overline{\alpha_1}\langle\overline{\vec{A}_1},\vec{C}_1\rangle - \langle\overline{\vec{A}_3},\vec{C}_1\rangle\right)\vec{A}_0.\\
\vec{B}_5 = -\left(\frac{(\theta_0+2)}{4}\langle\overline{\vec{C}_1},\vec{C}_2\rangle + \frac{2}{\theta_0-3}\overline{\alpha_1}\zeta_2\right)\overline{\vec{A}_0} + \frac{2\,\zeta_2}{\theta_0-3}\overline{\vec{A}_2} - 2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\vec{A}_1\\
\quad + \left(8\theta_0(\theta_0+1)|\vec{A}_1|^2\alpha_2 - 2\langle\overline{\vec{A}_2},\vec{C}_2\rangle - \frac{2}{\theta_0-3}\overline{\zeta_0}\zeta_2\right)\vec{A}_0\\
\vec{B}_6 = \frac{2\,\zeta_5}{\theta_0-4}\overline{\vec{A}_1} - \frac{4}{\theta_0-3}|\vec{A}_1|^2\zeta_2\overline{\vec{A}_0} + \left(-2\langle\overline{\vec{A}_1},\vec{C}_3\rangle + 4\theta_0(\theta_0+1)\alpha_1\alpha_2\right)\vec{A}_0 - 4\theta_0(\theta_0+1)\alpha_2\vec{A}_2 - 2\langle\overline{\vec{A}_1},C_2\rangle\vec{A}_1.
\end{cases}$$

$$
\partial_z \vec{\Phi} = \begin{pmatrix}
1 & A_0 & \theta_0 - 1 & 0 \\
1 & A_1 & \theta_0 & 0 \\
1 & A_2 & \theta_0 + 1 & 0 \\
1 & A_3 & \theta_0 + 2 & 0 \\
1 & A_4 & \theta_0 + 3 & 0 \\
1 & A_5 & \theta_0 + 4 & 0 \\
\frac{1}{4\,\theta_0} & C_1 & 1 & \theta_0 \\
\frac{1}{4\,(\theta_0+1)} & B_1 & 1 & \theta_0 + 1 \\
\frac{1}{4\,(\theta_0+2)} & B_2 & 1 & \theta_0 + 2 \\
\frac{\overline{\alpha_1}}{4\,(\theta_0+2)} & C_1 & 1 & \theta_0 + 2 \\
\frac{1}{4\,(\theta_0+3)} & B_4 & 1 & \theta_0 + 3 \\
\frac{\overline{\alpha_3}}{4\,(\theta_0+3)} & C_1 & 1 & \theta_0 + 3 \\
\frac{\overline{\alpha_1}}{4\,(\theta_0+3)} & B_1 & 1 & \theta_0 + 3 \\
\frac{\alpha_2}{4\,(\theta_0+3)} & \overline{C_1} & 1 & \theta_0 + 3 \\
\frac{1}{4\,\theta_0} & C_2 & 2 & \theta_0 \\
\frac{1}{4\,(\theta_0+1)} & B_3 & 2 & \theta_0 + 1 \\
\frac{|A_1|^2}{2\,(\theta_0+1)} & C_1 & 2 & \theta_0 + 1 \\
\frac{1}{4\,(\theta_0+2)} & B_5 & 2 & \theta_0 + 2 \\
\frac{\overline{\alpha_5}}{4\,(\theta_0+2)} & C_1 & 2 & \theta_0 + 2 \\
\frac{\overline{\alpha_1}}{4\,(\theta_0+2)} & C_2 & 2 & \theta_0 + 2 \\
\frac{|A_1|^2}{2\,(\theta_0+2)} & B_1 & 2 & \theta_0 + 2 \\
\frac{1}{4\,\theta_0} & C_3 & 3 & \theta_0 \\
\frac{\alpha_1}{4\,\theta_0} & C_1 & 3 & \theta_0 \\
\frac{1}{4\,(\theta_0+1)} & B_6 & 3 & \theta_0 + 1 \\
\frac{\alpha_5}{4\,(\theta_0+1)} & C_1 & 3 & \theta_0 + 1 \\
\frac{\alpha_1}{4\,(\theta_0+1)} & B_1 & 3 & \theta_0 + 1
\end{pmatrix}
\begin{pmatrix}
\frac{|A_1|^2}{2\,(\theta_0+1)} & C_2 & 3 & \theta_0 + 1 \\
\frac{1}{4\,\theta_0} & C_4 & 4 & \theta_0 \\
\frac{\alpha_3}{4\,\theta_0} & C_1 & 4 & \theta_0 \\
\frac{\alpha_1}{4\,\theta_0} & C_2 & 4 & \theta_0 \\
\frac{1}{4\,(2\,\theta_0+1)} & E_2 & -\theta_0 + 3 & 2\,\theta_0 + 1 \\
\frac{\alpha_2}{4\,(2\,\theta_0+1)} & C_1 & -\theta_0 + 3 & 2\,\theta_0 + 1 \\
\frac{1}{8\,\theta_0} & E_1 & -\theta_0 + 3 & 2\,\theta_0 \\
\frac{1}{8\,\theta_0} & E_3 & -\theta_0 + 4 & 2\,\theta_0 \\
\frac{1}{8} & \overline{B_1} & \theta_0 & 2 \\
\frac{1}{12} & \overline{B_3} & \theta_0 & 3 \\
\frac{1}{6}\,|A_1|^2 & \overline{C_1} & \theta_0 & 3 \\
\frac{1}{16} & \overline{B_6} & \theta_0 & 4 \\
\frac{1}{16}\,\overline{\alpha_5} & \overline{C_1} & \theta_0 & 4 \\
\frac{1}{16}\,\overline{\alpha_1} & \overline{B_1} & \theta_0 & 4 \\
\frac{1}{8}\,|A_1|^2 & \overline{C_2} & \theta_0 & 4 \\
\frac{1}{8} & \overline{C_1} & \theta_0 - 1 & 2 \\
\frac{1}{12} & \overline{C_2} & \theta_0 - 1 & 3 \\
\frac{1}{16} & \overline{C_3} & \theta_0 - 1 & 4 \\
\frac{1}{16}\,\overline{\alpha_1} & \overline{C_1} & \theta_0 - 1 & 4 \\
\frac{1}{20} & \overline{C_4} & \theta_0 - 1 & 5 \\
\frac{1}{20}\,\overline{\alpha_3} & \overline{C_1} & \theta_0 - 1 & 5 \\
\frac{1}{20}\,\overline{\alpha_1} & \overline{C_2} & \theta_0 - 1 & 5 \\
\frac{1}{8} & \overline{B_2} & \theta_0 + 1 & 2 \\
\frac{1}{8}\,\alpha_1 & \overline{C_1} & \theta_0 + 1 & 2 \\
\frac{1}{12} & \overline{B_5} & \theta_0 + 1 & 3 \\
\frac{1}{12}\,\alpha_5 & \overline{C_1} & \theta_0 + 1 & 3 \\
\frac{1}{12}\,\alpha_1 & \overline{C_2} & \theta_0 + 1 & 3
\end{pmatrix}
$$

$$\begin{pmatrix} \frac{1}{6}\,|A_1|^2 & \overline{B_1} & \theta_0+1 & 3 \\ \frac{1}{8} & \overline{B_4} & \theta_0+2 & 2 \\ \frac{1}{8}\,\overline{\alpha_2} & C_1 & \theta_0+2 & 2 \\ \frac{1}{8}\,\alpha_3 & \overline{C_1} & \theta_0+2 & 2 \\ \frac{1}{8}\,\alpha_1 & \overline{B_1} & \theta_0+2 & 2 \\ -\frac{1}{4\,(\theta_0-4)} & \overline{E_1} & 2\,\theta_0-1 & -\theta_0+4 \\ -\frac{1}{4\,(\theta_0-5)} & \overline{E_3} & 2\,\theta_0-1 & -\theta_0+5 \\ -\frac{1}{4\,(\theta_0-4)} & \overline{E_2} & 2\,\theta_0 & -\theta_0+4 \\ -\frac{\overline{\alpha_2}}{4\,(\theta_0-4)} & \overline{C_1} & 2\,\theta_0 & -\theta_0+4 \end{pmatrix} \tag{2.7.4}$$

$0 = \langle \partial_z \vec{\Phi}, \partial_z \vec{\Phi} \rangle =$

$$
\begin{pmatrix}
\frac{4\left(\theta_0^2+\theta_0\right)A_0E_2+\left(4\left(\alpha_2\theta_0^2+\alpha_2\theta_0\right)A_0+B_1(2\,\theta_0+1)\right)C_1}{8\,(2\,\theta_0^3+3\,\theta_0^2+\theta_0)} & 2 & 2\,\theta_0+1 \\
\frac{4\,A_0E_1\theta_0+C_1^2}{16\,\theta_0^2} & 2 & 2\,\theta_0 \\
\frac{2\,A_1E_1\theta_0+2\,A_0E_3\theta_0+C_1C_2}{8\,\theta_0^2} & 3 & 2\,\theta_0 \\
\frac{A_0C_1}{2\,\theta_0} & \theta_0 & \theta_0 \\
\frac{A_0B_1}{2\,(\theta_0+1)} & \theta_0 & \theta_0+1 \\
\frac{8\,A_0C_1\theta_0\overline{\alpha_1}+8\,A_0B_2\theta_0+C_1(\theta_0+2)\overline{C_1}}{16\,(\theta_0^2+2\,\theta_0)} & \theta_0 & \theta_0+2 \\
\lambda_1 & \theta_0 & \theta_0+3 \\
\frac{A_1C_1+A_0C_2}{2\,\theta_0} & \theta_0+1 & \theta_0 \\
\frac{2\,A_0C_1|A_1|^2+A_1B_1+A_0B_3}{2\,(\theta_0+1)} & \theta_0+1 & \theta_0+1 \\
\lambda_2 & \theta_0+1 & \theta_0+2 \\
\frac{(A_0\alpha_1+A_2)C_1+A_1C_2+A_0C_3}{2\,\theta_0} & \theta_0+2 & \theta_0 \\
\frac{2\,A_0C_2|A_1|^2+(A_0\alpha_1+A_2)B_1+A_1B_3+A_0B_6+\left(2\,A_1|A_1|^2+A_0\alpha_5\right)C_1}{2\,(\theta_0+1)} & \theta_0+2 & \theta_0+1 \\
\frac{(A_1\alpha_1+A_0\alpha_3+A_3)C_1+(A_0\alpha_1+A_2)C_2+A_1C_3+A_0C_4}{2\,\theta_0} & \theta_0+3 & \theta_0 \\
A_0^2 & 2\,\theta_0-2 & 0 \\
\frac{1}{4}\,A_0\overline{C_1} & 2\,\theta_0-2 & 2 \\
\frac{1}{6}\,A_0\overline{C_2} & 2\,\theta_0-2 & 3 \\
\frac{1}{8}\,A_0\overline{C_1}\,\overline{\alpha_1}+\frac{1}{64}\,\overline{C_1}^2+\frac{1}{8}\,A_0\overline{C_3} & 2\,\theta_0-2 & 4 \\
\frac{1}{10}\,A_0\overline{C_1}\,\overline{\alpha_3}+\frac{1}{240}\left(24\,A_0\overline{\alpha_1}+5\,\overline{C_1}\right)\overline{C_2}+\frac{1}{10}\,A_0\overline{C_4} & 2\,\theta_0-2 & 5 \\
2\,A_0A_1 & 2\,\theta_0-1 & 0 \\
\frac{1}{4}\,A_0\overline{B_1}+\frac{1}{4}\,A_1\overline{C_1} & 2\,\theta_0-1 & 2 \\
\frac{1}{3}\,A_0|A_1|^2\overline{C_1}+\frac{1}{6}\,A_0\overline{B_3}+\frac{1}{6}\,A_1\overline{C_2} & 2\,\theta_0-1 & 3 \\
\frac{1}{4}\,A_0|A_1|^2\overline{C_2}+\frac{1}{8}\,A_0\overline{B_1}\,\overline{\alpha_1}+\frac{1}{8}\,A_0\overline{B_6}+\frac{1}{32}\left(4\,A_1\overline{\alpha_1}+4\,A_0\overline{\alpha_5}+\overline{B_1}\right)\overline{C_1}+\frac{1}{8}\,A_1\overline{C_3} & 2\,\theta_0-1 & 4 \\
2\,A_1A_2+2\,A_0A_3 & 2\,\theta_0+1 & 0 \\
\frac{1}{4}\,A_0C_1\overline{\alpha_2}+\frac{1}{4}\,(A_0\alpha_1+A_2)\overline{B_1}+\frac{1}{4}\,A_1\overline{B_2}+\frac{1}{4}\,A_0\overline{B_4}+\frac{1}{4}\,(A_1\alpha_1+A_0\alpha_3+A_3)\overline{C_1} & 2\,\theta_0+1 & 2 \\
A_2^2+2\,A_1A_3+2\,A_0A_4 & 2\,\theta_0+2 & 0
\end{pmatrix}
$$

$$\begin{pmatrix} 2\,A_2A_3 + 2\,A_1A_4 + 2\,A_0A_5 & 2\,\theta_0 + 3 & 0 \\ -\frac{A_0\overline{E_1}}{2\,(\theta_0 - 4)} & 3\,\theta_0 - 2 & -\theta_0 + 4 \\ -\frac{A_0\overline{E_3}}{2\,(\theta_0 - 5)} & 3\,\theta_0 - 2 & -\theta_0 + 5 \\ -\frac{A_0\overline{C_1}\,\overline{\alpha_2} + A_1\overline{E_1} + A_0\overline{E_2}}{2\,(\theta_0 - 4)} & 3\,\theta_0 - 1 & -\theta_0 + 4 \\ A_1^2 + 2\,A_0A_2 & 2\,\theta_0 & 0 \\ \frac{1}{4}\,A_1\overline{B_1} + \frac{1}{4}\,A_0\overline{B_2} + \frac{1}{4}\,(A_0\alpha_1 + A_2)\overline{C_1} & 2\,\theta_0 & 2 \\ \frac{1}{3}\,A_0|A_1|^2\overline{B_1} + \frac{1}{6}\,A_1\overline{B_3} + \frac{1}{6}\,A_0\overline{B_5} + \frac{1}{6}\left(2\,A_1|A_1|^2 + A_0\alpha_5\right)\overline{C_1} + \frac{1}{6}\,(A_0\alpha_1 + A_2)\overline{C_2} & 2\,\theta_0 & 3 \end{pmatrix}$$

where

$$\lambda_1 = \frac{1}{48\,(\theta_0^3 + 4\,\theta_0^2 + 3\,\theta_0)}\Big\{24\left(\theta_0^2\overline{\alpha_1} + \theta_0\overline{\alpha_1}\right)A_0B_1 + 24\left(\theta_0^2 + \theta_0\right)A_0B_4 + 24\left(\theta_0^2\overline{\alpha_3} + \theta_0\overline{\alpha_3}\right)A_0C_1$$

$$+ 2\left(\theta_0^2 + 4\,\theta_0 + 3\right)C_1\overline{C_2} + 3\left(8\left(\alpha_2\theta_0^2 + \alpha_2\theta_0\right)A_0 + \left(\theta_0^2 + 3\,\theta_0\right)B_1\right)\overline{C_1}\Big\}$$

$$\lambda_2 = \frac{1}{16\,(\theta_0^2 + 2\,\theta_0)}\Big\{16\,A_0B_1\theta_0|A_1|^2 + 8\,A_1B_2\theta_0 + 8\,A_0B_5\theta_0 + \left(8\,A_1\theta_0\overline{\alpha_1} + 8\,A_0\theta_0\overline{\alpha_5} + (\theta_0 + 2)\overline{B_1}\right)C_1$$

$$+ \left(8\,A_0\theta_0\overline{\alpha_1} + (\theta_0 + 2)\overline{C_1}\right)C_2\Big\}$$

or

$$
e^{2\lambda} = \begin{pmatrix}
\frac{C_1\overline{A_0}}{2\,\theta_0} & 1 & 2\,\theta_0 - 1 \\
\frac{\left(\theta_0^2 + \theta_0\right)B_2\overline{A_0} + \left(\theta_0^2 + 2\,\theta_0\right)B_1\overline{A_1} + \left(\left(\theta_0^2\overline{\alpha_1} + \theta_0\overline{\alpha_1}\right)\overline{A_0} + \left(\theta_0^2 + 3\,\theta_0 + 2\right)\overline{A_2}\right)C_1}{2\,(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0)} & 1 & 2\,\theta_0 + 1 \\
\mu_1 & 1 & 2\,\theta_0 + 2 \\
\frac{B_1\theta_0\overline{A_0} + C_1(\theta_0 + 1)\overline{A_1}}{2\,(\theta_0^2 + \theta_0)} & 1 & 2\,\theta_0 \\
\frac{C_2\overline{A_0}}{2\,\theta_0} & 2 & 2\,\theta_0 - 1 \\
\mu_2 & 2 & 2\,\theta_0 + 1 \\
\frac{2\,C_1\theta_0|A_1|^2\overline{A_0} + B_3\theta_0\overline{A_0} + C_2(\theta_0 + 1)\overline{A_1}}{2\,(\theta_0^2 + \theta_0)} & 2 & 2\,\theta_0 \\
\frac{C_1^2(\theta_0 - 4) - 8\,A_0E_1\theta_0 + 8\,(\alpha_1\theta_0 - 4\,\alpha_1)C_1\overline{A_0} + 8\,C_3(\theta_0 - 4)\overline{A_0}}{16\,(\theta_0^2 - 4\,\theta_0)} & 3 & 2\,\theta_0 - 1 \\
\mu_3 & 3 & 2\,\theta_0 \\
\mu_4 & 4 & 2\,\theta_0 - 1 \\
\frac{E_1\overline{A_0}}{4\,\theta_0} & -\theta_0 + 3 & 3\,\theta_0 - 1 \\
\frac{2\,C_1\alpha_2\theta_0\overline{A_0} + 2\,E_2\theta_0\overline{A_0} + E_1(2\,\theta_0 + 1)\overline{A_1}}{4\,(2\,\theta_0^2 + \theta_0)} & -\theta_0 + 3 & 3\,\theta_0 \\
\frac{E_3\overline{A_0}}{4\,\theta_0} & -\theta_0 + 4 & 3\,\theta_0 - 1 \\
2\,A_1\overline{A_1} & \theta_0 & \theta_0 \\
2\,A_1\overline{A_0} & \theta_0 & \theta_0 - 1 \\
2\,A_1\overline{A_2} + \frac{1}{4}\,\overline{A_0B_1} & \theta_0 & \theta_0 + 1 \\
\frac{1}{3}\,|A_1|^2\overline{A_0C_1} + 2\,A_1\overline{A_3} + \frac{1}{4}\,\overline{A_1B_1} + \frac{1}{6}\,\overline{A_0B_3} & \theta_0 & \theta_0 + 2 \\
\frac{1}{4}\,|A_1|^2\overline{A_0C_2} + 2\,A_1\overline{A_4} + \frac{1}{8}\,\left(\overline{A_0}\overline{\alpha_1} + 2\,\overline{A_2}\right)\overline{B_1} + \frac{1}{6}\,\overline{A_1B_3} + \frac{1}{8}\,\overline{A_0B_6} + \frac{1}{24}\,\left(8\,|A_1|^2\overline{A_1} + 3\,\overline{A_0}\overline{\alpha_5}\right)\overline{C_1} & \theta_0 & \theta_0 + 3 \\
2\,A_0\overline{A_1} & \theta_0 - 1 & \theta_0 \\
2\,A_0\overline{A_0} & \theta_0 - 1 & \theta_0 - 1 \\
2\,A_0\overline{A_2} + \frac{1}{4}\,\overline{A_0C_1} & \theta_0 - 1 & \theta_0 + 1 \\
2\,A_0\overline{A_3} + \frac{1}{4}\,\overline{A_1C_1} + \frac{1}{6}\,\overline{A_0C_2} & \theta_0 - 1 & \theta_0 + 2 \\
2\,A_0\overline{A_4} + \frac{1}{8}\,\left(\overline{A_0}\overline{\alpha_1} + 2\,\overline{A_2}\right)\overline{C_1} + \frac{1}{6}\,\overline{A_1C_2} + \frac{1}{8}\,\overline{A_0C_3} & \theta_0 - 1 & \theta_0 + 3 \\
2\,A_0\overline{A_5} + \frac{1}{40}\,\left(5\,\overline{A_1}\overline{\alpha_1} + 4\,\overline{A_0}\overline{\alpha_3} + 10\,\overline{A_3}\right)\overline{C_1} + \frac{1}{30}\,\left(3\,\overline{A_0}\overline{\alpha_1} + 5\,\overline{A_2}\right)\overline{C_2} + \frac{1}{8}\,\overline{A_1C_3} + \frac{1}{10}\,\overline{A_0C_4} & \theta_0 - 1 & \theta_0 + 4 \\
\frac{1}{4}\,A_0B_1 + 2\,A_2\overline{A_1} & \theta_0 + 1 & \theta_0 \\
\frac{1}{4}\,A_0C_1 + 2\,A_2\overline{A_0} & \theta_0 + 1 & \theta_0 - 1 \\
\frac{8\,A_0C_1\theta_0^2\overline{\alpha_1} + 8\,A_0B_2\theta_0^2 + 64\,A_2\theta_0^2\overline{A_2} + 8\,\theta_0^2\overline{A_0B_2} + \left(8\,\alpha_1\theta_0^2\overline{A_0} + \left(\theta_0^2 + 4\right)C_1\right)\overline{C_1}}{32\,\theta_0^2} & \theta_0 + 1 & \theta_0 + 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
\mu_5 & \theta_0 + 1 & \theta_0 + 2 \\
\frac{1}{3}\,A_0 C_1 |A_1|^2 + \frac{1}{4}\,A_1 B_1 + \frac{1}{6}\,A_0 B_3 + 2\,A_3 \overline{A_1} & \theta_0 + 2 & \theta_0 \\
\frac{1}{4}\,A_1 C_1 + \frac{1}{6}\,A_0 C_2 + 2\,A_3 \overline{A_0} & \theta_0 + 2 & \theta_0 - 1 \\
\mu_6 & \theta_0 + 2 & \theta_0 + 1 \\
\frac{1}{4}\,A_0 C_2 |A_1|^2 + \frac{1}{8}\,(A_0\alpha_1 + 2\,A_2) B_1 + \frac{1}{6}\,A_1 B_3 + \frac{1}{8}\,A_0 B_6 + \frac{1}{24}\,\left(8\,A_1 |A_1|^2 + 3\,A_0 \alpha_5\right) C_1 + 2\,A_4 \overline{A_1} & \theta_0 + 3 & \theta_0 \\
\frac{1}{8}\,(A_0\alpha_1 + 2\,A_2) C_1 + \frac{1}{6}\,A_1 C_2 + \frac{1}{8}\,A_0 C_3 + 2\,A_4 \overline{A_0} & \theta_0 + 3 & \theta_0 - 1 \\
\frac{1}{40}\,(5\,A_1\alpha_1 + 4\,A_0\alpha_3 + 10\,A_3) C_1 + \frac{1}{30}\,(3\,A_0\alpha_1 + 5\,A_2) C_2 + \frac{1}{8}\,A_1 C_3 + \frac{1}{10}\,A_0 C_4 + 2\,A_5 \overline{A_0} & \theta_0 + 4 & \theta_0 - 1 \\
\frac{A_0 \overline{C_1}}{2\,\theta_0} & 2\,\theta_0 - 1 & 1 \\
\frac{A_0 \overline{C_2}}{2\,\theta_0} & 2\,\theta_0 - 1 & 2 \\
\frac{8\,(\theta_0 \overline{\alpha_1} - 4\,\overline{\alpha_1}) A_0 \overline{C_1} + (\theta_0 - 4) \overline{C_1}^2 + 8\,A_0 (\theta_0 - 4) \overline{C_3} - 8\,\theta_0 \overline{A_0 E_1}}{16\,(\theta_0^2 - 4\,\theta_0)} & 2\,\theta_0 - 1 & 3 \\
\mu_7 & 2\,\theta_0 - 1 & 4 \\
\frac{\left(\theta_0^2 + 2\,\theta_0\right) A_1 \overline{B_1} + \left(\theta_0^2 + \theta_0\right) A_0 \overline{B_2} + \left(\left(\alpha_1\theta_0^2 + \alpha_1\theta_0\right) A_0 + \left(\theta_0^2 + 3\,\theta_0 + 2\right) A_2\right) \overline{C_1}}{2\,(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0)} & 2\,\theta_0 + 1 & 1 \\
\mu_8 & 2\,\theta_0 + 1 & 2 \\
\mu_9 & 2\,\theta_0 + 2 & 1 \\
\frac{A_0 \overline{E_1}}{4\,\theta_0} & 3\,\theta_0 - 1 & -\theta_0 + 3 \\
\frac{A_0 \overline{E_3}}{4\,\theta_0} & 3\,\theta_0 - 1 & -\theta_0 + 4 \\
\frac{A_0 \theta_0 \overline{B_1} + A_1 (\theta_0 + 1) \overline{C_1}}{2\,(\theta_0^2 + \theta_0)} & 2\,\theta_0 & 1 \\
\frac{2\,A_0 \theta_0 |A_1|^2 \overline{C_1} + A_0 \theta_0 \overline{B_3} + A_1 (\theta_0 + 1) \overline{C_2}}{2\,(\theta_0^2 + \theta_0)} & 2\,\theta_0 & 2 \\
\mu_{10} & 2\,\theta_0 & 3 \\
\frac{2\,A_0 \theta_0 \overline{C_1}\,\overline{\alpha_2} + A_1 (2\,\theta_0 + 1) \overline{E_1} + 2\,A_0 \theta_0 \overline{E_2}}{4\,(2\,\theta_0^2 + \theta_0)} & 3\,\theta_0 & -\theta_0 + 3
\end{pmatrix}
$$

where

$$
\mu_1 = \frac{1}{2\,(\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0)} \Bigg\{ \left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right) B_4 \overline{A_0} + \left(\theta_0^3 + 4\,\theta_0^2 + 3\,\theta_0\right) B_2 \overline{A_1} + \left(\alpha_2\theta_0^3 + 3\,\alpha_2\theta_0^2 + 2\,\alpha_2\theta_0\right) \overline{A_0 C_1}
$$

$$
+ \left(\left(\theta_0^3 \overline{\alpha_1} + 3\,\theta_0^2 \overline{\alpha_1} + 2\,\theta_0 \overline{\alpha_1}\right) \overline{A_0} + \left(\theta_0^3 + 5\,\theta_0^2 + 6\,\theta_0\right) \overline{A_2}\right) B_1 + \Bigg(\left(\theta_0^3 \overline{\alpha_3} + 3\,\theta_0^2 \overline{\alpha_3} + 2\,\theta_0 \overline{\alpha_3}\right) \overline{A_0}
$$

$$
+ \left(\theta_0^3 \overline{\alpha_1} + 4\,\theta_0^2 \overline{\alpha_1} + 3\,\theta_0 \overline{\alpha_1}\right) \overline{A_1} + \left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right) \overline{A_3} \Bigg) C_1 \Bigg\}
$$

$$
\mu_2 = \frac{1}{2\,(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0)} \Bigg\{ 2\,\left(\theta_0^2 + \theta_0\right) B_1 |A_1|^2 \overline{A_0} + \left(\theta_0^2 + \theta_0\right) B_5 \overline{A_0} + \left(\theta_0^2 + 2\,\theta_0\right) B_3 \overline{A_1}
$$

$$
+ \left(2\,\left(\theta_0^2 + 2\,\theta_0\right) |A_1|^2 \overline{A_1} + \left(\theta_0^2 \overline{\alpha_5} + \theta_0 \overline{\alpha_5}\right) \overline{A_0}\right) C_1 + \left(\left(\theta_0^2 \overline{\alpha_1} + \theta_0 \overline{\alpha_1}\right) \overline{A_0} + \left(\theta_0^2 + 3\,\theta_0 + 2\right) \overline{A_2}\right) C_2 \Bigg\}
$$

$$
\mu_3 = \frac{1}{16\,(\theta_0^3 - 3\,\theta_0^2 - 4\,\theta_0)} \Bigg\{ 16\,\left(\theta_0^2 - 4\,\theta_0\right) C_2 |A_1|^2 \overline{A_0} - 8\,\left(\theta_0^2 + \theta_0\right) A_0 E_2 + 8\,\left(\alpha_1\theta_0^2 - 4\,\alpha_1\theta_0\right) B_1 \overline{A_0} + 8\,\left(\theta_0^2 - 4\,\theta_0\right) B_6 \overline{A_0}
$$

$$
+ 8\,\left(\theta_0^2 - 3\,\theta_0 - 4\right) C_3 \overline{A_1} - \left(8\,\left(\alpha_2\theta_0^2 + \alpha_2\theta_0\right) A_0 - \left(2\,\theta_0^2 - 7\,\theta_0 - 4\right) B_1 - 8\,\left(\alpha_5\theta_0^2 - 4\,\alpha_5\theta_0\right) \overline{A_0} - 8\,\left(\alpha_1\theta_0^2 - 3\,\alpha_1\theta_0 - 4\,\alpha_1\right) \overline{A_1}\right) C_1 \Bigg\}
$$

$$\mu_4 = -\frac{1}{48\left(\theta_0^3 - 9\,\theta_0^2 + 20\,\theta_0\right)}\Big\{24\left(\theta_0^2 - 5\,\theta_0\right)A_1E_1 + 24\left(\theta_0^2 - 4\,\theta_0\right)A_0E_3 - 24\left(\alpha_3\theta_0^2 - 9\,\alpha_3\theta_0 + 20\,\alpha_3\right)C_1\overline{A_0}$$

$$- 24\left(\theta_0^2 - 9\,\theta_0 + 20\right)C_4\overline{A_0} - \left(5\left(\theta_0^2 - 9\,\theta_0 + 20\right)C_1 + 24\left(\alpha_1\theta_0^2 - 9\,\alpha_1\theta_0 + 20\,\alpha_1\right)\overline{A_0}\right)C_2\Big\}$$

$$\mu_5 = \frac{1}{96\left(\theta_0^3 + \theta_0^2\right)}\Big\{32\left(\theta_0^3 + \theta_0^2\right)|A_1|^2\overline{A_0B_1} + 24\left(\theta_0^3\overline{\alpha_1} + \theta_0^2\overline{\alpha_1}\right)A_0B_1 + 24\left(\theta_0^3 + \theta_0^2\right)A_0B_4 + 24\left(\theta_0^3\overline{\alpha_3} + \theta_0^2\overline{\alpha_3}\right)A_0C_1$$

$$+ 192\left(\theta_0^3 + \theta_0^2\right)A_2\overline{A_3} + 24\left(\theta_0^3 + \theta_0^2\right)\overline{A_1B_2} + 16\left(\theta_0^3 + \theta_0^2\right)\overline{A_0B_5} + \Big(24\left(\alpha_2\theta_0^3 + \alpha_2\theta_0^2\right)A_0 + 3\left(\theta_0^3 + \theta_0^2 + 4\,\theta_0\right)B_1$$

$$+ 16\left(\alpha_5\theta_0^3 + \alpha_5\theta_0^2\right)\overline{A_0} + 24\left(\alpha_1\theta_0^3 + \alpha_1\theta_0^2\right)\overline{A_1}\Big)\overline{C_1} + 2\left(\left(\theta_0^3 + \theta_0^2 + 6\,\theta_0 + 6\right)C_1 + 8\left(\alpha_1\theta_0^3 + \alpha_1\theta_0^2\right)\overline{A_0}\right)\overline{C_2}\Big\}$$

$$\mu_6 = \frac{1}{96\left(\theta_0^3 + \theta_0^2\right)}\Big\{32\left(\theta_0^3 + \theta_0^2\right)A_0B_1|A_1|^2 + 24\left(\theta_0^3 + \theta_0^2\right)A_1B_2 + 16\left(\theta_0^3 + \theta_0^2\right)A_0B_5 + 192\left(\theta_0^3 + \theta_0^2\right)A_3\overline{A_2}$$

$$+ 24\left(\alpha_1\theta_0^3 + \alpha_1\theta_0^2\right)\overline{A_0B_1} + 24\left(\theta_0^3 + \theta_0^2\right)\overline{A_0B_4} + 24\left(\alpha_3\theta_0^3 + \alpha_3\theta_0^2\right)\overline{A_0C_1} + \Big(16\left(\theta_0^3\overline{\alpha_5} + \theta_0^2\overline{\alpha_5}\right)A_0$$

$$+ 24\left(\theta_0^3\overline{\alpha_1} + \theta_0^2\overline{\alpha_1}\right)A_1 + 24\left(\theta_0^3\overline{\alpha_2} + \theta_0^2\overline{\alpha_2}\right)\overline{A_0} + 3\left(\theta_0^3 + \theta_0^2 + 4\,\theta_0\right)\overline{B_1}\Big)C_1$$

$$+ 2\left(8\left(\theta_0^3\overline{\alpha_1} + \theta_0^2\overline{\alpha_1}\right)A_0 + \left(\theta_0^3 + \theta_0^2 + 6\,\theta_0 + 6\right)\overline{C_1}\right)C_2\Big\}$$

$$\mu_7 = \frac{1}{48\left(\theta_0^3 - 9\,\theta_0^2 + 20\,\theta_0\right)}\Big\{24\left(\theta_0^2\overline{\alpha_3} - 9\,\theta_0\overline{\alpha_3} + 20\,\overline{\alpha_3}\right)A_0\overline{C_1} + 24\left(\theta_0^2 - 9\,\theta_0 + 20\right)A_0\overline{C_4} - 24\left(\theta_0^2 - 5\,\theta_0\right)\overline{A_1E_1}$$

$$- 24\left(\theta_0^2 - 4\,\theta_0\right)\overline{A_0E_3} + \left(24\left(\theta_0^2\overline{\alpha_1} - 9\,\theta_0\overline{\alpha_1} + 20\,\overline{\alpha_1}\right)A_0 + 5\left(\theta_0^2 - 9\,\theta_0 + 20\right)\overline{C_1}\right)\overline{C_2}\Big\}$$

$$\mu_8 = \frac{1}{2\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)}\Big\{2\left(\theta_0^2 + \theta_0\right)A_0|A_1|^2\overline{B_1} + \left(\theta_0^2 + 2\,\theta_0\right)A_1\overline{B_3} + \left(\theta_0^2 + \theta_0\right)A_0\overline{B_5}$$

$$+ \left(2\left(\theta_0^2 + 2\,\theta_0\right)A_1|A_1|^2 + \left(\alpha_5\theta_0^2 + \alpha_5\theta_0\right)A_0\right)\overline{C_1} + \left(\left(\alpha_1\theta_0^2 + \alpha_1\theta_0\right)A_0 + \left(\theta_0^2 + 3\,\theta_0 + 2\right)A_2\right)\overline{C_2}\Big\}$$

$$\mu_9 = \frac{1}{2\left(\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0\right)}\Big\{\left(\theta_0^3\overline{\alpha_2} + 3\,\theta_0^2\overline{\alpha_2} + 2\,\theta_0\overline{\alpha_2}\right)A_0C_1 + \left(\theta_0^3 + 4\,\theta_0^2 + 3\,\theta_0\right)A_1\overline{B_2} + \left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)A_0\overline{B_4}$$

$$+ \left(\left(\alpha_1\theta_0^3 + 3\,\alpha_1\theta_0^2 + 2\,\alpha_1\theta_0\right)A_0 + \left(\theta_0^3 + 5\,\theta_0^2 + 6\,\theta_0\right)A_2\right)\overline{B_1} + \Big(\left(\alpha_3\theta_0^3 + 3\,\alpha_3\theta_0^2 + 2\,\alpha_3\theta_0\right)A_0$$

$$+ \left(\alpha_1\theta_0^3 + 4\,\alpha_1\theta_0^2 + 3\,\alpha_1\theta_0\right)A_1 + \left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)A_3\Big)\overline{C_1}\Big\}$$

$$\mu_{10} = \frac{1}{16\left(\theta_0^3 - 3\,\theta_0^2 - 4\,\theta_0\right)}\Big\{16\left(\theta_0^2 - 4\,\theta_0\right)A_0|A_1|^2\overline{C_2} + 8\left(\theta_0^2\overline{\alpha_1} - 4\,\theta_0\overline{\alpha_1}\right)A_0\overline{B_1} + 8\left(\theta_0^2 - 4\,\theta_0\right)A_0\overline{B_6}$$

$$+ 8\left(\theta_0^2 - 3\,\theta_0 - 4\right)A_1\overline{C_3} - 8\left(\theta_0^2 + \theta_0\right)\overline{A_0E_2} + \Big(8\left(\theta_0^2\overline{\alpha_5} - 4\,\theta_0\overline{\alpha_5}\right)A_0 + 8\left(\theta_0^2\overline{\alpha_1} - 3\,\theta_0\overline{\alpha_1} - 4\,\overline{\alpha_1}\right)A_1$$

$$- 8\left(\theta_0^2\overline{\alpha_2} + \theta_0\overline{\alpha_2}\right)\overline{A_0} + \left(2\,\theta_0^2 - 7\,\theta_0 - 4\right)\overline{B_1}\Big)\overline{C_1}\Big\}$$

The precise expression of these μ is irrelevant. The only strange new powers of order $2\theta_0 + 3$ are

$$\begin{pmatrix} \dfrac{2\,E_2\theta_0\overline{A_0} + E_1(2\,\theta_0 + 1)\overline{A_1}}{4\left(2\,\theta_0^2 + \theta_0\right)} & -\theta_0 + 3 & 3\,\theta_0 \\ \dfrac{E_3\overline{A_0}}{4\,\theta_0} & -\theta_0 + 4 & 3\,\theta_0 - 1 \end{pmatrix}$$

and their conjugate. As $\vec{E}_1 \in \mathrm{Span}(\overrightarrow{A_0})$, $\vec{E}_2 \in \mathrm{Span}(\overrightarrow{A_1}, \vec{C}_1)$ and $\vec{E}_3 \in \mathrm{Span}(\overrightarrow{A_0})$, they vanish and there exists $\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \in \mathbb{C}$ such that

$$e^{2\lambda} = |z|^{2\theta_0 - 2} + 2|\vec{A}_1|^2|z|^{2\theta_0} + \beta|z|^{2\theta_0 + 2} + 2\,\mathrm{Re}\,\Big(\alpha_0 z^{\theta_0}\overline{z}^{\theta_0 - 1} + \alpha_1 z^{\theta_0 + 1}\overline{z}^{\theta_0 - 2} + \alpha_2 z\overline{z}^{2\theta_0} + \alpha_3 z^{\theta_0 + 2}\overline{z}^{\theta_0 - 1} + \alpha_4 z^{\theta_0 + 3}\overline{z}^{\theta_0 - 1}$$

$$+ \alpha_5 z^{\theta_0 + 1}\overline{z}^{\theta_0} + \alpha_6 z^{\theta_0 + 2}\overline{z}^{\theta_0} + \alpha_7 z^3\overline{z}^{2\theta_0 - 1} + \alpha_8 z^2\overline{z}^{2\theta_0} + \alpha_9 z\overline{z}^{2\theta_0 + 1}$$

$$+\alpha_{10}z\overline{z}^{2\theta_0+2}+\alpha_{11}z^2\overline{z}^{2\theta_0+1}+\alpha_{12}z^3\overline{z}^{2\theta_0}+\alpha_{13}z^4\overline{z}^{2\theta_0-1}+\alpha_{14}z^{\theta_0-1}\overline{z}^{\theta_0+4}+\alpha_{15}z^{\theta_0}\overline{z}^{\theta_0+3}+\alpha_{16}z^{\theta_0+1}\overline{z}^{\theta_0+2}\Big)+O(|z|^{2\theta_0+4-\varepsilon})\tag{2.7.5}$$

which gives in Sage

$$e^{2\lambda}=\begin{pmatrix} 1 & \theta_0-1 & \theta_0-1 \\ 2\,|A_1|^2 & \theta_0 & \theta_0 \\ \beta & \theta_0+1 & \theta_0+1 \\ \alpha_1 & \theta_0+1 & \theta_0-1 \\ \alpha_2 & 1 & 2\,\theta_0 \\ \alpha_3 & \theta_0+2 & \theta_0-1 \\ \alpha_4 & \theta_0+3 & \theta_0-1 \\ \alpha_5 & \theta_0+1 & \theta_0 \\ \alpha_6 & \theta_0+2 & \theta_0 \\ \alpha_7 & 3 & 2\,\theta_0-1 \\ \alpha_8 & 2 & 2\,\theta_0 \\ \alpha_9 & 1 & 2\,\theta_0+1 \\ \alpha_{10} & 1 & 2\,\theta_0+2 \\ \alpha_{11} & 2 & 2\,\theta_0+1 \\ \alpha_{12} & 3 & 2\,\theta_0 \\ \alpha_{13} & 4 & 2\,\theta_0-1 \\ \alpha_{14} & \theta_0-1 & \theta_0+4 \\ \alpha_{15} & \theta_0 & \theta_0+3 \\ \alpha_{16} & \theta_0+1 & \theta_0+2 \end{pmatrix}\begin{pmatrix} \overline{\alpha_1} & \theta_0-1 & \theta_0+1 \\ \overline{\alpha_2} & 2\,\theta_0 & 1 \\ \overline{\alpha_3} & \theta_0-1 & \theta_0+2 \\ \overline{\alpha_4} & \theta_0-1 & \theta_0+3 \\ \overline{\alpha_5} & \theta_0 & \theta_0+1 \\ \overline{\alpha_6} & \theta_0 & \theta_0+2 \\ \overline{\alpha_7} & 2\,\theta_0-1 & 3 \\ \overline{\alpha_8} & 2\,\theta_0 & 2 \\ \overline{\alpha_9} & 2\,\theta_0+1 & 1 \\ \overline{\alpha_{10}} & 2\,\theta_0+2 & 1 \\ \overline{\alpha_{11}} & 2\,\theta_0+1 & 2 \\ \overline{\alpha_{12}} & 2\,\theta_0 & 3 \\ \overline{\alpha_{13}} & 2\,\theta_0-1 & 4 \\ \overline{\alpha_{14}} & \theta_0+4 & \theta_0-1 \\ \overline{\alpha_{15}} & \theta_0+3 & \theta_0 \\ \overline{\alpha_{16}} & \theta_0+2 & \theta_0+1 \end{pmatrix}$$

$$
\vec{h}_0 = \begin{pmatrix}
-\frac{\theta_0 - 2}{2\,\theta_0} & C_1 & 0 & \theta_0 \\
-\frac{\theta_0 - 2}{2\,(\theta_0 + 1)} & B_1 & 0 & \theta_0 + 1 \\
2\,\alpha_2\theta_0 - 4\,\alpha_2 & A_0 & 0 & \theta_0 + 1 \\
-\frac{\theta_0 - 2}{2\,(\theta_0 + 2)} & B_2 & 0 & \theta_0 + 2 \\
-\frac{\theta_0\overline{\alpha_1} - 2\,\overline{\alpha_1}}{2\,(\theta_0 + 2)} & C_1 & 0 & \theta_0 + 2 \\
2\,\alpha_9\theta_0 - 4\,\alpha_9 & A_0 & 0 & \theta_0 + 2 \\
-\frac{\theta_0 - 2}{2\,(\theta_0 + 3)} & B_4 & 0 & \theta_0 + 3 \\
-\frac{\theta_0\overline{\alpha_3} - 2\,\overline{\alpha_3}}{2\,(\theta_0 + 3)} & C_1 & 0 & \theta_0 + 3 \\
-\frac{\theta_0\overline{\alpha_1} - 2\,\overline{\alpha_1}}{2\,(\theta_0 + 3)} & B_1 & 0 & \theta_0 + 3 \\
\frac{\alpha_2\theta_0^2 - \alpha_2\theta_0 - 2\,\alpha_2}{4\,(\theta_0 + 3)} & \overline{C_1} & 0 & \theta_0 + 3 \\
-2\,(\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0 + 4\,\alpha_2\overline{\alpha_1} - 4\,\alpha_{10} & A_0 & 0 & \theta_0 + 3 \\
-\frac{\theta_0 - 3}{2\,\theta_0} & C_2 & 1 & \theta_0 \\
-\frac{\theta_0 - 3}{2\,(\theta_0 + 1)} & B_3 & 1 & \theta_0 + 1 \\
-\frac{\left(\theta_0^2 - 2\,\theta_0 + 1\right)|A_1|^2}{\theta_0^2 + \theta_0} & C_1 & 1 & \theta_0 + 1 \\
2\,\alpha_8\theta_0 - 6\,\alpha_8 & A_0 & 1 & \theta_0 + 1 \\
2\,\alpha_2\theta_0 - 4\,\alpha_2 & A_1 & 1 & \theta_0 + 1 \\
-\frac{\theta_0 - 3}{2\,(\theta_0 + 2)} & B_5 & 1 & \theta_0 + 2 \\
-\frac{\theta_0\overline{\alpha_1} - 3\,\overline{\alpha_1}}{2\,(\theta_0 + 2)} & C_2 & 1 & \theta_0 + 2 \\
-\frac{\theta_0^2\overline{\alpha_5} - 2\,\theta_0\overline{\alpha_5} + 2\,\overline{\alpha_5}}{2\,(\theta_0^2 + 2\,\theta_0)} & C_1 & 1 & \theta_0 + 2 \\
-\frac{\left(\theta_0^2 - \theta_0 - 1\right)|A_1|^2}{\theta_0^2 + 3\,\theta_0 + 2} & B_1 & 1 & \theta_0 + 2 \\
2\,\alpha_9\theta_0 - 4\,\alpha_9 & A_1 & 1 & \theta_0 + 2 \\
-4\,(\alpha_2\theta_0 - 3\,\alpha_2)|A_1|^2 + 2\,\alpha_{11}\theta_0 - 6\,\alpha_{11} & A_0 & 1 & \theta_0 + 2 \\
-\frac{\theta_0 - 4}{2\,\theta_0} & C_3 & 2 & \theta_0 \\
-\frac{\alpha_1\theta_0 - 2\,\alpha_1}{2\,\theta_0} & C_1 & 2 & \theta_0 \\
2\,\alpha_7\theta_0 - 8\,\alpha_7 & A_0 & 2 & \theta_0
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{\theta_0 - 4}{2\,(\theta_0 + 1)} & B_6 & 2 & \theta_0 + 1 \\
-\frac{\alpha_1\theta_0 - 2\,\alpha_1}{2\,(\theta_0 + 1)} & B_1 & 2 & \theta_0 + 1 \\
-\frac{\alpha_5\theta_0^2 - 2\,\alpha_5\theta_0 + 2\,\alpha_5}{2\,(\theta_0^2 + \theta_0)} & C_1 & 2 & \theta_0 + 1 \\
-\frac{\left(\theta_0^2 - 3\,\theta_0 + 1\right)|A_1|^2}{\theta_0^2 + \theta_0} & C_2 & 2 & \theta_0 + 1 \\
2\,\alpha_8\theta_0 - 6\,\alpha_8 & A_1 & 2 & \theta_0 + 1 \\
2\,\alpha_2\theta_0 - 4\,\alpha_2 & A_2 & 2 & \theta_0 + 1 \\
8\,\alpha_1\alpha_2 - 2\,(\alpha_1\alpha_2 - \alpha_{12})\theta_0 - 8\,\alpha_{12} & A_0 & 2 & \theta_0 + 1 \\
-\frac{\theta_0 - 5}{2\,\theta_0} & C_4 & 3 & \theta_0 \\
-\frac{\alpha_3\theta_0 - 2\,\alpha_3}{2\,\theta_0} & C_1 & 3 & \theta_0 \\
-\frac{\alpha_1\theta_0 - 3\,\alpha_1}{2\,\theta_0} & C_2 & 3 & \theta_0 \\
2\,\alpha_7\theta_0 - 8\,\alpha_7 & A_1 & 3 & \theta_0 \\
2\,\alpha_{13}\theta_0 - 10\,\alpha_{13} & A_0 & 3 & \theta_0 \\
-\frac{\theta_0 - 2}{2\,\theta_0 + 1} & E_2 & -\theta_0 + 2 & 2\,\theta_0 + 1 \\
\frac{\alpha_2\theta_0 - 2\,\alpha_2}{2\,(2\,\theta_0^2 + \theta_0)} & C_1 & -\theta_0 + 2 & 2\,\theta_0 + 1 \\
-\frac{\theta_0 - 2}{2\,\theta_0} & E_1 & -\theta_0 + 2 & 2\,\theta_0 \\
-\frac{2\,\theta_0 - 5}{4\,\theta_0} & E_3 & -\theta_0 + 3 & 2\,\theta_0 \\
4 & A_2 & \theta_0 & 0 \\
-4\,\alpha_1 & A_0 & \theta_0 & 0 \\
-4\,\alpha_5 & A_0 & \theta_0 & 1 \\
-4\,|A_1|^2 & A_1 & \theta_0 & 1 \\
\frac{1}{2} & \overline{B_2} & \theta_0 & 2 \\
-2\,\overline{\alpha_5} & A_1 & \theta_0 & 2 \\
8\,|A_1|^4 + 4\,\alpha_1\overline{\alpha_1} - 4\,\beta & A_0 & \theta_0 & 2 \\
\frac{1}{3} & \overline{B_5} & \theta_0 & 3 \\
-\frac{1}{6}\,\alpha_5 & \overline{C_1} & \theta_0 & 3 \\
\frac{1}{6}\,|A_1|^2 & \overline{B_1} & \theta_0 & 3 \\
8\,|A_1|^2\overline{\alpha_5} + 4\,\alpha_5\overline{\alpha_1} + 4\,\alpha_1\overline{\alpha_3} - 4\,\alpha_{16} & A_0 & \theta_0 & 3 \\
4\,|A_1|^2\overline{\alpha_1} - 2\,\overline{\alpha_6} & A_1 & \theta_0 & 3
\end{pmatrix}
$$

$$\begin{pmatrix}
2 & A_1 & \theta_0 - 1 & 0 \\
-4\,|A_1|^2 & A_0 & \theta_0 - 1 & 1 \\
\frac{1}{4} & \overline{B_1} & \theta_0 - 1 & 2 \\
-2\,\overline{\alpha_5} & A_0 & \theta_0 - 1 & 2 \\
\frac{1}{6} & \overline{B_3} & \theta_0 - 1 & 3 \\
-\frac{1}{6}\,|A_1|^2 & \overline{C_1} & \theta_0 - 1 & 3 \\
4\,|A_1|^2\overline{\alpha_1} - 2\,\overline{\alpha_6} & A_0 & \theta_0 - 1 & 3 \\
\frac{1}{8} & \overline{B_6} & \theta_0 - 1 & 4 \\
-\frac{1}{8}\,\overline{\alpha_5} & \overline{C_1} & \theta_0 - 1 & 4 \\
\frac{1}{8}\,\overline{\alpha_1} & \overline{B_1} & \theta_0 - 1 & 4 \\
-\frac{1}{12}\,|A_1|^2 & \overline{C_2} & \theta_0 - 1 & 4 \\
4\,|A_1|^2\overline{\alpha_3} + 2\,\overline{\alpha_1\alpha_5} - 2\,\alpha_{15} & A_0 & \theta_0 - 1 & 4 \\
6 & A_3 & \theta_0 + 1 & 0 \\
-6\,\alpha_3 & A_0 & \theta_0 + 1 & 0 \\
-4\,\alpha_1 & A_1 & \theta_0 + 1 & 0 \\
-4\,\alpha_5 & A_1 & \theta_0 + 1 & 1 \\
-4\,|A_1|^2 & A_2 & \theta_0 + 1 & 1 \\
12\,\alpha_1|A_1|^2 - 6\,\alpha_6 & A_0 & \theta_0 + 1 & 1 \\
\frac{3}{4} & \overline{B_4} & \theta_0 + 1 & 2 \\
\frac{\theta_0\overline{\alpha_2} - 2\,\overline{\alpha_2}}{4\,\theta_0} & C_1 & \theta_0 + 1 & 2 \\
-2\,\overline{\alpha_5} & A_2 & \theta_0 + 1 & 2 \\
\frac{1}{4}\,\alpha_1 & \overline{B_1} & \theta_0 + 1 & 2 \\
12\,\alpha_5|A_1|^2 + 6\,\alpha_3\overline{\alpha_1} + 6\,\alpha_1\overline{\alpha_5} - 6\,\overline{\alpha_{16}} & A_0 & \theta_0 + 1 & 2 \\
8\,|A_1|^4 + 4\,\alpha_1\overline{\alpha_1} - 4\,\beta & A_1 & \theta_0 + 1 & 2 \\
8 & A_4 & \theta_0 + 2 & 0 \\
-6\,\alpha_3 & A_1 & \theta_0 + 2 & 0 \\
-4\,\alpha_1 & A_2 & \theta_0 + 2 & 0 \\
4\,\alpha_1^2 - 8\,\alpha_4 & A_0 & \theta_0 + 2 & 0
\end{pmatrix}$$

$$
\begin{pmatrix}
-4\,\alpha_5 & A_2 & \theta_0+2 & 1 \\
-4\,|A_1|^2 & A_3 & \theta_0+2 & 1 \\
16\,\alpha_3|A_1|^2+8\,\alpha_1\alpha_5-8\,\overline{\alpha_{15}} & A_0 & \theta_0+2 & 1 \\
12\,\alpha_1|A_1|^2-6\,\alpha_6 & A_1 & \theta_0+2 & 1 \\
10 & A_5 & \theta_0+3 & 0 \\
-6\,\alpha_3 & A_2 & \theta_0+3 & 0 \\
-4\,\alpha_1 & A_3 & \theta_0+3 & 0 \\
10\,\alpha_1\alpha_3-10\,\overline{\alpha_{14}} & A_0 & \theta_0+3 & 0 \\
4\,\alpha_1^2-8\,\alpha_4 & A_1 & \theta_0+3 & 0 \\
-\frac{\theta_0}{2\,(\theta_0-4)} & \overline{E_1} & 2\,\theta_0-2 & -\theta_0+4 \\
-2\,\theta_0\overline{\alpha_7} & A_0 & 2\,\theta_0-2 & -\theta_0+4 \\
-\frac{\theta_0}{2\,(\theta_0-5)} & \overline{E_3} & 2\,\theta_0-2 & -\theta_0+5 \\
-2\,\theta_0\overline{\alpha_{13}} & A_0 & 2\,\theta_0-2 & -\theta_0+5 \\
-2\,\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2} & A_0 & 2\,\theta_0-1 & -\theta_0+2 \\
-2\,\theta_0\overline{\alpha_8}-2\,\overline{\alpha_8} & A_0 & 2\,\theta_0-1 & -\theta_0+3 \\
-\frac{\theta_0+1}{2\,(\theta_0-4)} & \overline{E_2} & 2\,\theta_0-1 & -\theta_0+4 \\
-\frac{\theta_0^2\overline{\alpha_2}-\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2}}{4\,(\theta_0-4)} & \overline{C_1} & 2\,\theta_0-1 & -\theta_0+4 \\
-2\,\theta_0\overline{\alpha_7} & A_1 & 2\,\theta_0-1 & -\theta_0+4 \\
2\,(\overline{\alpha_1\alpha_2}-\overline{\alpha_{12}})\theta_0+2\,\overline{\alpha_1\alpha_2}-2\,\overline{\alpha_{12}} & A_0 & 2\,\theta_0-1 & -\theta_0+4 \\
-2\,\theta_0\overline{\alpha_9}-4\,\overline{\alpha_9} & A_1 & 2\,\theta_0+1 & -\theta_0+2 \\
-2\,\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2} & A_2 & 2\,\theta_0+1 & -\theta_0+2 \\
2\,(\alpha_1\overline{\alpha_2}-\overline{\alpha_{10}})\theta_0+6\,\alpha_1\overline{\alpha_2}-6\,\overline{\alpha_{10}} & A_0 & 2\,\theta_0+1 & -\theta_0+2 \\
-2\,\theta_0\overline{\alpha_9}-4\,\overline{\alpha_9} & A_0 & 2\,\theta_0 & -\theta_0+2 \\
-2\,\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2} & A_1 & 2\,\theta_0 & -\theta_0+2 \\
-2\,\theta_0\overline{\alpha_8}-2\,\overline{\alpha_8} & A_1 & 2\,\theta_0 & -\theta_0+3 \\
4\,(\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2})|A_1|^2-2\,\theta_0\overline{\alpha_{11}}-4\,\overline{\alpha_{11}} & A_0 & 2\,\theta_0 & -\theta_0+3
\end{pmatrix}
$$

2.8 Order 2 terms in the quartic form

We have

$$g_0^{-1} \otimes \vec{Q}(\vec{h}_0)|_{\theta_0,2-\theta_0} = \frac{2}{\theta_0}\Big\{32\,A_1^2\theta_0|A_1|^4 + 48\,A_0A_1\alpha_5\theta_0|A_1|^2 - 24\,A_0^2\alpha_3\theta_0\overline{\alpha_5} + 8\,(3\,\alpha_3\overline{\alpha_1} + \alpha_1\overline{\alpha_5} - 3\,\overline{\alpha_{16}})A_0A_1\theta_0$$
$$+ 16\,(\alpha_1\overline{\alpha_1} - \beta)A_1^2\theta_0 - 8\,A_1A_2\theta_0\overline{\alpha_5} + 24\,A_0A_3\theta_0\overline{\alpha_5} - 2\left(\theta_0^4\overline{\alpha_2} - 4\,\theta_0^3\overline{\alpha_2} + 2\,\theta_0^2\overline{\alpha_2} + 4\,\theta_0\overline{\alpha_2} - 3\,\overline{\alpha_2}\right)A_0C_2$$
$$+ 3\,A_1\theta_0\overline{B_4} - \left(2\left(\theta_0^4\overline{\alpha_9} - \theta_0^3\overline{\alpha_9} - 4\,\theta_0^2\overline{\alpha_9} + 4\,\theta_0\overline{\alpha_9}\right)A_0 + \left(2\,\theta_0^4\overline{\alpha_2} - 4\,\theta_0^3\overline{\alpha_2} - 2\,\theta_0^2\overline{\alpha_2} + 3\,\theta_0\overline{\alpha_2} + 2\,\overline{\alpha_2}\right)A_1\right)C_1$$
$$+ 3\,(A_1\alpha_1\theta_0 + A_0\alpha_3\theta_0 - A_3\theta_0)\overline{B_1}\Big\}$$

Then

$$g_0^{-1} \otimes Q(\vec{h}_0) = \begin{pmatrix} \left(\theta_0^2 - 5\,\theta_0 + 6\right)A_1C_2 - 2\left(\left(\alpha_1\theta_0^2 - 2\,\alpha_1\theta_0\right)A_0 - \left(\theta_0^2 - 2\,\theta_0\right)A_2\right)C_1 & 0 & 0 \\ 16\,A_0A_2|A_1|^2 - 8\,A_0A_1\alpha_5 - 8\left(2\,A_0^2\alpha_1 + A_1^2\right)|A_1|^2 & \theta_0 - 1 & -\theta_0 + 1 \\ \omega_{-1} & -1 & 1 \\ 4\left(\theta_0^3\overline{\alpha_2} - \theta_0^2\overline{\alpha_2} - 2\,\theta_0\overline{\alpha_2}\right)A_0A_1 & 2\,\theta_0 - 2 & -2\,\theta_0 + 2 \\ \left(\theta_0^2 - 3\,\theta_0 + 2\right)A_1C_1 & -1 & 0 \end{pmatrix}$$

where

$$\omega_{-1} = -\frac{2\left(\theta_0^3 - 4\,\theta_0^2 + 5\,\theta_0 - 2\right)A_0C_1|A_1|^2 + 4\left(\alpha_2\theta_0^4 - 2\,\alpha_2\theta_0^3 - \alpha_2\theta_0^2 + 2\,\alpha_2\theta_0\right)A_0A_1 - \left(\theta_0^3 - 3\,\theta_0^2 + 2\,\theta_0\right)A_1B_1}{\theta_0} = 0$$

$$e^{-2u} - 1 = \begin{pmatrix} -2\,|A_1|^2 & 1 & 1 \\ -\alpha_1 & 2 & 0 \\ -\alpha_2 & -\theta_0 + 2 & \theta_0 + 1 \\ -\alpha_3 & 3 & 0 \\ -\alpha_5 & 2 & 1 \\ -\overline{\alpha_1} & 0 & 2 \\ -\overline{\alpha_2} & \theta_0 + 1 & -\theta_0 + 2 \\ -\overline{\alpha_3} & 0 & 3 \\ -\overline{\alpha_5} & 1 & 2 \end{pmatrix}, \quad e^{-2\lambda} = \begin{pmatrix} 1 & -\theta_0 + 1 & -\theta_0 + 1 \\ -2\,|A_1|^2 & -\theta_0 + 2 & -\theta_0 + 2 \\ -\alpha_1 & -\theta_0 + 3 & -\theta_0 + 1 \\ -\alpha_2 & -2\,\theta_0 + 3 & 2 \\ -\alpha_3 & -\theta_0 + 4 & -\theta_0 + 1 \\ -\alpha_5 & -\theta_0 + 3 & -\theta_0 + 2 \\ -\overline{\alpha_1} & -\theta_0 + 1 & -\theta_0 + 3 \\ -\overline{\alpha_2} & 2 & -2\,\theta_0 + 3 \\ -\overline{\alpha_3} & -\theta_0 + 1 & -\theta_0 + 4 \\ -\overline{\alpha_5} & -\theta_0 + 2 & -\theta_0 + 3 \end{pmatrix}$$

$$(g^{-1} - g_0^{-1}) \otimes Q(\vec{h}_0) =$$

$$\begin{pmatrix}
\omega_1 & 1 & 1 \\
-32\,A_0A_2|A_1|^4+16\,A_0A_1\alpha_5|A_1|^2+16\left(2\,A_0^2\alpha_1+A_1^2\right)|A_1|^4-\left(\theta_0^2\overline{\alpha_2}-3\,\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2}\right)A_1C_1 & \theta_0 & -\theta_0+2 \\
\omega_2 & 0 & 2 \\
-8\left(\theta_0^3\overline{\alpha_2}-\theta_0^2\overline{\alpha_2}-2\,\theta_0\overline{\alpha_2}\right)A_0A_1|A_1|^2 & 2\,\theta_0-1 & -2\,\theta_0+3 \\
-2\left(\theta_0^2-3\,\theta_0+2\right)A_1C_1|A_1|^2 & 0 & 1 \\
\omega_3 & 2 & 0 \\
-16\,A_0A_2\alpha_1|A_1|^2+8\,A_0A_1\alpha_1\alpha_5+8\left(2\,A_0^2\alpha_1^2+A_1^2\alpha_1\right)|A_1|^2 & \theta_0+1 & -\theta_0+1 \\
-4\left(\alpha_1\theta_0^3\overline{\alpha_2}-\alpha_1\theta_0^2\overline{\alpha_2}-2\,\alpha_1\theta_0\overline{\alpha_2}\right)A_0A_1 & 2\,\theta_0 & -2\,\theta_0+2 \\
-\left(\alpha_1\theta_0^2-3\,\alpha_1\theta_0+2\,\alpha_1\right)A_1C_1 & 1 & 0 \\
-\left(\alpha_2\theta_0^2-3\,\alpha_2\theta_0+2\,\alpha_2\right)A_1C_1 & -\theta_0+1 & \theta_0+1 \\
-16\,A_0A_2|A_1|^2\overline{\alpha_1}+8\,A_0A_1\alpha_5\overline{\alpha_1}+8\left(2\,A_0^2\alpha_1\overline{\alpha_1}+A_1^2\overline{\alpha_1}\right)|A_1|^2 & \theta_0-1 & -\theta_0+3 \\
\omega_3 & -1 & 3 \\
-4\left(\theta_0^3\overline{\alpha_1\alpha_2}-\theta_0^2\overline{\alpha_1\alpha_2}-2\,\theta_0\overline{\alpha_1\alpha_2}\right)A_0A_1 & 2\,\theta_0-2 & -2\,\theta_0+4 \\
-\left(\theta_0^2\overline{\alpha_1}-3\,\theta_0\overline{\alpha_1}+2\,\overline{\alpha_1}\right)A_1C_1 & -1 & 2
\end{pmatrix}$$

where

$$\begin{aligned}
\omega_1 &= -\frac{1}{\theta_0}\Big\{2\left(\theta_0^3-5\,\theta_0^2+6\,\theta_0\right)A_1C_2|A_1|^2-4\left(\alpha_1\alpha_2\theta_0^4-2\,\alpha_1\alpha_2\theta_0^3-\alpha_1\alpha_2\theta_0^2+2\,\alpha_1\alpha_2\theta_0\right)A_0A_1 \\
&\quad+\left(\alpha_1\theta_0^3-3\,\alpha_1\theta_0^2+2\,\alpha_1\theta_0\right)A_1B_1 \\
&\quad-\Big\{2\left(3\,\alpha_1\theta_0^3-8\,\alpha_1\theta_0^2+5\,\alpha_1\theta_0-2\,\alpha_1\right)A_0|A_1|^2-4\left(\theta_0^3-2\,\theta_0^2\right)A_2|A_1|^2-\left(\alpha_5\theta_0^3-3\,\alpha_5\theta_0^2+2\,\alpha_5\theta_0\right)A_1\Big\}C_1\Big\} \\
\omega_2 &= \frac{1}{\theta_0}\Big\{8\left(\alpha_2\theta_0^4-2\,\alpha_2\theta_0^3-\alpha_2\theta_0^2+2\,\alpha_2\theta_0\right)A_0A_1|A_1|^2-2\left(\theta_0^3-3\,\theta_0^2+2\,\theta_0\right)A_1B_1|A_1|^2 \\
&\quad-\left(\theta_0^3\overline{\alpha_1}-5\,\theta_0^2\overline{\alpha_1}+6\,\theta_0\overline{\alpha_1}\right)A_1C_2+\Big\{4\left(\theta_0^3-4\,\theta_0^2+5\,\theta_0-2\right)A_0|A_1|^4+2\left(\alpha_1\theta_0^3\overline{\alpha_1}-2\,\alpha_1\theta_0^2\overline{\alpha_1}\right)A_0 \\
&\quad-\left(\theta_0^3\overline{\alpha_5}-3\,\theta_0^2\overline{\alpha_5}+2\,\theta_0\overline{\alpha_5}\right)A_1-2\left(\theta_0^3\overline{\alpha_1}-2\,\theta_0^2\overline{\alpha_1}\right)A_2\Big\}C_1\Big\} \\
\omega_3 &= -\left(\alpha_1\theta_0^2-5\,\alpha_1\theta_0+6\,\alpha_1\right)A_1C_2+\left(2\left(\alpha_1^2\theta_0^2-2\,\alpha_1^2\theta_0\right)A_0-\left(\alpha_3\theta_0^2-3\,\alpha_3\theta_0+2\,\alpha_3\right)A_1-2\left(\alpha_1\theta_0^2-2\,\alpha_1\theta_0\right)A_2\right)C_1 \\
\omega_4 &= \frac{1}{\theta_0}\Big\{4\left(\alpha_2\theta_0^4\overline{\alpha_1}-2\,\alpha_2\theta_0^3\overline{\alpha_1}-\alpha_2\theta_0^2\overline{\alpha_1}+2\,\alpha_2\theta_0\overline{\alpha_1}\right)A_0A_1-\left(\theta_0^3\overline{\alpha_1}-3\,\theta_0^2\overline{\alpha_1}+2\,\theta_0\overline{\alpha_1}\right)A_1B_1 \\
&\quad+\left(2\left(\theta_0^3\overline{\alpha_1}-4\,\theta_0^2\overline{\alpha_1}+5\,\theta_0\overline{\alpha_1}-2\,\overline{\alpha_1}\right)A_0|A_1|^2-\left(\theta_0^3\overline{\alpha_3}-3\,\theta_0^2\overline{\alpha_3}+2\,\theta_0\overline{\alpha_3}\right)A_1\right)C_1\Big\}
\end{aligned}$$

then

$$\frac{5}{4}|\vec{H}|^2\vec{h}_0\,\dot{\otimes}\,\vec{h}_0=\begin{pmatrix}
\left(4\,A_1^2\right)\left(\frac{5}{16}\,C_1^2\right) & 2 & 0 \\
\left(4\,A_1^2\right)\left(\frac{5}{16}\,C_1\overline{C_1}\right) & \theta_0 & -\theta_0+2 \\
\left(4\,A_1^2\right)\left(\frac{5}{16}\,C_1\overline{C_1}\right) & \theta_0 & -\theta_0+2 \\
\left(4\,A_1^2\right)\left(\frac{5}{16}\,\overline{C_1}^2\right) & 2\,\theta_0-2 & -2\,\theta_0+4
\end{pmatrix} \qquad (2.8.1)$$

$$\langle\vec{H},\vec{h}_0\rangle^2 = \begin{pmatrix} (A_1C_1)\,(A_1C_1) & 2 & 0 \\ (A_1C_1)\left(A_1\overline{C_1}\right) & \theta_0 & -\theta_0+2 \\ (A_1C_1)\left(A_1\overline{C_1}\right) & \theta_0 & -\theta_0+2 \\ \left(A_1\overline{C_1}\right)\left(A_1\overline{C_1}\right) & 2\,\theta_0-2 & -2\,\theta_0+4 \end{pmatrix} \tag{2.8.2}$$

and finally

$$-K_g\vec{h}_0\,\dot{\otimes}\,\vec{h}_0 = \begin{pmatrix} 16\,A_1^2|A_1|^2 & \theta_0-1 & -\theta_0+1 \\ -64\,A_0A_1\alpha_1|A_1|^2+64\,A_1A_2|A_1|^2+16\,A_1^2\alpha_5 & \theta_0 & -\theta_0+1 \\ \kappa_1 & \theta_0+1 & -\theta_0+1 \\ -8\,A_1^2\alpha_2(\theta_0+1)(\theta_0-2)-\dfrac{8\,A_1C_1(\theta_0-2)|A_1|^2}{\theta_0} & 0 & 1 \\ \kappa_2 & 0 & 2 \\ \kappa_3 & 1 & 1 \\ 64\,A_0^2|A_1|^6-16\,A_1^2|A_1|^2\overline{\alpha_1}-96\,A_0A_1|A_1|^2\overline{\alpha_5}+4\,A_1|A_1|^2\overline{B_1}-24\left(2\,|A_1|^2\overline{\alpha_1}-\overline{\alpha_6}\right)A_1^2 & \theta_0-1 & -\theta_0+3 \\ -64\,A_0A_1|A_1|^4+16\,A_1^2\overline{\alpha_5} & \theta_0-1 & -\theta_0+2 \\ \kappa_4 & \theta_0 & -\theta_0+2 \\ 32\,A_0A_1(\theta_0+1)(\theta_0-2)|A_1|^2\overline{\alpha_2}-32\,(\theta_0\overline{\alpha_2}+\overline{\alpha_2})A_0A_1|A_1|^2-8\,A_1^2(\theta_0+1)(\theta_0-3)\overline{\alpha_8} & 2\,\theta_0-1 & -2\,\theta_0+3 \\ \dfrac{4\,A_1C_1\alpha_2(\theta_0+1)(\theta_0-2)^2}{\theta_0}+\dfrac{C_1^2(\theta_0-2)^2|A_1|^2}{\theta_0^2} & -\theta_0+1 & \theta_0+1 \\ -8\,A_1^2\alpha_7(\theta_0-4)\theta_0 & 2 & 0 \\ -8\,A_1^2(\theta_0+1)(\theta_0-2)\overline{\alpha_2} & 2\,\theta_0-1 & -2\,\theta_0+2 \\ 32\,A_0A_1\alpha_1(\theta_0+1)(\theta_0-2)\overline{\alpha_2}-32\,A_1A_2(\theta_0+1)(\theta_0-2)\overline{\alpha_2}-8\,A_1^2(\theta_0+2)(\theta_0-2)\overline{\alpha_9} & 2\,\theta_0 & -2\,\theta_0+2 \\ -8\,A_1^2(\theta_0-4)\theta_0\overline{\alpha_7} & 2\,\theta_0-2 & -2\,\theta_0+4 \end{pmatrix}$$

$$\begin{aligned}
\kappa_1 &= 64\,A_0^2\alpha_1^2|A_1|^2-80\,A_1^2\alpha_1|A_1|^2-128\,A_0A_2\alpha_1|A_1|^2-96\,A_0A_1\alpha_3|A_1|^2-64\,A_0A_1\alpha_1\alpha_5+64\,A_2^2|A_1|^2 \\
&\quad+96\,A_1A_3|A_1|^2-24\left(2\,\alpha_1|A_1|^2-\alpha_6\right)A_1^2+64\,A_1A_2\alpha_5 \\
\kappa_2 &= 32\,A_0A_1\alpha_2(\theta_0+1)(\theta_0-2)|A_1|^2+\frac{16\,A_0C_1(\theta_0-2)|A_1|^4}{\theta_0}-8\,A_1^2\alpha_9(\theta_0+2)(\theta_0-2)+32\,(\alpha_2\theta_0-2\,\alpha_2)A_0A_1|A_1|^2 \\
&\quad-\frac{8\,A_1B_1(\theta_0-2)|A_1|^2}{\theta_0+1}-\frac{8\,A_1C_1(\theta_0-2)\overline{\alpha_5}}{\theta_0} \\
\kappa_3 &= 32\,A_0A_1\alpha_1\alpha_2(\theta_0+1)(\theta_0-2)-32\,A_1A_2\alpha_2(\theta_0+1)(\theta_0-2)-8\,A_1^2\alpha_8(\theta_0+1)(\theta_0-3)+\frac{16\,A_0C_1\alpha_1(\theta_0-2)|A_1|^2}{\theta_0} \\
&\quad-\frac{16\,A_2C_1(\theta_0-2)|A_1|^2}{\theta_0}-\frac{8\,A_1C_2(\theta_0-3)|A_1|^2}{\theta_0}-\frac{8\,A_1C_1\alpha_5(\theta_0-2)}{\theta_0} \\
\kappa_4 &= 128\,A_0^2\alpha_1|A_1|^4-96\,A_1^2|A_1|^4-128\,A_0A_2|A_1|^4-128\,A_0A_1\alpha_5|A_1|^2+\frac{4\,A_1C_1(\theta_0+1)(\theta_0-2)^2\overline{\alpha_2}}{\theta_0}-64\,A_0A_1\alpha_1\overline{\alpha_5} \\
&\quad-32\left(2\,|A_1|^4+\alpha_1\overline{\alpha_1}-\beta\right)A_1^2+64\,A_1A_2\overline{\alpha_5}
\end{aligned}$$

Finally, the coefficient in $z^{\theta_0}\overline{z}^{2-\theta_0}dz^4$ in the Taylor expansion of the quartic form $\mathscr{Q}_{\vec{\Phi}}$ is

$$
\begin{aligned}
\Omega_0 = &\frac{2}{\theta_0}\bigg\{32\,A_1^2\theta_0|A_1|^4+48\,A_0A_1\alpha_5\theta_0|A_1|^2-24\,A_0^2\alpha_3\theta_0\overline{\alpha_5}+8\,(3\,\alpha_3\overline{\alpha_1}+\alpha_1\overline{\alpha_5}-3\,\overline{\alpha_{16}})A_0A_1\theta_0\\
&+16\,(\alpha_1\overline{\alpha_1}-\beta)A_1^2\theta_0-8\,A_1A_2\theta_0\overline{\alpha_5}+24\,A_0A_3\theta_0\overline{\alpha_5}-2\left(\theta_0^4\overline{\alpha_2}-4\,\theta_0^3\overline{\alpha_2}+2\,\theta_0^2\overline{\alpha_2}+4\,\theta_0\overline{\alpha_2}-3\,\overline{\alpha_2}\right)A_0C_2\\
&+3\,A_1\theta_0\overline{B_4}-\bigg(2\left(\theta_0^4\overline{\alpha_9}-\theta_0^3\overline{\alpha_9}-4\,\theta_0^2\overline{\alpha_9}+4\,\theta_0\overline{\alpha_9}\right)A_0+\left(2\,\theta_0^4\overline{\alpha_2}-4\,\theta_0^3\overline{\alpha_2}-2\,\theta_0^2\overline{\alpha_2}+3\,\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2}\right)A_1\bigg)C_1\\
&+3\,(A_1\alpha_1\theta_0+A_0\alpha_3\theta_0-A_3\theta_0)\overline{B_1}\bigg\}\\
&-32\,A_0A_2|A_1|^4+16\,A_0A_1\alpha_5|A_1|^2+16\left(2\,A_0^2\alpha_1+A_1^2\right)|A_1|^4-\left(\theta_0^2\overline{\alpha_2}-3\,\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2}\right)A_1C_1\\
&+\frac{5}{2}|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_1,\vec{C}_1\rangle\langle\overline{\vec{A}_1},\overline{\vec{C}_1}\rangle\\
&+128\,A_0^2\alpha_1|A_1|^4-96\,A_1^2|A_1|^4-128\,A_0A_2|A_1|^4-128\,A_0A_1\alpha_5|A_1|^2+\frac{4\,A_1C_1(\theta_0+1)(\theta_0-2)^2\overline{\alpha_2}}{\theta_0}-64\,A_0A_1\alpha_1\alpha_5\\
&-32\left(2\,|A_1|^4+\alpha_1\overline{\alpha_1}-\beta\right)A_1^2+64\,A_1A_2\overline{\alpha_5}
\end{aligned}\tag{2.8.3}
$$

$$
\begin{aligned}
&=\frac{2}{\theta_0}\bigg\{32\,A_1^2\theta_0|A_1|^4+16\left(|\alpha_1|^2-\beta\right)A_1^2\theta_0-8\,A_1A_2\theta_0\overline{\alpha_5}+24\,A_0A_3\theta_0\overline{\alpha_5}-2\left(\theta_0^4-4\,\theta_0^3+2\,\theta_0^2+4\,\theta_0-3\right)\overline{\alpha_2}A_0C_2\\
&+3\,A_1\theta_0\overline{B_4}-\left(\left(2\,\theta_0^4-4\,\theta_0^3-2\,\theta_0^2+3\,\theta_0+2\right)\overline{\alpha_2}A_1\right)C_1+3\,(A_0\alpha_3\theta_0-A_3\theta_0)\overline{B_1}\bigg\}\\
&-32\,A_0A_2|A_1|^4+16\left(A_1^2\right)|A_1|^4-\left(\theta_0^2-3\,\theta_0+2\right)\overline{\alpha_2}A_1C_1\\
&+\frac{5}{2}|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_1,\vec{C}_1\rangle\langle\overline{\vec{A}_1},\overline{\vec{C}_1}\rangle\\
&-96\,A_1^2|A_1|^4-128\,A_0A_2|A_1|^4+\frac{4\,A_1C_1(\theta_0+1)(\theta_0-2)^2\overline{\alpha_2}}{\theta_0}\\
&-32\left(2\,|A_1|^4+|\alpha_1|^2-\beta\right)A_1^2+64\,A_1A_2\overline{\alpha_5}.
\end{aligned}\tag{2.8.4}
$$

Let us first look at the coefficients

$$|\vec{A}_1|^4\langle\vec{A}_1,\vec{A}_1\rangle\quad\text{and}\ |\vec{A}_1|^4\langle\vec{A}_0,\vec{A}_2\rangle.$$

We see that these are

$$
\begin{aligned}
&\frac{2}{\theta_0}\left(32\theta_0|\vec{A}_1|^4\langle\vec{A}_1,\vec{A}_1\rangle\right)-32|\vec{A}_1|^4\langle\vec{A}_0,\vec{A}_2\rangle+16|\vec{A}_1|^4\langle\vec{A}_1,\vec{A}_1\rangle-96|\vec{A}_1|^4\langle\vec{A}_1,\vec{A}_1\rangle-128|\vec{A}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle-64|\vec{A}_1|^4\langle\vec{A}_1,\vec{A}_1\rangle\\
&=-80|\vec{A}_1|^4\left(\langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_0,\vec{A}_2\rangle\right)=0.
\end{aligned}
$$

Then, the coefficient involving $|\alpha_1|^2-\beta$ is

$$\frac{2}{\theta_0}\left(16\theta_0|\alpha_1|^2-\beta\right)\langle\vec{A}_1,\vec{A}_1\rangle-32\left(|\alpha_1|^2-\beta\right)\langle\vec{A}_1,\vec{A}_1\rangle=0.$$

Finally, the coefficient involving $\overline{\alpha_5}$ is

$$\frac{2}{\theta_0}\left(-8\theta_0\overline{\alpha_5}\langle\vec{A}_1,\vec{A}_2\rangle+24\theta_0\overline{\alpha_5}\langle\vec{A}_0,\vec{A}_3\rangle\right)+64\overline{\alpha_5}\langle\vec{A}_1,\vec{A}_2\rangle=48\overline{\alpha_5}\left(\langle\vec{A}_1,\vec{A}_2\rangle+\langle\vec{A}_0,\vec{A}_3\rangle\right)=0.$$

Therefore, we deduce that (using $\langle\vec{A}_1,\vec{C}_1\rangle+\langle\vec{A}_0,\vec{C}_2\rangle=0$)

$$
\begin{aligned}
\Omega_0 = &\frac{2}{\theta_0}\bigg\{32\,A_1^2\theta_0|A_1|^4+16\left(|\alpha_1|^2-\beta\right)A_1^2\theta_0-8\,A_1A_2\theta_0\overline{\alpha_5}+24\,A_0A_3\theta_0\overline{\alpha_5}-2\left(\theta_0^4-4\,\theta_0^3+2\,\theta_0^2+4\,\theta_0-3\right)\overline{\alpha_2}A_0C_2\\
&+3\,A_1\theta_0\overline{B_4}-\left(\left(2\,\theta_0^4-4\,\theta_0^3-2\,\theta_0^2+3\,\theta_0+2\right)\overline{\alpha_2}A_1\right)C_1+3\,(A_0\alpha_3\theta_0-A_3\theta_0)\overline{B_1}\bigg\}\\
&-32\,A_0A_2|A_1|^4+16\left(A_1^2\right)|A_1|^4-\left(\theta_0^2-3\,\theta_0+2\right)\overline{\alpha_2}A_1C_1\\
&+\frac{5}{2}|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_1,\vec{C}_1\rangle\langle\overline{\vec{A}_1},\overline{\vec{C}_1}\rangle\\
&-96\,A_1^2|A_1|^4-128\,A_0A_2|A_1|^4+\frac{4\,A_1C_1(\theta_0+1)(\theta_0-2)^2\overline{\alpha_2}}{\theta_0}
\end{aligned}
$$

$$
\begin{aligned}
&-32\left(2\,|A_1|^4+|\alpha_1|^2-\beta\right)A_1^2+64\,A_1A_2\overline{\alpha_5}\\
&=\frac{2}{\theta_0}\Big\{2\left(\theta_0^4-4\,\theta_0^3+2\,\theta_0^2+4\,\theta_0-3\right)\overline{\alpha_2}A_1C_1+3\,A_1\theta_0\overline{B_4}-\left(\left(2\,\theta_0^4-4\,\theta_0^3-2\,\theta_0^2+3\,\theta_0+2\right)\overline{\alpha_2}A_1\right)C_1\\
&+3\left(A_0\alpha_3\theta_0-A_3\theta_0\right)\overline{B_1}\Big\}-\left(\theta_0^2-3\,\theta_0+2\right)\overline{\alpha_2}A_1C_1+\frac{5}{2}|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_1,\vec{C}_1\rangle\langle\overline{\vec{A}_1},\overline{\vec{C}_1}\rangle\\
&+\frac{4\,A_1C_1(\theta_0+1)(\theta_0-2)^2\overline{\alpha_2}}{\theta_0}
\end{aligned}
\tag{2.8.5}
$$

Now, recall that

$$
\frac{1}{2}\vec{B}_4=\begin{pmatrix}
-\dfrac{\left(\theta_0^2+4\,\theta_0+3\right)C_1\overline{C_2}-\left(12\left(\alpha_2\theta_0^2+\alpha_2\theta_0\right)A_0-\left(\theta_0^2+4\,\theta_0\right)B_1\right)\overline{C_1}}{12\left(\theta_0^2+\theta_0\right)} & \overline{A_0} & -\theta_0+2 & 3\\
-\dfrac{\overline{A_1C_1}}{12\,\theta_0} & C_1 & -\theta_0+2 & 3\\
-\dfrac{C_1(\theta_0+3)\overline{C_1}}{12\,\theta_0} & \overline{A_1} & -\theta_0+2 & 3\\
-\dfrac{1}{24}\,C_1\overline{A_1} & \overline{C_1} & -\theta_0+2 & 3\\
\dfrac{1}{3}\left(\alpha_2\theta_0+\alpha_2\right)\overline{A_0C_1}+\dfrac{2}{3}\left(\overline{A_0\alpha_1}-\overline{A_2}\right)B_1+\left(\overline{A_1\alpha_1}+\overline{A_0\alpha_3}-\overline{A_3}\right)C_1-\dfrac{1}{3}\,B_2\overline{A_1} & A_0 & -\theta_0+2 & 3
\end{pmatrix}
$$

and for some unimportant $\lambda_1,\lambda_2\in\vec{C}$, we have

$$
\vec{B}_4=\lambda_1\vec{A}_0+\lambda_2\overline{\vec{A}_0}-\frac{1}{6\theta_0}\overline{\langle\vec{A}_1,\vec{C}_1\rangle}\vec{C}_1-\frac{(\theta_0+3)}{6\theta_0}|\vec{C}_1|^2\overline{\vec{A}_1}-\frac{1}{12}\langle\overline{\vec{A}_1},\vec{C}_1\rangle\overline{\vec{C}_1}. \tag{2.8.6}
$$

As $\langle\vec{A}_0,\vec{A}_1\rangle=\langle\vec{A}_0,\overline{\vec{A}_1}\rangle$, we obtain

$$
\langle\overline{\vec{A}_1},\vec{B}_4\rangle=-\frac{(\theta_0+2)}{12\theta_0}\overline{\langle\vec{A}_1,\vec{C}_1\rangle}\vec{C}_1-\frac{(\theta_0+3)}{6\theta_0}|\vec{C}_1|^2\overline{\langle\vec{A}_1,\vec{A}_1\rangle}. \tag{2.8.7}
$$

Now, as $\alpha_3\in\vec{C}$ is defined by

$$
\alpha_3=\frac{1}{12}\langle\vec{A}_1,\vec{C}_1\rangle+2\langle\overline{\vec{A}_0},\vec{A}_3\rangle, \tag{2.8.8}
$$

we have

$$
\langle\alpha_3\vec{A}_0-\vec{A}_3,\overline{\vec{A}_0}\rangle=\frac{\alpha_3}{2}-\langle\overline{\vec{A}_0},\vec{A}_3\rangle=\frac{1}{24}\langle\vec{A}_1,\vec{C}_1\rangle,
$$

so

$$
\langle\alpha_3\vec{A}_0-\vec{A}_3,\overline{\vec{B}_1}\rangle=-2\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\langle\alpha_3\vec{A}_0-\vec{A}_3,\overline{\vec{A}_0}\rangle=-\frac{1}{12}\langle\vec{A}_1,\vec{C}_1\rangle\langle\vec{A}_1,\overline{\vec{C}_1}\rangle. \tag{2.8.9}
$$

Finally, as

$$
\alpha_2=\frac{1}{2\theta_0(\theta_0+1)}\langle\vec{A}_1,\overline{\vec{C}_1}\rangle, \tag{2.8.10}
$$

we get

$$
\begin{aligned}
\Omega_0&=\left\{\frac{2}{\theta_0}\left(\frac{(\theta_0^4-4\theta_0^3+2\theta_0^2+4\theta_0-3)}{\theta_0(\theta_0+1)}-\frac{(\theta_0+2)}{4}-\frac{2\theta_0^4-4\theta_0^3-2\theta_0^2+3\theta_0+2}{2\theta_0(\theta_0+1)}-\frac{\theta_0}{4}\right)\right.\\
&\left.-\frac{\theta_0^2-3\theta_0+2}{2\theta_0(\theta_0+1)}+2+\frac{2(\theta_0-2)^2}{\theta_0^2}\right\}\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\langle\vec{A}_1,\vec{C}_1\rangle+\left\{\frac{2}{\theta_0}\left(-\frac{(\theta_0+3)}{2}\right)+\frac{5}{2}\right\}|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle\\
&=-\frac{3(\theta_0-2)}{2\theta_0}\langle\vec{A}_1,\vec{C}_1\rangle\langle\vec{A}_1,\overline{\vec{C}_1}\rangle+\frac{3(\theta_0-2)}{2\theta_0}|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3(\theta_0-2)}{2\theta_0}\left(|\vec{C}_1|^2\langle \vec{A}_1,\vec{A}_1\rangle - \langle \vec{A}_1,\overline{\vec{C}_1}\rangle\langle \vec{A}_1,\vec{C}_1\rangle\right)\\
&= 0
\end{aligned} \tag{2.8.11}
$$

so

$$
|\vec{C}_1|^2\langle \vec{A}_1,\vec{A}_1\rangle = \langle \vec{A}_1,\overline{\vec{C}_1}\rangle\langle \vec{A}_1,\vec{C}_1\rangle \tag{2.8.12}
$$

If we suppose that $\mathscr{Q}_{\vec{\Phi}}$ is meromorphic, then we obtain

$$
\begin{cases}
|\vec{A}_1|^2\langle \vec{A}_1,\vec{C}_1\rangle = \langle \overline{\vec{A}_1},\vec{C}_1\rangle\langle \vec{A}_1,\vec{A}_1\rangle\\
|\vec{C}_1|^2\langle \vec{A}_1,\vec{A}_1\rangle = \langle \vec{A}_1,\overline{\vec{C}_1}\rangle\langle \vec{A}_1,\vec{C}_1\rangle,
\end{cases} \tag{2.8.13}
$$

Remarking that is a linear system in $(\langle \vec{A}_1,\vec{C}_1\rangle,\langle \vec{A}_1,\vec{A}_1\rangle)$, we can recast (2.8.13) as

$$
\begin{pmatrix}
|\vec{A}_1|^2 & -\langle \overline{\vec{A}_1},\vec{C}_1\rangle\\
-\langle \vec{A}_1,\overline{\vec{C}_1}\rangle & |\vec{C}_1|^2
\end{pmatrix}
\begin{pmatrix}
\langle \vec{A}_1,\vec{C}_1\rangle\\
\langle \vec{A}_1,\vec{A}_1\rangle
\end{pmatrix} = 0. \tag{2.8.14}
$$

Thanks of Cauchy-Schwarz inequality, we obtain

$$
\det\begin{pmatrix}
|\vec{A}_1|^2 & -\langle \overline{\vec{A}_1},\vec{C}_1\rangle\\
-\langle \vec{A}_1,\overline{\vec{C}_1}\rangle & |\vec{C}_1|^2
\end{pmatrix} = |\vec{A}_1|^2|\vec{C}_1|^2 - |\langle \vec{A}_1,\overline{\vec{C}_1}\rangle|^2 \geq 0. \tag{2.8.15}
$$

Therefore, if the determinant is positive, we obtain

$$
\langle \vec{A}_1,\vec{C}_1\rangle = 0,
$$

and the holomorphy of the quartic form, and if the determinant vanishes,

$$
\vec{A}_1 \;\text{ and }\; \vec{C}_1 \;\text{ are proportional.} \tag{2.8.16}
$$

Chapter 3

Return to the invariance by inversions

To obtain the next order development, we need to develop

$$\vec{\beta} = \mathscr{I}_{\vec{\Phi}}(\vec{\alpha}) - g^{-1} \otimes \left(\vec{h}_0 \otimes \overline{\partial}|\vec{\Phi}|^2 - 2\langle \vec{\Phi}, \vec{h}_0\rangle \otimes \overline{\partial}\vec{\Phi}\right)$$

where

$$\begin{aligned}\vec{\alpha} &= \partial\vec{H} + |\vec{H}|^2\partial\vec{\Phi} + 2\, g^{-1} \otimes \langle \vec{H}, \vec{h}_0\rangle \otimes \overline{\partial}\vec{\Phi}\\ \mathscr{I}_{\vec{\Phi}}(\vec{X}) &= |\vec{\Phi}|^2\vec{X} - 2\langle \vec{\Phi}, \vec{X}\rangle\vec{\Phi},\end{aligned}$$

up to an error of order $\theta + 4$.

3.1 Computation of the order of development of tensors

We first need to see at which order we need to develop tensors to obtain the next order development of $\vec{\beta}$. We should obtain an error in $O(|z|^{\theta_0+4})$.

To obtain such error, we need to develop $\vec{\alpha}$ and $\vec{\Phi}$ up to order 3, for the component

$$\mathscr{I}_{\vec{\Phi}}(\vec{\alpha}) = \mathscr{I}_{\vec{\Phi}}\left(\partial\vec{H} + |\vec{H}|^2\partial\vec{\Phi} + 2\, g^{-1} \otimes \langle \vec{H}, \vec{h}_0\rangle \otimes \overline{\partial}\vec{\Phi}\right)$$

where for all $\vec{X} \in \mathbb{C}^n$, we have

$$\mathscr{I}_{\vec{\Phi}}(\vec{X}) = |\vec{\Phi}|^2\vec{X} - 2\langle \vec{\Phi}, \vec{X}\rangle\vec{\Phi}.$$

As we perform this order 3 development, we get

$$\vec{\alpha} = -\frac{(\theta_0 - 2)}{2}\frac{\vec{C}_1}{z^{\theta_0 - 1}} + \cdots + O(|z|^{4-\theta_0})$$

and

$$|\vec{\Phi}(z)|^2 = \frac{1}{\theta_0^2}|z|^{2\theta_0}\left(1 + \cdots + O(|z|^3)\right)$$

so

$$|\vec{\Phi}|^2\vec{\alpha} = -\frac{(\theta_0 - 2)}{2\theta_0^2}\vec{C}_1 z\overline{z}^{\theta_0} + \cdots + O(|z|^{\theta_0+4})$$

and likewise

$$\langle\vec{\Phi},\vec{\alpha}\rangle\vec{\Phi} = \cdots + O(|z|^{\theta_0+4})$$

so the developments of $\vec{\Phi}$ and $\vec{\alpha}$ are sufficient for the part

$$\mathscr{I}_{\vec{\Phi}}(\vec{\alpha}) = |\vec{\Phi}|^2\vec{\alpha} - 2\langle\vec{\Phi},\vec{\alpha}\rangle\vec{\Phi}.$$

Furthermore, we have

$$\langle\vec{\alpha},\vec{\Phi}\rangle = \left\langle -\frac{(\theta_0-2)}{2}\frac{\vec{C}_1}{z^{\theta_0-1}} + \cdots + O(|z|^{4-\theta_0}), \mathrm{Re}\left(\frac{2}{\theta_0}\vec{A}_0 z^{\theta_0}\right) + O(|z|^{\theta_0+3})\right\rangle = \cdots + O(|z|^4).$$

We indicate the order of the errors in the partial developments so that one can check when we throw higher order terms with the `throw` function that the order is correct.

However, we need to develop $\vec{\Phi}$, g and $\vec{h}_0$ up to order 4 for the other part

$$-g^{-1}\otimes\left(\overline{\partial}|\vec{\Phi}|^2\otimes\vec{h}_0 - 2\langle\vec{\Phi},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}\right).$$

Indeed, with

$$\begin{aligned}
\vec{h}_0 &= 2\,\vec{A}_1 z^{\theta_0-1}dz^2 + \cdots + O(|z|^{\theta_0+3})\\
g^{-1} &= |z|^{2-2\theta_0}\left(1 + \cdots + O(|z|^4)\right)\\
|\vec{\Phi}|^2 &= \frac{1}{\theta_0^2}|z|^{2\theta_0}\left(1 + \cdots + O(|z|^4)\right)
\end{aligned}$$

we obtain

$$\overline{\partial}|\vec{\Phi}|^2 = \frac{1}{\theta_0}|z|^{2\theta_0-2}\left(z\,d\overline{z} + \cdots + O(|z|^5)\right)$$

so

$$g^{-1}\otimes\vec{h}_0 = 2\vec{A}_1\overline{z}^{1-\theta_0}\frac{dz}{d\overline{z}} + \cdots + O(|z|^{5-\theta_0})$$

and

$$g^{-1}\otimes\overline{\partial}|\vec{\Phi}|^2\otimes\vec{h}_0 = \frac{2}{\theta_0}\vec{A}_1 z^{\theta_0}\,dz + \cdots + O(|z|^{\theta_0+4}).$$

We also have

$$g^{-1}\otimes\langle\vec{h}_0,\vec{\Phi}\rangle = \left\langle 2\vec{A}_1\overline{z}^{1-\theta_0}\frac{dz}{d\overline{z}} + \cdots + O(|z|^{5-\theta_0}), \mathrm{Re}\left(\frac{2}{\theta_0}\vec{A}_0 z^{\theta_0}\right) + O(|z|^{\theta_0+4})\right\rangle = \cdots + O(|z|^5).$$

3.2 Development of $\vec{\Phi}$ and $|\vec{\Phi}|^2$ up to order 4

We first recall that

$$\vec{\Phi}(z) = \begin{pmatrix} \dfrac{1}{\theta_0} & \overline{A_0} & 0 & \theta_0 \\ \dfrac{1}{\theta_0+1} & \overline{A_1} & 0 & \theta_0+1 \\ \dfrac{1}{\theta_0+2} & \overline{A_2} & 0 & \theta_0+2 \\ \dfrac{1}{\theta_0+3} & \overline{A_3} & 0 & \theta_0+3 \\ \dfrac{1}{\theta_0} & A_0 & \theta_0 & 0 \\ \dfrac{1}{\theta_0+1} & A_1 & \theta_0+1 & 0 \\ \dfrac{1}{\theta_0+2} & A_2 & \theta_0+2 & 0 \\ \dfrac{1}{\theta_0+3} & A_3 & \theta_0+3 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{1}{8\,\theta_0} & C_1 & 2 & \theta_0 \\ \dfrac{1}{8\,(\theta_0+1)} & B_1 & 2 & \theta_0+1 \\ \dfrac{1}{12\,\theta_0} & C_2 & 3 & \theta_0 \\ \dfrac{1}{8\,(\theta_0+1)} & \overline{B_1} & \theta_0+1 & 2 \\ \dfrac{1}{8\,\theta_0} & \overline{C_1} & \theta_0 & 2 \\ \dfrac{1}{12\,\theta_0} & \overline{C_2} & \theta_0 & 3 \end{pmatrix} + O(|z|^{\theta_0+4})$$

$$= \mathrm{Re}\left(\frac{2}{\theta_0}\vec{A}_0 z^{\theta_0} + \frac{2}{\theta_0+1}\vec{A}_1 z^{\theta_0+1} + \frac{2}{\theta_0+2}\vec{A}_2 z^{\theta_0+2} + \frac{2}{\theta_0+3}\vec{A}_3 z^{\theta_0+3} + \frac{1}{4\theta_0}\vec{C}_1 z^2\overline{z}^{\theta_0} + \frac{1}{6\theta_0}\vec{C}_2 z^3\overline{z}^{\theta_0} + \frac{1}{4(\theta_0+1)}\vec{B}_1 z^2\overline{z}^{\theta_0+1}\right) + O(|z|^{\theta_0+4})$$

Recall that

$$\alpha_2 = \frac{1}{2\theta_0(\theta_0+1)}\langle\overline{\vec{A}_1}, \vec{C}_1\rangle$$

so

$$\vec{B}_1 = -2\langle\overline{\vec{A}_1}, \vec{C}_1\rangle\vec{A}_0 = -4\theta_0(\theta_0+1)\alpha_2\vec{A}_0. \tag{3.2.1}$$

Now we compute

$$|\vec{\Phi}(z)|^2 = \begin{pmatrix}
\frac{2\,\overline{A_0 A_1}}{\theta_0^2+\theta_0} & 0 & 2\,\theta_0+1 \\
\frac{(\theta_0^2+2\,\theta_0)\overline{A_1}^2+2\,(\theta_0^2+2\,\theta_0+1)\overline{A_0 A_2}}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 0 & 2\,\theta_0+2 \\
\frac{2\,((\theta_0^2+3\,\theta_0)\overline{A_1 A_2}+(\theta_0^2+3\,\theta_0+2)\overline{A_0 A_3})}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 0 & 2\,\theta_0+3 \\
\frac{\overline{A_0}^2}{\theta_0^2} & 0 & 2\,\theta_0 \\
\frac{B_1\overline{A_0}+C_1\overline{A_1}}{4\,(\theta_0^2+\theta_0)} & 2 & 2\,\theta_0+1 \\
\frac{C_1\overline{A_0}}{4\,\theta_0^2} & 2 & 2\,\theta_0 \\
\frac{C_2\overline{A_0}}{6\,\theta_0^2} & 3 & 2\,\theta_0 \\
\frac{2\,A_0\overline{A_0}}{\theta_0^2} & \theta_0 & \theta_0 \\
\frac{2\,A_0\overline{A_1}}{\theta_0^2+\theta_0} & \theta_0 & \theta_0+1 \\
\frac{8\,A_0\theta_0\overline{A_2}+(\theta_0+2)\overline{A_0 C_1}}{4\,(\theta_0^3+2\,\theta_0^2)} & \theta_0 & \theta_0+2 \\
\frac{24\,(\theta_0^2+\theta_0)A_0\overline{A_3}+3\,(\theta_0^2+3\,\theta_0)\overline{A_1 C_1}+2\,(\theta_0^2+4\,\theta_0+3)\overline{A_0 C_2}}{12\,(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0 & \theta_0+3 \\
\frac{2\,A_1\overline{A_0}}{\theta_0^2+\theta_0} & \theta_0+1 & \theta_0 \\
\frac{2\,A_1\overline{A_1}}{\theta_0^2+2\,\theta_0+1} & \theta_0+1 & \theta_0+1 \\
\frac{8\,A_1\theta_0\overline{A_2}+(\theta_0+2)\overline{A_0 B_1}}{4\,(\theta_0^3+3\,\theta_0^2+2\,\theta_0)} & \theta_0+1 & \theta_0+2 \\
\frac{A_0C_1(\theta_0+2)+8\,A_2\theta_0\overline{A_0}}{4\,(\theta_0^3+2\,\theta_0^2)} & \theta_0+2 & \theta_0 \\
\frac{A_0B_1(\theta_0+2)+8\,A_2\theta_0\overline{A_1}}{4\,(\theta_0^3+3\,\theta_0^2+2\,\theta_0)} & \theta_0+2 & \theta_0+1 \\
\frac{3\,(\theta_0^2+3\,\theta_0)A_1C_1+2\,(\theta_0^2+4\,\theta_0+3)A_0C_2+24\,(\theta_0^2+\theta_0)A_3\overline{A_0}}{12\,(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0+3 & \theta_0 \\
\frac{2\,A_0A_1}{\theta_0^2+\theta_0} & 2\,\theta_0+1 & 0 \\
\frac{A_0\overline{B_1}+A_1\overline{C_1}}{4\,(\theta_0^2+\theta_0)} & 2\,\theta_0+1 & 2 \\
\frac{(\theta_0^2+2\,\theta_0)A_1^2+2\,(\theta_0^2+2\,\theta_0+1)A_0A_2}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 2\,\theta_0+2 & 0 \\
\frac{2\,((\theta_0^2+3\,\theta_0)A_1A_2+(\theta_0^2+3\,\theta_0+2)A_0A_3)}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 2\,\theta_0+3 & 0 \\
\frac{A_0^2}{\theta_0^2} & 2\,\theta_0 & 0 \\
\frac{A_0\overline{C_1}}{4\,\theta_0^2} & 2\,\theta_0 & 2 \\
\frac{A_0\overline{C_2}}{6\,\theta_0^2} & 2\,\theta_0 & 3
\end{pmatrix}$$

$$
= \begin{pmatrix}
\frac{(\theta_0^2 + 2\,\theta_0)\overline{A_1}^2 + 2\,(\theta_0^2 + 2\,\theta_0 + 1)\overline{A_0 A_2}}{\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0} & 0 & 2\,\theta_0 + 2 & \mathbf{(1)} \\
\frac{2\,((\theta_0^2 + 3\,\theta_0)\overline{A_1 A_2} + (\theta_0^2 + 3\,\theta_0 + 2)\overline{A_0 A_3})}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0} & 0 & 2\,\theta_0 + 3 & \mathbf{(2)} \\
\frac{B_1 \overline{A_0} + C_1 \overline{A_1}}{4\,(\theta_0^2 + \theta_0)} & 2 & 2\,\theta_0 + 1 & \mathbf{(3)} \\
\frac{2\,A_0 \overline{A_0}}{\theta_0^2} & \theta_0 & \theta_0 & \mathbf{(4)} \\
\frac{8\,A_0 \overline{A_2}}{4\,(\theta_0^2 + 2\,\theta_0)} & \theta_0 & \theta_0 + 2 & \mathbf{(5)} \\
\frac{24\,(\theta_0^2 + \theta_0)A_0 \overline{A_3} + 3\,(\theta_0^2 + 3\,\theta_0)\overline{A_1 C_1} + 2\,(\theta_0^2 + 4\,\theta_0 + 3)\overline{A_0 C_2}}{12\,(\theta_0^4 + 4\,\theta_0^3 + 3\,\theta_0^2)} & \theta_0 & \theta_0 + 3 & \mathbf{(6)} \\
\frac{2\,A_1 \overline{A_1}}{\theta_0^2 + 2\,\theta_0 + 1} & \theta_0 + 1 & \theta_0 + 1 & \mathbf{(7)} \\
\frac{8\,A_1 \theta_0 \overline{A_2}}{4\,(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0)} & \theta_0 + 1 & \theta_0 + 2 & \mathbf{(8)} \\
\frac{8\,A_2 \theta_0 \overline{A_0}}{4\,(\theta_0^3 + 2\,\theta_0^2)} & \theta_0 + 2 & \theta_0 & \mathbf{(9)} \\
\frac{8\,A_2 \theta_0 \overline{A_1}}{4\,(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0)} & \theta_0 + 2 & \theta_0 + 1 & \mathbf{(10)} \\
\frac{3\,(\theta_0^2 + 3\,\theta_0)A_1 C_1 + 2\,(\theta_0^2 + 4\,\theta_0 + 3)A_0 C_2 + 24\,(\theta_0^2 + \theta_0)A_3 \overline{A_0}}{12\,(\theta_0^4 + 4\,\theta_0^3 + 3\,\theta_0^2)} & \theta_0 + 3 & \theta_0 & \mathbf{(11)} \\
\frac{A_0 \overline{B_1} + A_1 \overline{C_1}}{4\,(\theta_0^2 + \theta_0)} & 2\,\theta_0 + 1 & 2 & \mathbf{(12)} \\
\frac{(\theta_0^2 + 2\,\theta_0)A_1^2 + 2\,(\theta_0^2 + 2\,\theta_0 + 1)A_0 A_2}{\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0} & 2\,\theta_0 + 2 & 0 & \mathbf{(13)} \\
\frac{2\,((\theta_0^2 + 3\,\theta_0)A_1 A_2 + (\theta_0^2 + 3\,\theta_0 + 2)A_0 A_3)}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0} & 2\,\theta_0 + 3 & 0 & \mathbf{(14)}
\end{pmatrix}
$$

Furthermore, we can further simplify this expression with

$$\langle \vec{A}_1, \vec{A}_1 \rangle + 2\langle \vec{A}_0, \vec{A}_2 \rangle = 0, \quad \langle \vec{A}_1, \vec{A}_2 \rangle + \langle \vec{A}_0, \vec{A}_3 \rangle = 0, \quad \langle \vec{A}_1, \vec{C}_1 \rangle + \langle \vec{A}_0, \vec{C}_2 \rangle = 0.$$

We now define $\zeta_0, \zeta_1, \zeta_2 \in \mathbb{C}$ such that

$$\begin{cases} \zeta_0 = \langle \vec{A}_1, \vec{A}_1 \rangle \\ \zeta_1 = \langle \vec{A}_1, \vec{A}_2 \rangle \\ \zeta_2 = \langle \vec{A}_1, \vec{C}_1 \rangle \end{cases} \tag{3.2.2}$$

which gives

$$
\begin{aligned}
\mathbf{(13)} &= \frac{(\theta_0^2 + 2\,\theta_0)A_1^2 + 2\,(\theta_0^2 + 2\,\theta_0 + 1)A_0 A_2}{\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0} = \frac{2\langle \vec{A}_0, \vec{A}_2 \rangle}{\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0} = -\frac{\langle \vec{A}_1, \vec{A}_1 \rangle}{\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0} \\
&= \frac{-\zeta_0}{\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0} \\
\mathbf{(14)} &= \frac{2\,((\theta_0^2 + 3\,\theta_0)A_1 A_2 + (\theta_0^2 + 3\,\theta_0 + 2)A_0 A_3)}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0} = \frac{4\langle \vec{A}_0, \vec{A}_3 \rangle}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0} = \frac{-4\langle \vec{A}_1, \vec{A}_2 \rangle}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0} \\
&= \frac{-4\,\zeta_1}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0}
\end{aligned} \tag{3.2.3}
$$

Now, recall that

$$\begin{cases} \alpha_1 = 2\langle \overline{\vec{A}_0}, \vec{A}_2 \rangle \\ \alpha_2 = \dfrac{1}{2\theta_0(\theta_0+1)} \langle \overline{\vec{A}_1}, \vec{C}_1 \rangle \\ \alpha_3 = \dfrac{1}{12} \langle \vec{A}_1, \vec{C}_1 \rangle + 2\langle \overline{\vec{A}_0}, \vec{A}_3 \rangle \\ \alpha_5 = 2\langle \overline{\vec{A}_1}, \vec{A}_2 \rangle \\ \alpha_6 = 2\langle \overline{\vec{A}_1}, \vec{A}_3 \rangle \\ \alpha_7 = \dfrac{1}{8\theta_0(\theta_0-4)} \langle \vec{C}_1, \vec{C}_1 \rangle \end{cases} \tag{3.2.4}$$

so

$$\begin{aligned} \mathbf{(11)} &= \frac{3\left(\theta_0^2+3\,\theta_0\right)A_1C_1 + 2\left(\theta_0^2+4\,\theta_0+3\right)A_0C_2 + 24\left(\theta_0^2+\theta_0\right)A_3\overline{A_0}}{12\left(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2\right)} \\ &= \frac{3\left(\theta_0^2+3\,\theta_0\right)A_1C_1 - 2\left(\theta_0^2+4\,\theta_0+3\right)A_1C_1 + 24\left(\theta_0^2+\theta_0\right)A_3\overline{A_0}}{12\left(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2\right)} \\ &= \frac{\left(3\theta_0^2+9\,\theta_0-2\,\theta_0^2-8\theta_0-6\right)A_1C_1 + 12\left(\theta_0(\theta_0+1)\right)(\alpha_3-\frac{1}{12}A_1\,C_1)}{12\left(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2\right)} \\ &= \frac{-6\,A_1C_1+12\theta_0(\theta_0+1)\alpha_3}{12\left(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2\right)} = \frac{-A_1C_1+2\theta_0(\theta_0+1)\alpha_3}{2(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} = \frac{-\zeta_2+2\theta_0(\theta_0+1)\alpha_3}{2(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} \end{aligned} \tag{3.2.5}$$

Then

$$\langle \vec{A}_0, \overline{\vec{B}_1} \rangle = \langle \vec{A}_0, -2\langle \vec{A}_1, \overline{\vec{C}_1} \rangle \vec{A}_0 \rangle = -\langle \vec{A}_1, \overline{\vec{C}_1} \rangle$$

so

$$\mathbf{(12)} = \frac{A_0\overline{B_1} + A_1\overline{C_1}}{4\left(\theta_0^2+\theta_0\right)} = 0$$

Now we directly have as $\alpha_5 = 2\langle \overline{\vec{A}_1}, \vec{A}_2 \rangle$ the identity

$$\mathbf{(10)} = \frac{8\,A_2\theta_0\overline{A_1}}{4\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)} = \frac{\alpha_5}{\theta_0^2+3\theta_0+2} = \frac{\alpha_5}{(\theta_0+1)(\theta_0+2)}.$$

Likewise,

$$\mathbf{(9)} = \frac{8\,A_2\theta_0\overline{A_0}}{4\left(\theta_0^3+2\,\theta_0^2\right)} = \frac{\alpha_1}{\theta_0(\theta_0+2)}.$$

Therefore, we finally obtain

$$|\vec{\Phi}(z)|^2 = \begin{pmatrix} \dfrac{-\overline{\zeta_0}}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 0 & 2\,\theta_0+2 & \textbf{(1)} \\ \dfrac{-4\overline{\zeta_1}}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 0 & 2\,\theta_0+3 & \textbf{(2)} \\ \dfrac{1}{\theta_0^2} & \theta_0 & \theta_0 & \textbf{(4)} \\ \dfrac{\overline{\alpha_1}}{\theta_0(\theta_0+2)} & \theta_0 & \theta_0+2 & \textbf{(5)} \\ \dfrac{-\overline{\zeta_2}+2\theta_0(\theta_0+1)\overline{\alpha_3}}{2(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0 & \theta_0+3 & \textbf{(6)} \\ \dfrac{2\,|A_1|^2}{(\theta_0+1)^2} & \theta_0+1 & \theta_0+1 & \textbf{(7)} \\ \dfrac{\overline{\alpha_5}}{(\theta_0+1)(\theta_0+2)} & \theta_0+1 & \theta_0+2 & \textbf{(8)} \\ \dfrac{\alpha_1}{\theta_0(\theta_0+2)} & \theta_0+2 & \theta_0 & \textbf{(9)} \\ \dfrac{\alpha_5}{(\theta_0+1)(\theta_0+2)} & \theta_0+2 & \theta_0+1 & \textbf{(10)} \\ \dfrac{-\zeta_2+2\theta_0(\theta_0+1)\alpha_3}{2(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0+3 & \theta_0 & \textbf{(11)} \\ \dfrac{-\zeta_0}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 2\,\theta_0+2 & 0 & \textbf{(13)} \\ \dfrac{-4\zeta_1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 2\,\theta_0+3 & 0 & \textbf{(14)} \end{pmatrix}$$

$$= \frac{1}{\theta_0^2}|z|^{2\theta_0} + \frac{2|\vec{A}_1|^2}{(\theta_0+1)^2}|z|^{2\theta_0+2} + 2\,\mathrm{Re}\left(\frac{\alpha_1}{\theta_0(\theta_0+2)}z^{\theta_0+2}\overline{z}^{\theta_0} + \frac{\alpha_5}{(\theta_0+1)(\theta_0+2)}z^{\theta_0+2}\overline{z}^{\theta_0+1} + \right.$$

$$\left. + \frac{-\zeta_2+2\theta_0(\theta_0+1)\alpha_3}{2(\theta_0^4+4\theta_0^3+3\theta_0^2)}z^{\theta_0+3}\overline{z}^{\theta_0} - \frac{\zeta_0}{\theta_0^4+4\theta_0^3+5\theta_0^2+2\theta_0}z^{2\theta_0+2} - \frac{4\zeta_1}{\theta_0^4+6\theta_0^3+11\theta_0^2+6\theta_0}z^{2\theta_0+3}\right) + O(|z|^{2\theta_0+4})$$

$$= |z|^{2\theta_0}\left\{\frac{1}{\theta_0^2} + \frac{2|\vec{A}_1|^2}{(\theta_0+1)^2}|z|^2 + 2\,\mathrm{Re}\left(\frac{\alpha_1}{\theta_0(\theta_0+2)}z^2 + \frac{\alpha_5}{(\theta_0+1)(\theta_0+2)}z^2\overline{z} + + \frac{-\zeta_2+2\theta_0(\theta_0+1)\alpha_3}{2(\theta_0^4+4\theta_0^3+3\theta_0^2)}z^3\right.\right.$$

$$\left.\left. - \frac{\zeta_0}{\theta_0^4+4\theta_0^3+5\theta_0^2+2\theta_0}z^{\theta_0+2}\overline{z}^{-\theta_0} - \frac{4\zeta_1}{\theta_0^4+6\theta_0^3+11\theta_0^2+6\theta_0}z^{\theta_0+3}\overline{z}^{-\theta_0}\right) + O(|z|^4)\right\}$$

to be compare with the right-hand size in the next equation

$$|\vec{\Phi}(z)|^2 = \begin{pmatrix} \frac{-\overline{\zeta_0}}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 0 & 2\,\theta_0+2 & \mathbf{(1)} \\ \frac{-4\overline{\zeta_1}}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 0 & 2\,\theta_0+3 & \mathbf{(2)} \\ \frac{1}{\theta_0^2} & \theta_0 & \theta_0 & \mathbf{(4)} \\ \frac{\overline{\alpha_1}}{\theta_0(\theta_0+2)} & \theta_0 & \theta_0+2 & \mathbf{(5)} \\ \frac{-\overline{\zeta_2}+2\theta_0(\theta_0+1)\overline{\alpha_3}}{2(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0 & \theta_0+3 & \mathbf{(6)} \\ \frac{2\,|A_1|^2}{(\theta_0+1)^2} & \theta_0+1 & \theta_0+1 & \mathbf{(7)} \\ \frac{\overline{\alpha_5}}{(\theta_0+1)(\theta_0+2)} & \theta_0+1 & \theta_0+2 & \mathbf{(8)} \\ \frac{\alpha_1}{\theta_0(\theta_0+2)} & \theta_0+2 & \theta_0 & \mathbf{(9)} \\ \frac{\alpha_5}{(\theta_0+1)(\theta_0+2)} & \theta_0+2 & \theta_0+1 & \mathbf{(10)} \\ \frac{-\zeta_2+2\theta_0(\theta_0+1)\alpha_3}{2(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0+3 & \theta_0 & \mathbf{(11)} \\ \frac{-\zeta_0}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 2\,\theta_0+2 & 0 & \mathbf{(13)} \\ \frac{-4\zeta_1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 2\,\theta_0+3 & 0 & \mathbf{(14)} \end{pmatrix} = \begin{pmatrix} -\frac{\overline{\zeta_0}}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 0 & 2\,\theta_0+2 \\ -\frac{4\,\overline{\zeta_1}}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 0 & 2\,\theta_0+3 \\ \frac{1}{\theta_0^2} & \theta_0 & \theta_0 \\ \frac{\overline{\alpha_1}}{(\theta_0+2)\theta_0} & \theta_0 & \theta_0+2 \\ \frac{2\,(\theta_0+1)\theta_0\overline{\alpha_3}-\overline{\zeta_2}}{2\,(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0 & \theta_0+3 \\ \frac{2\,|A_1|^2}{(\theta_0+1)^2} & \theta_0+1 & \theta_0+1 \\ \frac{\overline{\alpha_5}}{(\theta_0+2)(\theta_0+1)} & \theta_0+1 & \theta_0+2 \\ \frac{\alpha_1}{(\theta_0+2)\theta_0} & \theta_0+2 & \theta_0 \\ \frac{\alpha_5}{(\theta_0+2)(\theta_0+1)} & \theta_0+2 & \theta_0+1 \\ \frac{2\,\alpha_3(\theta_0+1)\theta_0-\zeta_2}{2\,(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0+3 & \theta_0 \\ -\frac{\zeta_0}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 2\,\theta_0+2 & 0 \\ -\frac{4\,\zeta_1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 2\,\theta_0+3 & 0 \end{pmatrix}$$

Another test to check that our quantity is real.

$$|\vec{\Phi}(z)|^2 = \begin{pmatrix} \frac{-\overline{\zeta_0}}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 0 & 2\,\theta_0+2 & \mathbf{(1)} \\ \frac{-4\overline{\zeta_1}}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 0 & 2\,\theta_0+3 & \mathbf{(2)} \\ \frac{1}{\theta_0^2} & \theta_0 & \theta_0 & \mathbf{(4)} \\ \frac{\overline{\alpha_1}}{\theta_0(\theta_0+2)} & \theta_0 & \theta_0+2 & \mathbf{(5)} \\ \frac{-\overline{\zeta_2}+2\theta_0(\theta_0+1)\overline{\alpha_3}}{2(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0 & \theta_0+3 & \mathbf{(6)} \\ \frac{2\,|A_1|^2}{(\theta_0+1)^2} & \theta_0+1 & \theta_0+1 & \mathbf{(7)} \\ \frac{\overline{\alpha_5}}{(\theta_0+1)(\theta_0+2)} & \theta_0+1 & \theta_0+2 & \mathbf{(8)} \\ \frac{\alpha_1}{\theta_0(\theta_0+2)} & \theta_0+2 & \theta_0 & \mathbf{(9)} \\ \frac{\alpha_5}{(\theta_0+1)(\theta_0+2)} & \theta_0+2 & \theta_0+1 & \mathbf{(10)} \\ \frac{-\zeta_2+2\theta_0(\theta_0+1)\alpha_3}{2(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0+3 & \theta_0 & \mathbf{(11)} \\ \frac{-\zeta_0}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 2\,\theta_0+2 & 0 & \mathbf{(13)} \\ \frac{-4\zeta_1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 2\,\theta_0+3 & 0 & \mathbf{(14)} \end{pmatrix} = \begin{pmatrix} -\frac{\overline{\zeta_0}}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 0 & 2\,\theta_0+2 \\ -\frac{4\,\overline{\zeta_1}}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 0 & 2\,\theta_0+3 \\ \frac{1}{\theta_0^2} & \theta_0 & \theta_0 \\ \frac{\overline{\alpha_1}}{\theta_0^2+2\,\theta_0} & \theta_0 & \theta_0+2 \\ \frac{2\,\theta_0^2\overline{\alpha_3}+2\,\theta_0\overline{\alpha_3}-\overline{\zeta_2}}{2\,(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0 & \theta_0+3 \\ \frac{2\,|A_1|^2}{\theta_0^2+2\,\theta_0+1} & \theta_0+1 & \theta_0+1 \\ \frac{\overline{\alpha_5}}{\theta_0^2+3\,\theta_0+2} & \theta_0+1 & \theta_0+2 \\ \frac{\alpha_1}{\theta_0^2+2\,\theta_0} & \theta_0+2 & \theta_0 \\ \frac{\alpha_5}{\theta_0^2+3\,\theta_0+2} & \theta_0+2 & \theta_0+1 \\ \frac{2\,\alpha_3\theta_0^2+2\,\alpha_3\theta_0-\zeta_2}{2\,(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0+3 & \theta_0 \\ -\frac{\zeta_0}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 2\,\theta_0+2 & 0 \\ -\frac{4\,\zeta_1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 2\,\theta_0+3 & 0 \end{pmatrix}$$

3.3 Development of $g^{-1} \otimes \vec{h}_0$ up to order 4

We directly compute

$$
g^{-1} \otimes \vec{h}_0 = \begin{pmatrix}
2 & A_1 & 0 & -\theta_0 + 1 \\
-4\,|A_1|^2 & A_0 & 0 & -\theta_0 + 2 \\
\frac{1}{4} & \overline{B_1} & 0 & -\theta_0 + 3 \\
-2\,\overline{\alpha_5} & A_0 & 0 & -\theta_0 + 3 \\
-2\,\overline{\alpha_1} & A_1 & 0 & -\theta_0 + 3 \\
\frac{1}{6} & \overline{B_3} & 0 & -\theta_0 + 4 \\
-2\,\overline{\alpha_3} & A_1 & 0 & -\theta_0 + 4 \\
-\frac{1}{6}\,|A_1|^2 & \overline{C_1} & 0 & -\theta_0 + 4 \\
8\,|A_1|^2\overline{\alpha_1} - 2\,\overline{\alpha_6} & A_0 & 0 & -\theta_0 + 4 \\
4 & A_2 & 1 & -\theta_0 + 1 \\
-4\,\alpha_1 & A_0 & 1 & -\theta_0 + 1 \\
-4\,\alpha_5 & A_0 & 1 & -\theta_0 + 2 \\
-8\,|A_1|^2 & A_1 & 1 & -\theta_0 + 2 \\
\frac{1}{2} & \overline{B_2} & 1 & -\theta_0 + 3 \\
-4\,\overline{\alpha_5} & A_1 & 1 & -\theta_0 + 3 \\
-4\,\overline{\alpha_1} & A_2 & 1 & -\theta_0 + 3 \\
16\,|A_1|^4 + 8\,\alpha_1\overline{\alpha_1} - 4\,\beta & A_0 & 1 & -\theta_0 + 3 \\
6 & A_3 & 2 & -\theta_0 + 1 \\
-6\,\alpha_3 & A_0 & 2 & -\theta_0 + 1 \\
-6\,\alpha_1 & A_1 & 2 & -\theta_0 + 1 \\
-6\,\alpha_5 & A_1 & 2 & -\theta_0 + 2 \\
-12\,|A_1|^2 & A_2 & 2 & -\theta_0 + 2 \\
24\,\alpha_1|A_1|^2 - 6\,\alpha_6 & A_0 & 2 & -\theta_0 + 2
\end{pmatrix}
\begin{pmatrix}
8 & A_4 & 3 & -\theta_0 + 1 \\
-8\,\alpha_3 & A_1 & 3 & -\theta_0 + 1 \\
-8\,\alpha_1 & A_2 & 3 & -\theta_0 + 1 \\
8\,\alpha_1^2 - 8\,\alpha_4 & A_0 & 3 & -\theta_0 + 1 \\
-\frac{\theta_0 - 2}{2\,\theta_0} & E_1 & -2\,\theta_0 + 3 & \theta_0 + 1 \\
-\frac{\theta_0 - 2}{2\,\theta_0} & C_1 & -\theta_0 + 1 & 1 \\
-\frac{\theta_0 - 2}{2\,(\theta_0 + 1)} & B_1 & -\theta_0 + 1 & 2 \\
2\,\alpha_2\theta_0 - 4\,\alpha_2 & A_0 & -\theta_0 + 1 & 2 \\
-\frac{\theta_0 - 2}{2\,(\theta_0 + 2)} & B_2 & -\theta_0 + 1 & 3 \\
\frac{\theta_0\overline{\alpha_1} - 2\,\overline{\alpha_1}}{\theta_0^2 + 2\,\theta_0} & C_1 & -\theta_0 + 1 & 3 \\
2\,\alpha_9\theta_0 - 4\,\alpha_9 & A_0 & -\theta_0 + 1 & 3 \\
-\frac{\theta_0 - 3}{2\,\theta_0} & C_2 & -\theta_0 + 2 & 1 \\
-\frac{\theta_0 - 3}{2\,(\theta_0 + 1)} & B_3 & -\theta_0 + 2 & 2 \\
\frac{(\theta_0 - 3)|A_1|^2}{\theta_0^2 + \theta_0} & C_1 & -\theta_0 + 2 & 2 \\
2\,\alpha_8\theta_0 - 6\,\alpha_8 & A_0 & -\theta_0 + 2 & 2 \\
2\,\alpha_2\theta_0 - 6\,\alpha_2 & A_1 & -\theta_0 + 2 & 2 \\
-\frac{\theta_0 - 4}{2\,\theta_0} & C_3 & -\theta_0 + 3 & 1 \\
2\,\alpha_7\theta_0 - 8\,\alpha_7 & A_0 & -\theta_0 + 3 & 1 \\
-2\,\theta_0\overline{\alpha_2} - 2\,\overline{\alpha_2} & A_0 & \theta_0 & -2\,\theta_0 + 3 \\
-2\,\theta_0\overline{\alpha_8} - 2\,\overline{\alpha_8} & A_0 & \theta_0 & -2\,\theta_0 + 4 \\
-\frac{\theta_0}{2\,(\theta_0 - 4)} & \overline{E_1} & \theta_0 - 1 & -2\,\theta_0 + 5 \\
-2\,\theta_0\overline{\alpha_7} & A_0 & \theta_0 - 1 & -2\,\theta_0 + 5 \\
-2\,\theta_0\overline{\alpha_9} - 4\,\overline{\alpha_9} & A_0 & \theta_0 + 1 & -2\,\theta_0 + 3 \\
-2\,\theta_0\overline{\alpha_2} - 4\,\overline{\alpha_2} & A_1 & \theta_0 + 1 & -2\,\theta_0 + 3
\end{pmatrix}
$$

and indeed we have a development of $g^{-1} \otimes \vec{h}_0$ up to an error in $O(|z|^{5-\theta_0})$.

3.4 Development of $\vec{\alpha}$ up to order 3

Now we need to do some preformating, and we first compute

$$\vec{\alpha}=\begin{pmatrix}
\frac{1}{2} & \overline{B_1} & 0 & -\theta_0+2\\
2\,A_1\overline{C_1} & \overline{A_0} & 0 & -\theta_0+2\\
\frac{1}{2} & \overline{B_3} & 0 & -\theta_0+3\\
2\,A_1\overline{C_1} & \overline{A_1} & 0 & -\theta_0+3\\
\cancel{-4\,A_0|A_1|^2\overline{C_1}}+2\,A_1\overline{C_2} & \overline{A_0} & 0 & -\theta_0+3\\
1 & \overline{B_2} & 1 & -\theta_0+2\\
\frac{1}{2}\,C_1\overline{C_1} & A_0 & 1 & -\theta_0+2\\
\cancel{2\,A_1\overline{B_1}}-4\,(\cancel{A_0\alpha_1}-A_2)\overline{C_1} & \overline{A_0} & 1 & -\theta_0+2\\
-\theta_0+2 & E_1 & -2\,\theta_0+3 & \theta_0\\
-\frac{C_1^2(\theta_0-2)}{2\,\theta_0} & \overline{A_0} & -2\,\theta_0+3 & \theta_0\\
-\frac{1}{2}\,\theta_0+1 & C_1 & -\theta_0+1 & 0\\
-\frac{1}{2}\,\theta_0+1 & B_1 & -\theta_0+1 & 1\\
-\frac{1}{2}\,\theta_0+1 & B_2 & -\theta_0+1 & 2\\
-\frac{C_1(\theta_0-2)\overline{C_1}}{2\,\theta_0} & \overline{A_0} & -\theta_0+1 & 2\\
-\frac{1}{2}\,\theta_0+\frac{3}{2} & C_2 & -\theta_0+2 & 0\\
2\,A_1C_1 & \overline{A_0} & -\theta_0+2 & 0\\
-\frac{1}{2}\,\theta_0+\frac{3}{2} & B_3 & -\theta_0+2 & 1\\
2\,A_1C_1 & \overline{A_1} & -\theta_0+2 & 1\\
\cancel{-4\,A_0C_1|A_1|^2}+\cancel{2\,A_1B_1} & \overline{A_0} & -\theta_0+2 & 1\\
-\frac{1}{2}\,\theta_0+2 & C_3 & -\theta_0+3 & 0\\
\frac{1}{4}\,C_1^2 & A_0 & -\theta_0+3 & 0\\
-4\,(\cancel{A_0\alpha_1}-A_2)C_1+2\,A_1C_2 & \overline{A_0} & -\theta_0+3 & 0\\
\frac{1}{2}\,\theta_0 & \overline{E_1} & \theta_0-1 & -2\,\theta_0+4\\
\frac{1}{4}\,\overline{C_1}^2 & A_0 & \theta_0-1 & -2\,\theta_0+4
\end{pmatrix}
=\begin{pmatrix}
\frac{1}{2} & \overline{B_1} & 0 & -\theta_0+2 & \textbf{(1)}\\
2\,A_1\overline{C_1} & \overline{A_0} & 0 & -\theta_0+2 & \textbf{(2)}\\
\frac{1}{2} & \overline{B_3} & 0 & -\theta_0+3 & \textbf{(3)}\\
2\,A_1\overline{C_1} & \overline{A_1} & 0 & -\theta_0+3 & \textbf{(4)}\\
2\,A_1\overline{C_2} & \overline{A_0} & 0 & -\theta_0+3 & \textbf{(5)}\\
1 & \overline{B_2} & 1 & -\theta_0+2 & \textbf{(6)}\\
\frac{1}{2}\,C_1\overline{C_1} & A_0 & 1 & -\theta_0+2 & \textbf{(7)}\\
4\,A_2\overline{C_1} & \overline{A_0} & 1 & -\theta_0+2 & \textbf{(8)}\\
-\theta_0+2 & E_1 & -2\,\theta_0+3 & \theta_0 & \textbf{(9)}\\
-\frac{C_1^2(\theta_0-2)}{2\,\theta_0} & \overline{A_0} & -2\,\theta_0+3 & \theta_0 & \textbf{(10)}\\
-\frac{1}{2}\,\theta_0+1 & C_1 & -\theta_0+1 & 0 & \textbf{(11)}\\
-\frac{1}{2}\,\theta_0+1 & B_1 & -\theta_0+1 & 1 & \textbf{(12)}\\
-\frac{1}{2}\,\theta_0+1 & B_2 & -\theta_0+1 & 2 & \textbf{(13)}\\
-\frac{C_1(\theta_0-2)\overline{C_1}}{2\,\theta_0} & \overline{A_0} & -\theta_0+1 & 2 & \textbf{(14)}\\
-\frac{1}{2}\,\theta_0+\frac{3}{2} & C_2 & -\theta_0+2 & 0 & \textbf{(15)}\\
2\,A_1C_1 & \overline{A_0} & -\theta_0+2 & 0 & \textbf{(16)}\\
-\frac{1}{2}\,\theta_0+\frac{3}{2} & B_3 & -\theta_0+2 & 1 & \textbf{(17)}\\
2\,A_1C_1 & \overline{A_1} & -\theta_0+2 & 1 & \textbf{(18)}\\
-\frac{1}{2}\,\theta_0+2 & C_3 & -\theta_0+3 & 0 & \textbf{(19)}\\
\frac{1}{4}\,C_1^2 & A_0 & -\theta_0+3 & 0 & \textbf{(20)}\\
4\,A_2C_1+2\,A_1C_2 & \overline{A_0} & -\theta_0+3 & 0 & \textbf{(21)}\\
\frac{1}{2}\,\theta_0 & \overline{E_1} & \theta_0-1 & -2\,\theta_0+4 & \textbf{(22)}\\
\frac{1}{4}\,\overline{C_1}^2 & A_0 & \theta_0-1 & -2\,\theta_0+4 & \textbf{(23)}
\end{pmatrix}$$

Most of the coefficients here cancel. Indeed, recall that

$$\begin{cases}
\vec{B}_1=-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0\\
\vec{B}_2=-\dfrac{(\theta_0+2)}{4\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0}-2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\vec{A}_0\\
\vec{B}_3=-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_1+\dfrac{2}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_1}-2\langle\overline{\vec{A}_1},\vec{C}_2\rangle\vec{A}_0\\
\vec{E}_1=-\dfrac{1}{2\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_0}.
\end{cases}\tag{3.4.1}$$

Therefore, we have

$$\begin{pmatrix} \frac{1}{2} & \overline{B_1} & 0 & -\theta_0+2 & \mathbf{(1)} \\ 2\,A_1\overline{C_1} & \overline{A_0} & 0 & -\theta_0+2 & \mathbf{(2)} \end{pmatrix} = \begin{pmatrix} -A_1\overline{C_1} & \overline{A_0} & 0 & -\theta_0+2 & \mathbf{(1)} \\ 2\,A_1\overline{C_1} & \overline{A_0} & 0 & -\theta_0+2 & \mathbf{(2)} \end{pmatrix} = \begin{pmatrix} A_1\overline{C_1} & \overline{A_0} & 0 & -\theta_0+2 & \mathbf{(1)} \end{pmatrix}$$

$$= \begin{pmatrix} 2\theta_0(\theta_0+1)\overline{\alpha_2} & \overline{A_0} & 0 & -\theta_0+2 & \mathbf{(1)} \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{1}{2} & \overline{B_3} & 0 & -\theta_0+3 & \mathbf{(3)} \\ 2\,A_1\overline{C_1} & \overline{A_1} & 0 & -\theta_0+3 & \mathbf{(4)} \\ 2\,A_1\overline{C_2} & \overline{A_0} & 0 & -\theta_0+3 & \mathbf{(5)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\theta_0-3}\overline{\langle \vec{A}_1, \vec{C}_1\rangle} & A_1 & 0 & -\theta_0+3 & \mathbf{(3)} \\ A_1\overline{C_1} & \overline{A_1} & 0 & -\theta_0+3 & \mathbf{(4)} \\ A_1\overline{C_2} & \overline{A_0} & 0 & -\theta_0+3 & \mathbf{(5)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\overline{\zeta_2}}{\theta_0-3} & A_1 & 0 & -\theta_0+3 & \mathbf{(3)} \\ 2\theta_0(\theta_0+1)\overline{\alpha_2} & \overline{A_1} & 0 & -\theta_0+3 & \mathbf{(4)} \\ \overline{\zeta_3} & \overline{A_0} & 0 & -\theta_0+3 & \mathbf{(5)} \end{pmatrix}$$

if

$$\zeta_3 = \langle \overline{\vec{A}_1}, \vec{C}_2\rangle. \tag{3.4.2}$$

Then

$$\begin{pmatrix} 1 & \overline{B_2} & 1 & -\theta_0+2 & \mathbf{(6)} \\ \frac{1}{2}\,C_1\overline{C_1} & A_0 & 1 & -\theta_0+2 & \mathbf{(7)} \\ 4\,A_2\overline{C_1} & \overline{A_0} & 1 & -\theta_0+2 & \mathbf{(8)} \end{pmatrix} = \left(-\frac{(\theta_0+2)}{4\theta_0}|\vec{C}_1|^2\vec{A}_0 - 2\langle \vec{A}_2, \overline{\vec{C}_1}\rangle\overline{\vec{A}_0} + \frac{1}{2}|\vec{C}_1|^2\vec{A}_0 + 4\langle \vec{A}_2, \overline{\vec{C}_1}\rangle\overline{\vec{A}_0}\right) z\overline{z}^{2-\theta_0}$$

$$= \left(\frac{(\theta_0-2)}{4\theta_0}|\vec{C}_1|^2\vec{A}_0 + 2\langle \vec{A}_2, \overline{\vec{C}_1}\rangle\overline{\vec{A}_0}\right) z\overline{z}^{2-\theta_0}$$

$$= \begin{pmatrix} \frac{(\theta_0-2)}{4\theta_0}|C_1|^2 & A_0 & 1 & -\theta_0+2 & \mathbf{(6)} \\ 2\,\overline{\zeta_4} & \overline{A_0} & 1 & -\theta_0+2 & \mathbf{(8)} \end{pmatrix}$$

if

$$\zeta_4 = \langle \overline{\vec{A}_2}, \vec{C}_1\rangle.$$

Now as $\vec{E}_1 = -\frac{1}{2\theta_0}\langle \vec{C}_1, \vec{C}_1\rangle\overline{\vec{A}_0}$, we have

$$\begin{pmatrix} -\theta_0+2 & E_1 & -2\,\theta_0+3 & \theta_0 & \mathbf{(9)} \\ -\frac{C_1^2(\theta_0-2)}{2\,\theta_0} & \overline{A_0} & -2\,\theta_0+3 & \theta_0 & \mathbf{(10)} \end{pmatrix} = 0, \quad \begin{pmatrix} \frac{1}{2}\,\theta_0 & \overline{E_1} & \theta_0-1 & -2\,\theta_0+4 & \mathbf{(22)} \\ \frac{1}{4}\,\overline{C_1}^2 & A_0 & \theta_0-1 & -2\,\theta_0+4 & \mathbf{(23)} \end{pmatrix} = 0$$

Then, we have

$$\begin{pmatrix} -\frac{1}{2}\,\theta_0+1 & C_1 & -\theta_0+1 & 0 & \mathbf{(11)} \\ -\frac{1}{2}\,\theta_0+1 & B_1 & -\theta_0+1 & 1 & \mathbf{(12)} \\ -\frac{1}{2}\,\theta_0+1 & B_2 & -\theta_0+1 & 2 & \mathbf{(13)} \\ -\frac{C_1(\theta_0-2)\overline{C_1}}{2\,\theta_0} & \overline{A_0} & -\theta_0+1 & 2 & \mathbf{(14)} \end{pmatrix} = \begin{pmatrix} -\frac{(\theta_0-2)}{2} & C_1 & -\theta_0+1 & 0 & \mathbf{(11)} \\ -\frac{(\theta_0-2)}{2} & B_1 & -\theta_0+1 & 1 & \mathbf{(12)} \\ -\frac{(\theta_0-2)}{2} & B_2 & -\theta_0+1 & 2 & \mathbf{(13)} \\ -\frac{(\theta_0-2)|C_1|^2}{2\,\theta_0} & \overline{A_0} & -\theta_0+1 & 2 & \mathbf{(14)} \end{pmatrix}.$$

Finally, if

$$\zeta_5 = 2\langle \vec{A}_2, \vec{C}_1\rangle + \langle \vec{A}_1, \vec{C}_2\rangle \tag{3.4.3}$$

recalling that

$$\alpha_7 = \frac{1}{8\theta_0(\theta_0-4)}\langle \vec{C}_1, \vec{C}_1\rangle$$

we obtain

$$\begin{pmatrix} -\frac{1}{2}\,\theta_0+\frac{3}{2} & C_2 & -\theta_0+2 & 0 & \mathbf{(15)} \\ 2\,A_1C_1 & \overline{A_0} & -\theta_0+2 & 0 & \mathbf{(16)} \\ -\frac{1}{2}\,\theta_0+\frac{3}{2} & B_3 & -\theta_0+2 & 1 & \mathbf{(17)} \\ 2\,A_1C_1 & \overline{A_1} & -\theta_0+2 & 1 & \mathbf{(18)} \\ -\frac{1}{2}\,\theta_0+2 & C_3 & -\theta_0+3 & 0 & \mathbf{(19)} \\ \frac{1}{4}\,C_1^2 & A_0 & -\theta_0+3 & 0 & \mathbf{(20)} \\ 4\,A_2C_1+2\,A_1C_2 & \overline{A_0} & -\theta_0+3 & 0 & \mathbf{(21)} \end{pmatrix} = \begin{pmatrix} -\frac{\theta_0-3}{2} & C_2 & -\theta_0+2 & 0 & \mathbf{(15)} \\ 2\,\zeta_2 & \overline{A_0} & -\theta_0+2 & 0 & \mathbf{(16)} \\ -\frac{\theta_0-3}{2} & B_3 & -\theta_0+2 & 1 & \mathbf{(17)} \\ 2\,\zeta_2 & \overline{A_1} & -\theta_0+2 & 1 & \mathbf{(18)} \\ -\frac{\theta_0-4}{2} & C_3 & -\theta_0+3 & 0 & \mathbf{(19)} \\ 2\theta_0(\theta_0-4)\alpha_7 & A_0 & -\theta_0+3 & 0 & \mathbf{(20)} \\ 2\zeta_5 & \overline{A_0} & -\theta_0+3 & 0 & \mathbf{(21)} \end{pmatrix}$$

We sum up her the new coefficients ζ that we have introduced

$$\begin{cases} \zeta_0 = \langle \vec{A}_1, \vec{A}_1\rangle \\ \zeta_1 = \langle \vec{A}_1, \vec{A}_2\rangle \\ \zeta_2 = \langle \vec{A}_1, \vec{C}_1\rangle \\ \zeta_3 = \overline{\langle \vec{A}_1}, \vec{C}_2\rangle \\ \zeta_4 = \overline{\langle \vec{A}_2}, \vec{C}_1\rangle \\ \zeta_5 = 2\langle \vec{A}_2, \vec{C}_1\rangle + \langle \vec{A}_1, \vec{C}_2\rangle \end{cases} \tag{3.4.4}$$

Finally, we obtain

$$
\vec{\alpha} = \begin{pmatrix}
\frac{1}{2} & \overline{B_1} & 0 & -\theta_0+2 & \mathbf{(1)} \\
2\,A_1\overline{C_1} & \overline{A_0} & 0 & -\theta_0+2 & \mathbf{(2)} \\
\frac{1}{2} & \overline{B_3} & 0 & -\theta_0+3 & \mathbf{(3)} \\
2\,A_1\overline{C_1} & \overline{A_1} & 0 & -\theta_0+3 & \mathbf{(4)} \\
2\,A_1\overline{C_2} & \overline{A_0} & 0 & -\theta_0+3 & \mathbf{(5)} \\
1 & \overline{B_2} & 1 & -\theta_0+2 & \mathbf{(6)} \\
\frac{1}{2}\,C_1\overline{C_1} & A_0 & 1 & -\theta_0+2 & \mathbf{(7)} \\
4\,A_2\overline{C_1} & \overline{A_0} & 1 & -\theta_0+2 & \mathbf{(8)} \\
\cancel{-\theta_0+2} & E_1 & -2\,\theta_0+3 & \theta_0 & \mathbf{(9)} \\
\cancel{-\frac{C_1^2(\theta_0-2)}{2\,\theta_0}} & \overline{A_0} & -2\,\theta_0+3 & \theta_0 & \mathbf{(10)} \\
-\frac{1}{2}\,\theta_0+1 & C_1 & -\theta_0+1 & 0 & \mathbf{(11)} \\
-\frac{1}{2}\,\theta_0+1 & B_1 & -\theta_0+1 & 1 & \mathbf{(12)} \\
-\frac{1}{2}\,\theta_0+1 & B_2 & -\theta_0+1 & 2 & \mathbf{(13)} \\
-\frac{C_1(\theta_0-2)\overline{C_1}}{2\,\theta_0} & \overline{A_0} & -\theta_0+1 & 2 & \mathbf{(14)} \\
-\frac{1}{2}\,\theta_0+\frac{3}{2} & C_2 & -\theta_0+2 & 0 & \mathbf{(15)} \\
2\,A_1C_1 & \overline{A_0} & -\theta_0+2 & 0 & \mathbf{(16)} \\
-\frac{1}{2}\,\theta_0+\frac{3}{2} & B_3 & -\theta_0+2 & 1 & \mathbf{(17)} \\
2\,A_1C_1 & \overline{A_1} & -\theta_0+2 & 1 & \mathbf{(18)} \\
-\frac{1}{2}\,\theta_0+2 & C_3 & -\theta_0+3 & 0 & \mathbf{(19)} \\
\frac{1}{4}\,C_1^2 & A_0 & -\theta_0+3 & 0 & \mathbf{(20)} \\
4\,A_2C_1+2\,A_1C_2 & \overline{A_0} & -\theta_0+3 & 0 & \mathbf{(21)} \\
\cancel{\frac{1}{2}\theta_0} & \overline{E_1} & \theta_0-1 & -2\,\theta_0+4 & \mathbf{(22)} \\
\cancel{\frac{1}{4}\overline{C_1}^2} & A_0 & \theta_0-1 & -2\,\theta_0+4 & \mathbf{(23)}
\end{pmatrix}
= \begin{pmatrix}
2\theta_0(\theta_0+1)\overline{\alpha_2} & \overline{A_0} & 0 & -\theta_0+2 & \mathbf{(1)} \\
\frac{\overline{\zeta_2}}{\theta_0-3} & A_1 & 0 & -\theta_0+3 & \mathbf{(3)} \\
2\theta_0(\theta_0+1)\overline{\alpha_2} & \overline{A_1} & 0 & -\theta_0+3 & \mathbf{(4)} \\
\overline{\zeta_3} & \overline{A_0} & 0 & -\theta_0+3 & \mathbf{(5)} \\
\frac{(\theta_0-2)}{4\theta_0}|C_1|^2 & A_0 & 1 & -\theta_0+2 & \mathbf{(6)} \\
2\,\overline{\zeta_4} & \overline{A_0} & 1 & -\theta_0+2 & \mathbf{(8)} \\
-\frac{(\theta_0-2)}{2} & C_1 & -\theta_0+1 & 0 & \mathbf{(11)} \\
-\frac{(\theta_0-2)}{2} & B_1 & -\theta_0+1 & 1 & \mathbf{(12)} \\
-\frac{(\theta_0-2)}{2} & B_2 & -\theta_0+1 & 2 & \mathbf{(13)} \\
-\frac{(\theta_0-2)|C_1|^2}{2\,\theta_0} & \overline{A_0} & -\theta_0+1 & 2 & \mathbf{(14)} \\
-\frac{(\theta_0-3)}{2} & C_2 & -\theta_0+2 & 0 & \mathbf{(15)} \\
2\,\zeta_2 & \overline{A_0} & -\theta_0+2 & 0 & \mathbf{(16)} \\
-\frac{(\theta_0-3)}{2} & B_3 & -\theta_0+2 & 1 & \mathbf{(17)} \\
2\,\zeta_2 & \overline{A_1} & -\theta_0+2 & 1 & \mathbf{(18)} \\
-\frac{(\theta_0-4)}{2} & C_3 & -\theta_0+3 & 0 & \mathbf{(19)} \\
2\theta_0(\theta_0-4)\alpha_7 & A_0 & -\theta_0+3 & 0 & \mathbf{(20)} \\
2\,\zeta_5 & \overline{A_0} & -\theta_0+3 & 0 & \mathbf{(21)}
\end{pmatrix}
$$

which once translated again into code yields

$$\vec{\alpha} = \begin{pmatrix} 2\theta_0(\theta_0+1)\overline{\alpha_2} & \overline{A_0} & 0 & -\theta_0+2 & \mathbf{(1)} \\ \frac{\overline{\zeta_2}}{\theta_0-3} & A_1 & 0 & -\theta_0+3 & \mathbf{(3)} \\ 2\theta_0(\theta_0+1)\overline{\alpha_2} & \overline{A_1} & 0 & -\theta_0+3 & \mathbf{(4)} \\ \overline{\zeta_3} & \overline{A_0} & 0 & -\theta_0+3 & \mathbf{(5)} \\ \frac{(\theta_0-2)}{4\theta_0}|C_1|^2 & A_0 & 1 & -\theta_0+2 & \mathbf{(6)} \\ 2\,\overline{\zeta_4} & \overline{A_0} & 1 & -\theta_0+2 & \mathbf{(8)} \\ -\frac{(\theta_0-2)}{2} & C_1 & -\theta_0+1 & 0 & \mathbf{(11)} \\ -\frac{(\theta_0-2)}{2} & B_1 & -\theta_0+1 & 1 & \mathbf{(12)} \\ -\frac{(\theta_0-2)}{2} & B_2 & -\theta_0+1 & 2 & \mathbf{(13)} \\ -\frac{(\theta_0-2)|C_1|^2}{2\,\theta_0} & \overline{A_0} & -\theta_0+1 & 2 & \mathbf{(14)} \\ -\frac{(\theta_0-3)}{2} & C_2 & -\theta_0+2 & 0 & \mathbf{(15)} \\ 2\,\zeta_2 & \overline{A_0} & -\theta_0+2 & 0 & \mathbf{(16)} \\ -\frac{(\theta_0-3)}{2} & B_3 & -\theta_0+2 & 1 & \mathbf{(17)} \\ 2\,\zeta_2 & \overline{A_1} & -\theta_0+2 & 1 & \mathbf{(18)} \\ -\frac{(\theta_0-4)}{2} & C_3 & -\theta_0+3 & 0 & \mathbf{(19)} \\ 2\theta_0(\theta_0-4)\alpha_7 & A_0 & -\theta_0+3 & 0 & \mathbf{(20)} \\ 2\,\zeta_5 & \overline{A_0} & -\theta_0+3 & 0 & \mathbf{(21)} \end{pmatrix} \begin{pmatrix} 2\,(\theta_0+1)\theta_0\overline{\alpha_2} & \overline{A_0} & 0 & -\theta_0+2 \\ \frac{\overline{\zeta_2}}{\theta_0-3} & A_1 & 0 & -\theta_0+3 \\ 2\,(\theta_0+1)\theta_0\overline{\alpha_2} & \overline{A_1} & 0 & -\theta_0+3 \\ \overline{\zeta_3} & \overline{A_0} & 0 & -\theta_0+3 \\ \frac{(\theta_0-2)|C_1|^2}{4\,\theta_0} & A_0 & 1 & -\theta_0+2 \\ 2\,\overline{\zeta_4} & \overline{A_0} & 1 & -\theta_0+2 \\ -\frac{1}{2}\,\theta_0+1 & C_1 & -\theta_0+1 & 0 \\ -\frac{1}{2}\,\theta_0+1 & B_1 & -\theta_0+1 & 1 \\ -\frac{1}{2}\,\theta_0+1 & B_2 & -\theta_0+1 & 2 \\ -\frac{(\theta_0-2)|C_1|^2}{2\,\theta_0} & \overline{A_0} & -\theta_0+1 & 2 \\ -\frac{1}{2}\,\theta_0+\frac{3}{2} & C_2 & -\theta_0+2 & 0 \\ 2\,\zeta_2 & \overline{A_0} & -\theta_0+2 & 0 \\ -\frac{1}{2}\,\theta_0+\frac{3}{2} & B_3 & -\theta_0+2 & 1 \\ 2\,\zeta_2 & \overline{A_1} & -\theta_0+2 & 1 \\ -\frac{1}{2}\,\theta_0+2 & C_3 & -\theta_0+3 & 0 \\ 2\,\alpha_7(\theta_0-4)\theta_0 & A_0 & -\theta_0+3 & 0 \\ 2\,\zeta_5 & \overline{A_0} & -\theta_0+3 & 0 \end{pmatrix}$$

Left TEX right Sage.

$$
\langle \vec{\alpha}, \vec{\Phi} \rangle = \begin{pmatrix}
2\left(\theta_0\overline{\alpha_2} + \overline{\alpha_2}\right)\overline{A_0}^2 & 0 & 2 & \textbf{(1)} \\
\dfrac{(\theta_0 - 3)\overline{A_0}^2\zeta_3 + 2\left(2\,\theta_0^3\overline{\alpha_2} - 5\,\theta_0^2\overline{\alpha_2} - 3\,\theta_0\overline{\alpha_2}\right)\overline{A_0 A_1} + A_1\overline{A_0\zeta_2}}{\theta_0^2 - 3\,\theta_0} & 0 & 3 & \textbf{(2)} \\
-\dfrac{A_0 C_1(\theta_0 - 2)}{2\,\theta_0} & 1 & 0 & \textbf{(3)} \\
-\dfrac{A_0 B_1(\theta_0 - 2)}{2\,\theta_0} & 1 & 1 & \textbf{(4)} \\
-\dfrac{4\,A_0(\theta_0 - 2)|C_1|^2\overline{A_0} - 32\,\theta_0\overline{A_0}^2\zeta_4 + 8\left(\theta_0^2 - 2\,\theta_0\right)A_0 B_2 + \left(\theta_0^2 - 2\,\theta_0\right)C_1\overline{C_1}}{16\,\theta_0^2} & 1 & 2 & \textbf{(5)} \\
\dfrac{4\,A_0(\theta_0 + 1)\zeta_2\overline{A_0} - \left(\theta_0^2 - 2\,\theta_0\right)A_1 C_1 - \left(\theta_0^2 - 2\,\theta_0 - 3\right)A_0 C_2}{2\left(\theta_0^2 + \theta_0\right)} & 2 & 0 & \textbf{(6)} \\
\dfrac{4\,A_0(\theta_0 + 1)\zeta_2\overline{A_1} - \left(\theta_0^2 - 2\,\theta_0\right)A_1 B_1 - \left(\theta_0^2 - 2\,\theta_0 - 3\right)A_0 B_3}{2\left(\theta_0^2 + \theta_0\right)} & 2 & 1 & \textbf{(7)} \\
\lambda_1 & 3 & 0 & \textbf{(8)} \\
-\dfrac{C_1(\theta_0 - 2)\overline{A_0}}{2\,\theta_0} & -\theta_0 + 1 & \theta_0 & \textbf{(9)} \\
-\dfrac{\left(\theta_0^2 - \theta_0 - 2\right)B_1\overline{A_0} + \left(\theta_0^2 - 2\,\theta_0\right)C_1\overline{A_1}}{2\left(\theta_0^2 + \theta_0\right)} & -\theta_0 + 1 & \theta_0 + 1 & \textbf{(10)} \\
\lambda_2 & -\theta_0 + 1 & \theta_0 + 2 & \textbf{(11)} \\
-\dfrac{C_2(\theta_0 - 3)\overline{A_0} - 4\,\zeta_2\overline{A_0}^2}{2\,\theta_0} & -\theta_0 + 2 & \theta_0 & \textbf{(12)} \\
\dfrac{4\left(2\,\theta_0 + 1\right)\zeta_2\overline{A_0}\,\overline{A_1} - \left(\theta_0^2 - 2\,\theta_0 - 3\right)B_3\overline{A_0} - \left(\theta_0^2 - 3\,\theta_0\right)C_2\overline{A_1}}{2\left(\theta_0^2 + \theta_0\right)} & -\theta_0 + 2 & \theta_0 + 1 & \textbf{(13)} \\
-\dfrac{C_1^2(\theta_0 - 2) - 32\left(\alpha_7\theta_0^2 - 4\,\alpha_7\theta_0\right)A_0\overline{A_0} + 8\,C_3(\theta_0 - 4)\overline{A_0} - 32\,\zeta_5\overline{A_0}^2}{16\,\theta_0} & -\theta_0 + 3 & \theta_0 & \textbf{(14)} \\
2\left(\theta_0\overline{\alpha_2} + \overline{\alpha_2}\right)A_0\overline{A_0} & \theta_0 & -\theta_0 + 2 & \textbf{(15)} \\
\dfrac{A_0(\theta_0 - 3)\overline{A_0\zeta_3} + 2\left(\theta_0^3\overline{\alpha_2} - 2\,\theta_0^2\overline{\alpha_2} - 3\,\theta_0\overline{\alpha_2}\right)A_0\overline{A_1} + A_0 A_1\overline{\zeta_2}}{\theta_0^2 - 3\,\theta_0} & \theta_0 & -\theta_0 + 3 & \textbf{(16)} \\
\dfrac{8\,A_1\theta_0^3\overline{A_0\alpha_2} + A_0^2(\theta_0 - 2)|C_1|^2 + 8\,A_0\theta_0\overline{A_0\zeta_4}}{4\,\theta_0^2} & \theta_0 + 1 & -\theta_0 + 2 & \textbf{(17)}
\end{pmatrix}
$$

where

$$
\lambda_1 = \frac{1}{2\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)}\Big\{4\left(\theta_0^2 + 2\,\theta_0\right)A_1\zeta_2\overline{A_0} + 4\left(\theta_0^2 + 3\,\theta_0 + 2\right)A_0\zeta_5\overline{A_0} + 4\left(\alpha_7\theta_0^4 - \alpha_7\theta_0^3 - 10\,\alpha_7\theta_0^2 - 8\,\alpha_7\theta_0\right)A_0^2
$$
$$
- \left(\theta_0^3 - \theta_0^2 - 2\,\theta_0\right)A_2 C_1 - \left(\theta_0^3 - \theta_0^2 - 6\,\theta_0\right)A_1 C_2 - \left(\theta_0^3 - \theta_0^2 - 10\,\theta_0 - 8\right)A_0 C_3\Big\}
$$

$$
\lambda_2 = -\frac{\left(\theta_0^3 + \theta_0^2 - 4\,\theta_0 - 4\right)|C_1|^2\overline{A_0}^2 + \left(\theta_0^4 + \theta_0^3 - 4\,\theta_0^2 - 4\,\theta_0\right)B_2\overline{A_0} + \left(\theta_0^4 - 4\,\theta_0^2\right)B_1\overline{A_1} + \left(\theta_0^4 - \theta_0^3 - 2\,\theta_0^2\right)C_1\overline{A_2}}{2\left(\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2\right)}.
$$

Therefore, we have

$$
\textbf{(1)} = \textbf{(2)} = \textbf{(3)} = \textbf{(4)} = \textbf{(7)} = \textbf{(9)} = \textbf{(12)}.
$$

Now, recall that by (2.3.7) and (2.3.11)

$$
\langle \vec{C}_1, \vec{C}_1 \rangle + 8\langle \overline{\vec{A}_0}, \vec{C}_3 \rangle = 0, \quad \alpha_7 = \frac{1}{8\theta_0(\theta_0 - 4)}\langle \vec{C}_1, \vec{C}_1 \rangle
$$

so as $|\vec{A}_0|^2 = \dfrac{1}{2}$

$$
C_1^2(\theta_0 - 2) - 32\left(\alpha_7\theta_0^2 - 4\,\alpha_7\theta_0\right)A_0\overline{A_0} + 8\,C_3(\theta_0 - 4)\overline{A_0} - 32\,\zeta_5\overline{A_0}^2
$$

$$
\begin{aligned}
&= C_1^2(\theta_0-2) - 16\,\theta_0\,(\theta_0-4\,)\alpha_7 + 8\,C_3(\theta_0-4)\overline{A_0}\\
&= (\theta_0-4)\langle \vec{C}_1,\vec{C}_1\rangle + 8(\theta_0-4)\langle \overline{\vec{A}_0},\vec{C}_3\rangle = 0
\end{aligned}
$$

so

$$
\textbf{(14)} = 0. \tag{3.4.5}
$$

Now, we compute

$$
\begin{aligned}
&4\,A_0(\theta_0-2)|C_1|^2\overline{A_0} - 32\,\theta_0\overline{A_0}^2\zeta_4 + 8\left(\theta_0^2-2\,\theta_0\right)A_0B_2 + \left(\theta_0^2-2\,\theta_0\right)C_1\overline{C_1}\\
&= 2(\theta_0-2)|\vec{C}_1|^2 + 8\theta_0(\theta_0-2)\langle \vec{A}_0,\vec{B}_2\rangle + \theta_0(\theta_0-2)|\vec{C}_1|^2\\
&= (\theta_0-2)\Bigg((\theta_0+2)|\vec{C}_1|^2 + 8\theta_0\left\langle \vec{A}_0, -\frac{(\theta_0+2)}{4\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0} - 2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\vec{A}_0\right\rangle\Bigg)\\
&= 0
\end{aligned}
$$

as $|\vec{A}_0|^2 = \dfrac{1}{2}$. Therefore, we have

$$
\textbf{(5)} = 0.
$$

Now, we have as $\zeta_2 = \langle \vec{A}_1,\vec{C}_1\rangle$ and $\langle \vec{A}_1,\vec{C}_1\rangle + \langle \vec{A}_0,\vec{C}_2\rangle = 0$ the identity

$$
\begin{aligned}
&4\,A_0(\theta_0+1)\zeta_2\overline{A_0} - \left(\theta_0^2-2\,\theta_0\right)A_1C_1 - \left(\theta_0^2-2\,\theta_0-3\right)A_0C_2\\
&= 2(\theta_0+1)A_1C_1 - \left(\theta_0^2-2\,\theta_0\right)A_1C_1 + \left(\theta_0^2-2\,\theta_0-3\right)A_1C_1\\
&= (2\theta_0-1)\langle \vec{A}_1,\vec{C}_1\rangle\\
&= (2\theta_0-1)\zeta_2.
\end{aligned}
$$

and

$$
\textbf{(6)} = \frac{4\,A_0(\theta_0+1)\zeta_2\overline{A_0} - \left(\theta_0^2-2\,\theta_0\right)A_1C_1 - \left(\theta_0^2-2\,\theta_0-3\right)A_0C_2}{2\left(\theta_0^2+\theta_0\right)} = \frac{(2\theta_0-1)\zeta_2}{2\theta_0(\theta_0+1)} \tag{3.4.6}
$$

Now, recall

$$
\begin{cases}
\zeta_0 = \langle \vec{A}_1,\vec{A}_1\rangle\\
\zeta_1 = \langle \vec{A}_1,\vec{A}_2\rangle\\
\zeta_2 = \langle \vec{A}_1,\vec{C}_1\rangle\\
\zeta_3 = \langle \overline{\vec{A}_1},\vec{C}_2\rangle\\
\zeta_4 = \langle \overline{\vec{A}_2},\vec{C}_1\rangle\\
\zeta_5 = 2\langle \vec{A}_2,\vec{C}_1\rangle + \langle \vec{A}_1,\vec{C}_2\rangle
\end{cases} \tag{3.4.7}
$$

In particular, as

$$
\langle \vec{A}_2,\vec{C}_1\rangle + \langle \vec{A}_1,\vec{C}_2\rangle + \langle \vec{A}_0,\vec{C}_3\rangle = 0,
$$

we have

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)}\Big\{4\left(\theta_0^2+2\,\theta_0\right)A_1\zeta_2\overline{A_0} + 4\left(\theta_0^2+3\,\theta_0+2\right)A_0\zeta_5\overline{A_0} + 4\left(\alpha_7\theta_0^4 - \alpha_7\theta_0^3 - 10\,\alpha_7\theta_0^2 - 8\,\alpha_7\theta_0\right)A_0^2\\
&\quad - \left(\theta_0^3-\theta_0^2-2\,\theta_0\right)A_2C_1 - \left(\theta_0^3-\theta_0^2-6\,\theta_0\right)A_1C_2 - \left(\theta_0^3-\theta_0^2-10\,\theta_0-8\right)A_0C_3\Big\}\\
&= \frac{1}{2\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)}\Big\{2(\theta_0^2+3\theta_0+2)\zeta_5\\
&\quad - \left(\theta_0^3-\theta_0^2-2\,\theta_0\right)A_2C_1 - \left(\theta_0^3-\theta_0^2-6\,\theta_0\right)A_1C_2 + \left(\theta_0^3-\theta_0^2-10\,\theta_0-8\right)(A_2C_1+A_1C_2)\Big\}\\
&= \frac{1}{2\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)}\Big\{2(\theta_0^2+3\theta_0+2)\zeta_5 - 8(\theta_0+1)A_2C_1 - 4(\theta_0+2)A_1C_2\Big\}
\end{aligned}
$$

$$= \frac{1}{2\,(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0)}\left\{2(\theta_0^2 + 3\theta_0 + 2)\zeta_5 - 4(\theta_0 + 1)\zeta_5 - 4A_1C_2\right\}$$
$$= \frac{1}{2\,(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0)}\left\{2\theta_0(\theta_0 + 1)\zeta_5 - 4\zeta_6\right\}$$

where

$$\zeta_6 = \langle \vec{A}_1, \vec{C}_2 \rangle.$$

Therefore,

$$\textbf{(8)} = \frac{1}{2\,(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0)}\left\{2\theta_0(\theta_0 + 1)\zeta_5 - 4\zeta_6\right\}.$$

Then, we compute as $\vec{B}_1 = -2\langle \overline{\vec{A}_1}, \vec{C}_1 \rangle \vec{A}_0$

$$\textbf{(10)} = -\frac{\left(\theta_0^2 - \theta_0 - 2\right)B_1\overline{A_0} + \left(\theta_0^2 - 2\,\theta_0\right)C_1\overline{A_1}}{2\,(\theta_0^2 + \theta_0)} = \frac{\left(\left(\theta_0^2 - \theta_0 - 2\right)\overline{A_1}C_1 - \left(\theta_0^2 - 2\,\theta_0\right)C_1\overline{A_1}\right)}{2\,(\theta_0^2 + \theta_0)}$$
$$= \frac{(\theta_0 - 2)\langle \overline{\vec{A}_1}, \vec{C}_1 \rangle}{2\theta_0(\theta_0 + 1)} = (\theta_0 - 2)\alpha_2.$$

Then, we have

$$\langle \overline{\vec{A}_0}, \vec{B}_2 \rangle = \left\langle \overline{\vec{A}_0}, -\frac{(\theta_0 + 2)}{4\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0} - 2\langle \overline{\vec{A}_2}, \vec{C}_1 \rangle \vec{A}_0 \right\rangle = -\langle \overline{\vec{A}_2}, \vec{C}_1 \rangle$$

so

$$\textbf{(11)} = \lambda_2 = -\frac{\left(\theta_0^3 + \theta_0^2 - 4\,\theta_0 - 4\right)|C_1|^2\overline{A_0}^2 + \left(\theta_0^4 + \theta_0^3 - 4\,\theta_0^2 - 4\,\theta_0\right)B_2\overline{A_0} + \left(\theta_0^4 - 4\,\theta_0^2\right)B_1\overline{A_1} + \left(\theta_0^4 - \theta_0^3 - 2\,\theta_0^2\right)C_1\overline{A_2}}{2\,(\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2)}$$
$$= \frac{\left(\theta_0^4 + \theta_0^3 - 4\,\theta_0^2 - 4\,\theta_0\right)\overline{A_2}\,C_1 - \left(\theta_0^4 - \theta_0^3 - 2\,\theta_0^2\right)C_1\overline{A_2}}{2\,(\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2)}$$
$$= \frac{\left(2\theta_0^3 - 2\,\theta_0^2 - 4\,\theta_0\right)\overline{A_2}\,C_1}{2\,(\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2)} = \frac{(\theta_0 + 1)(\theta_0 - 2)\overline{A_2}\,C_1}{(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0)} = \frac{(\theta_0 + 1)(\theta_0 - 2)\overline{A_2}\,C_1}{\theta_0(\theta_0 + 1)(\theta_0 + 2)} = \frac{(\theta_0 - 2)\overline{A_2}C_1}{\theta_0(\theta_0 + 2)}$$
$$= \frac{(\theta_0 - 2)\zeta_4}{\theta_0(\theta_0 + 2)}.$$

Then, we have

$$\langle \overline{\vec{A}_0}, \vec{B}_3 \rangle = \left\langle \overline{\vec{A}_0}, -2\langle \overline{\vec{A}_1}, \vec{C}_1 \rangle \vec{A}_1 + \frac{2}{\theta_0 - 3}\langle \vec{A}_1, \vec{C}_1 \rangle \overline{\vec{A}_1} - 2\langle \overline{\vec{A}_1}, \vec{C}_2 \rangle \vec{A}_0 \right\rangle = -\langle \overline{\vec{A}_1}, \vec{C}_2 \rangle$$

so

$$\textbf{(13)} = \frac{4\,(2\,\theta_0 + 1)\zeta_2\overline{A_0 A_1} - \left(\theta_0^2 - 2\,\theta_0 - 3\right)B_3\overline{A_0} - \left(\theta_0^2 - 3\,\theta_0\right)C_2\overline{A_1}}{2\,(\theta_0^2 + \theta_0)}$$
$$= \frac{\left(\theta_0^2 - 2\,\theta_0 - 3\right)\overline{A_1}C_2 - \left(\theta_0^2 - 3\,\theta_0\right)C_2\overline{A_1}}{2\,(\theta_0^2 + \theta_0)} = \frac{(\theta_0 - 3)\overline{A_1}C_2}{2\theta_0(\theta_0 + 1)}$$
$$= \frac{(\theta_0 - 3)\zeta_3}{2\theta_0(\theta_0 + 1)}.$$

Finally, we trivially have

$$\textbf{(15)} = 2\,(\theta_0\overline{\alpha_2} + \overline{\alpha_2})A_0\overline{A_0} = (\theta_0 + 1)\overline{\alpha_2}. \tag{3.4.8}$$

then

$$\textbf{(16)} = \frac{A_0(\theta_0 - 3)\overline{A_0\zeta_3} + 2\left(\theta_0^3\overline{\alpha_2} - 2\,\theta_0^2\overline{\alpha_2} - 3\,\theta_0\overline{\alpha_2}\right)A_0\overline{A_1} + A_0A_1\overline{\zeta_2}}{\theta_0^2 - 3\,\theta_0} = \frac{\overline{\zeta_3}}{2\theta_0} \tag{3.4.9}$$

while

$$\textbf{(17)} = \frac{8\,A_1\theta_0^3\overline{A_0\alpha_2} + A_0^2(\theta_0 - 2)|C_1|^2 + 8\,A_0\theta_0\overline{A_0\zeta_4}}{4\,\theta_0^2} = \frac{\overline{\zeta_4}}{\theta_0}. \tag{3.4.10}$$

Finally

$$\langle \vec{\alpha}, \vec{\Phi} \rangle = \begin{pmatrix} \dfrac{(2\theta_0-1)\zeta_2}{2\theta_0(\theta_0+1)} & 2 & 0 & \textbf{(6)} \\ \dfrac{1}{2\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)}\left\{2\theta_0(\theta_0+1)\zeta_5-4\zeta_6\right\} & 3 & 0 & \textbf{(8)} \\ (\theta_0-2)\alpha_2 & -\theta_0+1 & \theta_0+1 & \textbf{(10)} \\ \dfrac{(\theta_0-2)\zeta_4}{\theta_0(\theta_0+2)} & -\theta_0+1 & \theta_0+2 & \textbf{(11)} \\ \dfrac{(\theta_0-3)\zeta_3}{2\theta_0(\theta_0+1)} & -\theta_0+2 & \theta_0+1 & \textbf{(13)} \\ (\theta_0+1)\overline{\alpha_2} & \theta_0 & -\theta_0+2 & \textbf{(15)} \\ \dfrac{\overline{\zeta_3}}{2\theta_0} & \theta_0 & -\theta_0+3 & \textbf{(16)} \\ \dfrac{\overline{\zeta_4}}{\theta_0} & \theta_0+1 & -\theta_0+2 & \textbf{(17)} \end{pmatrix} =$$

In particular, we see easily that if

$$\langle \vec{\alpha}, \vec{\Phi} \rangle = f(z)dz,$$

Indeed, see `test_scaling` in the code. We can also check it directly. Indeed, we have

$$\begin{aligned} \partial_{\overline{z}} f(z) &= \begin{pmatrix} (\theta_0+1)(\theta_0-2)\alpha_2 & -\theta_0+1 & \theta_0 & \textbf{(10)} \\ \dfrac{(\theta_0-2)\zeta_4}{\theta_0} & -\theta_0+1 & \theta_0+1 & \textbf{(11)} \\ \dfrac{(\theta_0-3)\zeta_3}{2\theta_0} & -\theta_0+2 & \theta_0 & \textbf{(13)} \\ -(\theta_0-2)(\theta_0+1)\overline{\alpha_2} & \theta_0 & -\theta_0+1 & \textbf{(15)} \\ \dfrac{-(\theta_0-3)\overline{\zeta_3}}{2\theta_0} & \theta_0 & -\theta_0+2 & \textbf{(16)} \\ \dfrac{-(\theta_0-2)\overline{\zeta_4}}{\theta_0} & \theta_0+1 & -\theta_0+1 & \textbf{(17)} \end{pmatrix} \\ &= 2i\,\mathrm{Im}\left((\theta_0+1)(\theta_0-2)\alpha_2\frac{\overline{z}^{\theta_0}}{z}+\frac{(\theta_0-2)\zeta_4}{\theta_0}\frac{\overline{z}^{\theta_0+1}}{z^{\theta_0-1}}+\frac{(\theta_0-3)\zeta_3}{2\theta_0}\frac{\overline{z}^{\theta_0}}{z^{\theta_0-2}}\right) \end{aligned}$$

so $\partial_{\overline{z}} f(z)$ is purely imaginary, as expected.

Then

$$\mathrm{Re}\left(\partial_{\overline{z}} f(z)\right) = 0,$$

as expected. We also define

$$\zeta_7 = \frac{1}{2\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)}\left\{2\theta_0(\theta_0+1)\zeta_5-4\zeta_6\right\}$$

so that

$$\langle\vec{\Phi},\vec{\alpha}\rangle = \begin{pmatrix} \dfrac{(2\theta_0-1)\zeta_2}{2\theta_0(\theta_0+1)} & 2 & 0 & \textbf{(6)} \\ \zeta_7 & 3 & 0 & \textbf{(8)} \\ (\theta_0-2)\alpha_2 & -\theta_0+1 & \theta_0+1 & \textbf{(10)} \\ \dfrac{(\theta_0-2)\zeta_4}{\theta_0(\theta_0+2)} & -\theta_0+1 & \theta_0+2 & \textbf{(11)} \\ \dfrac{(\theta_0-3)\zeta_3}{2\theta_0(\theta_0+1)} & -\theta_0+2 & \theta_0+1 & \textbf{(13)} \\ (\theta_0+1)\overline{\alpha_2} & \theta_0 & -\theta_0+2 & \textbf{(15)} \\ \dfrac{\overline{\zeta_3}}{2\theta_0} & \theta_0 & -\theta_0+3 & \textbf{(16)} \\ \dfrac{\overline{\zeta_4}}{\theta_0} & \theta_0+1 & -\theta_0+2 & \textbf{(17)} \end{pmatrix} = \begin{pmatrix} \dfrac{(2\,\theta_0-1)\zeta_2}{2\,(\theta_0+1)\theta_0} & 2 & 0 \\ \zeta_7 & 3 & 0 \\ \alpha_2(\theta_0-2) & -\theta_0+1 & \theta_0+1 \\ \dfrac{(\theta_0-2)\zeta_4}{(\theta_0+2)\theta_0} & -\theta_0+1 & \theta_0+2 \\ \dfrac{(\theta_0-3)\zeta_3}{2\,(\theta_0+1)\theta_0} & -\theta_0+2 & \theta_0+1 \\ (\theta_0+1)\overline{\alpha_2} & \theta_0 & -\theta_0+2 \\ \dfrac{\overline{\zeta_3}}{2\,\theta_0} & \theta_0 & -\theta_0+3 \\ \dfrac{\overline{\zeta_4}}{\theta_0} & \theta_0+1 & -\theta_0+2 \end{pmatrix}$$

3.5 Development of $\vec{h}_0$ up to order 4

$$g^{-1}\otimes\langle\vec{h}_0,\vec{\Phi}\rangle =$$

$$
\begin{pmatrix}
\frac{2\,A_1\overline{A_0}}{\theta_0} & 0 & 1 \\
-\frac{2\left(2\,A_0(\theta_0+1)|A_1|^2\overline{A_0} - A_1\theta_0\overline{A_1}\right)}{\theta_0^2+\theta_0} = -\frac{2|A_1|^2}{\theta_0(\theta_0+1)} & 0 & 2 \\
\mu_1 & 0 & 3 \\
\mu_2 & 0 & 4 \\
-\frac{8\,A_0\alpha_1\theta_0\overline{A_0} + A_0C_1(\theta_0-2) - 8\,A_2\theta_0\overline{A_0}}{2\,\theta_0^2} & 1 & 1 \\
\mu_3 & 1 & 2 \\
\mu_4 & 1 & 3 \\
\mu_5 & 2 & 1 \\
\mu_6 & 2 & 2 \\
\mu_7 & 3 & 1 \\
-\frac{E_1(\theta_0-2)\overline{A_0}}{2\,\theta_0^2} & -2\,\theta_0+3 & 2\,\theta_0+1 \\
-\frac{C_1(\theta_0-2)\overline{A_0}}{2\,\theta_0^2} & -\theta_0+1 & \theta_0+1 \\
\frac{4\left(\alpha_2\theta_0^2-\alpha_2\theta_0-2\,\alpha_2\right)A_0\overline{A_0} - B_1(\theta_0-2)\overline{A_0} - C_1(\theta_0-2)\overline{A_1}}{2\,(\theta_0^2+\theta_0)} & -\theta_0+1 & \theta_0+2 \\
\mu_8 & -\theta_0+1 & \theta_0+3 \\
-\frac{C_2(\theta_0-3)\overline{A_0}}{2\,\theta_0^2} & -\theta_0+2 & \theta_0+1 \\
\mu_9 & -\theta_0+2 & \theta_0+2 \\
-\frac{C_1^2(\theta_0-2)+8\,A_0E_1(\theta_0-2)-32\left(\alpha_7\theta_0^2-4\,\alpha_7\theta_0\right)A_0\overline{A_0}+8\,C_3(\theta_0-4)\overline{A_0}}{16\,\theta_0^2} & -\theta_0+3 & \theta_0+1 \\
\frac{2\,A_0A_1}{\theta_0} & \theta_0 & -\theta_0+1 \\
-\frac{4\,A_0^2|A_1|^2}{\theta_0} & \theta_0 & -\theta_0+2 \\
-\frac{8\,(\theta_0\overline{\alpha_2}+\overline{\alpha_2})A_0\overline{A_0}+8\,A_0A_1\overline{\alpha_1}+8\,A_0^2\overline{\alpha_5}-A_0\overline{B_1}-A_1\overline{C_1}}{4\,\theta_0} & \theta_0 & -\theta_0+3 \\
\mu_{10} & \theta_0 & -\theta_0+4 \\
-\frac{4\,(\theta_0\overline{\alpha_7}-4\,\overline{\alpha_7})A_0\overline{A_0}+\overline{A_0E_1}}{2\,(\theta_0-4)} & \theta_0-1 & -\theta_0+5 \\
-\frac{2\left(2\,(\alpha_1\theta_0+\alpha_1)A_0^2-2\,A_0A_2(\theta_0+1)-A_1^2\theta_0\right)}{\theta_0^2+\theta_0} & \theta_0+1 & -\theta_0+1 \\
-\frac{4\left(A_0A_1(3\,\theta_0+2)|A_1|^2+(\alpha_5\theta_0+\alpha_5)A_0^2\right)}{\theta_0^2+\theta_0} & \theta_0+1 & -\theta_0+2 \\
\mu_{11} & \theta_0+1 & -\theta_0+3 \\
\mu_{12} & \theta_0+2 & -\theta_0+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
\mu_{13} & \theta_0 + 2 & -\theta_0 + 2 \\
\mu_{14} & \theta_0 + 3 & -\theta_0 + 1 \\
-\frac{4\,(\theta_0 \overline{\alpha_7} - 4\,\overline{\alpha_7}) A_0^2 + A_0 \overline{E_1}}{2\,(\theta_0 - 4)} & 2\,\theta_0 - 1 & -2\,\theta_0 + 5 \\
-\frac{2\left((\theta_0 \overline{\alpha_9} + 2\,\overline{\alpha_9}) A_0^2 + 2\,(\theta_0 \overline{\alpha_2} + \overline{\alpha_2}) A_0 A_1\right)}{\theta_0} & 2\,\theta_0 + 1 & -2\,\theta_0 + 3 \\
-\frac{2\,(\theta_0 \overline{\alpha_2} + \overline{\alpha_2}) A_0^2}{\theta_0} & 2\,\theta_0 & -2\,\theta_0 + 3 \\
-\frac{2\,(\theta_0 \overline{\alpha_8} + \overline{\alpha_8}) A_0^2}{\theta_0} & 2\,\theta_0 & -2\,\theta_0 + 4
\end{pmatrix}
$$

$$
= \begin{pmatrix}
-\frac{2|A_1|^2}{\theta_0(\theta_0 + 1)} & 0 & 2 & \textbf{(1)} \\
\mu_1 & 0 & 3 & \textbf{(2)} \\
\mu_2 & 0 & 4 & \textbf{(3)} \\
\mu_3 & 1 & 2 & \textbf{(4)} \\
\mu_4 & 1 & 3 & \textbf{(5)} \\
\mu_5 & 2 & 1 & \textbf{(6)} \\
\mu_6 & 2 & 2 & \textbf{(7)} \\
\mu_7 & 3 & 1 & \textbf{(8)} \\
\frac{(\theta_0 - 2)\alpha_2}{\theta_0} & -\theta_0 + 1 & \theta_0 + 2 & \textbf{(9)} \\
\mu_8 & -\theta_0 + 1 & \theta_0 + 3 & \textbf{(10)} \\
\mu_9 & -\theta_0 + 2 & \theta_0 + 2 & \textbf{(11)} \\
-\frac{C_1^2(\theta_0 - 2) + 8\,A_0 E_1(\theta_0 - 2) - 32\left(\alpha_7 \theta_0^2 - 4\,\alpha_7 \theta_0\right) A_0 \overline{A_0} + 8\,C_3(\theta_0 - 4)\overline{A_0}}{16\,\theta_0^2} & -\theta_0 + 3 & \theta_0 + 1 & \textbf{(12)} \\
-\frac{(\theta_0 + 1)\overline{\alpha_2}}{\theta_0} & \theta_0 & -\theta_0 + 3 & \textbf{(13)} \\
\mu_{10} & \theta_0 & -\theta_0 + 4 & \textbf{(14)} \\
-\frac{4\,(\theta_0 \overline{\alpha_7} - 4\,\overline{\alpha_7}) A_0 \overline{A_0} + \overline{A_0 E_1}}{2\,(\theta_0 - 4)} & \theta_0 - 1 & -\theta_0 + 5 & \textbf{(15)} \\
-\frac{2A_1^2}{\theta_0(\theta_0 + 1)} & \theta_0 + 1 & -\theta_0 + 1 & \textbf{(16)} \\
\mu_{11} & \theta_0 + 1 & -\theta_0 + 3 & \textbf{(17)} \\
\mu_{12} & \theta_0 + 2 & -\theta_0 + 1 & \textbf{(18)} \\
\mu_{13} & \theta_0 + 2 & -\theta_0 + 2 & \textbf{(19)} \\
\mu_{14} & \theta_0 + 3 & -\theta_0 + 1 & \textbf{(20)}
\end{pmatrix}
$$

where

$$
\mu_1 = -\frac{1}{4\,(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0)} \Big\{ 16\left(\theta_0^2 + 2\,\theta_0\right) A_0 |A_1|^2 \overline{A_1} + 8\left(\theta_0^2 \overline{\alpha_5} + 3\,\theta_0 \overline{\alpha_5} + 2\,\overline{\alpha_5}\right) A_0 \overline{A_0} + 8\left(\theta_0^2 \overline{\alpha_1} + 3\,\theta_0 \overline{\alpha_1} + 2\,\overline{\alpha_1}\right) A_1 \overline{A_0}
$$
$$
- 8\left(\theta_0^2 + \theta_0\right) A_1 \overline{A_2} - \left(\theta_0^2 + 3\,\theta_0 + 2\right) \overline{A_0 B_1} \Big\}
$$

$$\mu_2 = \frac{1}{12\left(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0\right)}\Big\{96\left(\theta_0^3\overline{\alpha_1}+6\,\theta_0^2\overline{\alpha_1}+11\,\theta_0\overline{\alpha_1}+6\,\overline{\alpha_1}\right)A_0|A_1|^2\overline{A_0}-48\left(\theta_0^3+4\,\theta_0^2+3\,\theta_0\right)A_0|A_1|^2\overline{A_2}$$
$$-2\left(\theta_0^3+6\,\theta_0^2+11\,\theta_0+6\right)|A_1|^2\overline{A_0C_1}-24\left(\theta_0^3\overline{\alpha_6}+6\,\theta_0^2\overline{\alpha_6}+11\,\theta_0\overline{\alpha_6}+6\,\overline{\alpha_6}\right)A_0\overline{A_0}$$
$$-24\left(\theta_0^3\overline{\alpha_3}+6\,\theta_0^2\overline{\alpha_3}+11\,\theta_0\overline{\alpha_3}+6\,\overline{\alpha_3}\right)A_1\overline{A_0}+24\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)A_1\overline{A_3}+3\left(\theta_0^3+5\,\theta_0^2+6\,\theta_0\right)\overline{A_1B_1}$$
$$+2\left(\theta_0^3+6\,\theta_0^2+11\,\theta_0+6\right)\overline{A_0B_3}-24\left(\left(\theta_0^3\overline{\alpha_5}+5\,\theta_0^2\overline{\alpha_5}+6\,\theta_0\overline{\alpha_5}\right)A_0+\left(\theta_0^3\overline{\alpha_1}+5\,\theta_0^2\overline{\alpha_1}+6\,\theta_0\overline{\alpha_1}\right)A_1\right)\overline{A_1}\Big\}$$

$$\mu_3 = -\frac{16\,A_1(\theta_0+1)|A_1|^2\overline{A_0}+8\,A_0\alpha_1\theta_0\overline{A_1}-4\left(\alpha_2\theta_0^2-\alpha_2\theta_0-2\,\alpha_2\right)A_0^2+A_0B_1(\theta_0-2)+8\left(\alpha_5\theta_0+\alpha_5\right)A_0\overline{A_0}-8\,A_2\theta_0\overline{A_1}}{2\left(\theta_0^2+\theta_0\right)}$$

$$\mu_4 = \frac{1}{16\left(\theta_0^4+3\,\theta_0^3+2\,\theta_0^2\right)}\Big\{256\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)A_0|A_1|^4\overline{A_0}-128\left(\theta_0^3+2\,\theta_0^2\right)A_1|A_1|^2\overline{A_1}$$
$$+32\left(\alpha_9\theta_0^4+\alpha_9\theta_0^3-4\,\alpha_9\theta_0^2-4\,\alpha_9\theta_0\right)A_0^2-8\left(\theta_0^3-\theta_0^2-2\,\theta_0\right)A_0B_2+16\left(\theta_0^2\overline{\alpha_1}-\theta_0\overline{\alpha_1}-2\,\overline{\alpha_1}\right)A_0C_1$$
$$+64\left((2\,\alpha_1\overline{\alpha_1}-\beta)\theta_0^3+3\,(2\,\alpha_1\overline{\alpha_1}-\beta)\theta_0^2+2\,(2\,\alpha_1\overline{\alpha_1}-\beta)\theta_0\right)A_0\overline{A_0}-64\left(\theta_0^3\overline{\alpha_5}+3\,\theta_0^2\overline{\alpha_5}+2\,\theta_0\overline{\alpha_5}\right)A_1\overline{A_0}$$
$$-64\left(\theta_0^3\overline{\alpha_1}+3\,\theta_0^2\overline{\alpha_1}+2\,\theta_0\overline{\alpha_1}\right)A_2\overline{A_0}-64\left(\alpha_5\theta_0^3+2\,\alpha_5\theta_0^2\right)A_0\overline{A_1}+8\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)\overline{A_0B_2}-\left(\theta_0^3+\theta_0^2-4\,\theta_0-4\right)C_1\overline{C_1}$$
$$-64\left(\left(\alpha_1\theta_0^3+\alpha_1\theta_0^2\right)A_0-\left(\theta_0^3+\theta_0^2\right)A_2\right)\overline{A_2}\Big\}$$

$$\mu_5 = -\frac{\left(\theta_0^2-5\,\theta_0\right)A_1C_1+2\left(\theta_0^2-2\,\theta_0-3\right)A_0C_2+24\left(\alpha_3\theta_0^2+\alpha_3\theta_0\right)A_0\overline{A_0}+24\left(\alpha_1\theta_0^2+\alpha_1\theta_0\right)A_1\overline{A_0}-24\left(\theta_0^2+\theta_0\right)A_3\overline{A_0}}{4\left(\theta_0^3+\theta_0^2\right)}$$

$$\mu_6 = -\frac{1}{4\left(\theta_0^4+2\,\theta_0^3+\theta_0^2\right)}\Big\{2\left(\theta_0^3+5\,\theta_0+6\right)A_0C_1|A_1|^2-96\left(\alpha_1\theta_0^3+2\,\alpha_1\theta_0^2+\alpha_1\theta_0\right)A_0|A_1|^2\overline{A_0} \tag{3.5.1}$$
$$+48\left(\theta_0^3+2\,\theta_0^2+\theta_0\right)A_2|A_1|^2\overline{A_0}-8\left(\alpha_8\theta_0^4-\alpha_8\theta_0^3-5\,\alpha_8\theta_0^2-3\,\alpha_8\theta_0\right)A_0^2+\left(\theta_0^3-5\,\theta_0^2\right)A_1B_1+2\left(\theta_0^3-2\,\theta_0^2-3\,\theta_0\right)A_0B_3$$
$$+24\left(\alpha_6\theta_0^3+2\,\alpha_6\theta_0^2+\alpha_6\theta_0\right)A_0\overline{A_0}-24\left(\theta_0^3+\theta_0^2\right)A_3\overline{A_1}-8\Bigg(\left(2\,\alpha_2\theta_0^4-2\,\alpha_2\theta_0^3-7\,\alpha_2\theta_0^2-3\,\alpha_2\theta_0\right)A_0$$
$$-3\left(\alpha_5\theta_0^3+2\,\alpha_5\theta_0^2+\alpha_5\theta_0\right)\overline{A_0}\Bigg)A_1+24\left(\left(\alpha_3\theta_0^3+\alpha_3\theta_0^2\right)A_0+\left(\alpha_1\theta_0^3+\alpha_1\theta_0^2\right)A_1\right)\overline{A_1}\Big\}$$

$$\mu_7 = \frac{1}{6\left(\theta_0^4+3\,\theta_0^3+2\,\theta_0^2\right)}\Big\{12\left(\alpha_7\theta_0^4-\alpha_7\theta_0^3-10\,\alpha_7\theta_0^2-8\,\alpha_7\theta_0\right)A_0^2-2\left(\theta_0^3-3\,\theta_0^2-10\,\theta_0\right)A_1C_2$$
$$-3\left(\theta_0^3-\theta_0^2-10\,\theta_0-8\right)A_0C_3+48\left(\left(\alpha_1^2-\alpha_4\right)\theta_0^3+3\left(\alpha_1^2-\alpha_4\right)\theta_0^2+2\left(\alpha_1^2-\alpha_4\right)\theta_0\right)A_0\overline{A_0} \tag{3.5.2}$$
$$-48\left(\alpha_3\theta_0^3+3\,\alpha_3\theta_0^2+2\,\alpha_3\theta_0\right)A_1\overline{A_0}-48\left(\alpha_1\theta_0^3+3\,\alpha_1\theta_0^2+2\,\alpha_1\theta_0\right)A_2\overline{A_0}+48\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)A_4\overline{A_0}$$
$$-3\left(\left(\alpha_1\theta_0^3+3\,\alpha_1\theta_0^2+2\,\alpha_1\theta_0\right)A_0-4\left(\theta_0^2+\theta_0\right)A_2\right)C_1\Big\}$$

$$\mu_8 = \frac{1}{2\left(\theta_0^5+4\,\theta_0^4+5\,\theta_0^3+2\,\theta_0^2\right)}\Big\{4\left(\alpha_9\theta_0^5+2\,\alpha_9\theta_0^4-3\,\alpha_9\theta_0^3-8\,\alpha_9\theta_0^2-4\,\alpha_9\theta_0\right)A_0\overline{A_0}-\left(\theta_0^4-3\,\theta_0^2-2\,\theta_0\right)B_2\overline{A_0}$$
$$+4\left(\alpha_2\theta_0^5+\alpha_2\theta_0^4-4\,\alpha_2\theta_0^3-4\,\alpha_2\theta_0^2\right)A_0\overline{A_1}-\left(\theta_0^4-4\,\theta_0^2\right)B_1\overline{A_1}+\left(2\left(\theta_0^3\overline{\alpha_1}-3\,\theta_0\overline{\alpha_1}-2\,\overline{\alpha_1}\right)\overline{A_0}-\left(\theta_0^4-3\,\theta_0^2-2\,\theta_0\right)\overline{A_2}\right)C_1\Big\}$$

$$\mu_9 = \frac{1}{2\left(\theta_0^3+\theta_0^2\right)}\Big\{2\,C_1(\theta_0-3)|A_1|^2\overline{A_0}+4\left(\alpha_8\theta_0^3-2\,\alpha_8\theta_0^2-3\,\alpha_8\theta_0\right)A_0\overline{A_0}+4\left(\alpha_2\theta_0^3-2\,\alpha_2\theta_0^2-3\,\alpha_2\theta_0\right)A_1\overline{A_0} \tag{3.5.3}$$
$$-\left(\theta_0^2-3\,\theta_0\right)B_3\overline{A_0}-\left(\theta_0^2-3\,\theta_0\right)C_2\overline{A_1}\Big\}$$

$$\mu_{10} = \frac{48\,A_0^2|A_1|^2\overline{\alpha_1}-4\,A_0|A_1|^2\overline{C_1}-12\,A_0\theta_0\overline{A_1}\overline{\alpha_2}-12\left(\theta_0\overline{\alpha_8}+\overline{\alpha_8}\right)A_0\overline{A_0}-12\,A_0A_1\overline{\alpha_3}-12\,A_0^2\overline{\alpha_6}+A_0\overline{B_3}+A_1\overline{C_2}}{6\,\theta_0}$$

$$\mu_{11} = \frac{1}{2\left(\theta_0^2+\theta_0\right)}\Big\{32\,A_0^2(\theta_0+1)|A_1|^4-4\,A_1^2\theta_0\overline{\alpha_1}+8\left((2\,\alpha_1\overline{\alpha_1}-\beta)\theta_0+2\,\alpha_1\overline{\alpha_1}-\beta\right)A_0^2-8\left(\theta_0\overline{\alpha_1}+\overline{\alpha_1}\right)A_0A_2$$
$$-4\left(\theta_0^2\overline{\alpha_9}+3\,\theta_0\overline{\alpha_9}+2\,\overline{\alpha_9}\right)A_0\overline{A_0}+A_1\theta_0\overline{B_1}+A_0(\theta_0+1)\overline{B_2}-4\Bigg(\left(3\,\theta_0\overline{\alpha_5}+2\,\overline{\alpha_5}\right)A_0$$
$$+\left(\theta_0^2\overline{\alpha_2}+3\,\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2}\right)\overline{A_0}\Bigg)A_1-\left((\alpha_1\theta_0+\alpha_1)A_0-A_2(\theta_0+1)\right)\overline{C_1}\Big\}$$

$$\mu_{12}=-\frac{2\left(3\left(\alpha_3\theta_0^2+3\,\alpha_3\theta_0+2\,\alpha_3\right)A_0^2+\left(5\,\alpha_1\theta_0^2+13\,\alpha_1\theta_0+6\,\alpha_1\right)A_0A_1-\left(3\,\theta_0^2+5\,\theta_0\right)A_1A_2-3\left(\theta_0^2+3\,\theta_0+2\right)A_0A_3\right)}{\theta_0^3+3\,\theta_0^2+2\,\theta_0}$$

$$\mu_{13}=-\frac{2}{\theta_0^3+3\,\theta_0^2+2\,\theta_0}\left\{4\left(2\,\theta_0^2+5\,\theta_0+3\right)A_0A_2|A_1|^2+3\left(\alpha_6\theta_0^2+3\,\alpha_6\theta_0+2\,\alpha_6\right)A_0^2+\left(5\,\alpha_5\theta_0^2+13\,\alpha_5\theta_0+6\,\alpha_5\right)A_0A_1\right.$$
$$\left.-4\left(3\left(\alpha_1\theta_0^2+3\,\alpha_1\theta_0+2\,\alpha_1\right)A_0^2-\left(\theta_0^2+2\,\theta_0\right)A_1^2\right)|A_1|^2\right\}$$
$$\mu_{14}=\frac{2}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0}\left\{4\left(\left(\alpha_1^2-\alpha_4\right)\theta_0^3+6\left(\alpha_1^2-\alpha_4\right)\theta_0^2+6\,\alpha_1^2+11\left(\alpha_1^2-\alpha_4\right)\theta_0-6\,\alpha_4\right)A_0^2\right.$$
$$-\left(7\,\alpha_3\theta_0^3+39\,\alpha_3\theta_0^2+62\,\alpha_3\theta_0+24\,\alpha_3\right)A_0A_1-3\left(\alpha_1\theta_0^3+5\,\alpha_1\theta_0^2+6\,\alpha_1\theta_0\right)A_1^2$$
$$-2\left(3\,\alpha_1\theta_0^3+16\,\alpha_1\theta_0^2+25\,\alpha_1\theta_0+12\,\alpha_1\right)A_0A_2+2\left(\theta_0^3+4\,\theta_0^2+3\,\theta_0\right)A_2^2+2\left(2\,\theta_0^3+9\,\theta_0^2+10\,\theta_0\right)A_1A_3$$
$$\left.+4\left(\theta_0^3+6\,\theta_0^2+11\,\theta_0+6\right)A_0A_4\right\}\tag{3.5.4}$$

We will not need all these μ at all.

Now, recall that

$$\alpha_7=\frac{1}{8\theta_0(\theta_0-4)}\langle\vec{C}_1,\vec{C}_1\rangle,\quad \vec{E}_1=-\frac{1}{2\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_0},\quad \langle\vec{C}_1,\vec{C}_1\rangle+8\langle\overline{\vec{A}_0},\vec{C}_3\rangle=0$$

so that

$$\langle\vec{A}_0,\vec{E}_1\rangle=-\frac{1}{4\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle=-2(\theta_0-4)\alpha_7$$

Therefore, we have

$$\begin{aligned}\mathbf{(12)}&=-\frac{C_1^2(\theta_0-2)+8\,A_0E_1(\theta_0-2)-32\left(\alpha_7\theta_0^2-4\,\alpha_7\theta_0\right)A_0\overline{A_0}+8\,C_3(\theta_0-4)\overline{A_0}}{16\,\theta_0^2}\\&=-\frac{1}{16\theta_0^2}\left((\theta_0-2)\langle\vec{C}_1,\vec{C}_1\rangle-\frac{2(\theta_0-2)}{\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle-16\theta_0(\theta_0-4)\alpha_7-(\theta_0-4)\langle\vec{C}_1,\vec{C}_1\rangle\right)\\&=-\frac{1}{16\theta_0^2}\left((\theta_0-2)\langle\vec{C}_1,\vec{C}_1\rangle-\frac{2(\theta_0-2)}{\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle-2\langle\vec{C}_1,\vec{C}_1\rangle-(\theta_0-4)\langle\vec{C}_1,\vec{C}_1\rangle\right)\\&=\frac{(\theta_0-2)}{8\theta_0^3}\langle\vec{C}_1,\vec{C}_1\rangle=\frac{(\theta_0-2)(\theta_0-4)}{\theta_0^2}\alpha_7.\end{aligned}\tag{3.5.5}$$

as

$$(\theta_0-2)-\frac{2(\theta_0-2)}{\theta_0}-\frac{2}{\theta_0}-(\theta_0-4)=2-2+\frac{4}{\theta_0}-\frac{2}{\theta_0}=\frac{2}{\theta_0}$$

Then

$$\mathbf{(15)}=-\frac{4\left(\theta_0\overline{\alpha_7}-4\,\overline{\alpha_7}\right)A_0\overline{A_0}+\overline{A_0E_1}}{2\left(\theta_0-4\right)}=-\frac{1}{2(\theta_0-4)}\left(2(\theta_0-4)\overline{\alpha_7}-2(\theta_0-4)\alpha_7\right)=0$$

Finally, we have (the μ coefficients do not seem to vanish so easily, even if there are numerous cancellations)

$$g^{-1}\otimes\langle\vec{h}_0,\vec{\Phi}\rangle=\begin{pmatrix} -\dfrac{2|A_1|^2}{\theta_0(\theta_0+1)} & 0 & 2 & \textbf{(1)}\\ \mu_1 & 0 & 3 & \textbf{(2)}\\ \mu_2 & 0 & 4 & \textbf{(3)}\\ \mu_3 & 1 & 2 & \textbf{(4)}\\ \mu_4 & 1 & 3 & \textbf{(5)}\\ \mu_5 & 2 & 1 & \textbf{(6)}\\ \mu_6 & 2 & 2 & \textbf{(7)}\\ \mu_7 & 3 & 1 & \textbf{(8)}\\ \dfrac{(\theta_0-2)\alpha_2}{\theta_0} & -\theta_0+1 & \theta_0+2 & \textbf{(9)}\\ \mu_8 & -\theta_0+1 & \theta_0+3 & \textbf{(10)}\\ \mu_9 & -\theta_0+2 & \theta_0+2 & \textbf{(11)}\\ \dfrac{(\theta_0-2)(\theta_0-4)\alpha_7}{\theta_0^2} & -\theta_0+3 & \theta_0+1 & \textbf{(12)}\\ -\dfrac{(\theta_0+1)\overline{\alpha_2}}{\theta_0} & \theta_0 & -\theta_0+3 & \textbf{(13)}\\ \mu_{10} & \theta_0 & -\theta_0+4 & \textbf{(14)}\\ -\dfrac{2\zeta_0}{\theta_0(\theta_0+1)} & \theta_0+1 & -\theta_0+1 & \textbf{(16)}\\ \mu_{11} & \theta_0+1 & -\theta_0+3 & \textbf{(17)}\\ \mu_{12} & \theta_0+2 & -\theta_0+1 & \textbf{(18)}\\ \mu_{13} & \theta_0+2 & -\theta_0+2 & \textbf{(19)}\\ \mu_{14} & \theta_0+3 & -\theta_0+1 & \textbf{(20)} \end{pmatrix}=\begin{pmatrix} -\dfrac{2\,|A_1|^2}{(\theta_0+1)\theta_0} & 0 & 2\\ \mu_1 & 0 & 3\\ \mu_2 & 0 & 4\\ \mu_3 & 1 & 2\\ \mu_4 & 1 & 3\\ \mu_5 & 2 & 1\\ \mu_6 & 2 & 2\\ \mu_7 & 3 & 1\\ \dfrac{\alpha_2(\theta_0-2)}{\theta_0} & -\theta_0+1 & \theta_0+2\\ \mu_8 & -\theta_0+1 & \theta_0+3\\ \mu_9 & -\theta_0+2 & \theta_0+2\\ \dfrac{\alpha_7(\theta_0-2)(\theta_0-4)}{\theta_0^2} & -\theta_0+3 & \theta_0+1\\ -\dfrac{(\theta_0+1)\overline{\alpha_2}}{\theta_0} & \theta_0 & -\theta_0+3\\ \mu_{10} & \theta_0 & -\theta_0+4\\ -\dfrac{2\,\zeta_0}{(\theta_0+1)\theta_0} & \theta_0+1 & -\theta_0+1\\ \mu_{11} & \theta_0+1 & -\theta_0+3\\ \mu_{12} & \theta_0+2 & -\theta_0+1\\ \mu_{13} & \theta_0+2 & -\theta_0+2\\ \mu_{14} & \theta_0+3 & -\theta_0+1 \end{pmatrix} \tag{3.5.6}$$

Code check the right. Observe that Sage enjoys mirror symmetry.

3.6 Second order development of the cancellation law

Finally, if

$$\vec{\beta}=\mathscr{I}_{\vec{\Phi}}(\vec{\alpha})-g^{-1}\otimes\left(\overline{\partial}|\vec{\Phi}|^2\vec{h}_0-2\langle\vec{\Phi},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}\right)=\vec{F}(z)dz$$

where as usual

$$\vec{\alpha}=\partial\vec{H}+|\vec{H}|^2\partial\vec{\Phi}+2\,g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle$$

we have

$$\mathrm{Re}\left(\partial_{\overline{z}}\vec{F}(z)\right)=0,$$

and

$\mathrm{Re}\,(\partial_{\overline{z}}\vec{F}(z)) =$

$$\begin{pmatrix}
\frac{\mu_1\theta_0^2 + 2\,\mu_1\theta_0 + 2\,\overline{\alpha_5}}{\theta_0} & \overline{A_0} & 0 & \theta_0 + 1 \\
\frac{\mu_1\theta_0^3 + 5\,\mu_1\theta_0^2 + 2\,(3\,\mu_1 + \overline{\alpha_5})\theta_0 + 6\,\overline{\alpha_5}}{\theta_0^2 + 2\,\theta_0} & \overline{A_1} & 0 & \theta_0 + 2 \\
\frac{\theta_0^3\overline{\mu_{13}} - 4\,(\theta_0 + 3)|A_1|^2\overline{\zeta_0} + 3\,\theta_0^2\overline{\mu_{13}} + 2\,\theta_0\overline{\mu_{13}}}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_0 & 0 & \theta_0 + 2 \\
\frac{\theta_0^3\overline{\mu_{12}} + 3\,\theta_0^2\overline{\mu_{12}} + 2\,\theta_0\overline{\mu_{12}} + 4\,(2\,\theta_0 + 3)\overline{\zeta_1}}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_1 & 0 & \theta_0 + 2 \\
\frac{\mu_2\theta_0^4 + 6\,\mu_2\theta_0^3 + (11\,\mu_2 + 3\,\overline{\alpha_6})\theta_0^2 - 2\left(3\,\theta_0^2\overline{\alpha_1} + 13\,\theta_0\overline{\alpha_1} + 12\,\overline{\alpha_1}\right)|A_1|^2 + 3\,(2\,\mu_2 + 3\,\overline{\alpha_6})\theta_0 + 6\,\overline{\alpha_6}}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & \overline{A_0} & 0 & \theta_0 + 2 \\
-\frac{\theta_0 - 1}{2\,\theta_0^2} & B_1 & 1 & \theta_0 \\
\frac{\mu_3\theta_0^2 + \mu_3\theta_0 + 2\,\alpha_5}{\theta_0} & \overline{A_0} & 1 & \theta_0 \\
-\frac{2\left(\alpha_2\theta_0^2 - \alpha_2\right)}{\theta_0} & A_0 & 1 & \theta_0 \\
-\frac{\theta_0 - 1}{\theta_0^2} & B_2 & 1 & \theta_0 + 1 \\
\frac{\mu_3\theta_0^3 + 3\,\mu_3\theta_0^2 + 2\,(\alpha_5 + \mu_3)\theta_0 + 4\,\alpha_5}{\theta_0^2 + \theta_0} & \overline{A_1} & 1 & \theta_0 + 1 \\
-\frac{\alpha_9\theta_0^4 + (\alpha_9 - 2\,\overline{\mu_{11}})\theta_0^3 - 2\,(2\,\alpha_9 + \overline{\mu_{11}})\theta_0^2 + 4\,\alpha_1\theta_0\overline{\zeta_0} - 4\,\alpha_9\theta_0 + \left(\theta_0^2 - \theta_0 - 2\right)\zeta_4}{\theta_0^3 + \theta_0^2} & A_0 & 1 & \theta_0 + 1 \\
\frac{4\,\mu_4\theta_0^5 + 12\,\mu_4\theta_0^4 - 32\left(\theta_0^3 + 2\,\theta_0^2\right)|A_1|^4 - 8\,(2\,\alpha_1\overline{\alpha_1} - 2\,\beta - \mu_4)\theta_0^3 - 16\,(\alpha_1\overline{\alpha_1} - \beta)\theta_0^2 - \left(\theta_0^3 - 3\,\theta_0 - 2\right)|C_1|^2}{4\left(\theta_0^4 + \theta_0^3\right)} & \overline{A_0} & 1 & \theta_0 + 1 \\
-\frac{2\,\theta_0 - 3}{4\,\theta_0^2} & B_3 & 2 & \theta_0 \\
-\frac{2\,\alpha_8\theta_0^3 - 2\,(2\,\alpha_8 + 3\,\overline{\mu_{10}})\theta_0^2 - 6\,\alpha_8\theta_0 + (\theta_0 - 3)\zeta_3}{2\,\theta_0^2} & A_0 & 2 & \theta_0 \\
\frac{\mu_6\theta_0^2 - 6\,\alpha_1|A_1|^2 + \mu_6\theta_0 + 3\,\alpha_6}{\theta_0} & \overline{A_0} & 2 & \theta_0 \\
-\frac{2\,\alpha_2\theta_0^2 - \alpha_2\theta_0 - 3\,\alpha_2}{\theta_0} & A_1 & 2 & \theta_0 \\
\frac{2\,\mu_5\theta_0^4 - 4\,\mu_5\theta_0^3 - 6\,\mu_5\theta_0^2 + (5\,\theta_0 - 12)\zeta_2}{2\left(\theta_0^3 - 3\,\theta_0^2\right)} & \overline{A_1} & 2 & \theta_0 \\
\frac{2\,\mu_5\theta_0^3 + 2\,\mu_5\theta_0^2 + 3\,\zeta_2}{2\left(\theta_0^2 + \theta_0\right)} & \overline{A_0} & 2 & \theta_0 - 1 \\
\frac{\mu_7\theta_0^2 - \theta_0\zeta_7 + \zeta_5}{\theta_0} & \overline{A_0} & 3 & \theta_0 - 1 \\
-\frac{(\alpha_9 - 2\,\mu_8)\theta_0^3 + 2\,(\alpha_9 - 3\,\mu_8)\theta_0^2 - 4\,(\alpha_9 + \mu_8)\theta_0 + (\theta_0 - 2)\zeta_4 - 8\,\alpha_9}{\theta_0^2 + 2\,\theta_0} & \overline{A_0} & -\theta_0 + 1 & 2\,\theta_0 + 1 \\
-\frac{2\,(\alpha_8 - 2\,\mu_9)\theta_0^3 - 2\,(\alpha_8 + 3\,\mu_9)\theta_0^2 - 2\,(5\,\alpha_8 + \mu_9)\theta_0 + (\theta_0 - 3)\zeta_3 - 6\,\alpha_8}{2\left(\theta_0^2 + \theta_0\right)} & \overline{A_0} & -\theta_0 + 2 & 2\,\theta_0 \\
\frac{\theta_0 - 4}{4\,\theta_0} & E_1 & -\theta_0 + 3 & 2\,\theta_0 - 1 \\
\frac{\alpha_7\theta_0^2 - 8\,\alpha_7\theta_0 + 16\,\alpha_7}{\theta_0} & \overline{A_0} & -\theta_0 + 3 & 2\,\theta_0 - 1 \\
-\frac{\theta_0 - 1}{2\,\theta_0^2} & \overline{B_1} & \theta_0 & 1 \\
\frac{\theta_0^2\overline{\mu_3} + \theta_0\overline{\mu_3} + 2\,\overline{\alpha_5}}{\theta_0} & A_0 & \theta_0 & 1 \\
-\frac{2\left(\theta_0^2\overline{\alpha_2} - \overline{\alpha_2}\right)}{\theta_0} & \overline{A_0} & \theta_0 & 1
\end{pmatrix}$$

$$
\begin{pmatrix}
-\frac{2\,\theta_0-3}{4\,\theta_0^2} & \overline{B_3} & \theta_0 & 2 \\
-\frac{2\,\theta_0^3\overline{\alpha_8}-2\,(3\,\mu_{10}+2\,\overline{\alpha_8})\theta_0^2-6\,\theta_0\overline{\alpha_8}+(\theta_0-3)\overline{\zeta_3}}{2\,\theta_0^2} & \overline{A_0} & \theta_0 & 2 \\
-\frac{2\,\theta_0^2\overline{\alpha_2}-\theta_0\overline{\alpha_2}-3\,\overline{\alpha_2}}{\theta_0} & \overline{A_1} & \theta_0 & 2 \\
-\frac{6\,|A_1|^2\overline{\alpha_1}-\theta_0^2\overline{\mu_6}-\theta_0\overline{\mu_6}-3\,\overline{\alpha_6}}{\theta_0} & A_0 & \theta_0 & 2 \\
\frac{2\,\theta_0^4\overline{\mu_5}-4\,\theta_0^3\overline{\mu_5}-6\,\theta_0^2\overline{\mu_5}+(5\,\theta_0-12)\overline{\zeta_2}}{2\left(\theta_0^3-3\,\theta_0^2\right)} & A_1 & \theta_0 & 2 \\
\frac{2\,\theta_0^3\overline{\mu_5}+2\,\theta_0^2\overline{\mu_5}+3\,\overline{\zeta_2}}{2\left(\theta_0^2+\theta_0\right)} & A_0 & \theta_0-1 & 2 \\
\frac{\theta_0^2\overline{\mu_7}-\theta_0\overline{\zeta_7}+\overline{\zeta_5}}{\theta_0} & A_0 & \theta_0-1 & 3 \\
\frac{\theta_0^2\overline{\mu_1}+2\,\theta_0\overline{\mu_1}+2\,\alpha_5}{\theta_0} & A_0 & \theta_0+1 & 0 \\
-\frac{\theta_0-1}{\theta_0^2} & \overline{B_2} & \theta_0+1 & 1 \\
\frac{\theta_0^3\overline{\mu_3}+3\,\theta_0^2\overline{\mu_3}+2\,\theta_0(\overline{\alpha_5}+\overline{\mu_3})+4\,\overline{\alpha_5}}{\theta_0^2+\theta_0} & A_1 & \theta_0+1 & 1 \\
-\frac{\theta_0^4\overline{\alpha_9}-(2\,\mu_{11}-\overline{\alpha_9})\theta_0^3-2\,(\mu_{11}+2\,\overline{\alpha_9})\theta_0^2+4\,\theta_0\zeta_0\overline{\alpha_1}-4\,\theta_0\overline{\alpha_9}+\left(\theta_0^2-\theta_0-2\right)\overline{\zeta_4}}{\theta_0^3+\theta_0^2} & \overline{A_0} & \theta_0+1 & 1 \\
\frac{4\,\theta_0^5\overline{\mu_4}-32\left(\theta_0^3+2\,\theta_0^2\right)|A_1|^4+12\,\theta_0^4\overline{\mu_4}-8\,(2\,\alpha_1\overline{\alpha_1}-2\,\beta-\overline{\mu_4})\theta_0^3-16\,(\alpha_1\overline{\alpha_1}-\beta)\theta_0^2-\left(\theta_0^3-3\,\theta_0-2\right)|C_1|^2}{4\left(\theta_0^4+\theta_0^3\right)} & A_0 & \theta_0+1 & 1 \\
\frac{\theta_0^3\overline{\mu_1}+5\,\theta_0^2\overline{\mu_1}+2\,(\alpha_5+3\,\overline{\mu_1})\theta_0+6\,\alpha_5}{\theta_0^2+2\,\theta_0} & A_1 & \theta_0+2 & 0 \\
\frac{\mu_{13}\theta_0^3-4\,(\theta_0+3)\zeta_0|A_1|^2+3\,\mu_{13}\theta_0^2+2\,\mu_{13}\theta_0}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & \overline{A_0} & \theta_0+2 & 0 \\
\frac{\mu_{12}\theta_0^3+3\,\mu_{12}\theta_0^2+2\,\mu_{12}\theta_0+4\,(2\,\theta_0+3)\zeta_1}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & \overline{A_1} & \theta_0+2 & 0 \\
\frac{\theta_0^4\overline{\mu_2}+6\,\theta_0^3\overline{\mu_2}+(3\,\alpha_6+11\,\overline{\mu_2})\theta_0^2-2\left(3\,\alpha_1\theta_0^2+13\,\alpha_1\theta_0+12\,\alpha_1\right)|A_1|^2+3\,(3\,\alpha_6+2\,\overline{\mu_2})\theta_0+6\,\alpha_6}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & A_0 & \theta_0+2 & 0 \\
\frac{\theta_0-4}{4\,\theta_0} & \overline{E_1} & 2\,\theta_0-1 & -\theta_0+3 \\
\frac{\theta_0^2\overline{\alpha_7}-8\,\theta_0\overline{\alpha_7}+16\,\overline{\alpha_7}}{\theta_0} & A_0 & 2\,\theta_0-1 & -\theta_0+3 \\
-\frac{\theta_0^3(\overline{\alpha_9}-2\,\overline{\mu_8})+2\,\theta_0^2(\overline{\alpha_9}-3\,\overline{\mu_8})-4\,\theta_0(\overline{\alpha_9}+\overline{\mu_8})+(\theta_0-2)\overline{\zeta_4}-8\,\overline{\alpha_9}}{\theta_0^2+2\,\theta_0} & A_0 & 2\,\theta_0+1 & -\theta_0+1 \\
-\frac{2\,\theta_0^3(\overline{\alpha_8}-2\,\overline{\mu_9})-2\,\theta_0^2(\overline{\alpha_8}+3\,\overline{\mu_9})-2\,\theta_0(5\,\overline{\alpha_8}+\overline{\mu_9})+(\theta_0-3)\overline{\zeta_3}-6\,\overline{\alpha_8}}{2\left(\theta_0^2+\theta_0\right)} & A_0 & 2\,\theta_0 & -\theta_0+2
\end{pmatrix}
$$

One can check that all relations of this order are trivial.

Chapter 4

Last order in the inversion

4.1 Development of $\vec{\Phi}$

$|\vec{\Phi}(z)|^2 =$

$$
\begin{pmatrix}
\frac{2\,\overline{A_0 A_1}}{\theta_0^2+\theta_0} & 0 & 2\,\theta_0+1 & \mathbf{(1)} \\
\frac{\left(\theta_0^2+2\,\theta_0\right)\overline{A_1}^2+2\left(\theta_0^2+2\,\theta_0+1\right)\overline{A_0 A_2}}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 0 & 2\,\theta_0+2 & \mathbf{(2)} \\
\frac{2\left(\left(\theta_0^2+3\,\theta_0\right)\overline{A_1 A_2}+\left(\theta_0^2+3\,\theta_0+2\right)\overline{A_0 A_3}\right)}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 0 & 2\,\theta_0+3 & \mathbf{(3)} \\
\lambda_1 & 0 & 2\,\theta_0+4 & \mathbf{(4)} \\
\frac{\overline{A_0}^2}{\theta_0^2} & 0 & 2\,\theta_0 & \mathbf{(5)} \\
\frac{B_1\overline{A_0}+C_1\overline{A_1}}{4\left(\theta_0^2+\theta_0\right)} & 2 & 2\,\theta_0+1 & \mathbf{(6)} \\
\frac{\left(\theta_0^2+2\,\theta_0+1\right)B_2\overline{A_0}+\left(\theta_0^2+2\,\theta_0\right)B_1\overline{A_1}+\left(\left(\theta_0^2\overline{\alpha_1}+2\,\theta_0\overline{\alpha_1}+\overline{\alpha_1}\right)\overline{A_0}+\left(\theta_0^2+2\,\theta_0+1\right)\overline{A_2}\right)C_1}{4\left(\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0\right)} & 2 & 2\,\theta_0+2 & \mathbf{(7)} \\
\frac{C_1\overline{A_0}}{4\,\theta_0^2} & 2 & 2\,\theta_0 & \mathbf{(8)} \\
\frac{2\,C_1|A_1|^2\overline{A_0}+B_3\overline{A_0}+C_2\overline{A_1}}{6\left(\theta_0^2+\theta_0\right)} & 3 & 2\,\theta_0+1 & \mathbf{(9)} \\
\frac{C_2\overline{A_0}}{6\,\theta_0^2} & 3 & 2\,\theta_0 & \mathbf{(10)} \\
\frac{C_1^2(\theta_0-4)+8\left(\alpha_1\theta_0-4\,\alpha_1\right)C_1\overline{A_0}+8\,C_3(\theta_0-4)\overline{A_0}-16\,A_0E_1}{64\left(\theta_0^3-4\,\theta_0^2\right)} & 4 & 2\,\theta_0 & \mathbf{(11)} \\
-\frac{E_1\overline{A_0}}{4\left(\theta_0^3-4\,\theta_0^2\right)} & -\theta_0+4 & 3\,\theta_0 & \mathbf{(12)} \\
\frac{2\,A_0\overline{A_0}}{\theta_0^2} & \theta_0 & \theta_0 & \mathbf{(13)} \\
\frac{2\,A_0\overline{A_1}}{\theta_0^2+\theta_0} & \theta_0 & \theta_0+1 & \mathbf{(14)} \\
\frac{8\,A_0\theta_0\overline{A_2}+(\theta_0+2)\overline{A_0C_1}}{4\left(\theta_0^3+2\,\theta_0^2\right)} & \theta_0 & \theta_0+2 & \mathbf{(15)} \\
\frac{24\left(\theta_0^2+\theta_0\right)A_0\overline{A_3}+3\left(\theta_0^2+3\,\theta_0\right)\overline{A_1C_1}+2\left(\theta_0^2+4\,\theta_0+3\right)\overline{A_0C_2}}{12\left(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2\right)} & \theta_0 & \theta_0+3 & \mathbf{(16)} \\
\lambda_2 & \theta_0 & \theta_0+4 & \mathbf{(17)} \\
\frac{2\,A_1\overline{A_0}}{\theta_0^2+\theta_0} & \theta_0+1 & \theta_0 & \mathbf{(18)}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\dfrac{2\,A_1\overline{A_1}}{\theta_0^2+2\,\theta_0+1} & \theta_0+1 & \theta_0+1 & \textbf{(19)}\\
\dfrac{8\,A_1\theta_0\overline{A_2}+(\theta_0+2)\overline{A_0B_1}}{4\,(\theta_0^3+3\,\theta_0^2+2\,\theta_0)} & \theta_0+1 & \theta_0+2 & \textbf{(20)}\\
\dfrac{4\left(\theta_0^2+4\,\theta_0+3\right)|A_1|^2\overline{A_0C_1}+24\left(\theta_0^2+\theta_0\right)A_1\overline{A_3}+3\left(\theta_0^2+3\,\theta_0\right)\overline{A_1B_1}+2\left(\theta_0^2+4\,\theta_0+3\right)\overline{A_0B_3}}{12\,(\theta_0^4+5\,\theta_0^3+7\,\theta_0^2+3\,\theta_0)} & \theta_0+1 & \theta_0+3 & \textbf{(21)}\\
\dfrac{A_0C_1(\theta_0+2)+8\,A_2\theta_0\overline{A_0}}{4\,(\theta_0^3+2\,\theta_0^2)} & \theta_0+2 & \theta_0 & \textbf{(22)}\\
\dfrac{A_0B_1(\theta_0+2)+8\,A_2\theta_0\overline{A_1}}{4\,(\theta_0^3+3\,\theta_0^2+2\,\theta_0)} & \theta_0+2 & \theta_0+1 & \textbf{(23)}\\
\lambda_3 & \theta_0+2 & \theta_0+2 & \textbf{(24)}\\
\dfrac{3\left(\theta_0^2+3\,\theta_0\right)A_1C_1+2\left(\theta_0^2+4\,\theta_0+3\right)A_0C_2+24\left(\theta_0^2+\theta_0\right)A_3\overline{A_0}}{12\,(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0+3 & \theta_0 & \textbf{(25)}\\
\dfrac{4\left(\theta_0^2+4\,\theta_0+3\right)A_0C_1|A_1|^2+3\left(\theta_0^2+3\,\theta_0\right)A_1B_1+2\left(\theta_0^2+4\,\theta_0+3\right)A_0B_3+24\left(\theta_0^2+\theta_0\right)A_3\overline{A_1}}{12\,(\theta_0^4+5\,\theta_0^3+7\,\theta_0^2+3\,\theta_0)} & \theta_0+3 & \theta_0+1 & \textbf{(26)}\\
\lambda_4 & \theta_0+4 & \theta_0 & \textbf{(27)}\\
\dfrac{2\,A_0A_1}{\theta_0^2+\theta_0} & 2\,\theta_0+1 & 0 & \textbf{(28)}\\
\dfrac{A_0\overline{B_1}+A_1\overline{C_1}}{4\,(\theta_0^2+\theta_0)} & 2\,\theta_0+1 & 2 & \textbf{(29)}\\
\dfrac{2\,A_0|A_1|^2\overline{C_1}+A_0\overline{B_3}+A_1\overline{C_2}}{6\,(\theta_0^2+\theta_0)} & 2\,\theta_0+1 & 3 & \textbf{(30)}\\
\dfrac{\left(\theta_0^2+2\,\theta_0\right)A_1^2+2\left(\theta_0^2+2\,\theta_0+1\right)A_0A_2}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 2\,\theta_0+2 & 0 & \textbf{(31)}\\
\dfrac{\left(\theta_0^2+2\,\theta_0\right)A_1\overline{B_1}+\left(\theta_0^2+2\,\theta_0+1\right)A_0\overline{B_2}+\left(\left(\alpha_1\theta_0^2+2\,\alpha_1\theta_0+\alpha_1\right)A_0+\left(\theta_0^2+2\,\theta_0+1\right)A_2\right)\overline{C_1}}{4\,(\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0)} & 2\,\theta_0+2 & 2 & \textbf{(32)}\\
\dfrac{2\left(\left(\theta_0^2+3\,\theta_0\right)A_1A_2+\left(\theta_0^2+3\,\theta_0+2\right)A_0A_3\right)}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 2\,\theta_0+3 & 0 & \textbf{(33)}\\
\lambda_5 & 2\,\theta_0+4 & 0 & \textbf{(34)}\\
\dfrac{A_0^2}{\theta_0^2} & 2\,\theta_0 & 0 & \textbf{(35)}\\
\dfrac{A_0\overline{C_1}}{4\,\theta_0^2} & 2\,\theta_0 & 2 & \textbf{(36)}\\
\dfrac{A_0\overline{C_2}}{6\,\theta_0^2} & 2\,\theta_0 & 3 & \textbf{(37)}\\
\dfrac{8\,(\theta_0\overline{\alpha_1}-4\,\overline{\alpha_1})A_0\overline{C_1}+(\theta_0-4)\overline{C_1}^2+8\,A_0(\theta_0-4)\overline{C_3}-16\,\overline{A_0E_1}}{64\,(\theta_0^3-4\,\theta_0^2)} & 2\,\theta_0 & 4 & \textbf{(38)}\\
-\dfrac{A_0\overline{E_1}}{4\,(\theta_0^3-4\,\theta_0^2)} & 3\,\theta_0 & -\theta_0+4 & \textbf{(39)}
\end{pmatrix}
$$

where

$$\lambda_1=\frac{\left(\theta_0^4+8\,\theta_0^3+19\,\theta_0^2+12\,\theta_0\right)\overline{A_2}^2+2\left(\theta_0^4+8\,\theta_0^3+20\,\theta_0^2+16\,\theta_0\right)\overline{A_1A_3}+2\left(\theta_0^4+8\,\theta_0^3+23\,\theta_0^2+28\,\theta_0+12\right)\overline{A_0A_4}}{\theta_0^6+12\,\theta_0^5+55\,\theta_0^4+120\,\theta_0^3+124\,\theta_0^2+48\,\theta_0} \tag{4.1.1}$$

$$\lambda_2=\frac{1}{24\,(\theta_0^5+7\,\theta_0^4+14\,\theta_0^3+8\,\theta_0^2)}\Big\{48\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)A_0\overline{A_4}+4\left(\theta_0^3+6\,\theta_0^2+8\,\theta_0\right)\overline{A_1C_2}+3\left(\theta_0^3+7\,\theta_0^2+14\,\theta_0+8\right)\overline{A_0C_3} \tag{4.1.2}$$

$$+3\left(\left(\theta_0^3\overline{\alpha_1}+7\,\theta_0^2\overline{\alpha_1}+14\,\theta_0\overline{\alpha_1}+8\,\overline{\alpha_1}\right)\overline{A_0}+2\left(\theta_0^3+5\,\theta_0^2+4\,\theta_0\right)\overline{A_2}\right)\overline{C_1}\Big\} \tag{4.1.3}$$

$$\lambda_3 = \frac{1}{32\,(\theta_0^4+4\,\theta_0^3+4\,\theta_0^2)}\Big\{64\,A_2\theta_0^2\overline{A_2}+8\left(\theta_0^2+2\,\theta_0\right)A_0B_2+8\left(\theta_0^2\overline{\alpha_1}+2\,\theta_0\overline{\alpha_1}\right)A_0C_1+8\left(\theta_0^2+2\,\theta_0\right)\overline{A_0B_2} \tag{4.1.4}$$

$$+\left(\left(\theta_0^2+4\,\theta_0+4\right)C_1+8\left(\alpha_1\theta_0^2+2\,\alpha_1\theta_0\right)\overline{A_0}\right)\overline{C_1}\Big\} \tag{4.1.5}$$

$$\lambda_4 = \frac{1}{24\,(\theta_0^5+7\,\theta_0^4+14\,\theta_0^3+8\,\theta_0^2)}\Big\{4\left(\theta_0^3+6\,\theta_0^2+8\,\theta_0\right)A_1C_2+3\left(\theta_0^3+7\,\theta_0^2+14\,\theta_0+8\right)A_0C_3+48\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)A_4\overline{A_0} \tag{4.1.6}$$

$$+3\left(\left(\alpha_1\theta_0^3+7\,\alpha_1\theta_0^2+14\,\alpha_1\theta_0+8\,\alpha_1\right)A_0+2\left(\theta_0^3+5\,\theta_0^2+4\,\theta_0\right)A_2\right)C_1\Big\} \tag{4.1.7}$$

$$\lambda_5 = \frac{\left(\theta_0^4+8\,\theta_0^3+19\,\theta_0^2+12\,\theta_0\right)A_2^2+2\left(\theta_0^4+8\,\theta_0^3+20\,\theta_0^2+16\,\theta_0\right)A_1A_3+2\left(\theta_0^4+8\,\theta_0^3+23\,\theta_0^2+28\,\theta_0+12\right)A_0A_4}{\theta_0^6+12\,\theta_0^5+55\,\theta_0^4+120\,\theta_0^3+124\,\theta_0^2+48\,\theta_0} \tag{4.1.8}$$

Recall by REF that

$$|\vec{\Phi}(z)|^2 = \begin{pmatrix}
\dfrac{-\overline{\zeta_0}}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 0 & 2\,\theta_0+2 & \mathbf{(1)}\\
\dfrac{-4\overline{\zeta_1}}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 0 & 2\,\theta_0+3 & \mathbf{(2)}\\
\dfrac{1}{\theta_0^2} & \theta_0 & \theta_0 & \mathbf{(4)}\\
\dfrac{\overline{\alpha_1}}{\theta_0(\theta_0+2)} & \theta_0 & \theta_0+2 & \mathbf{(5)}\\
\dfrac{-\overline{\zeta_2}+2\theta_0(\theta_0+1)\overline{\alpha_3}}{2(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0 & \theta_0+3 & \mathbf{(6)}\\
\dfrac{2\,|A_1|^2}{(\theta_0+1)^2} & \theta_0+1 & \theta_0+1 & \mathbf{(7)}\\
\dfrac{\overline{\alpha_5}}{(\theta_0+1)(\theta_0+2)} & \theta_0+1 & \theta_0+2 & \mathbf{(8)}\\
\dfrac{\alpha_1}{\theta_0(\theta_0+2)} & \theta_0+2 & \theta_0 & \mathbf{(9)}\\
\dfrac{\alpha_5}{(\theta_0+1)(\theta_0+2)} & \theta_0+2 & \theta_0+1 & \mathbf{(10)}\\
\dfrac{-\zeta_2+2\theta_0(\theta_0+1)\alpha_3}{2(\theta_0^4+4\,\theta_0^3+3\,\theta_0^2)} & \theta_0+3 & \theta_0 & \mathbf{(11)}\\
\dfrac{-\zeta_0}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & 2\,\theta_0+2 & 0 & \mathbf{(13)}\\
\dfrac{-4\zeta_1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & 2\,\theta_0+3 & 0 & \mathbf{(14)}
\end{pmatrix}$$

$$= \frac{1}{\theta_0^2}|z|^{2\theta_0}+\frac{2|\vec{A}_1|^2}{(\theta_0+1)^2}|z|^{2\theta_0+2}+2\,\mathrm{Re}\left(\frac{\alpha_1}{\theta_0(\theta_0+2)}z^{\theta_0+2}\overline{z}^{\theta_0}+\frac{\alpha_5}{(\theta_0+1)(\theta_0+2)}z^{\theta_0+2}\overline{z}^{\theta_0+1}+\right.$$
$$\left.+\frac{-\zeta_2+2\theta_0(\theta_0+1)\alpha_3}{2(\theta_0^4+4\theta_0^3+3\theta_0^2)}z^{\theta_0+3}\overline{z}^{\theta_0}-\frac{\zeta_0}{\theta_0^4+4\theta_0^3+5\theta_0^2+2\theta_0}z^{2\theta_0+2}-\frac{4\zeta_1}{\theta_0^4+6\theta_0^3+11\theta_0^2+6\theta_0}z^{2\theta_0+3}\right)+O(|z|^{2\theta_0+4})$$
$$= |z|^{2\theta_0}\left\{\frac{1}{\theta_0^2}+\frac{2|\vec{A}_1|^2}{(\theta_0+1)^2}|z|^2+2\,\mathrm{Re}\left(\frac{\alpha_1}{\theta_0(\theta_0+2)}z^2+\frac{\alpha_5}{(\theta_0+1)(\theta_0+2)}z^2\overline{z}++\frac{-\zeta_2+2\theta_0(\theta_0+1)\alpha_3}{2(\theta_0^4+4\theta_0^3+3\theta_0^2)}z^3\right.\right.$$
$$\left.\left.-\frac{\zeta_0}{\theta_0^4+4\theta_0^3+5\theta_0^2+2\theta_0}z^{\theta_0+2}\overline{z}^{-\theta_0}-\frac{4\zeta_1}{\theta_0^4+6\theta_0^3+11\theta_0^2+6\theta_0}z^{\theta_0+3}\overline{z}^{-\theta_0}\right)+O(|z|^4)\right\}$$

where

$$\begin{cases}\zeta_0=\langle\vec{A}_1,\vec{A}_1\rangle\\ \zeta_1=\langle\vec{A}_1,\vec{A}_2\rangle\\ \zeta_2=\langle\vec{A}_1,\vec{C}_1\rangle\end{cases}$$

so we need only consider powers of order equal to $2\theta_0+4$, which correspond to the lines

$$\textbf{(4)},\textbf{(7)},\textbf{(9)},\textbf{(11)},\textbf{(12)},\textbf{(17)},\textbf{(21)},\textbf{(24)},\textbf{(26)},\textbf{(27)},\textbf{(30)},\textbf{(32)},\textbf{(34)},\textbf{(38)},\textbf{(39)}.$$

and this is thanks of

$$\begin{cases} \vec{B}_1 = -2\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \vec{A}_0 \\ \vec{B}_2 = -\dfrac{(\theta_0+2)}{4\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0} - 2\langle \overline{\vec{A}_2}, \vec{C}_1\rangle \vec{A}_0 \\ \vec{B}_3 = -2\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \vec{A}_1 + \dfrac{2}{\theta_0-3}\langle \vec{A}_1, \vec{C}_1\rangle \overline{\vec{A}_1} - 2\langle \overline{\vec{A}_1}, \vec{C}_2\rangle \vec{A}_0 \\ \vec{E}_1 = -\dfrac{1}{2\theta_0}\langle \vec{C}_1, \vec{C}_1\rangle \overline{\vec{A}_0}. \end{cases} \tag{4.1.9}$$

some words

$$\begin{pmatrix}
\lambda_1 & 0 & 2\,\theta_0+4 & \textbf{(4)} \\
\dfrac{\left(\theta_0^2+2\,\theta_0+1\right)B_2\overline{A_0}+\left(\theta_0^2+2\,\theta_0\right)B_1\overline{A_1}+\left(\left(\theta_0^2\overline{\alpha_1}+2\,\theta_0\overline{\alpha_1}+\overline{\alpha_1}\right)\overline{A_0}+\left(\theta_0^2+2\,\theta_0+1\right)\overline{A_2}\right)C_1}{4\left(\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0\right)} & 2 & 2\,\theta_0+2 & \textbf{(7)} \\
\dfrac{2\,C_1|A_1|^2\overline{A_0}+B_3\overline{A_0}+C_2\overline{A_1}}{6\left(\theta_0^2+\theta_0\right)} & 3 & 2\,\theta_0+1 & \textbf{(9)} \\
\dfrac{C_1^2(\theta_0-4)+8\left(\alpha_1\theta_0-4\,\alpha_1\right)C_1\overline{A_0}+8\,C_3(\theta_0-4)\overline{A_0}-16\,A_0E_1}{64\left(\theta_0^3-4\,\theta_0^2\right)} & 4 & 2\,\theta_0 & \textbf{(11)} \\
-\dfrac{E_1\overline{A_0}}{4\left(\theta_0^3-4\,\theta_0^2\right)} & -\theta_0+4 & 3\,\theta_0 & \textbf{(12)} \\
\lambda_2 & \theta_0 & \theta_0+4 & \textbf{(17)} \\
\dfrac{4\left(\theta_0^2+4\,\theta_0+3\right)|A_1|^2\overline{A_0C_1}+24\left(\theta_0^2+\theta_0\right)A_1\overline{A_3}+3\left(\theta_0^2+3\,\theta_0\right)\overline{A_1B_1}+2\left(\theta_0^2+4\,\theta_0+3\right)\overline{A_0}B_3}{12\left(\theta_0^4+5\,\theta_0^3+7\,\theta_0^2+3\,\theta_0\right)} & \theta_0+1 & \theta_0+3 & \textbf{(21)} \\
\lambda_3 & \theta_0+2 & \theta_0+2 & \textbf{(24)} \\
\dfrac{4\left(\theta_0^2+4\,\theta_0+3\right)A_0C_1|A_1|^2+3\left(\theta_0^2+3\,\theta_0\right)A_1B_1+2\left(\theta_0^2+4\,\theta_0+3\right)A_0B_3+24\left(\theta_0^2+\theta_0\right)A_3\overline{A_1}}{12\left(\theta_0^4+5\,\theta_0^3+7\,\theta_0^2+3\,\theta_0\right)} & \theta_0+3 & \theta_0+1 & \textbf{(26)} \\
\lambda_4 & \theta_0+4 & \theta_0 & \textbf{(27)} \\
\dfrac{2\,A_0|A_1|^2\overline{C_1}+A_0\overline{B_3}+A_1\overline{C_2}}{6\left(\theta_0^2+\theta_0\right)} & 2\,\theta_0+1 & 3 & \textbf{(30)} \\
\dfrac{\left(\theta_0^2+2\,\theta_0\right)A_1\overline{B_1}+\left(\theta_0^2+2\,\theta_0+1\right)A_0\overline{B_2}+\left(\left(\alpha_1\theta_0^2+2\,\alpha_1\theta_0+\alpha_1\right)A_0+\left(\theta_0^2+2\,\theta_0+1\right)A_2\right)\overline{C_1}}{4\left(\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0\right)} & 2\,\theta_0+2 & 2 & \textbf{(32)} \\
\lambda_5 & 2\,\theta_0+4 & 0 & \textbf{(34)} \\
\dfrac{8\left(\theta_0\overline{\alpha_1}-4\,\overline{\alpha_1}\right)A_0\overline{C_1}+(\theta_0-4)\overline{C_1}^2+8\,A_0(\theta_0-4)\overline{C_3}-16\,\overline{A_0E_1}}{64\left(\theta_0^3-4\,\theta_0^2\right)} & 2\,\theta_0 & 4 & \textbf{(38)} \\
-\dfrac{A_0\overline{E_1}}{4\left(\theta_0^3-4\,\theta_0^2\right)} & 3\,\theta_0 & -\theta_0+4 & \textbf{(39)}
\end{pmatrix}$$

$$
= \begin{pmatrix}
\lambda_1 & 0 & 2\,\theta_0+4 & \mathbf{(4)} \\
\dfrac{-16\,A_0 E_1}{64\,(\theta_0^3-4\,\theta_0^2)} & 4 & 2\,\theta_0 & \mathbf{(11)} \\
\lambda_2 & \theta_0 & \theta_0+4 & \mathbf{(17)} \\
\dfrac{24\left(\theta_0^2+\theta_0\right)A_1\overline{A_3}}{12\,(\theta_0^4+5\,\theta_0^3+7\,\theta_0^2+3\,\theta_0)} & \theta_0+1 & \theta_0+3 & \mathbf{(21)} \\
\lambda_3 & \theta_0+2 & \theta_0+2 & \mathbf{(24)} \\
\dfrac{24\left(\theta_0^2+\theta_0\right)A_3\overline{A_1}}{12\,(\theta_0^4+5\,\theta_0^3+7\,\theta_0^2+3\,\theta_0)} & \theta_0+3 & \theta_0+1 & \mathbf{(26)} \\
\lambda_4 & \theta_0+4 & \theta_0 & \mathbf{(27)} \\
\lambda_5 & 2\,\theta_0+4 & 0 & \mathbf{(34)} \\
\dfrac{-16\,\overline{A_0 E_1}}{64\,(\theta_0^3-4\,\theta_0^2)} & 2\,\theta_0 & 4 & \mathbf{(38)}
\end{pmatrix}
$$

λ_1 does not simplify much and as $\vec{\Phi}$ is real, we have $\lambda_5 = \overline{\lambda_1}$. Then, we also have $\lambda_4 = \overline{\lambda_2}$, and finally,

$$
\begin{aligned}
\lambda_3 &= \frac{1}{32\,(\theta_0^4+4\,\theta_0^3+4\,\theta_0^2)}\Big\{64\,A_2\theta_0^2\overline{A_2} + 8\left(\theta_0^2+2\,\theta_0\right)A_0B_2 + 8\left(\theta_0^2\overline{\alpha_1}+2\,\theta_0\overline{\alpha_1}\right)A_0C_1 + 8\left(\theta_0^2+2\,\theta_0\right)\overline{A_0B_2} \\
&\quad + \Big(\left(\theta_0^2+4\,\theta_0+4\right)C_1 + 8\left(\alpha_1\theta_0^2+2\,\alpha_1\theta_0\right)\overline{A_0}\Big)\overline{C_1}\Big\} \\
&= \frac{1}{32\theta_0^2(\theta_0+1)^2}\left\{64\,\theta_0^2|\vec{A}_2|^2 + (\theta_0+2)^2|\vec{C}_1|^2 + 16\theta_0(\theta_0+2)\mathrm{Re}\left(\langle\vec{A}_0,\vec{B}_2\rangle\right)\right\}
\end{aligned}
$$

Now, we have

$$
\langle\vec{A}_0,\vec{B}_2\rangle = \left\langle \vec{A}_0, -\frac{(\theta_0+2)}{4\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0} - 2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\vec{A}_0\right\rangle = -\frac{(\theta_0+2)}{8\theta_0}|\vec{C}_1|^2 \in \mathbb{R}
$$

so

$$
16\theta_0(\theta_0+2)\mathrm{Re}\left(\langle\vec{A}_0,\vec{B}_2\rangle\right) = 16\theta_0(\theta_0+2)\langle\vec{A}_0,\vec{B}_2\rangle = 16\theta_0(\theta_0+2)\left(-\frac{(\theta_0+2)}{8\theta_0}|\vec{C}_1|^2\right) = -2(\theta_0+2)^2|\vec{C}_1|^2
$$

and finally

$$
\lambda_3 = \frac{1}{32\theta_0^2(\theta_0+2)^2}\left(64\theta_0^2|\vec{A}_2|^2 - (\theta_0+2)^2|\vec{C}_1|^2\right)
$$

Now, recall that

$$
\begin{aligned}
\alpha_7 &= \frac{1}{8\theta_0(\theta_0-4)}\langle\vec{C}_1,\vec{C}_1\rangle \\
\vec{E}_1 &= -\frac{1}{2\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_0} = -4(\theta_0-4)\alpha_7\vec{A}_0
\end{aligned}
$$

so as $|\vec{A}_0|^2 = \dfrac{1}{2}$

$$
\langle\vec{A}_0,\vec{E}_1\rangle = -2(\theta_0-4)\alpha_7
$$

and

$$
\mathbf{(11)} = \frac{-16\,A_0E_1}{64\,(\theta_0^3-4\,\theta_0^2)} = -\frac{1}{4\theta_0^2(\theta_0-4)}\left(-2(\theta_0-4)\alpha_7\right) = \frac{\alpha_7}{2\theta_0^2}.
$$

To keep consistent notations, we define

$$\begin{cases} \zeta_8 = \lambda_4 \\ \zeta_9 = \lambda_5 \\ \zeta_{10} = \lambda_3 = \dfrac{1}{32\theta_0^2(\theta_0+2)^2}\left(64\,\theta_0^2|\vec{A}_2|^2 - (\theta_0+2)^2\vec{C}_1|^2\right) \\ \zeta_{11} = \langle \overline{\vec{A}_1}, \vec{A}_3\rangle \end{cases}$$

where λ_4, λ_5 are given by (4.1.1), so that

$$\begin{pmatrix} \lambda_1 & 0 & 2\,\theta_0+4 & \mathbf{(4)} \\ \dfrac{-16\,A_0E_1}{64\,(\theta_0^3-4\,\theta_0^2)} & 4 & 2\,\theta_0 & \mathbf{(11)} \\ \lambda_2 & \theta_0 & \theta_0+4 & \mathbf{(17)} \\ \dfrac{24\left(\theta_0^2+\theta_0\right)A_1\overline{A_3}}{12\,(\theta_0^4+5\,\theta_0^3+7\,\theta_0^2+3\,\theta_0)} & \theta_0+1 & \theta_0+3 & \mathbf{(21)} \\ \lambda_3 & \theta_0+2 & \theta_0+2 & \mathbf{(24)} \\ \dfrac{24\left(\theta_0^2+\theta_0\right)A_3\overline{A_1}}{12\,(\theta_0^4+5\,\theta_0^3+7\,\theta_0^2+3\,\theta_0)} & \theta_0+3 & \theta_0+1 & \mathbf{(26)} \\ \lambda_4 & \theta_0+4 & \theta_0 & \mathbf{(27)} \\ \lambda_5 & 2\,\theta_0+4 & 0 & \mathbf{(34)} \\ \dfrac{-16\,\overline{A_0E_1}}{64\,(\theta_0^3-4\,\theta_0^2)} & 2\,\theta_0 & 4 & \mathbf{(38)} \end{pmatrix} = \begin{pmatrix} \overline{\zeta_9} & 0 & 2\,\theta_0+4 & \mathbf{(4)} \\ \dfrac{\alpha_7}{2\theta_0^2} & 4 & 2\,\theta_0 & \mathbf{(11)} \\ \overline{\zeta_8} & \theta_0 & \theta_0+4 & \mathbf{(17)} \\ \dfrac{2\,\overline{\zeta_{11}}}{(\theta_0+1)(\theta_0+3)} & \theta_0+1 & \theta_0+3 & \mathbf{(21)} \\ \zeta_{10} & \theta_0+2 & \theta_0+2 & \mathbf{(24)} \\ \dfrac{2\zeta_{11}}{(\theta_0+1)(\theta_0+3)} & \theta_0+3 & \theta_0+1 & \mathbf{(26)} \\ \zeta_8 & \theta_0+4 & \theta_0 & \mathbf{(27)} \\ \zeta_9 & 2\,\theta_0+4 & 0 & \mathbf{(34)} \\ \dfrac{\overline{\alpha_7}}{2\theta_0^2} & 2\,\theta_0 & 4 & \mathbf{(38)} \end{pmatrix}$$

as

$$\left(\theta_0^4+5\,\theta_0^3+7\,\theta_0^2+3\,\theta_0\right) = \theta_0(\theta_0+1)^2(\theta_0+3)$$

so that

$$\frac{24\left(\theta_0^2+\theta_0\right)A_3\overline{A_1}}{12\,(\theta_0^4+5\,\theta_0^3+7\,\theta_0^2+3\,\theta_0)} = \frac{2\theta_0(\theta_0+1)\,\zeta_{11}}{\theta_0(\theta_0+1)^2(\theta_0+3)} = \frac{2\zeta_{11}}{(\theta_0+1)(\theta_0+3)}$$

Now we compare the Sage transcription on the right to the expression we computed on the left

$$\begin{pmatrix} \overline{\zeta_9} & 0 & 2\,\theta_0+4 & \mathbf{(4)} \\ \dfrac{\alpha_7}{2\theta_0^2} & 4 & 2\,\theta_0 & \mathbf{(11)} \\ \overline{\zeta_8} & \theta_0 & \theta_0+4 & \mathbf{(17)} \\ \dfrac{2\,\overline{\zeta_{11}}}{(\theta_0+1)(\theta_0+3)} & \theta_0+1 & \theta_0+3 & \mathbf{(21)} \\ \zeta_{10} & \theta_0+2 & \theta_0+2 & \mathbf{(24)} \\ \dfrac{2\zeta_{11}}{(\theta_0+1)(\theta_0+3)} & \theta_0+3 & \theta_0+1 & \mathbf{(26)} \\ \zeta_8 & \theta_0+4 & \theta_0 & \mathbf{(27)} \\ \zeta_9 & 2\,\theta_0+4 & 0 & \mathbf{(34)} \\ \dfrac{\overline{\alpha_7}}{2\theta_0^2} & 2\,\theta_0 & 4 & \mathbf{(38)} \end{pmatrix}, \begin{pmatrix} \overline{\zeta_9} & 0 & 2\,\theta_0+4 \\ \dfrac{\alpha_7}{2\,\theta_0^2} & 4 & 2\,\theta_0 \\ \overline{\zeta_8} & \theta_0 & \theta_0+4 \\ \dfrac{2\,\overline{\zeta_{11}}}{(\theta_0+3)(\theta_0+1)} & \theta_0+1 & \theta_0+3 \\ \zeta_{10} & \theta_0+2 & \theta_0+2 \\ \dfrac{2\,\zeta_{11}}{(\theta_0+3)(\theta_0+1)} & \theta_0+3 & \theta_0+1 \\ \zeta_8 & \theta_0+4 & \theta_0 \\ \zeta_9 & 2\,\theta_0+4 & 0 \\ \dfrac{\overline{\alpha_7}}{2\,\theta_0^2} & 2\,\theta_0 & 4 \end{pmatrix}$$

and we see that both expressions coincide. Adding up with the previous computations, we obtain

$$|\vec{\Phi}(z)|^2 = \begin{pmatrix} -\frac{\overline{\zeta_0}}{\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0} & 0 & 2\,\theta_0 + 2 \\ -\frac{4\,\overline{\zeta_1}}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0} & 0 & 2\,\theta_0 + 3 \\ \frac{1}{\theta_0^2} & \theta_0 & \theta_0 \\ \frac{\overline{\alpha_1}}{(\theta_0 + 2)\theta_0} & \theta_0 & \theta_0 + 2 \\ \frac{2\,(\theta_0 + 1)\theta_0\overline{\alpha_3} - \overline{\zeta_2}}{2\,(\theta_0^4 + 4\,\theta_0^3 + 3\,\theta_0^2)} & \theta_0 & \theta_0 + 3 \\ \frac{2\,|A_1|^2}{(\theta_0 + 1)^2} & \theta_0 + 1 & \theta_0 + 1 \\ \frac{\overline{\alpha_5}}{(\theta_0 + 2)(\theta_0 + 1)} & \theta_0 + 1 & \theta_0 + 2 \\ \frac{\alpha_1}{(\theta_0 + 2)\theta_0} & \theta_0 + 2 & \theta_0 \\ \frac{\alpha_5}{(\theta_0 + 2)(\theta_0 + 1)} & \theta_0 + 2 & \theta_0 + 1 \\ \frac{2\,\alpha_3(\theta_0 + 1)\theta_0 - \zeta_2}{2\,(\theta_0^4 + 4\,\theta_0^3 + 3\,\theta_0^2)} & \theta_0 + 3 & \theta_0 \\ -\frac{\zeta_0}{\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0} & 2\,\theta_0 + 2 & 0 \\ -\frac{4\,\zeta_1}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0} & 2\,\theta_0 + 3 & 0 \end{pmatrix} \begin{pmatrix} \overline{\zeta_9} & 0 & 2\,\theta_0 + 4 \\ \frac{\alpha_7}{2\,\theta_0^2} & 4 & 2\,\theta_0 \\ \overline{\zeta_8} & \theta_0 & \theta_0 + 4 \\ \frac{2\,\overline{\zeta_{11}}}{(\theta_0 + 3)(\theta_0 + 1)} & \theta_0 + 1 & \theta_0 + 3 \\ \zeta_{10} & \theta_0 + 2 & \theta_0 + 2 \\ \frac{2\,\zeta_{11}}{(\theta_0 + 3)(\theta_0 + 1)} & \theta_0 + 3 & \theta_0 + 1 \\ \zeta_8 & \theta_0 + 4 & \theta_0 \\ \zeta_9 & 2\,\theta_0 + 4 & 0 \\ \frac{\overline{\alpha_7}}{2\,\theta_0^2} & 2\,\theta_0 & 4 \end{pmatrix}$$

4.2 Development of $\vec{\alpha}$

$\vec{\alpha} =$

$$
\begin{pmatrix}
\frac{1}{2} & \overline{B_1} & 0 & -\theta_0 + 2 & \textbf{(1)} \\
2\,A_1\overline{C_1} & \overline{A_0} & 0 & -\theta_0 + 2 & \textbf{(2)} \\
\frac{1}{2} & \overline{B_3} & 0 & -\theta_0 + 3 & \textbf{(3)} \\
2\,A_1\overline{C_1} & \overline{A_1} & 0 & -\theta_0 + 3 & \textbf{(4)} \\
-4\,A_0|A_1|^2\overline{C_1} + 2\,A_1\overline{C_2} & \overline{A_0} & 0 & -\theta_0 + 3 & \textbf{(5)} \\
\frac{1}{2} & \overline{B_6} & 0 & -\theta_0 + 4 & \textbf{(6)} \\
2\,A_1\overline{C_1} & \overline{A_2} & 0 & -\theta_0 + 4 & \textbf{(7)} \\
-4\,A_0|A_1|^2\overline{C_2} - 2\,A_1\overline{C_1}\,\overline{\alpha_1} - 2\,A_0\overline{C_1}\,\overline{\alpha_5} + \frac{1}{4}\,\overline{B_1C_1} + 2\,A_1\overline{C_3} & \overline{A_0} & 0 & -\theta_0 + 4 & \textbf{(8)} \\
-4\,A_0|A_1|^2\overline{C_1} + 2\,A_1\overline{C_2} & \overline{A_1} & 0 & -\theta_0 + 4 & \textbf{(9)} \\
1 & \overline{B_2} & 1 & -\theta_0 + 2 & \textbf{(10)} \\
\frac{1}{2}\,C_1\overline{C_1} & A_0 & 1 & -\theta_0 + 2 & \textbf{(11)} \\
-4\,A_0\alpha_1\overline{C_1} + 2\,A_1\overline{B_1} + 4\,A_2\overline{C_1} & \overline{A_0} & 1 & -\theta_0 + 2 & \textbf{(12)} \\
1 & \overline{B_5} & 1 & -\theta_0 + 3 & \textbf{(13)} \\
\frac{1}{2}\,B_1\overline{C_1} + \frac{1}{2}\,C_1\overline{C_2} & A_0 & 1 & -\theta_0 + 3 & \textbf{(14)} \\
-4\,A_0\alpha_1\overline{C_1} + 2\,A_1\overline{B_1} + 4\,A_2\overline{C_1} & \overline{A_1} & 1 & -\theta_0 + 3 & \textbf{(15)} \\
-4\,A_0|A_1|^2\overline{B_1} - 8\,A_1|A_1|^2\overline{C_1} - 4\,A_0\alpha_5\overline{C_1} - 4\,A_0\alpha_1\overline{C_2} + 2\,A_1\overline{B_3} + 4\,A_2\overline{C_2} & \overline{A_0} & 1 & -\theta_0 + 3 & \textbf{(16)} \\
\frac{3}{2} & \overline{B_4} & 2 & -\theta_0 + 2 & \textbf{(17)} \\
\frac{A_1C_1}{2\,\theta_0} & \overline{C_1} & 2 & -\theta_0 + 2 & \textbf{(18)} \\
\frac{1}{2}\,C_1\overline{C_1} & A_1 & 2 & -\theta_0 + 2 & \textbf{(19)} \\
\frac{1}{4}\,A_1\overline{C_1} & C_1 & 2 & -\theta_0 + 2 & \textbf{(20)} \\
\frac{1}{2}\,C_1\overline{B_1} + \frac{1}{2}\,C_2\overline{C_1} & A_0 & 2 & -\theta_0 + 2 & \textbf{(21)} \\
-2\,(\theta_0\overline{\alpha_2} + \overline{\alpha_2})A_0\overline{C_1} - 4\,A_0\alpha_1\overline{B_1} - 6\,A_1\alpha_1\overline{C_1} - 6\,A_0\alpha_3\overline{C_1} + 4\,A_2\overline{B_1} + 2\,A_1\overline{B_2} + 6\,A_3\overline{C_1} & \overline{A_0} & 2 & -\theta_0 + 2 & \textbf{(22)}
\end{pmatrix}
$$

$$
\left(\begin{array}{ccccc}
-\theta_0 + 2 & E_1 & -2\,\theta_0 + 3 & \theta_0 & \textbf{(23)} \\
-\frac{C_1^2(\theta_0 - 2)}{2\,\theta_0} & \overline{A_0} & -2\,\theta_0 + 3 & \theta_0 & \textbf{(24)} \\
-\theta_0 + 2 & E_2 & -2\,\theta_0 + 3 & \theta_0 + 1 & \textbf{(25)} \\
-\frac{C_1^2(\theta_0 - 2)}{2\,\theta_0} & \overline{A_1} & -2\,\theta_0 + 3 & \theta_0 + 1 & \textbf{(26)} \\
2\,(\alpha_2\theta_0 - 2\,\alpha_2)A_0C_1 - \frac{B_1C_1(\theta_0 - 2)}{2\,(\theta_0 + 1)} - \frac{B_1C_1(\theta_0 - 2)}{2\,\theta_0} & \overline{A_0} & -2\,\theta_0 + 3 & \theta_0 + 1 & \textbf{(27)} \\
-\theta_0 + \frac{5}{2} & E_3 & -2\,\theta_0 + 4 & \theta_0 & \textbf{(28)} \\
2\,A_1E_1 - \frac{C_1C_2(\theta_0 - 2)}{2\,\theta_0} - \frac{C_1C_2(\theta_0 - 3)}{2\,\theta_0} & \overline{A_0} & -2\,\theta_0 + 4 & \theta_0 & \textbf{(29)} \\
-\frac{1}{2}\,\theta_0 + 1 & C_1 & -\theta_0 + 1 & 0 & \textbf{(30)} \\
-\frac{1}{2}\,\theta_0 + 1 & B_1 & -\theta_0 + 1 & 1 & \textbf{(31)} \\
-\frac{1}{2}\,\theta_0 + 1 & B_2 & -\theta_0 + 1 & 2 & \textbf{(32)} \\
-\frac{C_1(\theta_0 - 2)\overline{C_1}}{2\,\theta_0} & \overline{A_0} & -\theta_0 + 1 & 2 & \textbf{(33)} \\
-\frac{1}{2}\,\theta_0 + 1 & B_4 & -\theta_0 + 1 & 3 & \textbf{(34)} \\
-\frac{C_1(\theta_0 - 2)\overline{C_1}}{2\,\theta_0} & \overline{A_1} & -\theta_0 + 1 & 3 & \textbf{(35)} \\
2\,(\alpha_2\theta_0 - 2\,\alpha_2)A_0\overline{C_1} - \frac{B_1(\theta_0 - 2)\overline{C_1}}{2\,(\theta_0 + 1)} - \frac{C_1(\theta_0 - 2)\overline{C_2}}{2\,\theta_0} & \overline{A_0} & -\theta_0 + 1 & 3 & \textbf{(36)} \\
-\frac{1}{2}\,\theta_0 + \frac{3}{2} & C_2 & -\theta_0 + 2 & 0 & \textbf{(37)} \\
2\,A_1C_1 & \overline{A_0} & -\theta_0 + 2 & 0 & \textbf{(38)} \\
-\frac{1}{2}\,\theta_0 + \frac{3}{2} & B_3 & -\theta_0 + 2 & 1 & \textbf{(39)} \\
2\,A_1C_1 & \overline{A_1} & -\theta_0 + 2 & 1 & \textbf{(40)} \\
-4\,A_0C_1|A_1|^2 + 2\,A_1B_1 & \overline{A_0} & -\theta_0 + 2 & 1 & \textbf{(41)} \\
-\frac{1}{2}\,\theta_0 + \frac{3}{2} & B_5 & -\theta_0 + 2 & 2 & \textbf{(42)} \\
2\,A_1C_1 & \overline{A_2} & -\theta_0 + 2 & 2 & \textbf{(43)} \\
-4\,A_0C_1|A_1|^2 + 2\,A_1B_1 & \overline{A_1} & -\theta_0 + 2 & 2 & \textbf{(44)} \\
\zeta_{12} & \overline{A_0} & -\theta_0 + 2 & 2 & \textbf{(45)} \\
-\frac{1}{2}\,\theta_0 + 2 & C_3 & -\theta_0 + 3 & 0 & \textbf{(46)} \\
\frac{1}{4}\,C_1^2 & A_0 & -\theta_0 + 3 & 0 & \textbf{(47)} \\
-4\,A_0C_1\alpha_1 + 4\,A_2C_1 + 2\,A_1C_2 & \overline{A_0} & -\theta_0 + 3 & 0 & \textbf{(48)}
\end{array}\right)
$$

$$
\left(\begin{array}{ccccc}
-\frac{1}{2}\,\theta_0 + 2 & B_6 & -\theta_0 + 3 & 1 & \textbf{(49)} \\
\frac{1}{2}\,B_1C_1 & A_0 & -\theta_0 + 3 & 1 & \textbf{(50)} \\
-4\,A_0C_1\alpha_1 + 4\,A_2C_1 + 2\,A_1C_2 & \overline{A_1} & -\theta_0 + 3 & 1 & \textbf{(51)} \\
-8\,A_1C_1|A_1|^2 - 4\,A_0C_2|A_1|^2 - 4\,A_0B_1\alpha_1 - 4\,A_0C_1\alpha_5 + 4\,A_2B_1 + 2\,A_1B_3 & \overline{A_0} & -\theta_0 + 3 & 1 & \textbf{(52)} \\
-\frac{1}{2}\,\theta_0 + \frac{5}{2} & C_4 & -\theta_0 + 4 & 0 & \textbf{(53)} \\
\frac{1}{2}\,C_1C_2 & A_0 & -\theta_0 + 4 & 0 & \textbf{(54)} \\
\frac{1}{4}\,C_1^2 & A_1 & -\theta_0 + 4 & 0 & \textbf{(55)} \\
\frac{1}{4}\,A_1C_1 & C_1 & -\theta_0 + 4 & 0 & \textbf{(56)} \\
-6\,A_1C_1\alpha_1 - 4\,A_0C_2\alpha_1 - 6\,A_0C_1\alpha_3 + 6\,A_3C_1 + 4\,A_2C_2 + 2\,A_1C_3 & \overline{A_0} & -\theta_0 + 4 & 0 & \textbf{(57)} \\
\frac{A_1\overline{C_1}}{2\,\theta_0} & \overline{C_1} & \theta_0 & -2\,\theta_0 + 4 & \textbf{(58)} \\
\frac{1}{2}\,\theta_0 + \frac{1}{2} & \overline{E_2} & \theta_0 & -2\,\theta_0 + 4 & \textbf{(59)} \\
\frac{1}{4}\,\overline{C_1}^2 & A_1 & \theta_0 & -2\,\theta_0 + 4 & \textbf{(60)} \\
\frac{1}{2}\,\overline{B_1C_1} & A_0 & \theta_0 & -2\,\theta_0 + 4 & \textbf{(61)} \\
-2\,(\theta_0\overline{\alpha_2} + \overline{\alpha_2})A_0\overline{C_1} + 2\,A_1\overline{E_1} & \overline{A_0} & \theta_0 & -2\,\theta_0 + 4 & \textbf{(62)} \\
\frac{1}{2}\,\theta_0 & \overline{E_1} & \theta_0 - 1 & -2\,\theta_0 + 4 & \textbf{(63)} \\
\frac{1}{4}\,\overline{C_1}^2 & A_0 & \theta_0 - 1 & -2\,\theta_0 + 4 & \textbf{(64)} \\
\frac{1}{2}\,\theta_0 & \overline{E_3} & \theta_0 - 1 & -2\,\theta_0 + 5 & \textbf{(65)} \\
\frac{1}{2}\,\overline{C_1C_2} & A_0 & \theta_0 - 1 & -2\,\theta_0 + 5 & \textbf{(66)}
\end{array}\right)
$$

where

$$
\begin{aligned}
\zeta_{12} &= -4\,A_0B_1|A_1|^2 - 2\,A_1C_1\overline{\alpha_1} - 2\,A_0C_1\alpha_5 + 2\,A_1B_2 + \frac{1}{4}\,C_1\overline{B_1} - \frac{C_1(\theta_0 - 2)\overline{B_1}}{2\,\theta_0} - \frac{C_2(\theta_0 - 3)\overline{C_1}}{2\,\theta_0} \\
&= -\frac{(\theta_0 - 3)}{2\theta_0}\langle \overline{\vec{C}_1}, \vec{C}_2 \rangle
\end{aligned}
$$

We have as

$$
\alpha_2 = \frac{1}{2\theta_0(\theta_0 + 1)}\langle \overline{\vec{A}_1}, \vec{C}_1 \rangle, \quad \alpha_7 = \frac{1}{8\theta_0(\theta_0 - 4)}\langle \vec{C}_1, \vec{C}_1 \rangle
$$

word

$$
\frac{1}{2}\vec{E}_2 = \left(\begin{array}{cccc}
\frac{(\alpha_2\theta_0 + \alpha_2)C_1\overline{A_0} - E_1\overline{A_1}}{\theta_0 + 1} & A_0 & -2\,\theta_0 + 4 & \theta_0 + 1 \\
-\frac{C_1\overline{A_1}}{4\,(\theta_0^2 + \theta_0)} & C_1 & -2\,\theta_0 + 4 & \theta_0 + 1 \\
\frac{\left(4\,(\alpha_2\theta_0^2 + \alpha_2\theta_0)A_0 - B_1(4\,\theta_0 + 1)\right)C_1}{8\,(\theta_0^2 + \theta_0)} & \overline{A_0} & -2\,\theta_0 + 4 & \theta_0 + 1 \\
-\frac{C_1^2(2\,\theta_0 + 1)}{8\,(\theta_0^2 + \theta_0)} & \overline{A_1} & -2\,\theta_0 + 4 & \theta_0 + 1
\end{array}\right)
$$

$$= -\frac{\alpha_2}{2}\vec{C}_1 - \frac{(2\theta_0+1)(\theta_0-4)}{\theta_0+1}\alpha_7\overline{\vec{A}_1}$$

so

$$\vec{E}_2 = -\alpha_2\vec{C}_1 - \frac{2(2\theta_0+1)(\theta_0-4)}{\theta_0+1}\alpha_7\overline{\vec{A}_1} \tag{4.2.1}$$

Finally,

$$\frac{1}{2}\vec{E}_3 = \begin{pmatrix} -\frac{C_1C_2(2\,\theta_0-5) - 2\,A_1E_1\theta_0}{2\,(2\,\theta_0^2-5\,\theta_0)} & \overline{A_0} & -2\,\theta_0+5 & \theta_0 \end{pmatrix}$$

$$= -\frac{1}{2\theta_0}\langle\vec{C}_1,\vec{C}_2\rangle\overline{\vec{A}_0}$$

So

$$\begin{cases} \vec{E}_2 = -\alpha_2\,\vec{C}_1 - \frac{2(2\theta_0+1)(\theta_0-4)}{\theta_0+1}\alpha_7\overline{\vec{A}_1} \\ \vec{E}_3 = -\frac{1}{\theta_0}\langle\vec{C}_1,\vec{C}_2\rangle\overline{\vec{A}_0} \end{cases}$$

we have Thanks of the previous development we need only compute the terms of degree $4-\theta_0$, and they correspond to the powers

$$\begin{pmatrix} \frac{1}{2}\,\theta_0 & \overline{E_3} & \theta_0-1 & -2\,\theta_0+5 & \mathbf{(65)} \\ \frac{1}{2}\,\overline{C_1C_2} & A_0 & \theta_0-1 & -2\,\theta_0+5 & \mathbf{(66)} \end{pmatrix}$$

$$= \frac{\theta_0}{2}\left(-\frac{1}{\theta_0}\overline{\langle\vec{C}_1,\vec{C}_2\rangle\vec{A}_0}\right) + \frac{1}{2}\overline{\langle\vec{C}_1,\vec{C}_2\rangle}\vec{A}_0 = 0$$

Then, we have

$$\begin{pmatrix} \frac{A_1\overline{C_1}}{2\,\theta_0} & \overline{C_1} & \theta_0 & -2\,\theta_0+4 & \mathbf{(58)} \\ \frac{1}{2}\,\theta_0+\frac{1}{2} & \overline{E_2} & \theta_0 & -2\,\theta_0+4 & \mathbf{(59)} \\ \frac{1}{4}\,\overline{C_1}^2 & A_1 & \theta_0 & -2\,\theta_0+4 & \mathbf{(60)} \\ \frac{1}{2}\,\overline{B_1C_1} & A_0 & \theta_0 & -2\,\theta_0+4 & \mathbf{(61)} \\ -2\,(\theta_0\overline{\alpha_2}+\overline{\alpha_2})A_0\overline{C_1}+2\,A_1\overline{E_1} & \overline{A_0} & \theta_0 & -2\,\theta_0+4 & \mathbf{(62)} \end{pmatrix}$$

$$= (\theta_0+1)\overline{\alpha_2}\overline{\vec{C}_1} + \frac{(\theta_0+1)}{2}\overline{\vec{E}_2} + 2\theta_0(\theta_0-4)\overline{\alpha_7}\vec{A}_1$$

$$= (\theta_0+1)\overline{\alpha_2}\overline{\vec{C}_1} + \frac{(\theta_0+1)}{2}\left(-\overline{\alpha_2}\overline{\vec{C}_1} - \frac{2(2\theta_0+1)(\theta_0-4)}{\theta_0+1}\overline{\alpha_7}\vec{A}_1\right) + 2\theta_0(\theta_0-4)\overline{\alpha_7}\vec{A}_1$$

$$= \frac{(\theta_0+1)}{2}\overline{\alpha_2}\overline{\vec{C}_1} - (2\theta_0+1)(\theta_0-4)\overline{\alpha_7}\vec{A}_1 + 2\theta_0(\theta_0-4)\overline{\alpha_7}\vec{A}_1$$

$$= \frac{(\theta_0+1)}{2}\overline{\alpha_2}\overline{\vec{C}_1} - (\theta_0-4)\overline{\alpha_7}\vec{A}_1$$

Then, we compute

$$\begin{pmatrix} -\theta_0+2 & E_2 & -2\,\theta_0+3 & \theta_0+1 & \mathbf{(25)} \\ -\frac{C_1^2(\theta_0-2)}{2\,\theta_0} & \overline{A_1} & -2\,\theta_0+3 & \theta_0+1 & \mathbf{(26)} \\ 2\,(\alpha_2\theta_0-2\,\alpha_2)A_0C_1 - \frac{B_1C_1(\theta_0-2)}{2\,(\theta_0+1)} - \frac{B_1C_1(\theta_0-2)}{2\,\theta_0} & \overline{A_0} & -2\,\theta_0+3 & \theta_0+1 & \mathbf{(27)} \end{pmatrix}$$

$$
\begin{aligned}
&= -(\theta_0-2)\left(-\alpha_2\vec{C}_1 - \frac{2(2\theta_0+1)(\theta_0-4)}{\theta_0+1}\alpha_7\overline{\vec{A}_1}\right) - 4(\theta_0-2)(\theta_0-4)\alpha_7\overline{\vec{A}_1} \\
&= (\theta_0-2)\alpha_2\vec{C}_1 - \frac{2(\theta_0-2)(\theta_0-4)}{\theta_0+1}\alpha_7\overline{\vec{A}_1}
\end{aligned}
$$

and finally

$$
\begin{pmatrix}
-\theta_0+\frac{5}{2} & E_3 & -2\,\theta_0+4 & \theta_0 & \mathbf{(28)} \\
2\,A_1E_1 - \frac{C_1C_2(\theta_0-2)}{2\,\theta_0} - \frac{C_1C_2(\theta_0-3)}{2\,\theta_0} & \overline{A_0} & -2\,\theta_0+4 & \theta_0 & \mathbf{(29)}
\end{pmatrix}
$$

$$
\begin{aligned}
&= -\frac{(2\theta_0-5)}{2}\vec{E}_3 - \frac{(2\theta_0-5)}{2\theta_0}\langle\vec{C}_1,\vec{C}_2\rangle\overline{\vec{A}_0} \\
&= -\frac{(2\theta_0-5)}{2}\left(-\frac{1}{\theta_0}\langle\vec{C}_1,\vec{C}_2\rangle\overline{\vec{A}_0}\right) - \frac{(2\theta_0-5)}{2\theta_0}\langle\vec{C}_1,\vec{C}_2\rangle\overline{\vec{A}_0} \\
&= 0
\end{aligned}
$$

Finally, we see that the new powers of order $4-\theta_0$ are

$$
\begin{pmatrix}
0 & -\theta_0+4 & \mathbf{(6)}-\mathbf{(9)} \\
1 & -\theta_0+3 & \mathbf{(13)}-\mathbf{(16)} \\
2 & -\theta_0+2 & \mathbf{(17)}-\mathbf{(22)} \\
-2\theta_0+3 & \theta_0+1 & \mathbf{(25)}-\mathbf{(27)} \\
-\theta_0+1 & 3 & \mathbf{(34)}-\mathbf{(36)} \\
-\theta_0+2 & 2 & \mathbf{(42)}-\mathbf{(45)} \\
-\theta_0+3 & 1 & \mathbf{(49)}-\mathbf{(52)} \\
-\theta_0+4 & 0 & \mathbf{(54)}-\mathbf{(57)} \\
\theta_0 & -2\theta_0+4 & \mathbf{(58)}-\mathbf{(62)}
\end{pmatrix}
$$

so for some $\vec{B}_7, \vec{B}_8, \vec{B}_9, \vec{B}_{10}, \vec{B}_{11}, \vec{B}_{12}, \vec{B}_{13} \in \mathbb{C}^n$, the new powers are

$$\begin{pmatrix}
1 & B_7 & 0 & -\theta_0+4 & \mathbf{(6)-(9)} \\
1 & B_8 & 1 & -\theta_0+3 & \mathbf{(13)-(16)} \\
1 & B_9 & 2 & -\theta_0+2 & \mathbf{(17)-(22)} \\
(\theta_0-2)\alpha_2 & C_1 & -2\theta_0+3 & \theta_0+1 & \mathbf{(25)-(27)} \\
-\dfrac{2(\theta_0-2)(\theta_0-4)}{\theta_0+1}\alpha_7 & \overline{A_1} & -2\theta_0+3 & \theta_0+1 & \mathbf{(25)-(27)} \\
1 & B_{10} & -\theta_0+1 & 3 & \mathbf{(34)-(36)} \\
1 & B_{11} & -\theta_0+2 & 2 & \mathbf{(42)-(45)} \\
1 & B_{12} & -\theta_0+3 & 1 & \mathbf{(49)-(52)} \\
1 & B_{13} & -\theta_0+4 & 0 & \mathbf{(54)-(57)} \\
\dfrac{(\theta_0+1)}{2}\overline{\alpha_2} & \overline{C_1} & \theta_0 & -2\theta_0+4 & \mathbf{(58)-(62)} \\
-(\theta_0-4)\overline{\alpha_7} & A_1 & \theta_0 & -2\theta_0+4 & \mathbf{(58)-(62)}.
\end{pmatrix}$$

to be compared with the Sage version

$$\begin{pmatrix}
1 & B_7 & 0 & -\theta_0+4 & \mathbf{(6)-(9)} \\
1 & B_8 & 1 & -\theta_0+3 & \mathbf{(13)-(16)} \\
1 & B_9 & 2 & -\theta_0+2 & \mathbf{(17)-(22)} \\
(\theta_0-2)\alpha_2 & C_1 & -2\theta_0+3 & \theta_0+1 & \mathbf{(25)-(27)} \\
-\dfrac{2(\theta_0-2)(\theta_0-4)}{\theta_0+1}\alpha_7 & \overline{A_1} & -2\theta_0+3 & \theta_0+1 & \mathbf{(25)-(27)} \\
1 & B_{10} & -\theta_0+1 & 3 & \mathbf{(34)-(36)} \\
1 & B_{11} & -\theta_0+2 & 2 & \mathbf{(42)-(45)} \\
1 & B_{12} & -\theta_0+3 & 1 & \mathbf{(49)-(52)} \\
1 & B_{13} & -\theta_0+4 & 0 & \mathbf{(54)-(57)} \\
\dfrac{(\theta_0+1)}{2}\overline{\alpha_2} & \overline{C_1} & \theta_0 & -2\theta_0+4 & \mathbf{(58)-(62)} \\
-(\theta_0-4)\overline{\alpha_7} & A_1 & \theta_0 & -2\theta_0+4 & \mathbf{(58)-(62)}.
\end{pmatrix}
\begin{pmatrix}
1 & B_7 & 0 & -\theta_0+4 \\
1 & B_8 & 1 & -\theta_0+3 \\
1 & B_9 & 2 & -\theta_0+2 \\
\alpha_2(\theta_0-2) & C_1 & -2\,\theta_0+3 & \theta_0+1 \\
-\dfrac{2\,\alpha_7(\theta_0-2)(\theta_0-4)}{\theta_0+1} & \overline{A_1} & -2\,\theta_0+3 & \theta_0+1 \\
1 & B_{10} & -\theta_0+1 & 3 \\
1 & B_{11} & -\theta_0+2 & 2 \\
1 & B_{12} & -\theta_0+3 & 1 \\
1 & B_{13} & -\theta_0+4 & 0 \\
\dfrac{1}{2}\,(\theta_0+1)\overline{\alpha_2} & \overline{C_1} & \theta_0 & -2\,\theta_0+4 \\
-(\theta_0-4)\overline{\alpha_7} & A_1 & \theta_0 & -2\,\theta_0+4
\end{pmatrix}$$

Using the previous development of $\vec{\alpha}$ up to order $4-\theta_0$, we obtain

$$\vec{\alpha} = \begin{pmatrix} 2\,(\theta_0+1)\theta_0\overline{\alpha_2} & \overline{A_0} & 0 & -\theta_0+2 \\ \frac{\overline{\zeta_2}}{\theta_0-3} & A_1 & 0 & -\theta_0+3 \\ 2\,(\theta_0+1)\theta_0\overline{\alpha_2} & \overline{A_1} & 0 & -\theta_0+3 \\ \overline{\zeta_3} & \overline{A_0} & 0 & -\theta_0+3 \\ \frac{(\theta_0-2)|C_1|^2}{4\,\theta_0} & A_0 & 1 & -\theta_0+2 \\ 2\,\overline{\zeta_4} & \overline{A_0} & 1 & -\theta_0+2 \\ -\frac{1}{2}\,\theta_0+1 & C_1 & -\theta_0+1 & 0 \\ -\frac{1}{2}\,\theta_0+1 & B_1 & -\theta_0+1 & 1 \\ -\frac{1}{2}\,\theta_0+1 & B_2 & -\theta_0+1 & 2 \\ -\frac{(\theta_0-2)|C_1|^2}{2\,\theta_0} & \overline{A_0} & -\theta_0+1 & 2 \\ -\frac{1}{2}\,\theta_0+\frac{3}{2} & C_2 & -\theta_0+2 & 0 \\ 2\,\zeta_2 & \overline{A_0} & -\theta_0+2 & 0 \\ -\frac{1}{2}\,\theta_0+\frac{3}{2} & B_3 & -\theta_0+2 & 1 \\ 2\,\zeta_2 & \overline{A_1} & -\theta_0+2 & 1 \\ -\frac{1}{2}\,\theta_0+2 & C_3 & -\theta_0+3 & 0 \\ 2\,\alpha_7(\theta_0-4)\theta_0 & A_0 & -\theta_0+3 & 0 \\ 2\,\zeta_5 & \overline{A_0} & -\theta_0+3 & 0 \end{pmatrix} \begin{pmatrix} 1 & B_7 & 0 & -\theta_0+4 \\ 1 & B_8 & 1 & -\theta_0+3 \\ 1 & B_9 & 2 & -\theta_0+2 \\ \alpha_2(\theta_0-2) & C_1 & -2\,\theta_0+3 & \theta_0+1 \\ -\frac{2\,\alpha_7(\theta_0-2)(\theta_0-4)}{\theta_0+1} & \overline{A_1} & -2\,\theta_0+3 & \theta_0+1 \\ 1 & B_{10} & -\theta_0+1 & 3 \\ 1 & B_{11} & -\theta_0+2 & 2 \\ 1 & B_{12} & -\theta_0+3 & 1 \\ 1 & B_{13} & -\theta_0+4 & 0 \\ \frac{1}{2}\,(\theta_0+1)\overline{\alpha_2} & \overline{C_1} & \theta_0 & -2\,\theta_0+4 \\ -(\theta_0-4)\overline{\alpha_7} & A_1 & \theta_0 & -2\,\theta_0+4 \end{pmatrix}$$

4.3 Development of $\langle\vec{\alpha},\vec{\Phi}\rangle$

We have

$$\langle\vec{\alpha},\vec{\Phi}\rangle =$$

$$
\begin{pmatrix}
2\,(\theta_0\overline{\alpha_2}+\overline{\alpha_2})\overline{A_0}^2 & 0 & 2 & \mathbf{(1)}\\
\frac{(\theta_0-3)\overline{A_0}^2\overline{\zeta_3}+2\left(2\,\theta_0^3\overline{\alpha_2}-5\,\theta_0^2\overline{\alpha_2}-3\,\theta_0\overline{\alpha_2}\right)\overline{A_0A_1}+A_1\overline{A_0\zeta_2}}{\theta_0^2-3\,\theta_0} & 0 & 3 & \mathbf{(2)}\\
\zeta_{13} & 0 & 4 & \mathbf{(3)}\\
-\frac{A_0C_1(\theta_0-2)}{2\,\theta_0} & 1 & 0 & \mathbf{(4)}\\
-\frac{A_0B_1(\theta_0-2)}{2\,\theta_0} & 1 & 1 & \mathbf{(5)}\\
-\frac{4\,A_0(\theta_0-2)|C_1|^2\overline{A_0}-32\,\theta_0\overline{A_0}^2\overline{\zeta_4}+8\left(\theta_0^2-2\,\theta_0\right)A_0B_2+\left(\theta_0^2-2\,\theta_0\right)C_1\overline{C_1}}{16\,\theta_0^2} & 1 & 2 & \mathbf{(6)}\\
\zeta_{14} & 1 & 3 & \mathbf{(7)}\\
\frac{4\,A_0(\theta_0+1)\zeta_2\overline{A_0}-\left(\theta_0^2-2\,\theta_0\right)A_1C_1-\left(\theta_0^2-2\,\theta_0-3\right)A_0C_2}{2\left(\theta_0^2+\theta_0\right)} & 2 & 0 & \mathbf{(8)}\\
\frac{4\,A_0(\theta_0+1)\zeta_2\overline{A_1}-\left(\theta_0^2-2\,\theta_0\right)A_1B_1-\left(\theta_0^2-2\,\theta_0-3\right)A_0B_3}{2\left(\theta_0^2+\theta_0\right)} & 2 & 1 & \mathbf{(9)}\\
\zeta_{15} & 2 & 2 & \mathbf{(10)}\\
\zeta_{7} & 3 & 0 & \mathbf{(11)}\\
\zeta_{16} & 3 & 1 & \mathbf{(12)}\\
\zeta_{17} & 4 & 0 & \mathbf{(13)}\\
\frac{\left(\alpha_2\theta_0^2-\alpha_2\theta_0-2\,\alpha_2\right)C_1\overline{A_0}-2\left(\alpha_7\theta_0^2-6\,\alpha_7\theta_0+8\,\alpha_7\right)\overline{A_0A_1}}{\theta_0^2+\theta_0} & -2\,\theta_0+3 & 2\,\theta_0+1 & \mathbf{(14)}\\
-\frac{C_1(\theta_0-2)\overline{A_0}}{2\,\theta_0} & -\theta_0+1 & \theta_0 & \mathbf{(15)}\\
-\frac{\left(\theta_0^2-\theta_0-2\right)B_1\overline{A_0}+\left(\theta_0^2-2\,\theta_0\right)C_1\overline{A_1}}{2\left(\theta_0^2+\theta_0\right)} & -\theta_0+1 & \theta_0+1 & \mathbf{(16)}\\
\lambda_2 & -\theta_0+1 & \theta_0+2 & \mathbf{(17)}\\
\zeta_{18} & -\theta_0+1 & \theta_0+3 & \mathbf{(18)}\\
-\frac{C_2(\theta_0-3)\overline{A_0}-4\,\zeta_2\overline{A_0}^2}{2\,\theta_0} & -\theta_0+2 & \theta_0 & \mathbf{(19)}\\
\frac{4\,(2\,\theta_0+1)\zeta_2\overline{A_0A_1}-\left(\theta_0^2-2\,\theta_0-3\right)B_3\overline{A_0}-\left(\theta_0^2-3\,\theta_0\right)C_2\overline{A_1}}{2\left(\theta_0^2+\theta_0\right)} & -\theta_0+2 & \theta_0+1 & \mathbf{(20)}\\
\zeta_{19} & -\theta_0+2 & \theta_0+2 & \mathbf{(21)}\\
-\frac{C_1^2(\theta_0-2)-32\left(\alpha_7\theta_0^2-4\,\alpha_7\theta_0\right)A_0\overline{A_0}+8\,C_3(\theta_0-4)\overline{A_0}-32\,\zeta_5\overline{A_0}^2}{16\,\theta_0} & -\theta_0+3 & \theta_0 & \mathbf{(22)}\\
\zeta_{20} & -\theta_0+3 & \theta_0+1 & \mathbf{(23)}\\
-\frac{C_1C_2(5\,\theta_0-13)-12\,C_1\zeta_2\overline{A_0}-48\,B_{13}\overline{A_0}}{48\,\theta_0} & -\theta_0+4 & \theta_0 & \mathbf{(24)}
\end{pmatrix}
$$

$$
\begin{pmatrix}
2\left(\theta_0\overline{\alpha_2}+\overline{\alpha_2}\right)A_0\overline{A_0} & \theta_0 & -\theta_0+2 & \mathbf{(25)}\\
\frac{A_0(\theta_0-3)\overline{A_0\zeta_3}+2\left(\theta_0^3\overline{\alpha_2}-2\,\theta_0^2\overline{\alpha_2}-3\,\theta_0\overline{\alpha_2}\right)A_0\overline{A_1}+A_0A_1\overline{\zeta_2}}{\theta_0^2-3\,\theta_0} & \theta_0 & -\theta_0+3 & \mathbf{(26)}\\
-\frac{4\left(\theta_0\overline{\alpha_7}-4\,\overline{\alpha_7}\right)A_1\overline{A_0}-\left(\theta_0^2\overline{\alpha_2}+3\,\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2}\right)\overline{A_0C_1}-4\,A_0B_7}{4\,\theta_0} & \theta_0 & -\theta_0+4 & \mathbf{(27)}\\
\frac{8\,A_1\theta_0^3\overline{A_0}\overline{\alpha_2}+A_0^2(\theta_0-2)|C_1|^2+8\,A_0\theta_0\overline{A_0\zeta_4}}{4\,\theta_0^2} & \theta_0+1 & -\theta_0+2 & \mathbf{(28)}\\
\frac{A_1^2\theta_0\overline{\zeta_2}+\left(\theta_0^2-3\,\theta_0\right)A_1\overline{A_0\zeta_3}+\left(\theta_0^2-2\,\theta_0-3\right)A_0B_8+2\left(\theta_0^4\overline{\alpha_2}-2\,\theta_0^3\overline{\alpha_2}-3\,\theta_0^2\overline{\alpha_2}\right)A_1\overline{A_1}}{\theta_0^3-2\,\theta_0^2-3\,\theta_0} & \theta_0+1 & -\theta_0+3 & \mathbf{(29)}\\
\frac{\left(\theta_0^2-4\right)A_0A_1|C_1|^2+8\left(\theta_0^2+2\,\theta_0\right)A_1\overline{A_0\zeta_4}+4\left(\theta_0^2+3\,\theta_0+2\right)A_0B_9+8\left(\theta_0^4\overline{\alpha_2}+2\,\theta_0^3\overline{\alpha_2}+\theta_0^2\overline{\alpha_2}\right)A_2\overline{A_0}}{4\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)} & \theta_0+2 & -\theta_0+2 & \mathbf{(30)}\\
-\frac{2\left(\theta_0\overline{\alpha_7}-4\,\overline{\alpha_7}\right)A_0A_1-\left(\theta_0\overline{\alpha_2}+\overline{\alpha_2}\right)A_0\overline{C_1}}{2\,\theta_0} & 2\,\theta_0 & -2\,\theta_0+4 & \mathbf{(31)}
\end{pmatrix}
\tag{4.3.1}
$$

where

$$
\zeta_{13}=\frac{1}{\theta_0^4-7\,\theta_0^2-6\,\theta_0}\Big\{\left(\theta_0^3-7\,\theta_0-6\right)B_7\overline{A_0}+2\left(\theta_0^5\overline{\alpha_2}-7\,\theta_0^3\overline{\alpha_2}-6\,\theta_0^2\overline{\alpha_2}\right)\overline{A_1}^2+2\left(\theta_0^5\overline{\alpha_2}-\theta_0^4\overline{\alpha_2}-5\,\theta_0^3\overline{\alpha_2}-3\,\theta_0^2\overline{\alpha_2}\right)\overline{A_0A_2}
$$

$$
+\left(\left(\theta_0^2+2\,\theta_0\right)A_1\overline{\zeta_2}+\left(\theta_0^3-\theta_0^2-6\,\theta_0\right)\overline{A_0\zeta_3}\right)\overline{A_1}\Big\}
$$

$$
\zeta_{14}=\frac{1}{48\left(\theta_0^2+\theta_0\right)}\Big\{12\,A_0(\theta_0-2)|C_1|^2\overline{A_1}+96\,\theta_0\overline{A_0A_1\zeta_4}+48\,A_0B_{10}(\theta_0+1)+48\,B_8(\theta_0+1)\overline{A_0}-3\left(\theta_0^2-\theta_0-2\right)B_1\overline{C_1}
$$

$$
-2\left(\theta_0^2-\theta_0-2\right)C_1\overline{C_2}\Big\}
$$

$$
\zeta_{15}=-\frac{1}{16\left(\theta_0^2+\theta_0\right)}\Big\{8\,A_1(\theta_0-2)|C_1|^2\overline{A_0}-4\left(\theta_0+1\right)\zeta_2\overline{A_0C_1}+8\left(\theta_0^2-2\,\theta_0\right)A_1B_2-16\,A_0B_{11}(\theta_0+1)
$$

$$
-16\,B_9(\theta_0+1)\overline{A_0}+\left(\theta_0^2-2\,\theta_0-3\right)C_2\overline{C_1}-\left(4\left(\theta_0^3\overline{\alpha_2}+2\,\theta_0^2\overline{\alpha_2}+\theta_0\overline{\alpha_2}\right)\overline{A_0}-\left(\theta_0^2-2\,\theta_0\right)\overline{B_1}\right)C_1\Big\}
$$

$$
\zeta_{16}=\frac{4\left(\theta_0^2+2\,\theta_0\right)A_1\zeta_2\overline{A_1}-\left(\theta_0^3-\theta_0^2-2\,\theta_0\right)A_2B_1+2\left(\theta_0^2+3\,\theta_0+2\right)A_0B_{12}-\left(\theta_0^3-\theta_0^2-6\,\theta_0\right)A_1B_3}{2\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)}
$$

$$
-\left(\theta_0^3-\theta_0^2-2\,\theta_0\right)A_2C_1-\left(\theta_0^3-\theta_0^2-6\,\theta_0\right)A_1C_2-\left(\theta_0^3-\theta_0^2-10\,\theta_0-8\right)A_0C_3\Big\}
$$

$$
\zeta_{17}=\frac{1}{2\left(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0\right)}\Big\{4\left(\theta_0^3+4\,\theta_0^2+3\,\theta_0\right)A_2\zeta_2\overline{A_0}+2\left(\theta_0^3+6\,\theta_0^2+11\,\theta_0+6\right)A_0B_{13}
$$

$$
-\left(\theta_0^4+\theta_0^3-4\,\theta_0^2-4\,\theta_0\right)A_3C_1-\left(\theta_0^4+\theta_0^3-9\,\theta_0^2-9\,\theta_0\right)A_2C_2-\left(\theta_0^4+\theta_0^3-14\,\theta_0^2-24\,\theta_0\right)A_1C_3
$$

$$
+4\left(\left(\theta_0^3+5\,\theta_0^2+6\,\theta_0\right)\zeta_5\overline{A_0}+\left(\alpha_7\theta_0^5+\alpha_7\theta_0^4-14\,\alpha_7\theta_0^3-24\,\alpha_7\theta_0^2\right)A_0\right)A_1\Big\}
$$

$$
\lambda_2=-\frac{\left(\theta_0^3+\theta_0^2-4\,\theta_0-4\right)|C_1|^2\overline{A_0}^2+\left(\theta_0^4+\theta_0^3-4\,\theta_0^2-4\,\theta_0\right)B_2\overline{A_0}+\left(\theta_0^4-4\,\theta_0^2\right)B_1\overline{A_1}+\left(\theta_0^4-\theta_0^3-2\,\theta_0^2\right)C_1\overline{A_2}}{2\left(\theta_0^4+3\,\theta_0^3+2\,\theta_0^2\right)}
$$

$$
\zeta_{18}=-\frac{1}{2\left(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0\right)}\Big\{\left(\theta_0^3+3\,\theta_0^2-4\,\theta_0-12\right)|C_1|^2\overline{A_0A_1}-2\left(\theta_0^3+6\,\theta_0^2+11\,\theta_0+6\right)B_{10}\overline{A_0}
$$

$$
+\left(\theta_0^4+3\,\theta_0^3-4\,\theta_0^2-12\,\theta_0\right)B_2\overline{A_1}+\left(\theta_0^4+2\,\theta_0^3-5\,\theta_0^2-6\,\theta_0\right)B_1\overline{A_2}+\left(\theta_0^4+\theta_0^3-4\,\theta_0^2-4\,\theta_0\right)C_1\overline{A_3}\Big\}
$$

$$
\zeta_{19}=\frac{4\left(\theta_0^2+2\,\theta_0\right)\zeta_2\overline{A_1}^2+4\left(\theta_0^2+\theta_0\right)\zeta_2\overline{A_0A_2}+2\left(\theta_0^2+3\,\theta_0+2\right)B_{11}\overline{A_0}-\left(\theta_0^3-\theta_0^2-6\,\theta_0\right)B_3\overline{A_1}-\left(\theta_0^3-2\,\theta_0^2-3\,\theta_0\right)C_2\overline{A_2}}{2\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)}
$$

$$
\zeta_{20}=\frac{1}{16\left(\theta_0^2+\theta_0\right)}\Big\{16\,B_{12}(\theta_0+1)\overline{A_0}-8\left(\theta_0^2-4\,\theta_0\right)C_3\overline{A_1}+\left(16\left(\alpha_2\theta_0^2-\alpha_2\theta_0-2\,\alpha_2\right)A_0-\left(2\,\theta_0^2-3\,\theta_0-2\right)B_1\right)C_1
$$

$$
+32\left(\theta_0\zeta_5\overline{A_0}+\left(\alpha_7\theta_0^3-5\,\alpha_7\theta_0^2+6\,\alpha_7\theta_0-8\,\alpha_7\right)A_0\right)\overline{A_1}\Big\}
$$

$$\zeta_{16} = \frac{4\left(\theta_0^2 + 2\,\theta_0\right)A_1\zeta_2\overline{A_1} - \left(\theta_0^3 - \theta_0^2 - 2\,\theta_0\right)A_2B_1 + 2\left(\theta_0^2 + 3\,\theta_0 + 2\right)A_0B_{12} - \left(\theta_0^3 - \theta_0^2 - 6\,\theta_0\right)A_1B_3}{2\,(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0)}$$
$$= \frac{4\left(\theta_0^2 + 2\,\theta_0\right)A_1\zeta_2\overline{A_1} - \left(\theta_0^3 - \theta_0^2 - 2\,\theta_0\right)A_2B_1 + 2\left(\theta_0^2 + 3\,\theta_0 + 2\right)A_0B_{12} - \left(\theta_0^3 - \theta_0^2 - 6\,\theta_0\right)A_1B_3}{2\,(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0)}$$

Now, recall that

$$\langle\vec{\alpha},\vec{\Phi}\rangle = \begin{pmatrix} \frac{(2\,\theta_0-1)\zeta_2}{2\,(\theta_0+1)\theta_0} & 2 & 0 \\ \zeta_7 & 3 & 0 \\ \alpha_2(\theta_0-2) & -\theta_0+1 & \theta_0+1 \\ \frac{(\theta_0-2)\zeta_4}{(\theta_0+2)\theta_0} & -\theta_0+1 & \theta_0+2 \\ \frac{(\theta_0-3)\zeta_3}{2\,(\theta_0+1)\theta_0} & -\theta_0+2 & \theta_0+1 \\ (\theta_0+1)\overline{\alpha_2} & \theta_0 & -\theta_0+2 \\ \frac{\overline{\zeta_3}}{2\,\theta_0} & \theta_0 & -\theta_0+3 \\ \frac{\overline{\zeta_4}}{\theta_0} & \theta_0+1 & -\theta_0+2 \end{pmatrix}$$

so we need only consider the order 4 in (4.3.1). Now, recall that

$$\begin{cases} \vec{B}_1 = -2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0 \\ \vec{B}_2 = -\frac{(\theta_0+2)}{4\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0} + \left(\overline{\alpha_0}\langle\overline{\vec{A}_1},\vec{C}_1\rangle - 2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\right)\vec{A}_0 \\ \vec{B}_3 = -2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_1 + \frac{2}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_1} + \frac{2\overline{\alpha_0}}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_0} + 2\left(\alpha_0\langle\overline{\vec{A}_1},\vec{C}_1\rangle - \langle\overline{\vec{A}_1},\vec{C}_2\rangle\right)\vec{A}_0 \\ \vec{B}_4 = -\frac{(\theta_0+3)}{6\theta_0}\langle\vec{C}_1,\overline{\vec{C}_2}\rangle\overline{\vec{A}_0} - \frac{\overline{\zeta_2}}{6\theta_0}\vec{C}_1 - \frac{(\theta_0+3)}{6\theta_0}|\vec{C}_1|^2\overline{\vec{A}_1} - \frac{\theta_0(\theta_0+1)}{6}\alpha_2\overline{\vec{C}_1} + 2\left(\overline{\alpha_1}\langle\overline{\vec{A}_1},\vec{C}_1\rangle - \langle\overline{\vec{A}_3},\vec{C}_1\rangle\right)\vec{A}_0. \\ \vec{B}_5 = -\left(\frac{(\theta_0+2)}{4}\langle\overline{\vec{C}_1},\vec{C}_2\rangle + \frac{2}{\theta_0-3}\overline{\alpha_1}\zeta_2\right)\overline{\vec{A}_0} + \frac{2\,\zeta_2}{\theta_0-3}\overline{\vec{A}_2} - 2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\vec{A}_1 \\ \quad + \left(8\theta_0(\theta_0+1)|\vec{A}_1|^2\alpha_2 - 2\langle\overline{\vec{A}_2},\vec{C}_2\rangle - \frac{2}{\theta_0-3}\overline{\zeta_0}\zeta_2\right)\vec{A}_0 \\ \vec{B}_6 = \frac{2\,\zeta_5}{\theta_0-4}\overline{\vec{A}_1} - \frac{4}{\theta_0-3}|\vec{A}_1|^2\zeta_2\overline{\vec{A}_0} + \left(-2\langle\overline{\vec{A}_1},\vec{C}_3\rangle + 4\theta_0(\theta_0+1)\alpha_1\alpha_2\right)\vec{A}_0 - 4\theta_0(\theta_0+1)\alpha_2\vec{A}_2 - 2\langle\overline{\vec{A}_1},C_2\rangle\vec{A}_1 \\ \vec{E}_1 = -\frac{1}{2\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_0} \\ \vec{E}_2 = -\alpha_2\,\vec{C}_1 - \frac{2(2\theta_0+1)(\theta_0-4)}{\theta_0+1}\alpha_7\overline{\vec{A}_1} \\ \vec{E}_3 = -\frac{1}{\theta_0}\langle\vec{C}_1,\vec{C}_2\rangle\overline{\vec{A}_0} \end{cases}$$

The new powers of order 4 are

$$\mathbf{(3)}, \mathbf{(7)}, \mathbf{(10)}, \mathbf{(12)} - \mathbf{(14)}, \mathbf{(18)}, \mathbf{(21)}, \mathbf{(23)} - \mathbf{(24)}, \mathbf{(27)}, \mathbf{(29)} - \mathbf{(31)}.$$

or

$$\begin{pmatrix}
\zeta_{13} & 0 & 4 & \mathbf{(3)} \\
\zeta_{14} & 1 & 3 & \mathbf{(7)} \\
\zeta_{15} & 2 & 2 & \mathbf{(10)} \\
\zeta_{16} & 3 & 1 & \mathbf{(12)} \\
\zeta_{17} & 4 & 0 & \mathbf{(13)} \\
\frac{\left(\alpha_2\theta_0^2 - \alpha_2\theta_0 - 2\,\alpha_2\right)C_1\overline{A_0} - 2\left(\alpha_7\theta_0^2 - 6\,\alpha_7\theta_0 + 8\,\alpha_7\right)\overline{A_0 A_1}}{\theta_0^2 + \theta_0} & -2\,\theta_0 + 3 & 2\,\theta_0 + 1 & \mathbf{(14)} \\
\zeta_{18} & -\theta_0 + 1 & \theta_0 + 3 & \mathbf{(18)} \\
\zeta_{19} & -\theta_0 + 2 & \theta_0 + 2 & \mathbf{(21)} \\
\zeta_{20} & -\theta_0 + 3 & \theta_0 + 1 & \mathbf{(23)} \\
-\frac{C_1C_2(5\,\theta_0 - 13) - 12\,C_1\zeta_2\overline{A_0} - 48\,B_{13}\overline{A_0}}{48\,\theta_0} & -\theta_0 + 4 & \theta_0 & \mathbf{(24)} \\
-\frac{4\,(\theta_0\overline{\alpha_7} - 4\,\overline{\alpha_7})A_1\overline{A_0} - \left(\theta_0^2\overline{\alpha_2} + 3\,\theta_0\overline{\alpha_2} + 2\,\overline{\alpha_2}\right)\overline{A_0C_1} - 4\,A_0B_7}{4\,\theta_0} & \theta_0 & -\theta_0 + 4 & \mathbf{(27)} \\
\frac{A_1^2\theta_0\overline{\zeta_2} + \left(\theta_0^2 - 3\,\theta_0\right)A_1\overline{A_0\zeta_3} + \left(\theta_0^2 - 2\,\theta_0 - 3\right)A_0B_8 + 2\left(\theta_0^4\overline{\alpha_2} - 2\,\theta_0^3\overline{\alpha_2} - 3\,\theta_0^2\overline{\alpha_2}\right)A_1\overline{A_1}}{\theta_0^3 - 2\,\theta_0^2 - 3\,\theta_0} & \theta_0 + 1 & -\theta_0 + 3 & \mathbf{(29)} \\
\frac{\left(\theta_0^2 - 4\right)A_0A_1|C_1|^2 + 8\left(\theta_0^2 + 2\,\theta_0\right)A_1\overline{A_0\zeta_4} + 4\left(\theta_0^2 + 3\,\theta_0 + 2\right)A_0B_9 + 8\left(\theta_0^4\overline{\alpha_2} + 2\,\theta_0^3\overline{\alpha_2} + \theta_0^2\overline{\alpha_2}\right)A_2\overline{A_0}}{4\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)} & \theta_0 + 2 & -\theta_0 + 2 & \mathbf{(30)} \\
-\frac{2\,(\theta_0\overline{\alpha_7} - 4\,\overline{\alpha_7})A_0A_1 - (\theta_0\overline{\alpha_2} + \overline{\alpha_2})A_0\overline{C_1}}{2\,\theta_0} & 2\,\theta_0 & -2\,\theta_0 + 4 & \mathbf{(31)}
\end{pmatrix}$$

$$= \begin{pmatrix}
\zeta_{13} & 0 & 4 & \mathbf{(3)} \\
\zeta_{14} & 1 & 3 & \mathbf{(7)} \\
\zeta_{15} & 2 & 2 & \mathbf{(10)} \\
\zeta_{16} & 3 & 1 & \mathbf{(12)} \\
\zeta_{17} & 4 & 0 & \mathbf{(13)} \\
\zeta_{18} & -\theta_0 + 1 & \theta_0 + 3 & \mathbf{(18)} \\
\zeta_{19} & -\theta_0 + 2 & \theta_0 + 2 & \mathbf{(21)} \\
\zeta_{20} & -\theta_0 + 3 & \theta_0 + 1 & \mathbf{(23)} \\
-\frac{C_1C_2(5\,\theta_0 - 13) - 48\,B_{13}\overline{A_0}}{48\,\theta_0} & -\theta_0 + 4 & \theta_0 & \mathbf{(24)} \\
\frac{A_0B_7}{\theta_0} & \theta_0 & -\theta_0 + 4 & \mathbf{(27)} \\
\frac{A_1^2\theta_0\overline{\zeta_2} + \left(\theta_0^2 - 2\,\theta_0 - 3\right)A_0B_8 + 2\left(\theta_0^4\overline{\alpha_2} - 2\,\theta_0^3\overline{\alpha_2} - 3\,\theta_0^2\overline{\alpha_2}\right)A_1\overline{A_1}}{\theta_0^3 - 2\,\theta_0^2 - 3\,\theta_0} & \theta_0 + 1 & -\theta_0 + 3 & \mathbf{(29)} \\
\frac{4\left(\theta_0^2 + 3\,\theta_0 + 2\right)A_0B_9 + 8\left(\theta_0^4\overline{\alpha_2} + 2\,\theta_0^3\overline{\alpha_2} + \theta_0^2\overline{\alpha_2}\right)A_2\overline{A_0}}{4\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)} & \theta_0 + 2 & -\theta_0 + 2 & \mathbf{(30)}
\end{pmatrix}$$

so for

$$\begin{cases} \zeta_{21} = -\dfrac{1}{48\theta_0}\Big((5\theta_0 - 13)\langle \vec{C}_1, \vec{C}_2\rangle + \langle \overline{\vec{A}_0}, \vec{B}_{13}\rangle\Big) \\ \zeta_{22} = \dfrac{1}{\theta_0}\langle \vec{A}_0, \vec{B}_7\rangle \\ \zeta_{23} = \dfrac{\zeta_0\overline{\zeta}_2}{(\theta_0+1)(\theta_0-3)} + \langle \vec{A}_0, \vec{B}_8\rangle + 2|\vec{A}_1|^2\overline{\alpha_2} \\ \zeta_{24} = \dfrac{1}{\theta_0}\langle \vec{A}_0, \vec{B}_9\rangle + \dfrac{\theta_0(\theta_0+2)}{\theta_0+1}\alpha_1\overline{\alpha_2} \end{cases}$$

we obtain as matrix of powers of order 4

$$\begin{pmatrix} \zeta_{13} & 0 & 4 & \mathbf{(3)} \\ \zeta_{14} & 1 & 3 & \mathbf{(7)} \\ \zeta_{15} & 2 & 2 & \mathbf{(10)} \\ \zeta_{16} & 3 & 1 & \mathbf{(12)} \\ \zeta_{17} & 4 & 0 & \mathbf{(13)} \\ \zeta_{18} & -\theta_0+1 & \theta_0+3 & \mathbf{(18)} \\ \zeta_{19} & -\theta_0+2 & \theta_0+2 & \mathbf{(21)} \\ \zeta_{20} & -\theta_0+3 & \theta_0+1 & \mathbf{(23)} \\ \zeta_{21} & -\theta_0+4 & \theta_0 & \mathbf{(24)} \\ \zeta_{22} & \theta_0 & -\theta_0+4 & \mathbf{(27)} \\ \zeta_{23} & \theta_0+1 & -\theta_0+3 & \mathbf{(29)} \\ \zeta_{24} & \theta_0+2 & -\theta_0+2 & \mathbf{(30)} \end{pmatrix}$$

At this point, no risk of mistakes...

TeX on the left and Sage on the right

$$\begin{pmatrix} \zeta_{13} & 0 & 4 & \textbf{(3)} \\ \zeta_{14} & 1 & 3 & \textbf{(7)} \\ \zeta_{15} & 2 & 2 & \textbf{(10)} \\ \zeta_{16} & 3 & 1 & \textbf{(12)} \\ \zeta_{17} & 4 & 0 & \textbf{(13)} \\ \zeta_{18} & -\theta_0+1 & \theta_0+3 & \textbf{(18)} \\ \zeta_{19} & -\theta_0+2 & \theta_0+2 & \textbf{(21)} \\ \zeta_{20} & -\theta_0+3 & \theta_0+1 & \textbf{(23)} \\ \zeta_{21} & -\theta_0+4 & \theta_0 & \textbf{(24)} \\ \zeta_{22} & \theta_0 & -\theta_0+4 & \textbf{(27)} \\ \zeta_{23} & \theta_0+1 & -\theta_0+3 & \textbf{(29)} \\ \zeta_{24} & \theta_0+2 & -\theta_0+2 & \textbf{(30)} \end{pmatrix} \begin{pmatrix} \zeta_{13} & 0 & 4 \\ \zeta_{14} & 1 & 3 \\ \zeta_{15} & 2 & 2 \\ \zeta_{16} & 3 & 1 \\ \zeta_{17} & 4 & 0 \\ \zeta_{18} & -\theta_0+1 & \theta_0+3 \\ \zeta_{19} & -\theta_0+2 & \theta_0+2 \\ \zeta_{20} & -\theta_0+3 & \theta_0+1 \\ \zeta_{21} & -\theta_0+4 & \theta_0 \\ \zeta_{22} & \theta_0 & -\theta_0+4 \\ \zeta_{23} & \theta_0+1 & -\theta_0+3 \\ \zeta_{24} & \theta_0+2 & -\theta_0+2 \end{pmatrix}$$

Finally, we have

$$\langle \alpha, \vec{\Phi} \rangle = \begin{pmatrix} \frac{(2\,\theta_0-1)\zeta_2}{2\,(\theta_0+1)\theta_0} & 2 & 0 \\ \zeta_7 & 3 & 0 \\ \alpha_2(\theta_0-2) & -\theta_0+1 & \theta_0+1 \\ \frac{(\theta_0-2)\zeta_4}{(\theta_0+2)\theta_0} & -\theta_0+1 & \theta_0+2 \\ \frac{(\theta_0-3)\zeta_3}{2\,(\theta_0+1)\theta_0} & -\theta_0+2 & \theta_0+1 \\ (\theta_0+1)\overline{\alpha_2} & \theta_0 & -\theta_0+2 \\ \frac{\overline{\zeta_3}}{2\,\theta_0} & \theta_0 & -\theta_0+3 \\ \frac{\overline{\zeta_4}}{\theta_0} & \theta_0+1 & -\theta_0+2 \end{pmatrix} \begin{pmatrix} \zeta_{13} & 0 & 4 \\ \zeta_{14} & 1 & 3 \\ \zeta_{15} & 2 & 2 \\ \zeta_{16} & 3 & 1 \\ \zeta_{17} & 4 & 0 \\ \zeta_{18} & -\theta_0+1 & \theta_0+3 \\ \zeta_{19} & -\theta_0+2 & \theta_0+2 \\ \zeta_{20} & -\theta_0+3 & \theta_0+1 \\ \zeta_{21} & -\theta_0+4 & \theta_0 \\ \zeta_{22} & \theta_0 & -\theta_0+4 \\ \zeta_{23} & \theta_0+1 & -\theta_0+3 \\ \zeta_{24} & \theta_0+2 & -\theta_0+2 \end{pmatrix}$$

4.4 Next order development of $\vec{h}_0$

We first have

$$g^{-1}\otimes\vec{h}_0=\begin{pmatrix}
2 & A_1 & 0 & -\theta_0+1\\
-4\,|A_1|^2 & A_0 & 0 & -\theta_0+2\\
\frac{1}{4} & \overline{B_1} & 0 & -\theta_0+3\\
-2\,\overline{\alpha_5} & A_0 & 0 & -\theta_0+3\\
-2\,\overline{\alpha_1} & A_1 & 0 & -\theta_0+3\\
\frac{1}{6} & \overline{B_3} & 0 & -\theta_0+4\\
-2\,\overline{\alpha_3} & A_1 & 0 & -\theta_0+4\\
-\frac{1}{6}\,|A_1|^2 & \overline{C_1} & 0 & -\theta_0+4\\
8\,|A_1|^2\overline{\alpha_1}-2\,\overline{\alpha_6} & A_0 & 0 & -\theta_0+4\\
\frac{1}{8} & \overline{B_6} & 0 & -\theta_0+5\\
-\frac{1}{8}\,\overline{\alpha_5} & \overline{C_1} & 0 & -\theta_0+5\\
-\frac{1}{8}\,\overline{\alpha_1} & \overline{B_1} & 0 & -\theta_0+5\\
2\,\overline{\alpha_1}^2-2\,\overline{\alpha_4} & A_1 & 0 & -\theta_0+5\\
-\frac{1}{12}\,|A_1|^2 & \overline{C_2} & 0 & -\theta_0+5\\
8\,|A_1|^2\overline{\alpha_3}+4\,\overline{\alpha_1\alpha_5}-2\,\alpha_{15} & A_0 & 0 & -\theta_0+5\\
4 & A_2 & 1 & -\theta_0+1\\
-4\,\alpha_1 & A_0 & 1 & -\theta_0+1\\
-4\,\alpha_5 & A_0 & 1 & -\theta_0+2\\
-8\,|A_1|^2 & A_1 & 1 & -\theta_0+2\\
\frac{1}{2} & \overline{B_2} & 1 & -\theta_0+3\\
-4\,\overline{\alpha_5} & A_1 & 1 & -\theta_0+3\\
-4\,\overline{\alpha_1} & A_2 & 1 & -\theta_0+3\\
16\,|A_1|^4+8\,\alpha_1\overline{\alpha_1}-4\,\beta & A_0 & 1 & -\theta_0+3
\end{pmatrix}$$

$$
\begin{pmatrix}
\frac{1}{3} & \overline{B_5} & 1 & -\theta_0+4 \\
-4\,\overline{\alpha_3} & A_2 & 1 & -\theta_0+4 \\
-\frac{1}{6}\,\alpha_5 & \overline{C_1} & 1 & -\theta_0+4 \\
-\frac{1}{3}\,|A_1|^2 & \overline{B_1} & 1 & -\theta_0+4 \\
16\,|A_1|^2\overline{\alpha_5}+8\,\alpha_5\overline{\alpha_1}+8\,\alpha_1\overline{\alpha_3}-4\,\alpha_{16} & A_0 & 1 & -\theta_0+4 \\
16\,|A_1|^2\overline{\alpha_1}-4\,\overline{\alpha_6} & A_1 & 1 & -\theta_0+4 \\
6 & A_3 & 2 & -\theta_0+1 \\
-6\,\alpha_3 & A_0 & 2 & -\theta_0+1 \\
-6\,\alpha_1 & A_1 & 2 & -\theta_0+1 \\
-6\,\alpha_5 & A_1 & 2 & -\theta_0+2 \\
-12\,|A_1|^2 & A_2 & 2 & -\theta_0+2 \\
24\,\alpha_1|A_1|^2-6\,\alpha_6 & A_0 & 2 & -\theta_0+2 \\
\frac{3}{4} & \overline{B_4} & 2 & -\theta_0+3 \\
\frac{3\,(\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2})}{4\,\theta_0} & C_1 & 2 & -\theta_0+3 \\
-6\,\overline{\alpha_5} & A_2 & 2 & -\theta_0+3 \\
-6\,\overline{\alpha_1} & A_3 & 2 & -\theta_0+3 \\
24\,\alpha_5|A_1|^2+12\,\alpha_3\overline{\alpha_1}+12\,\alpha_1\overline{\alpha_5}-6\,\overline{\alpha_{16}} & A_0 & 2 & -\theta_0+3 \\
24\,|A_1|^4+12\,\alpha_1\overline{\alpha_1}-6\,\beta & A_1 & 2 & -\theta_0+3 \\
8 & A_4 & 3 & -\theta_0+1 \\
-8\,\alpha_3 & A_1 & 3 & -\theta_0+1 \\
-8\,\alpha_1 & A_2 & 3 & -\theta_0+1 \\
8\,\alpha_1^2-8\,\alpha_4 & A_0 & 3 & -\theta_0+1 \\
-8\,\alpha_5 & A_2 & 3 & -\theta_0+2 \\
-16\,|A_1|^2 & A_3 & 3 & -\theta_0+2 \\
32\,\alpha_3|A_1|^2+16\,\alpha_1\alpha_5-8\,\overline{\alpha_{15}} & A_0 & 3 & -\theta_0+2 \\
32\,\alpha_1|A_1|^2-8\,\alpha_6 & A_1 & 3 & -\theta_0+2
\end{pmatrix}
$$

$$\begin{pmatrix}
10 & A_5 & 4 & -\theta_0 + 1 \\
-10\,\alpha_3 & A_2 & 4 & -\theta_0 + 1 \\
-10\,\alpha_1 & A_3 & 4 & -\theta_0 + 1 \\
20\,\alpha_1\alpha_3 - 10\,\overline{\alpha_{14}} & A_0 & 4 & -\theta_0 + 1 \\
10\,\alpha_1^2 - 10\,\alpha_4 & A_1 & 4 & -\theta_0 + 1 \\
-\frac{\theta_0 - 2}{2\,\theta_0} & E_1 & -2\,\theta_0 + 3 & \theta_0 + 1 \\
-\frac{\theta_0 - 2}{2\,\theta_0 + 1} & E_2 & -2\,\theta_0 + 3 & \theta_0 + 2 \\
\frac{\alpha_2\theta_0^2 - \alpha_2\theta_0 - 2\,\alpha_2}{2\,\theta_0^2 + \theta_0} & C_1 & -2\,\theta_0 + 3 & \theta_0 + 2 \\
-\frac{2\,\theta_0 - 5}{4\,\theta_0} & E_3 & -2\,\theta_0 + 4 & \theta_0 + 1 \\
-\frac{\theta_0 - 2}{2\,\theta_0} & C_1 & -\theta_0 + 1 & 1 \\
-\frac{\theta_0 - 2}{2\,(\theta_0 + 1)} & B_1 & -\theta_0 + 1 & 2 \\
2\,\alpha_2\theta_0 - 4\,\alpha_2 & A_0 & -\theta_0 + 1 & 2 \\
-\frac{\theta_0 - 2}{2\,(\theta_0 + 2)} & B_2 & -\theta_0 + 1 & 3 \\
\frac{\theta_0\overline{\alpha_1} - 2\,\overline{\alpha_1}}{\theta_0^2 + 2\,\theta_0} & C_1 & -\theta_0 + 1 & 3 \\
2\,\alpha_9\theta_0 - 4\,\alpha_9 & A_0 & -\theta_0 + 1 & 3 \\
-\frac{\theta_0 - 2}{2\,(\theta_0 + 3)} & B_4 & -\theta_0 + 1 & 4 \\
\frac{\alpha_2\theta_0^2 - \alpha_2\theta_0 - 2\,\alpha_2}{4\,(\theta_0 + 3)} & \overline{C_1} & -\theta_0 + 1 & 4 \\
\frac{3\,(\theta_0\overline{\alpha_3} - 2\,\overline{\alpha_3})}{2\,(\theta_0^2 + 3\,\theta_0)} & C_1 & -\theta_0 + 1 & 4 \\
\frac{\theta_0\overline{\alpha_1} - 2\,\overline{\alpha_1}}{\theta_0^2 + 4\,\theta_0 + 3} & B_1 & -\theta_0 + 1 & 4 \\
-2\,(2\,\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0 + 8\,\alpha_2\overline{\alpha_1} - 4\,\alpha_{10} & A_0 & -\theta_0 + 1 & 4 \\
-\frac{\theta_0 - 3}{2\,\theta_0} & C_2 & -\theta_0 + 2 & 1 \\
-\frac{\theta_0 - 3}{2\,(\theta_0 + 1)} & B_3 & -\theta_0 + 2 & 2 \\
\frac{(\theta_0 - 3)|A_1|^2}{\theta_0^2 + \theta_0} & C_1 & -\theta_0 + 2 & 2 \\
2\,\alpha_8\theta_0 - 6\,\alpha_8 & A_0 & -\theta_0 + 2 & 2 \\
2\,\alpha_2\theta_0 - 6\,\alpha_2 & A_1 & -\theta_0 + 2 & 2
\end{pmatrix}$$

$$\begin{pmatrix}
-\frac{\theta_0 - 3}{2\,(\theta_0 + 2)} & B_5 & -\theta_0 + 2 & 3 \\
\frac{\theta_0\overline{\alpha_5} - 3\,\overline{\alpha_5}}{\theta_0^2 + 2\,\theta_0} & C_1 & -\theta_0 + 2 & 3 \\
\frac{\theta_0\overline{\alpha_1} - 3\,\overline{\alpha_1}}{\theta_0^2 + 2\,\theta_0} & C_2 & -\theta_0 + 2 & 3 \\
\frac{(\theta_0 - 3)|A_1|^2}{\theta_0^2 + 3\,\theta_0 + 2} & B_1 & -\theta_0 + 2 & 3 \\
2\,\alpha_9\theta_0 - 6\,\alpha_9 & A_1 & -\theta_0 + 2 & 3 \\
-8\,(\alpha_2\theta_0 - 3\,\alpha_2)|A_1|^2 + 2\,\alpha_{11}\theta_0 - 6\,\alpha_{11} & A_0 & -\theta_0 + 2 & 3 \\
-\frac{\theta_0 - 4}{2\,\theta_0} & C_3 & -\theta_0 + 3 & 1 \\
2\,\alpha_7\theta_0 - 8\,\alpha_7 & A_0 & -\theta_0 + 3 & 1 \\
-\frac{\theta_0 - 4}{2\,(\theta_0 + 1)} & B_6 & -\theta_0 + 3 & 2 \\
\frac{\alpha_5\theta_0 - 4\,\alpha_5}{2\,(\theta_0^2 + \theta_0)} & C_1 & -\theta_0 + 3 & 2 \\
\frac{(\theta_0 - 4)|A_1|^2}{\theta_0^2 + \theta_0} & C_2 & -\theta_0 + 3 & 2 \\
2\,\alpha_8\theta_0 - 8\,\alpha_8 & A_1 & -\theta_0 + 3 & 2 \\
2\,\alpha_2\theta_0 - 8\,\alpha_2 & A_2 & -\theta_0 + 3 & 2 \\
16\,\alpha_1\alpha_2 - 2\,(2\,\alpha_1\alpha_2 - \alpha_{12})\theta_0 - 8\,\alpha_{12} & A_0 & -\theta_0 + 3 & 2 \\
-\frac{\theta_0 - 5}{2\,\theta_0} & C_4 & -\theta_0 + 4 & 1 \\
2\,\alpha_7\theta_0 - 10\,\alpha_7 & A_1 & -\theta_0 + 4 & 1 \\
2\,\alpha_{13}\theta_0 - 10\,\alpha_{13} & A_0 & -\theta_0 + 4 & 1 \\
-2\,\theta_0\overline{\alpha_2} - 2\,\overline{\alpha_2} & A_0 & \theta_0 & -2\,\theta_0 + 3 \\
-2\,\theta_0\overline{\alpha_8} - 2\,\overline{\alpha_8} & A_0 & \theta_0 & -2\,\theta_0 + 4 \\
-\frac{\theta_0 + 1}{2\,(\theta_0 - 4)} & \overline{E_2} & \theta_0 & -2\,\theta_0 + 5 \\
-\frac{\theta_0^2\overline{\alpha_2} - \theta_0\overline{\alpha_2} - 2\,\overline{\alpha_2}}{4\,(\theta_0 - 4)} & \overline{C_1} & \theta_0 & -2\,\theta_0 + 5 \\
-2\,\theta_0\overline{\alpha_7} - 2\,\overline{\alpha_7} & A_1 & \theta_0 & -2\,\theta_0 + 5 \\
2\,(2\,\overline{\alpha_1\alpha_2} - \overline{\alpha_{12}})\theta_0 + 4\,\overline{\alpha_1\alpha_2} - 2\,\overline{\alpha_{12}} & A_0 & \theta_0 & -2\,\theta_0 + 5
\end{pmatrix}$$

$$\begin{pmatrix}
-\frac{\theta_0}{2\,(\theta_0 - 4)} & \overline{E_1} & \theta_0 - 1 & -2\,\theta_0 + 5 \\
-2\,\theta_0\overline{\alpha_7} & A_0 & \theta_0 - 1 & -2\,\theta_0 + 5 \\
-\frac{\theta_0}{2\,(\theta_0 - 5)} & \overline{E_3} & \theta_0 - 1 & -2\,\theta_0 + 6 \\
-2\,\theta_0\overline{\alpha_{13}} & A_0 & \theta_0 - 1 & -2\,\theta_0 + 6 \\
-2\,\theta_0\overline{\alpha_9} - 4\,\overline{\alpha_9} & A_0 & \theta_0 + 1 & -2\,\theta_0 + 3 \\
-2\,\theta_0\overline{\alpha_2} - 4\,\overline{\alpha_2} & A_1 & \theta_0 + 1 & -2\,\theta_0 + 3 \\
-2\,\theta_0\overline{\alpha_8} - 4\,\overline{\alpha_8} & A_1 & \theta_0 + 1 & -2\,\theta_0 + 4 \\
8\,(\theta_0\overline{\alpha_2} + 2\,\overline{\alpha_2})|A_1|^2 - 2\,\theta_0\overline{\alpha_{11}} - 4\,\overline{\alpha_{11}} & A_0 & \theta_0 + 1 & -2\,\theta_0 + 4 \\
-2\,\theta_0\overline{\alpha_9} - 6\,\overline{\alpha_9} & A_1 & \theta_0 + 2 & -2\,\theta_0 + 3 \\
-2\,\theta_0\overline{\alpha_2} - 6\,\overline{\alpha_2} & A_2 & \theta_0 + 2 & -2\,\theta_0 + 3 \\
2\,(2\,\alpha_1\overline{\alpha_2} - \overline{\alpha_{10}})\theta_0 + 12\,\alpha_1\overline{\alpha_2} - 6\,\overline{\alpha_{10}} & A_0 & \theta_0 + 2 & -2\,\theta_0 + 3
\end{pmatrix}$$

$g^{-1} \otimes \langle \vec{\Phi}, \vec{h}_0 \rangle =$

$$
\begin{pmatrix}
\frac{2\,A_1\overline{A_0}}{\theta_0} & 0 & 1 \\
-\frac{2\left(2\,A_0(\theta_0+1)|A_1|^2\overline{A_0} - A_1\theta_0\overline{A_1}\right)}{\theta_0^2+\theta_0} & 0 & 2 \\
\nu_1 & 0 & 3 \\
\nu_2 & 0 & 4 \\
\nu_3 & 0 & 5 \\
-\frac{8\,A_0\alpha_1\theta_0\overline{A_0} + A_0C_1(\theta_0-2) - 8\,A_2\theta_0\overline{A_0}}{2\,\theta_0^2} & 1 & 1 \\
\nu_4 & 1 & 2 \\
\nu_5 & 1 & 3 \\
\nu_6 & 1 & 4 \\
\nu_7 & 2 & 1 \\
\nu_8 & 2 & 2 \\
\nu_9 & 2 & 3 \\
\nu_{10} & 3 & 1 \\
\nu_{11} & 3 & 2 \\
\nu_{12} & 4 & 1 \\
-\frac{E_1(\theta_0-2)\overline{A_0}}{2\,\theta_0^2} & -2\,\theta_0+3 & 2\,\theta_0+1 \\
\frac{2\left(\alpha_2\theta_0^3 - 3\,\alpha_2\theta_0 - 2\,\alpha_2\right)C_1\overline{A_0} - 2\left(\theta_0^3-\theta_0^2-2\,\theta_0\right)E_2\overline{A_0} - \left(2\,\theta_0^3-3\,\theta_0^2-2\,\theta_0\right)E_1\overline{A_1}}{2\,(2\,\theta_0^4+3\,\theta_0^3+\theta_0^2)} & -2\,\theta_0+3 & 2\,\theta_0+2 \\
-\frac{E_3(2\,\theta_0-5)\overline{A_0}}{4\,\theta_0^2} & -2\,\theta_0+4 & 2\,\theta_0+1 \\
-\frac{C_1(\theta_0-2)\overline{A_0}}{2\,\theta_0^2} & -\theta_0+1 & \theta_0+1 \\
\frac{4\left(\alpha_2\theta_0^2-\alpha_2\theta_0-2\,\alpha_2\right)A_0\overline{A_0} - B_1(\theta_0-2)\overline{A_0} - C_1(\theta_0-2)\overline{A_1}}{2\,(\theta_0^2+\theta_0)} & -\theta_0+1 & \theta_0+2 \\
\nu_{13} & -\theta_0+1 & \theta_0+3 \\
\nu_{14} & -\theta_0+1 & \theta_0+4 \\
-\frac{C_2(\theta_0-3)\overline{A_0}}{2\,\theta_0^2} & -\theta_0+2 & \theta_0+1 \\
\nu_{15} & -\theta_0+2 & \theta_0+2 \\
\nu_{16} & -\theta_0+2 & \theta_0+3 \\
-\frac{C_1^2(\theta_0-2) + 8\,A_0E_1(\theta_0-2) - 32\left(\alpha_7\theta_0^2-4\,\alpha_7\theta_0\right)A_0\overline{A_0} + 8\,C_3(\theta_0-4)\overline{A_0}}{16\,\theta_0^2} & -\theta_0+3 & \theta_0+1
\end{pmatrix}
$$

$$
\begin{pmatrix}
\nu_{17} & -\theta_0+3 & \theta_0+2 \\
\nu_{18} & -\theta_0+4 & \theta_0+1 \\
\frac{2\,A_0A_1}{\theta_0} & \theta_0 & -\theta_0+1 \\
-\frac{4\,A_0^2|A_1|^2}{\theta_0} & \theta_0 & -\theta_0+2 \\
-\frac{8\,(\theta_0\overline{\alpha_2}+\overline{\alpha_2})A_0\overline{A_0}+8\,A_0A_1\overline{\alpha_1}+8\,A_0^2\overline{\alpha_5}-A_0\overline{B_1}-A_1\overline{C_1}}{4\,\theta_0} & \theta_0 & -\theta_0+3 \\
\nu_{19} & \theta_0 & -\theta_0+4 \\
\nu_{20} & \theta_0 & -\theta_0+5 \\
-\frac{4\,(\theta_0\overline{\alpha_7}-4\,\overline{\alpha_7})A_0\overline{A_0}+\overline{A_0E_1}}{2\,(\theta_0-4)} & \theta_0-1 & -\theta_0+5 \\
\nu_{21} & \theta_0-1 & -\theta_0+6 \\
-\frac{2\left(2\,(\alpha_1\theta_0+\alpha_1)A_0^2-2\,A_0A_2(\theta_0+1)-A_1^2\theta_0\right)}{\theta_0^2+\theta_0} & \theta_0+1 & -\theta_0+1 \\
-\frac{4\left(A_0A_1(3\,\theta_0+2)|A_1|^2+(\alpha_5\theta_0+\alpha_5)A_0^2\right)}{\theta_0^2+\theta_0} & \theta_0+1 & -\theta_0+2 \\
\nu_{22} & \theta_0+1 & -\theta_0+3 \\
\nu_{23} & \theta_0+1 & -\theta_0+4 \\
\nu_{24} & \theta_0+2 & -\theta_0+1 \\
\nu_{25} & \theta_0+2 & -\theta_0+2 \\
\nu_{26} & \theta_0+2 & -\theta_0+3 \\
\nu_{27} & \theta_0+3 & -\theta_0+1 \\
\nu_{28} & \theta_0+3 & -\theta_0+2 \\
\nu_{29} & \theta_0+4 & -\theta_0+1 \\
-\frac{4\,(\theta_0\overline{\alpha_7}-4\,\overline{\alpha_7})A_0^2+A_0\overline{E_1}}{2\,(\theta_0-4)} & 2\,\theta_0-1 & -2\,\theta_0+5 \\
-\frac{4\,(\theta_0\overline{\alpha_{13}}-5\,\overline{\alpha_{13}})A_0^2+A_0\overline{E_3}}{2\,(\theta_0-5)} & 2\,\theta_0-1 & -2\,\theta_0+6 \\
-\frac{2\left((\theta_0\overline{\alpha_9}+2\,\overline{\alpha_9})A_0^2+2\,(\theta_0\overline{\alpha_2}+\overline{\alpha_2})A_0A_1\right)}{\theta_0} & 2\,\theta_0+1 & -2\,\theta_0+3 \\
\frac{2\left(4\,(\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2})A_0^2|A_1|^2-(\theta_0\overline{\alpha_{11}}+2\,\overline{\alpha_{11}})A_0^2-2\,(\theta_0\overline{\alpha_8}+\overline{\alpha_8})A_0A_1\right)}{\theta_0} & 2\,\theta_0+1 & -2\,\theta_0+4 \\
\nu_{30} & 2\,\theta_0+2 & -2\,\theta_0+3 \\
-\frac{2\,(\theta_0\overline{\alpha_2}+\overline{\alpha_2})A_0^2}{\theta_0} & 2\,\theta_0 & -2\,\theta_0+3 \\
-\frac{2\,(\theta_0\overline{\alpha_8}+\overline{\alpha_8})A_0^2}{\theta_0} & 2\,\theta_0 & -2\,\theta_0+4 \\
\nu_{31} & 2\,\theta_0 & -2\,\theta_0+5
\end{pmatrix}
$$

where

$$
\nu_1 = -\frac{1}{4\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)}\Big\{16\left(\theta_0^2+2\,\theta_0\right)A_0|A_1|^2\overline{A_1}+8\left(\theta_0^2\overline{\alpha_5}+3\,\theta_0\overline{\alpha_5}+2\,\overline{\alpha_5}\right)A_0\overline{A_0}+8\left(\theta_0^2\overline{\alpha_1}+3\,\theta_0\overline{\alpha_1}+2\,\overline{\alpha_1}\right)A_1\overline{A_0}
$$

$$
-8\left(\theta_0^2+\theta_0\right)A_1\overline{A_2}-\left(\theta_0^2+3\,\theta_0+2\right)\overline{A_0B_1}\Big\}
$$

$$\nu_2 = \frac{1}{12\left(\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0\right)}\Big\{96\left(\theta_0^3\overline{\alpha_1} + 6\,\theta_0^2\overline{\alpha_1} + 11\,\theta_0\overline{\alpha_1} + 6\,\overline{\alpha_1}\right)A_0|A_1|^2\overline{A_0} - 48\left(\theta_0^3 + 4\,\theta_0^2 + 3\,\theta_0\right)A_0|A_1|^2\overline{A_2}$$

$$- 2\left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)|A_1|^2\overline{A_0C_1} - 24\left(\theta_0^3\overline{\alpha_6} + 6\,\theta_0^2\overline{\alpha_6} + 11\,\theta_0\overline{\alpha_6} + 6\,\overline{\alpha_6}\right)A_0\overline{A_0} - 24\left(\theta_0^3\overline{\alpha_3} + 6\,\theta_0^2\overline{\alpha_3} + 11\,\theta_0\overline{\alpha_3} + 6\,\overline{\alpha_3}\right)A_1\overline{A_0}$$

$$+ 24\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)A_1\overline{A_3} + 3\left(\theta_0^3 + 5\,\theta_0^2 + 6\,\theta_0\right)\overline{A_1B_1} + 2\left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)\overline{A_0B_3}$$

$$- 24\left(\left(\theta_0^3\overline{\alpha_5} + 5\,\theta_0^2\overline{\alpha_5} + 6\,\theta_0\overline{\alpha_5}\right)A_0 + \left(\theta_0^3\overline{\alpha_1} + 5\,\theta_0^2\overline{\alpha_1} + 6\,\theta_0\overline{\alpha_1}\right)A_1\right)\overline{A_1}\Big\}$$

$$\nu_3 = -\frac{1}{24\left(\theta_0^5 + 10\,\theta_0^4 + 35\,\theta_0^3 + 50\,\theta_0^2 + 24\,\theta_0\right)}\Big\{96\left(\theta_0^4 + 7\,\theta_0^3 + 14\,\theta_0^2 + 8\,\theta_0\right)A_0|A_1|^2\overline{A_3}$$

$$+ 2\left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)|A_1|^2\overline{A_0C_2} - 192\Big\{\left(\theta_0^4\overline{\alpha_3} + 10\,\theta_0^3\overline{\alpha_3} + 35\,\theta_0^2\overline{\alpha_3} + 50\,\theta_0\overline{\alpha_3} + 24\,\overline{\alpha_3}\right)A_0\overline{A_0}$$

$$+ \left(\theta_0^4\overline{\alpha_1} + 9\,\theta_0^3\overline{\alpha_1} + 26\,\theta_0^2\overline{\alpha_1} + 24\,\theta_0\overline{\alpha_1}\right)A_0\overline{A_1}\Big\}|A_1|^2 - 48\Big\{(2\,\overline{\alpha_1\alpha_5} - \alpha_{15})\theta_0^4 + 10\,(2\,\overline{\alpha_1\alpha_5} - \alpha_{15})\theta_0^3 + 35\,(2\,\overline{\alpha_1\alpha_5} - \alpha_{15})\theta_0^2$$

$$+ 50\,(2\,\overline{\alpha_1\alpha_5} - \alpha_{15})\theta_0 + 48\,\overline{\alpha_1\alpha_5} - 24\,\alpha_{15}\Big\}A_0\overline{A_0} - 48\Big\{\left(\overline{\alpha_1}^2 - \overline{\alpha_4}\right)\theta_0^4 + 10\left(\overline{\alpha_1}^2 - \overline{\alpha_4}\right)\theta_0^3 + 35\left(\overline{\alpha_1}^2 - \overline{\alpha_4}\right)\theta_0^2 + 50\left(\overline{\alpha_1}^2 - \overline{\alpha_4}\right)\theta_0$$

$$+ 24\,\overline{\alpha_1}^2 - 24\,\overline{\alpha_4}\Big\}A_1\overline{A_0} - 48\left(\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0\right)A_1\overline{A_4} - 4\left(\theta_0^4 + 9\,\theta_0^3 + 26\,\theta_0^2 + 24\,\theta_0\right)\overline{A_1B_3}$$

$$- 3\left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)\overline{A_0B_6} + 48\Big\{\left(\theta_0^4\overline{\alpha_6} + 9\,\theta_0^3\overline{\alpha_6} + 26\,\theta_0^2\overline{\alpha_6} + 24\,\theta_0\overline{\alpha_6}\right)A_0$$

$$+ \left(\theta_0^4\overline{\alpha_3} + 9\,\theta_0^3\overline{\alpha_3} + 26\,\theta_0^2\overline{\alpha_3} + 24\,\theta_0\overline{\alpha_3}\right)A_1\Big\}\overline{A_1} + 48\Big\{\left(\theta_0^4\overline{\alpha_5} + 8\,\theta_0^3\overline{\alpha_5} + 19\,\theta_0^2\overline{\alpha_5} + 12\,\theta_0\overline{\alpha_5}\right)A_0$$

$$+ \left(\theta_0^4\overline{\alpha_1} + 8\,\theta_0^3\overline{\alpha_1} + 19\,\theta_0^2\overline{\alpha_1} + 12\,\theta_0\overline{\alpha_1}\right)A_1\Big\}\overline{A_2}$$

$$+ 3\Big\{\left(\theta_0^4\overline{\alpha_1} + 10\,\theta_0^3\overline{\alpha_1} + 35\,\theta_0^2\overline{\alpha_1} + 50\,\theta_0\overline{\alpha_1} + 24\,\overline{\alpha_1}\right)\overline{A_0} - 2\left(\theta_0^4 + 8\,\theta_0^3 + 19\,\theta_0^2 + 12\,\theta_0\right)\overline{A_2}\Big\}\overline{B_1}$$

$$+ \left(4\left(\theta_0^4 + 9\,\theta_0^3 + 26\,\theta_0^2 + 24\,\theta_0\right)|A_1|^2\overline{A_1} + 3\left(\theta_0^4\overline{\alpha_5} + 10\,\theta_0^3\overline{\alpha_5} + 35\,\theta_0^2\overline{\alpha_5} + 50\,\theta_0\overline{\alpha_5} + 24\,\overline{\alpha_5}\right)\overline{A_0}\right)\overline{C_1}\Big\}$$

$$\nu_4 = -\frac{16\,A_1(\theta_0 + 1)|A_1|^2\overline{A_0} + 8\,A_0\alpha_1\theta_0\overline{A_1} - 4\left(\alpha_2\theta_0^2 - \alpha_2\theta_0 - 2\,\alpha_2\right)A_0^2 + A_0B_1(\theta_0 - 2) + 8\,(\alpha_5\theta_0 + \alpha_5)A_0\overline{A_0} - 8\,A_2\theta_0\overline{A_1}}{2\left(\theta_0^2 + \theta_0\right)}$$

$$\nu_5 = \frac{1}{16\left(\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2\right)}\Big\{256\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)A_0|A_1|^4\overline{A_0} - 128\left(\theta_0^3 + 2\,\theta_0^2\right)A_1|A_1|^2\overline{A_1} + 32\left(\alpha_9\theta_0^4 + \alpha_9\theta_0^3 - 4\,\alpha_9\theta_0^2 - 4\,\alpha_9\theta_0\right)A_0^2$$

$$- 8\left(\theta_0^3 - \theta_0^2 - 2\,\theta_0\right)A_0B_2 + 16\left(\theta_0^2\overline{\alpha_1} - \theta_0\overline{\alpha_1} - 2\,\overline{\alpha_1}\right)A_0C_1 + 64\left((2\,\alpha_1\overline{\alpha_1} - \beta)\theta_0^3 + 3\,(2\,\alpha_1\overline{\alpha_1} - \beta)\theta_0^2 + 2\,(2\,\alpha_1\overline{\alpha_1} - \beta)\theta_0\right)A_0\overline{A_0}$$

$$- 64\left(\theta_0^3\overline{\alpha_5} + 3\,\theta_0^2\overline{\alpha_5} + 2\,\theta_0\overline{\alpha_5}\right)A_1\overline{A_0} - 64\left(\theta_0^3\overline{\alpha_1} + 3\,\theta_0^2\overline{\alpha_1} + 2\,\theta_0\overline{\alpha_1}\right)A_2\overline{A_0} - 64\left(\alpha_5\theta_0^3 + 2\,\alpha_5\theta_0^2\right)A_0\overline{A_1} + 8\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)\overline{A_0B_2}$$

$$- \left(\theta_0^3 + \theta_0^2 - 4\,\theta_0 - 4\right)C_1\overline{C_1} - 64\left(\left(\alpha_1\theta_0^3 + \alpha_1\theta_0^2\right)A_0 - \left(\theta_0^3 + \theta_0^2\right)A_2\right)\overline{A_2}\Big\}$$

$$\nu_6 = \frac{1}{48\left(\theta_0^5 + 6\,\theta_0^4 + 11\,\theta_0^3 + 6\,\theta_0^2\right)}\Big\{768\left(\theta_0^4 + 5\,\theta_0^3 + 6\,\theta_0^2\right)A_0|A_1|^4\overline{A_1} - 16\left(\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0\right)|A_1|^2\overline{A_0B_1}$$

$$- 96\left((2\,\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0^5 + 4\,(2\,\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0^4 - (2\,\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0^3 - 16\,(2\,\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0^2 - 12\,(2\,\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0\right)A_0^2$$

$$+ 48\left(\theta_0^3\overline{\alpha_1} - 4\,\theta_0\overline{\alpha_1}\right)A_0B_1 - 24\left(\theta_0^4 + \theta_0^3 - 4\,\theta_0^2 - 4\,\theta_0\right)A_0B_4 + 72\left(\theta_0^3\overline{\alpha_3} + \theta_0^2\overline{\alpha_3} - 4\,\theta_0\overline{\alpha_3} - 4\,\overline{\alpha_3}\right)A_0C_1$$

$$+ 768\left(\left(\theta_0^4\overline{\alpha_5} + 6\,\theta_0^3\overline{\alpha_5} + 11\,\theta_0^2\overline{\alpha_5} + 6\,\theta_0\overline{\alpha_5}\right)A_0\overline{A_0} + \left(\theta_0^4\overline{\alpha_1} + 6\,\theta_0^3\overline{\alpha_1} + 11\,\theta_0^2\overline{\alpha_1} + 6\,\theta_0\overline{\alpha_1}\right)A_1\overline{A_0}\right)|A_1|^2$$

$$+ 192\left((2\,\alpha_5\overline{\alpha_1} + 2\,\alpha_1\overline{\alpha_3} - \alpha_{16})\theta_0^4 + 6\,(2\,\alpha_5\overline{\alpha_1} + 2\,\alpha_1\overline{\alpha_3} - \alpha_{16})\theta_0^3 + 11\,(2\,\alpha_5\overline{\alpha_1} + 2\,\alpha_1\overline{\alpha_3} - \alpha_{16})\theta_0^2 + 6\,(2\,\alpha_5\overline{\alpha_1} + 2\,\alpha_1\overline{\alpha_3} - \alpha_{16})\theta_0\right)A_0\overline{A_0}$$

$$- 192\left(\theta_0^4\overline{\alpha_6} + 6\,\theta_0^3\overline{\alpha_6} + 11\,\theta_0^2\overline{\alpha_6} + 6\,\theta_0\overline{\alpha_6}\right)A_1\overline{A_0} + 24\left(\theta_0^4 + 5\,\theta_0^3 + 6\,\theta_0^2\right)\overline{A_1B_2} + 16\left(\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0\right)\overline{A_0B_5}$$

$$- 2\left(\theta_0^4 + 4\,\theta_0^3 - \theta_0^2 - 16\,\theta_0 - 12\right)C_1\overline{C_2} - 192\left(\left(\theta_0^4\overline{\alpha_3} + 6\,\theta_0^3\overline{\alpha_3} + 11\,\theta_0^2\overline{\alpha_3} + 6\,\theta_0\overline{\alpha_3}\right)\overline{A_0} + \left(\theta_0^4\overline{\alpha_1} + 5\,\theta_0^3\overline{\alpha_1} + 6\,\theta_0^2\overline{\alpha_1}\right)\overline{A_1}\right)A_2$$

$$+ 192\left(\left((2\,\alpha_1\overline{\alpha_1} - \beta)\theta_0^4 + 5\,(2\,\alpha_1\overline{\alpha_1} - \beta)\theta_0^3 + 6\,(2\,\alpha_1\overline{\alpha_1} - \beta)\theta_0^2\right)A_0 - \left(\theta_0^4\overline{\alpha_5} + 5\,\theta_0^3\overline{\alpha_5} + 6\,\theta_0^2\overline{\alpha_5}\right)A_1\right)\overline{A_1}$$

$$- 192\left(2\left(\theta_0^4 + 4\,\theta_0^3 + 3\,\theta_0^2\right)A_1|A_1|^2 + \left(\alpha_5\theta_0^4 + 4\,\alpha_5\theta_0^3 + 3\,\alpha_5\theta_0^2\right)A_0\right)\overline{A_2} - 192\left(\left(\alpha_1\theta_0^4 + 3\,\alpha_1\theta_0^3 + 2\,\alpha_1\theta_0^2\right)A_0 - \left(\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2\right)A_2\right)\overline{A_3}$$

$$+ \left(24\left(\alpha_2\theta_0^5 + 3\,\alpha_2\theta_0^4 - 2\,\alpha_2\theta_0^3 - 12\,\alpha_2\theta_0^2 - 8\,\alpha_2\theta_0\right)A_0 - 3\left(\theta_0^4 + 3\,\theta_0^3 - 4\,\theta_0^2 - 12\,\theta_0\right)B_1 - 8\left(\alpha_5\theta_0^4 + 6\,\alpha_5\theta_0^3 + 11\,\alpha_5\theta_0^2 + 6\,\alpha_5\theta_0\right)\overline{A_0}\right)\overline{C_1}\Big\}$$

$$\nu_7 = -\frac{\left(\theta_0^2 - 5\,\theta_0\right)A_1C_1 + 2\left(\theta_0^2 - 2\,\theta_0 - 3\right)A_0C_2 + 24\left(\alpha_3\theta_0^2 + \alpha_3\theta_0\right)A_0\overline{A_0} + 24\left(\alpha_1\theta_0^2 + \alpha_1\theta_0\right)A_1\overline{A_0} - 24\left(\theta_0^2 + \theta_0\right)A_3\overline{A_0}}{4\left(\theta_0^3 + \theta_0^2\right)}$$

$$\nu_8 = -\frac{1}{4\left(\theta_0^4 + 2\,\theta_0^3 + \theta_0^2\right)}\Big\{2\left(\theta_0^3 + 5\,\theta_0 + 6\right)A_0C_1|A_1|^2 - 96\left(\alpha_1\theta_0^3 + 2\,\alpha_1\theta_0^2 + \alpha_1\theta_0\right)A_0|A_1|^2\overline{A_0} + 48\left(\theta_0^3 + 2\,\theta_0^2 + \theta_0\right)A_2|A_1|^2\overline{A_0}$$

$$- 8 (\alpha_8\theta_0^4 - \alpha_8\theta_0^3 - 5\,\alpha_8\theta_0^2 - 3\,\alpha_8\theta_0) A_0^2 + (\theta_0^3 - 5\,\theta_0^2) A_1 B_1 + 2 (\theta_0^3 - 2\,\theta_0^2 - 3\,\theta_0) A_0 B_3 + 24 (\alpha_6\theta_0^3 + 2\,\alpha_6\theta_0^2 + \alpha_6\theta_0) A_0 \overline{A_0}$$

$$- 24 (\theta_0^3 + \theta_0^2) A_3 \overline{A_1} - 8 ((2\,\alpha_2\theta_0^4 - 2\,\alpha_2\theta_0^3 - 7\,\alpha_2\theta_0^2 - 3\,\alpha_2\theta_0) A_0 - 3 (\alpha_5\theta_0^3 + 2\,\alpha_5\theta_0^2 + \alpha_5\theta_0) \overline{A_0}) A_1$$

$$+ 24 ((\alpha_3\theta_0^3 + \alpha_3\theta_0^2) A_0 + (\alpha_1\theta_0^3 + \alpha_1\theta_0^2) A_1) \overline{A_1} \Big\}$$

$$\nu_9 = \frac{1}{32 (\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2)} \Big\{ 768 (\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0) A_1 |A_1|^4 \overline{A_0} - 16 (\theta_0^3 + 6\,\theta_0) A_0 B_1 |A_1|^2 + 64 (\alpha_{11}\theta_0^4 - 7\,\alpha_{11}\theta_0^2 - 6\,\alpha_{11}\theta_0) A_0^2$$

$$- 8 (\theta_0^3 - 5\,\theta_0^2) A_1 B_2 - 16 (\theta_0^3 - 2\,\theta_0^2 - 3\,\theta_0) A_0 B_5 - 256 \Big\{ (\alpha_2\theta_0^4 - 7\,\alpha_2\theta_0^2 - 6\,\alpha_2\theta_0) A_0^2 - 3 (\alpha_5\theta_0^3 + 3\,\alpha_5\theta_0^2 + 2\,\alpha_5\theta_0) A_0 \overline{A_0}$$

$$- 3 (\alpha_1\theta_0^3 + 2\,\alpha_1\theta_0^2) A_0 \overline{A_1} \Big\} |A_1|^2 + 192 \Big\{ (2\,\alpha_3\overline{\alpha_1} + 2\,\alpha_1\overline{\alpha_5} - \overline{\alpha_{16}})\theta_0^3 + 3\,(2\,\alpha_3\overline{\alpha_1} + 2\,\alpha_1\overline{\alpha_5} - \overline{\alpha_{16}})\theta_0^2$$

$$+ 2\,(2\,\alpha_3\overline{\alpha_1} + 2\,\alpha_1\overline{\alpha_5} - \overline{\alpha_{16}})\theta_0 \Big\} A_0 \overline{A_0} + 24 (\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0) \overline{A_0 B_4}$$

$$+ 64 ((2\,\alpha_9\theta_0^4 - 11\,\alpha_9\theta_0^2 - 6\,\alpha_9\theta_0) A_0 + 3 ((2\,\alpha_1\overline{\alpha_1} - \beta)\theta_0^3 + 3\,(2\,\alpha_1\overline{\alpha_1} - \beta)\theta_0^2 + 2\,(2\,\alpha_1\overline{\alpha_1} - \beta)\theta_0) \overline{A_0}) A_1$$

$$- 192 (2 (\theta_0^3 + 2\,\theta_0^2) |A_1|^2 \overline{A_1} + (\theta_0^3\overline{\alpha_5} + 3\,\theta_0^2\overline{\alpha_5} + 2\,\theta_0\overline{\alpha_5}) \overline{A_0}) A_2 - 192 ((\theta_0^3\overline{\alpha_1} + 3\,\theta_0^2\overline{\alpha_1} + 2\,\theta_0\overline{\alpha_1}) \overline{A_0} - (\theta_0^3 + \theta_0^2) \overline{A_2}) A_3$$

$$- (8 (\theta_0^3\overline{\alpha_5} - \theta_0^2\overline{\alpha_5} + 10\,\theta_0\overline{\alpha_5} + 12\,\overline{\alpha_5}) A_0 - 16 (\theta_0^2\overline{\alpha_1} - 5\,\theta_0\overline{\alpha_1}) A_1 - 24 (\theta_0^3\overline{\alpha_2} + \theta_0^2\overline{\alpha_2} - 4\,\theta_0\overline{\alpha_2} - 4\,\overline{\alpha_2}) \overline{A_0} + (\theta_0^3 - 3\,\theta_0^2 - 10\,\theta_0) \overline{B_1}) C_1$$

$$+ 2 (16 (\theta_0^2\overline{\alpha_1} - 2\,\theta_0\overline{\alpha_1} - 3\,\overline{\alpha_1}) A_0 - (\theta_0^3 - 7\,\theta_0 - 6) \overline{C_1}) C_2 - 192 ((\alpha_6\theta_0^3 + 2\,\alpha_6\theta_0^2) A_0 + (\alpha_5\theta_0^3 + 2\,\alpha_5\theta_0^2) A_1) \overline{A_1}$$

$$- 192 ((\alpha_3\theta_0^3 + \alpha_3\theta_0^2) A_0 + (\alpha_1\theta_0^3 + \alpha_1\theta_0^2) A_1) \overline{A_2} \Big\}$$

$$\nu_{10} = \frac{1}{6 (\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2)} \Big\{ 12 (\alpha_7\theta_0^4 - \alpha_7\theta_0^3 - 10\,\alpha_7\theta_0^2 - 8\,\alpha_7\theta_0) A_0^2 - 2 (\theta_0^3 - 3\,\theta_0^2 - 10\,\theta_0) A_1 C_2 - 3 (\theta_0^3 - \theta_0^2 - 10\,\theta_0 - 8) A_0 C_3$$

$$+ 48 ((\alpha_1^2 - \alpha_4)\theta_0^3 + 3 (\alpha_1^2 - \alpha_4)\theta_0^2 + 2 (\alpha_1^2 - \alpha_4)\theta_0) A_0 \overline{A_0} - 48 (\alpha_3\theta_0^3 + 3\,\alpha_3\theta_0^2 + 2\,\alpha_3\theta_0) A_1 \overline{A_0} - 48 (\alpha_1\theta_0^3 + 3\,\alpha_1\theta_0^2 + 2\,\alpha_1\theta_0) A_2 \overline{A_0}$$

$$+ 48 (\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0) A_4 \overline{A_0} - 3 ((\alpha_1\theta_0^3 + 3\,\alpha_1\theta_0^2 + 2\,\alpha_1\theta_0) A_0 - 4 (\theta_0^2 + \theta_0) A_2) C_1 \Big\}$$

$$\nu_{11} = -\frac{1}{6 (\theta_0^5 + 4\,\theta_0^4 + 5\,\theta_0^3 + 2\,\theta_0^2)} \Big\{ 2 (\theta_0^4 + \theta_0^3 + 8\,\theta_0^2 + 32\,\theta_0 + 24) A_0 C_2 |A_1|^2 + 96 (\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0) A_3 |A_1|^2 \overline{A_0}$$

$$+ 12 ((2\,\alpha_1\alpha_2 - \alpha_{12})\theta_0^5 - 11\,(2\,\alpha_1\alpha_2 - \alpha_{12})\theta_0^3 - 18\,(2\,\alpha_1\alpha_2 - \alpha_{12})\theta_0^2 - 8\,(2\,\alpha_1\alpha_2 - \alpha_{12})\theta_0) A_0^2 - 12 (\alpha_2\theta_0^5 - 7\,\alpha_2\theta_0^3 - 6\,\alpha_2\theta_0^2) A_1^2$$

$$+ 2 (\theta_0^4 - 3\,\theta_0^3 - 10\,\theta_0^2) A_1 B_3 + 3 (\theta_0^4 - \theta_0^3 - 10\,\theta_0^2 - 8\,\theta_0) A_0 B_6 - 192 \Big\{ (\alpha_3\theta_0^4 + 4\,\alpha_3\theta_0^3 + 5\,\alpha_3\theta_0^2 + 2\,\alpha_3\theta_0) A_0 \overline{A_0}$$

$$+ (\alpha_1\theta_0^4 + 4\,\alpha_1\theta_0^3 + 5\,\alpha_1\theta_0^2 + 2\,\alpha_1\theta_0) A_1 \overline{A_0} \Big\} |A_1|^2$$

$$- 48 ((2\,\alpha_1\alpha_5 - \overline{\alpha_{15}})\theta_0^4 + 4\,(2\,\alpha_1\alpha_5 - \overline{\alpha_{15}})\theta_0^3 + 5\,(2\,\alpha_1\alpha_5 - \overline{\alpha_{15}})\theta_0^2 + 2\,(2\,\alpha_1\alpha_5 - \overline{\alpha_{15}})\theta_0) A_0 \overline{A_0}$$

$$- 48 (\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2) A_4 \overline{A_1} - 24 ((\alpha_8\theta_0^5 - 9\,\alpha_8\theta_0^3 - 12\,\alpha_8\theta_0^2 - 4\,\alpha_8\theta_0) A_0 - 2 (\alpha_6\theta_0^4 + 4\,\alpha_6\theta_0^3 + 5\,\alpha_6\theta_0^2 + 2\,\alpha_6\theta_0) \overline{A_0}) A_1$$

$$- 24 ((\alpha_2\theta_0^5 - 7\,\alpha_2\theta_0^3 - 10\,\alpha_2\theta_0^2 - 4\,\alpha_2\theta_0) A_0 - 2 (\alpha_5\theta_0^4 + 4\,\alpha_5\theta_0^3 + 5\,\alpha_5\theta_0^2 + 2\,\alpha_5\theta_0) \overline{A_0} - 2 (\alpha_1\theta_0^4 + 3\,\alpha_1\theta_0^3 + 2\,\alpha_1\theta_0^2) \overline{A_1}) A_2$$

$$+ 3 ((\alpha_1\theta_0^4 + 3\,\alpha_1\theta_0^3 + 2\,\alpha_1\theta_0^2) A_0 - 4 (\theta_0^3 + \theta_0^2) A_2) B_1$$

$$+ (4 (\theta_0^4 + 3\,\theta_0^3 + 8\,\theta_0^2 + 12\,\theta_0) A_1 |A_1|^2 + 3 (\alpha_5\theta_0^4 + 3\,\alpha_5\theta_0^3 + 6\,\alpha_5\theta_0^2 + 12\,\alpha_5\theta_0 + 8\,\alpha_5) A_0) C_1$$

$$- 48 (((\alpha_1^2 - \alpha_4)\theta_0^4 + 3 (\alpha_1^2 - \alpha_4)\theta_0^3 + 2 (\alpha_1^2 - \alpha_4)\theta_0^2) A_0 - (\alpha_3\theta_0^4 + 3\,\alpha_3\theta_0^3 + 2\,\alpha_3\theta_0^2) A_1) \overline{A_1} \Big\}$$

$$\nu_{12} = \frac{1}{24 (\theta_0^5 + 6\,\theta_0^4 + 11\,\theta_0^3 + 6\,\theta_0^2)} \Big\{ 48 (\alpha_{13}\theta_0^5 + \alpha_{13}\theta_0^4 - 19\,\alpha_{13}\theta_0^3 - 49\,\alpha_{13}\theta_0^2 - 30\,\alpha_{13}\theta_0) A_0^2 - 3 (3\,\theta_0^4 - 2\,\theta_0^3 - 67\,\theta_0^2 - 102\,\theta_0) A_1 C_3$$

$$- 12 (\theta_0^4 + \theta_0^3 - 19\,\theta_0^2 - 49\,\theta_0 - 30) A_0 C_4 + 240 \Big\{ (2\,\alpha_1\alpha_3 - \overline{\alpha_{14}})\theta_0^4 + 6\,(2\,\alpha_1\alpha_3 - \overline{\alpha_{14}})\theta_0^3 + 11\,(2\,\alpha_1\alpha_3 - \overline{\alpha_{14}})\theta_0^2$$

$$+ 6\,(2\,\alpha_1\alpha_3 - \overline{\alpha_{14}})\theta_0 \Big\} A_0 \overline{A_0} - 240 (\alpha_3\theta_0^4 + 6\,\alpha_3\theta_0^3 + 11\,\alpha_3\theta_0^2 + 6\,\alpha_3\theta_0) A_2 \overline{A_0} - 240 (\alpha_1\theta_0^4 + 6\,\alpha_1\theta_0^3 + 11\,\alpha_1\theta_0^2 + 6\,\alpha_1\theta_0) A_3 \overline{A_0}$$

$$+ 240 (\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0) A_5 \overline{A_0} + 48 \Big\{ (2\,\alpha_7\theta_0^5 + 2\,\alpha_7\theta_0^4 - 33\,\alpha_7\theta_0^3 - 73\,\alpha_7\theta_0^2 - 30\,\alpha_7\theta_0) A_0$$

$$+ 5 ((\alpha_1^2 - \alpha_4)\theta_0^4 + 6 (\alpha_1^2 - \alpha_4)\theta_0^3 + 11 (\alpha_1^2 - \alpha_4)\theta_0^2 + 6 (\alpha_1^2 - \alpha_4)\theta_0) \overline{A_0} \Big\} A_1$$

$$- 3 (6 (\alpha_3\theta_0^4 + 6\,\alpha_3\theta_0^3 + 11\,\alpha_3\theta_0^2 + 6\,\alpha_3\theta_0) A_0 + 5 (\alpha_1\theta_0^4 + 6\,\alpha_1\theta_0^3 + 11\,\alpha_1\theta_0^2 + 6\,\alpha_1\theta_0) A_1 - 2 (\theta_0^4 + 16\,\theta_0^3 + 41\,\theta_0^2 + 26\,\theta_0) A_3) C_1$$

$$- 4 (2 (\alpha_1\theta_0^4 + 6\,\alpha_1\theta_0^3 + 11\,\alpha_1\theta_0^2 + 6\,\alpha_1\theta_0) A_0 + (\theta_0^4 - 9\,\theta_0^3 - 49\,\theta_0^2 - 39\,\theta_0) A_2) C_2 \Big\}$$

$$\nu_{13} = \frac{1}{2\left(\theta_0^5 + 4\,\theta_0^4 + 5\,\theta_0^3 + 2\,\theta_0^2\right)}\Big\{4\left(\alpha_9\theta_0^5 + 2\,\alpha_9\theta_0^4 - 3\,\alpha_9\theta_0^3 - 8\,\alpha_9\theta_0^2 - 4\,\alpha_9\theta_0\right)A_0\overline{A_0} - \left(\theta_0^4 - 3\,\theta_0^2 - 2\,\theta_0\right)B_2\overline{A_0}$$
$$+ 4\left(\alpha_2\theta_0^5 + \alpha_2\theta_0^4 - 4\,\alpha_2\theta_0^3 - 4\,\alpha_2\theta_0^2\right)A_0\overline{A_1} - \left(\theta_0^4 - 4\,\theta_0^2\right)B_1\overline{A_1} + \left(2\left(\theta_0^3\overline{\alpha_1} - 3\,\theta_0\overline{\alpha_1} - 2\,\overline{\alpha_1}\right)\overline{A_0} - \left(\theta_0^4 - 3\,\theta_0^2 - 2\,\theta_0\right)\overline{A_2}\right)C_1\Big\}$$
$$\nu_{14} = -\frac{1}{4\left(\theta_0^5 + 6\,\theta_0^4 + 11\,\theta_0^3 + 6\,\theta_0^2\right)}\Big\{8\Big\{(2\,\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0^5 + 4\,(2\,\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0^4 - (2\,\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0^3 - 16\,(2\,\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0^2$$
$$- 12\,(2\,\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0\Big\}A_0\overline{A_0} + 2\left(\theta_0^4 + \theta_0^3 - 4\,\theta_0^2 - 4\,\theta_0\right)B_4\overline{A_0} - 8\left(\alpha_9\theta_0^5 + 3\,\alpha_9\theta_0^4 - 4\,\alpha_9\theta_0^3 - 12\,\alpha_9\theta_0^2\right)A_0\overline{A_1}$$
$$+ 2\left(\theta_0^4 + \theta_0^3 - 6\,\theta_0^2\right)B_2\overline{A_1} - 8\left(\alpha_2\theta_0^5 + 2\,\alpha_2\theta_0^4 - 5\,\alpha_2\theta_0^3 - 6\,\alpha_2\theta_0^2\right)A_0\overline{A_2} - \left(\alpha_2\theta_0^5 + 2\,\alpha_2\theta_0^4 - 3\,\alpha_2\theta_0^3 - 8\,\alpha_2\theta_0^2 - 4\,\alpha_2\theta_0\right)\overline{A_0C_1}$$
$$- 2\left(2\left(\theta_0^3\overline{\alpha_1} - 4\,\theta_0\overline{\alpha_1}\right)\overline{A_0} - \left(\theta_0^4 + \theta_0^3 - 6\,\theta_0^2\right)\overline{A_2}\right)B_1 - 2\Big\{3\left(\theta_0^3\overline{\alpha_3} + \theta_0^2\overline{\alpha_3} - 4\,\theta_0\overline{\alpha_3} - 4\,\overline{\alpha_3}\right)\overline{A_0} + 2\left(\theta_0^3\overline{\alpha_1} + \theta_0^2\overline{\alpha_1} - 6\,\theta_0\overline{\alpha_1}\right)\overline{A_1}$$
$$- \left(\theta_0^4 + \theta_0^3 - 4\,\theta_0^2 - 4\,\theta_0\right)\overline{A_3}\Big\}C_1\Big\}$$
$$\nu_{15} = \frac{1}{2\left(\theta_0^3 + \theta_0^2\right)}\Big\{2\,C_1(\theta_0 - 3)|A_1|^2\overline{A_0} + 4\left(\alpha_8\theta_0^3 - 2\,\alpha_8\theta_0^2 - 3\,\alpha_8\theta_0\right)A_0\overline{A_0} + 4\left(\alpha_2\theta_0^3 - 2\,\alpha_2\theta_0^2 - 3\,\alpha_2\theta_0\right)A_1\overline{A_0}$$
$$- \left(\theta_0^2 - 3\,\theta_0\right)B_3\overline{A_0} - \left(\theta_0^2 - 3\,\theta_0\right)C_2\overline{A_1}\Big\}$$
$$\nu_{16} = -\frac{1}{2\left(\theta_0^5 + 4\,\theta_0^4 + 5\,\theta_0^3 + 2\,\theta_0^2\right)}\Big\{16\left(\alpha_2\theta_0^5 + \alpha_2\theta_0^4 - 7\,\alpha_2\theta_0^3 - 13\,\alpha_2\theta_0^2 - 6\,\alpha_2\theta_0\right)A_0|A_1|^2\overline{A_0} - 2\left(\theta_0^3 - 2\,\theta_0^2 - 3\,\theta_0\right)B_1|A_1|^2\overline{A_0}$$
$$- 4\left(\alpha_{11}\theta_0^5 + \alpha_{11}\theta_0^4 - 7\,\alpha_{11}\theta_0^3 - 13\,\alpha_{11}\theta_0^2 - 6\,\alpha_{11}\theta_0\right)A_0\overline{A_0} - 4\left(\alpha_9\theta_0^5 + \alpha_9\theta_0^4 - 7\,\alpha_9\theta_0^3 - 13\,\alpha_9\theta_0^2 - 6\,\alpha_9\theta_0\right)A_1\overline{A_0}$$
$$+ \left(\theta_0^4 - \theta_0^3 - 5\,\theta_0^2 - 3\,\theta_0\right)B_5\overline{A_0} + \left(\theta_0^4 - \theta_0^3 - 6\,\theta_0^2\right)B_3\overline{A_1} - 2\left(\left(\theta_0^3 - \theta_0^2 - 6\,\theta_0\right)|A_1|^2\overline{A_1} + \left(\theta_0^3\overline{\alpha_5} - \theta_0^2\overline{\alpha_5} - 5\,\theta_0\overline{\alpha_5} - 3\,\overline{\alpha_5}\right)\overline{A_0}\right)C_1$$
$$- \left(2\left(\theta_0^3\overline{\alpha_1} - \theta_0^2\overline{\alpha_1} - 5\,\theta_0\overline{\alpha_1} - 3\,\overline{\alpha_1}\right)\overline{A_0} - \left(\theta_0^4 - \theta_0^3 - 5\,\theta_0^2 - 3\,\theta_0\right)\overline{A_2}\right)C_2$$
$$- 4\left(\left(\alpha_8\theta_0^5 - 7\,\alpha_8\theta_0^3 - 6\,\alpha_8\theta_0^2\right)A_0 + \left(\alpha_2\theta_0^5 - 7\,\alpha_2\theta_0^3 - 6\,\alpha_2\theta_0^2\right)A_1\right)\overline{A_1}\Big\}$$
$$\nu_{17} = \frac{1}{8\left(2\,\theta_0^4 + 3\,\theta_0^3 + \theta_0^2\right)}\Big\{8\left(2\,\theta_0^2 - 7\,\theta_0 - 4\right)C_2|A_1|^2\overline{A_0} - 8\left(\theta_0^3 - \theta_0^2 - 2\,\theta_0\right)A_0E_2$$
$$- 16\left(2\,(2\,\alpha_1\alpha_2 - \alpha_{12})\theta_0^4 - 5\,(2\,\alpha_1\alpha_2 - \alpha_{12})\theta_0^3 - 11\,(2\,\alpha_1\alpha_2 - \alpha_{12})\theta_0^2 - 4\,(2\,\alpha_1\alpha_2 - \alpha_{12})\theta_0\right)A_0\overline{A_0}$$
$$+ 16\left(2\,\alpha_8\theta_0^4 - 5\,\alpha_8\theta_0^3 - 11\,\alpha_8\theta_0^2 - 4\,\alpha_8\theta_0\right)A_1\overline{A_0} + 16\left(2\,\alpha_2\theta_0^4 - 5\,\alpha_2\theta_0^3 - 11\,\alpha_2\theta_0^2 - 4\,\alpha_2\theta_0\right)A_2\overline{A_0} - 4\left(2\,\theta_0^3 - 7\,\theta_0^2 - 4\,\theta_0\right)B_6\overline{A_0}$$
$$+ 16\left(2\,\alpha_7\theta_0^4 - 7\,\alpha_7\theta_0^3 - 4\,\alpha_7\theta_0^2\right)A_0\overline{A_1} - 4\left(2\,\theta_0^3 - 7\,\theta_0^2 - 4\,\theta_0\right)C_3\overline{A_1}$$
$$+ \left(2\left(2\,\alpha_2\theta_0^4 + 3\,\alpha_2\theta_0^3 - 5\,\alpha_2\theta_0^2 - 14\,\alpha_2\theta_0 - 8\,\alpha_2\right)A_0 - \left(2\,\theta_0^3 - 3\,\theta_0^2 - 2\,\theta_0\right)B_1 + 4\left(2\,\alpha_5\theta_0^2 - 7\,\alpha_5\theta_0 - 4\,\alpha_5\right)\overline{A_0}\right)C_1\Big\}$$
$$\nu_{18} = -\frac{1}{48\left(\theta_0^4 - 3\,\theta_0^3 - 4\,\theta_0^2\right)}\Big\{\left(5\,\theta_0^3 - 28\,\theta_0^2 + 19\,\theta_0 + 52\right)C_1C_2 + 12\left(2\,\theta_0^3 - 11\,\theta_0^2 + 17\,\theta_0\right)A_1E_1 + 12\left(2\,\theta_0^3 - 11\,\theta_0^2 + 7\,\theta_0 + 20\right)A_0E_3$$
$$- 96\left(\alpha_{13}\theta_0^4 - 8\,\alpha_{13}\theta_0^3 + 11\,\alpha_{13}\theta_0^2 + 20\,\alpha_{13}\theta_0\right)A_0\overline{A_0} - 96\left(\alpha_7\theta_0^4 - 8\,\alpha_7\theta_0^3 + 11\,\alpha_7\theta_0^2 + 20\,\alpha_7\theta_0\right)A_1\overline{A_0} + 24\left(\theta_0^3 - 8\,\theta_0^2 + 11\,\theta_0 + 20\right)C_4\overline{A_0}\Big\}$$
$$\nu_{19} = \frac{48\,A_0^2|A_1|^2\overline{\alpha_1} - 4\,A_0|A_1|^2\overline{C_1} - 12\,A_0\theta_0\overline{A_1}\overline{\alpha_2} - 12\,(\theta_0\overline{\alpha_8} + \overline{\alpha_8})A_0\overline{A_0} - 12\,A_0A_1\overline{\alpha_3} - 12\,A_0^2\overline{\alpha_6} + A_0\overline{B_3} + A_1\overline{C_2}}{6\,\theta_0}$$
$$\nu_{20} = \frac{1}{96\left(\theta_0^3 - 2\,\theta_0^2 - 8\,\theta_0\right)}\Big\{768\left(\theta_0^2\overline{\alpha_3} - 2\,\theta_0\overline{\alpha_3} - 8\,\overline{\alpha_3}\right)A_0^2|A_1|^2 - 40\left(\theta_0^2 - 2\,\theta_0 - 8\right)A_0|A_1|^2\overline{C_2}$$
$$+ 192\left((2\,\overline{\alpha_1\alpha_5} - \alpha_{15})\theta_0^2 - 2\,(2\,\overline{\alpha_1\alpha_5} - \alpha_{15})\theta_0 - 16\,\overline{\alpha_1\alpha_5} + 8\,\alpha_{15}\right)A_0^2$$
$$+ 192\left((2\,\overline{\alpha_1\alpha_2} - \overline{\alpha_{12}})\theta_0^3 - (2\,\overline{\alpha_1\alpha_2} - \overline{\alpha_{12}})\theta_0^2 - 10\,(2\,\overline{\alpha_1\alpha_2} - \overline{\alpha_{12}})\theta_0 - 16\,\overline{\alpha_1\alpha_2} + 8\,\overline{\alpha_{12}}\right)A_0\overline{A_0}$$
$$- 192\left(\theta_0^3\overline{\alpha_8} - 2\,\theta_0^2\overline{\alpha_8} - 8\,\theta_0\overline{\alpha_8}\right)A_0\overline{A_1} - 192\left(\theta_0^3\overline{\alpha_2} - 3\,\theta_0^2\overline{\alpha_2} - 4\,\theta_0\overline{\alpha_2}\right)A_0\overline{A_2} - 12\left(\theta_0^2\overline{\alpha_1} - 2\,\theta_0\overline{\alpha_1} - 8\,\overline{\alpha_1}\right)A_0\overline{B_1}$$
$$+ 12\left(\theta_0^2 - 2\,\theta_0 - 8\right)A_0\overline{B_6} + 12\left(\theta_0^2 - 2\,\theta_0 - 8\right)A_1\overline{C_3} - 48\left(\theta_0^2 + 3\,\theta_0 + 2\right)\overline{A_0E_2}$$
$$+ 192\left(\left(\left(\overline{\alpha_1}^2 - \overline{\alpha_4}\right)\theta_0^2 - 2\left(\overline{\alpha_1}^2 - \overline{\alpha_4}\right)\theta_0 - 8\,\overline{\alpha_1}^2 + 8\,\overline{\alpha_4}\right)A_0 - \left(\theta_0^3\overline{\alpha_7} - \theta_0^2\overline{\alpha_7} - 10\,\theta_0\overline{\alpha_7} - 8\,\overline{\alpha_7}\right)\overline{A_0}\right)A_1$$
$$- 3\left(12\left(\theta_0^2\overline{\alpha_5} - 2\,\theta_0\overline{\alpha_5} - 8\,\overline{\alpha_5}\right)A_0 + 4\left(\theta_0^2\overline{\alpha_1} - 2\,\theta_0\overline{\alpha_1} - 8\,\overline{\alpha_1}\right)A_1 + 8\left(\theta_0^3\overline{\alpha_2} + \theta_0^2\overline{\alpha_2} - 4\,\theta_0\overline{\alpha_2} - 4\,\overline{\alpha_2}\right)\overline{A_0} - \left(\theta_0^2 - 2\,\theta_0 - 8\right)\overline{B_1}\right)\overline{C_1}\Big\}$$
$$\nu_{21} = -\frac{1}{2\left(\theta_0^3 - 8\,\theta_0^2 + 11\,\theta_0 + 20\right)}\Big\{4\left(\theta_0^3\overline{\alpha_{13}} - 8\,\theta_0^2\overline{\alpha_{13}} + 11\,\theta_0\overline{\alpha_{13}} + 20\,\overline{\alpha_{13}}\right)A_0\overline{A_0} + 4\left(\theta_0^3\overline{\alpha_7} - 9\,\theta_0^2\overline{\alpha_7} + 20\,\theta_0\overline{\alpha_7}\right)A_0\overline{A_1}$$
$$+ \left(\theta_0^2 - 5\,\theta_0\right)\overline{A_1E_1} + \left(\theta_0^2 - 3\,\theta_0 - 4\right)\overline{A_0E_3}\Big\}$$

$$\nu_{22} = \frac{1}{2\left(\theta_0^2+\theta_0\right)}\Big\{32\,A_0^2(\theta_0+1)|A_1|^4 - 4\,A_1^2\theta_0\overline{\alpha_1} + 8\left((2\,\alpha_1\overline{\alpha_1}-\beta)\theta_0 + 2\,\alpha_1\overline{\alpha_1}-\beta\right)A_0^2 - 8\left(\theta_0\overline{\alpha_1}+\overline{\alpha_1}\right)A_0A_2$$

$$-4\left(\theta_0^2\overline{\alpha_9}+3\,\theta_0\overline{\alpha_9}+2\,\overline{\alpha_9}\right)A_0\overline{A_0} + A_1\theta_0\overline{B_1} + A_0(\theta_0+1)\overline{B_2}$$

$$-4\left((3\,\theta_0\overline{\alpha_5}+2\,\overline{\alpha_5})A_0 + \left(\theta_0^2\overline{\alpha_2}+3\,\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2}\right)\overline{A_0}\right)A_1 - \left((\alpha_1\theta_0+\alpha_1)A_0 - A_2(\theta_0+1)\right)\overline{C_1}\Big\}$$

$$\nu_{23} = -\frac{1}{6\left(\theta_0^2+\theta_0\right)}\Big\{A_0(5\,\theta_0+2)|A_1|^2\overline{B_1} + 12\,A_1^2\theta_0\overline{\alpha_3} - 24\left((2\,\alpha_5\overline{\alpha_1}+2\,\alpha_1\overline{\alpha_3}-\alpha_{16})\theta_0 + 2\,\alpha_5\overline{\alpha_1}+2\,\alpha_1\overline{\alpha_3}-\alpha_{16}\right)A_0^2$$

$$+24\left(\theta_0\overline{\alpha_3}+\overline{\alpha_3}\right)A_0A_2 - 48\left(2\left(\theta_0\overline{\alpha_5}+\overline{\alpha_5}\right)A_0^2 + \left(3\,\theta_0\overline{\alpha_1}+2\,\overline{\alpha_1}\right)A_0A_1 + \left(\theta_0^2\overline{\alpha_2}+3\,\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2}\right)A_0\overline{A_0}\right)|A_1|^2$$

$$+12\left(\theta_0^2\overline{\alpha_{11}}+3\,\theta_0\overline{\alpha_{11}}+2\,\overline{\alpha_{11}}\right)A_0\overline{A_0} - 2\,A_1\theta_0\overline{B_3} - 2\,A_0(\theta_0+1)\overline{B_5} + 12\left((3\,\theta_0\overline{\alpha_6}+2\,\overline{\alpha_6})A_0 + \left(\theta_0^2\overline{\alpha_8}+3\,\theta_0\overline{\alpha_8}+2\,\overline{\alpha_8}\right)\overline{A_0}\right)A_1$$

$$+12\left(\left(\theta_0^2\overline{\alpha_9}+2\,\theta_0\overline{\alpha_9}\right)A_0 + \left(\theta_0^2\overline{\alpha_2}+2\,\theta_0\overline{\alpha_2}\right)A_1\right)\overline{A_1} + \left(A_1(5\,\theta_0+6)|A_1|^2 + 4\left(\alpha_5\theta_0+\alpha_5\right)A_0\right)\overline{C_1} + 2\left((\alpha_1\theta_0+\alpha_1)A_0 - A_2(\theta_0+1)\right)\overline{C_2}\Big\}$$

$$\nu_{24} = -\frac{2\left(3\left(\alpha_3\theta_0^2+3\,\alpha_3\theta_0+2\,\alpha_3\right)A_0^2 + \left(5\,\alpha_1\theta_0^2+13\,\alpha_1\theta_0+6\,\alpha_1\right)A_0A_1 - \left(3\,\theta_0^2+5\,\theta_0\right)A_1A_2 - 3\left(\theta_0^2+3\,\theta_0+2\right)A_0A_3\right)}{\theta_0^3+3\,\theta_0^2+2\,\theta_0}$$

$$\nu_{25} = -\frac{2}{\theta_0^3+3\,\theta_0^2+2\,\theta_0}\Big\{4\left(2\,\theta_0^2+5\,\theta_0+3\right)A_0A_2|A_1|^2 + 3\left(\alpha_6\theta_0^2+3\,\alpha_6\theta_0+2\,\alpha_6\right)A_0^2 + \left(5\,\alpha_5\theta_0^2+13\,\alpha_5\theta_0+6\,\alpha_5\right)A_0A_1$$

$$-4\left(3\left(\alpha_1\theta_0^2+3\,\alpha_1\theta_0+2\,\alpha_1\right)A_0^2 - \left(\theta_0^2+2\,\theta_0\right)A_1^2\right)|A_1|^2\Big\}$$

$$\nu_{26} = \frac{1}{4\left(\theta_0^4+3\,\theta_0^3+2\,\theta_0^2\right)}\Big\{32\left(5\,\theta_0^3+13\,\theta_0^2+6\,\theta_0\right)A_0A_1|A_1|^4 + 96\left(\alpha_5\theta_0^3+3\,\alpha_5\theta_0^2+2\,\alpha_5\theta_0\right)A_0^2|A_1|^2$$

$$+24\left((2\,\alpha_3\overline{\alpha_1}+2\,\alpha_1\overline{\alpha_5}-\overline{\alpha_{16}})\theta_0^3 + 3\,(2\,\alpha_3\overline{\alpha_1}+2\,\alpha_1\overline{\alpha_5}-\overline{\alpha_{16}})\theta_0^2 + 2\,(2\,\alpha_3\overline{\alpha_1}+2\,\alpha_1\overline{\alpha_5}-\overline{\alpha_{16}})\theta_0\right)A_0^2 - 16\left(\theta_0^3\overline{\alpha_5}+2\,\theta_0^2\overline{\alpha_5}\right)A_1^2$$

$$-24\left(\theta_0^3\overline{\alpha_1}+3\,\theta_0^2\overline{\alpha_1}+2\,\theta_0\overline{\alpha_1}\right)A_0A_3 - \left(\theta_0^4\overline{\alpha_2}+\theta_0^3\overline{\alpha_2}+2\,\theta_0^2\overline{\alpha_2}+14\,\theta_0\overline{\alpha_2}+12\,\overline{\alpha_2}\right)A_0C_1$$

$$+8\left((2\,\alpha_1\overline{\alpha_2}-\overline{\alpha_{10}})\theta_0^4 + 6\,(2\,\alpha_1\overline{\alpha_2}-\overline{\alpha_{10}})\theta_0^3 + 11\,(2\,\alpha_1\overline{\alpha_2}-\overline{\alpha_{10}})\theta_0^2 + 6\,(2\,\alpha_1\overline{\alpha_2}-\overline{\alpha_{10}})\theta_0\right)A_0\overline{A_0}$$

$$+\left(3\,\theta_0^3+5\,\theta_0^2\right)A_1\overline{B_2} + 3\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)A_0\overline{B_4} + 8\Big\{\left(5\,(2\,\alpha_1\overline{\alpha_1}-\beta)\theta_0^3 + 13\,(2\,\alpha_1\overline{\alpha_1}-\beta)\theta_0^2 + 6\,(2\,\alpha_1\overline{\alpha_1}-\beta)\theta_0\right)A_0$$

$$-\left(\theta_0^4\overline{\alpha_9}+6\,\theta_0^3\overline{\alpha_9}+11\,\theta_0^2\overline{\alpha_9}+6\,\theta_0\overline{\alpha_9}\right)\overline{A_0}\Big\}A_1 - 8\Big\{2\left(2\,\theta_0^3\overline{\alpha_5}+5\,\theta_0^2\overline{\alpha_5}+3\,\theta_0\overline{\alpha_5}\right)A_0 + \left(3\,\theta_0^3\overline{\alpha_1}+5\,\theta_0^2\overline{\alpha_1}\right)A_1$$

$$+\left(\theta_0^4\overline{\alpha_2}+6\,\theta_0^3\overline{\alpha_2}+11\,\theta_0^2\overline{\alpha_2}+6\,\theta_0\overline{\alpha_2}\right)\overline{A_0}\Big\}A_2 - \left(2\left(\alpha_1\theta_0^3+2\,\alpha_1\theta_0^2\right)A_0 - \left(3\,\theta_0^3+5\,\theta_0^2\right)A_2\right)\overline{B_1}$$

$$-\left(3\left(\alpha_3\theta_0^3+3\,\alpha_3\theta_0^2+2\,\alpha_3\theta_0\right)A_0 + 2\left(\alpha_1\theta_0^3+4\,\alpha_1\theta_0^2+3\,\alpha_1\theta_0\right)A_1 - 3\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)A_3\right)\overline{C_1}\Big\}$$

$$\nu_{27} = \frac{2}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0}\Big\{4\left(\left(\alpha_1^2-\alpha_4\right)\theta_0^3 + 6\left(\alpha_1^2-\alpha_4\right)\theta_0^2 + 6\,\alpha_1^2 + 11\left(\alpha_1^2-\alpha_4\right)\theta_0 - 6\,\alpha_4\right)A_0^2$$

$$-\left(7\,\alpha_3\theta_0^3+39\,\alpha_3\theta_0^2+62\,\alpha_3\theta_0+24\,\alpha_3\right)A_0A_1 - 3\left(\alpha_1\theta_0^3+5\,\alpha_1\theta_0^2+6\,\alpha_1\theta_0\right)A_1^2 - 2\left(3\,\alpha_1\theta_0^3+16\,\alpha_1\theta_0^2+25\,\alpha_1\theta_0+12\,\alpha_1\right)A_0A_2$$

$$+2\left(\theta_0^3+4\,\theta_0^2+3\,\theta_0\right)A_2^2 + 2\left(2\,\theta_0^3+9\,\theta_0^2+10\,\theta_0\right)A_1A_3 + 4\left(\theta_0^3+6\,\theta_0^2+11\,\theta_0+6\right)A_0A_4\Big\}$$

$$\nu_{28} = -\frac{2}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0}\Big\{2\left(5\,\theta_0^3+27\,\theta_0^2+46\,\theta_0+24\right)A_0A_3|A_1|^2 - 4\Big\{(2\,\alpha_1\alpha_5-\overline{\alpha_{15}})\theta_0^3 + 6\,(2\,\alpha_1\alpha_5-\overline{\alpha_{15}})\theta_0^2 + 12\,\alpha_1\alpha_5$$

$$+11\,(2\,\alpha_1\alpha_5-\overline{\alpha_{15}})\theta_0 - 6\,\overline{\alpha_{15}}\Big\}A_0^2 + \left(7\,\alpha_6\theta_0^3+39\,\alpha_6\theta_0^2+62\,\alpha_6\theta_0+24\,\alpha_6\right)A_0A_1 + 3\left(\alpha_5\theta_0^3+5\,\alpha_5\theta_0^2+6\,\alpha_5\theta_0\right)A_1^2$$

$$-4\left(4\left(\alpha_3\theta_0^3+6\,\alpha_3\theta_0^2+11\,\alpha_3\theta_0+6\,\alpha_3\right)A_0^2 + \left(7\,\alpha_1\theta_0^3+39\,\alpha_1\theta_0^2+62\,\alpha_1\theta_0+24\,\alpha_1\right)A_0A_1\right)|A_1|^2$$

$$+2\left(\left(5\,\theta_0^3+23\,\theta_0^2+24\,\theta_0\right)A_1|A_1|^2 + \left(3\,\alpha_5\theta_0^3+16\,\alpha_5\theta_0^2+25\,\alpha_5\theta_0+12\,\alpha_5\right)A_0\right)A_2\Big\}$$

$$\nu_{29} = \frac{2}{\theta_0^5+10\,\theta_0^4+35\,\theta_0^3+50\,\theta_0^2+24\,\theta_0}\Big\{5\Big\{(2\,\alpha_1\alpha_3-\overline{\alpha_{14}})\theta_0^4 + 10\,(2\,\alpha_1\alpha_3-\overline{\alpha_{14}})\theta_0^3 + 35\,(2\,\alpha_1\alpha_3-\overline{\alpha_{14}})\theta_0^2 + 48\,\alpha_1\alpha_3$$

$$+50\,(2\,\alpha_1\alpha_3-\overline{\alpha_{14}})\theta_0 - 24\,\overline{\alpha_{14}}\Big\}A_0^2 + \Big\{9\left(\alpha_1^2-\alpha_4\right)\theta_0^4 + 86\left(\alpha_1^2-\alpha_4\right)\theta_0^3 + 279\left(\alpha_1^2-\alpha_4\right)\theta_0^2 + 120\,\alpha_1^2 + 346\left(\alpha_1^2-\alpha_4\right)\theta_0$$

$$-120\,\alpha_4\Big\}A_0A_1 - 4\left(\alpha_3\theta_0^4+9\,\alpha_3\theta_0^3+26\,\alpha_3\theta_0^2+24\,\alpha_3\theta_0\right)A_1^2 + \left(5\,\theta_0^4+42\,\theta_0^3+115\,\theta_0^2+102\,\theta_0\right)A_1A_4$$

$$+5\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)A_0A_5 - \Big\{2\left(4\,\alpha_3\theta_0^4+37\,\alpha_3\theta_0^3+116\,\alpha_3\theta_0^2+143\,\alpha_3\theta_0+60\,\alpha_3\right)A_0$$

$$+\left(7\,\alpha_1\theta_0^4+60\,\alpha_1\theta_0^3+161\,\alpha_1\theta_0^2+132\,\alpha_1\theta_0\right)A_1\Big\}A_2 - \Big\{\left(7\,\alpha_1\theta_0^4+64\,\alpha_1\theta_0^3+203\,\alpha_1\theta_0^2+266\,\alpha_1\theta_0+120\,\alpha_1\right)A_0$$

$$- \left(5\,\theta_0^4 + 38\,\theta_0^3 + 85\,\theta_0^2 + 52\,\theta_0\right) A_2 \Big\} A_3 \Big\}$$

$$\nu_{30} = \frac{2}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} \Big\{ \left((2\,\alpha_1\overline{\alpha_2} - \overline{\alpha_{10}})\theta_0^3 + 6\,(2\,\alpha_1\overline{\alpha_2} - \overline{\alpha_{10}})\theta_0^2 + 11\,(2\,\alpha_1\overline{\alpha_2} - \overline{\alpha_{10}})\theta_0 + 12\,\alpha_1\overline{\alpha_2} - 6\,\overline{\alpha_{10}}\right) A_0^2$$
$$- \left(2\,\theta_0^3\overline{\alpha_9} + 10\,\theta_0^2\overline{\alpha_9} + 15\,\theta_0\overline{\alpha_9} + 6\,\overline{\alpha_9}\right) A_0 A_1 - \left(\theta_0^3\overline{\alpha_2} + 4\,\theta_0^2\overline{\alpha_2} + 4\,\theta_0\overline{\alpha_2}\right) A_1^2 - 2\,\left(\theta_0^3\overline{\alpha_2} + 4\,\theta_0^2\overline{\alpha_2} + 6\,\theta_0\overline{\alpha_2} + 3\,\overline{\alpha_2}\right) A_0 A_2 \Big\}$$

$$\nu_{31} = \frac{1}{4\left(\theta_0^3 - 3\,\theta_0^2 - 4\,\theta_0\right)} \Big\{ 8\,\left((2\,\overline{\alpha_1\alpha_2} - \overline{\alpha_{12}})\theta_0^3 - 2\,(2\,\overline{\alpha_1\alpha_2} - \overline{\alpha_{12}})\theta_0^2 - 7\,(2\,\overline{\alpha_1\alpha_2} - \overline{\alpha_{12}})\theta_0 - 8\,\overline{\alpha_1\alpha_2} + 4\,\overline{\alpha_{12}}\right) A_0^2$$
$$- 8\,\left(2\,\theta_0^3\overline{\alpha_7} - 6\,\theta_0^2\overline{\alpha_7} - 7\,\theta_0\overline{\alpha_7} - 4\,\overline{\alpha_7}\right) A_0 A_1 - 2\,\left(\theta_0^3\overline{\alpha_2} - \theta_0^2\overline{\alpha_2} - 5\,\theta_0\overline{\alpha_2} - 3\,\overline{\alpha_2}\right) A_0 \overline{C_1}$$
$$- \left(2\,\theta_0^2 + \theta_0 + 1\right) A_1 \overline{E_1} - 2\,\left(\theta_0^2 + 2\,\theta_0 + 1\right) A_0 \overline{E_2} \Big\}$$

Now, recall that

$$g^{-1} \otimes \langle \vec{h}_0, \vec{\Phi} \rangle = \begin{pmatrix} -\frac{2\,|A_1|^2}{(\theta_0+1)\theta_0} & 0 & 2 \\ \mu_1 & 0 & 3 \\ \mu_2 & 0 & 4 \\ \mu_3 & 1 & 2 \\ \mu_4 & 1 & 3 \\ \mu_5 & 2 & 1 \\ \mu_6 & 2 & 2 \\ \mu_7 & 3 & 1 \\ \frac{\alpha_2(\theta_0-2)}{\theta_0} & -\theta_0+1 & \theta_0+2 \\ \mu_8 & -\theta_0+1 & \theta_0+3 \end{pmatrix} \begin{pmatrix} \mu_9 & -\theta_0+2 & \theta_0+2 \\ \frac{\alpha_7(\theta_0-2)(\theta_0-4)}{\theta_0^2} & -\theta_0+3 & \theta_0+1 \\ -\frac{(\theta_0+1)\overline{\alpha_2}}{\theta_0} & \theta_0 & -\theta_0+3 \\ \mu_{10} & \theta_0 & -\theta_0+4 \\ -\frac{2\,\zeta_0}{(\theta_0+1)\theta_0} & \theta_0+1 & -\theta_0+1 \\ \mu_{11} & \theta_0+1 & -\theta_0+3 \\ \mu_{12} & \theta_0+2 & -\theta_0+1 \\ \mu_{13} & \theta_0+2 & -\theta_0+2 \\ \mu_{14} & \theta_0+3 & -\theta_0+1 \end{pmatrix}$$

so we need only look for powers of order 5. They are

$$\begin{pmatrix} \nu_3 & 0 & 5 \\ \nu_6 & 1 & 4 \\ \nu_9 & 2 & 3 \\ \nu_{11} & 3 & 2 \\ \nu_{12} & 4 & 1 \\ \nu_{14} & -\theta_0+1 & \theta_0+4 \\ \nu_{16} & -\theta_0+2 & \theta_0+3 \\ \nu_{17} & -\theta_0+3 & \theta_0+2 \\ \nu_{18} & -\theta_0+4 & \theta_0+1 \\ \nu_{20} & \theta_0 & -\theta_0+5 \\ \nu_{21} & \theta_0-1 & -\theta_0+6 \\ \nu_{23} & \theta_0+1 & -\theta_0+4 \\ \nu_{26} & \theta_0+2 & -\theta_0+3 \\ \nu_{28} & \theta_0+3 & -\theta_0+2 \\ \nu_{29} & \theta_0+4 & -\theta_0+1 \\ \nu_{30} & 2\theta_0+2 & -2\theta_0+3 \\ \nu_{31} & 2\theta_0 & -2\theta_0+5 \end{pmatrix} = \begin{pmatrix} \mu_{15} & 0 & 5 \\ \mu_{16} & 1 & 4 \\ \mu_{17} & 2 & 3 \\ \mu_{18} & 3 & 2 \\ \mu_{19} & 4 & 1 \\ \mu_{20} & -\theta_0+1 & \theta_0+4 \\ \mu_{21} & -\theta_0+2 & \theta_0+3 \\ \mu_{22} & -\theta_0+3 & \theta_0+2 \\ \mu_{23} & -\theta_0+4 & \theta_0+1 \\ \mu_{24} & \theta_0 & -\theta_0+5 \\ \mu_{25} & \theta_0-1 & -\theta_0+6 \\ \mu_{26} & \theta_0+1 & -\theta_0+4 \\ \mu_{27} & \theta_0+2 & -\theta_0+3 \\ \mu_{28} & \theta_0+3 & -\theta_0+2 \\ \mu_{29} & \theta_0+4 & -\theta_0+1 \\ \mu_{30} & 2\theta_0+2 & -2\theta_0+3 \\ \mu_{31} & 2\theta_0 & -2\theta_0+5 \end{pmatrix}$$

to preserve notations we write the new terms with $\{\mu_j\}_{15\leq j\leq 31}$. In Sage, we have

$$\begin{pmatrix} \mu_{15} & 0 & 5 \\ \mu_{16} & 1 & 4 \\ \mu_{17} & 2 & 3 \\ \mu_{18} & 3 & 2 \\ \mu_{19} & 4 & 1 \\ \mu_{20} & -\theta_0+1 & \theta_0+4 \\ \mu_{21} & -\theta_0+2 & \theta_0+3 \\ \mu_{22} & -\theta_0+3 & \theta_0+2 \\ \mu_{23} & -\theta_0+4 & \theta_0+1 \\ \mu_{24} & \theta_0 & -\theta_0+5 \\ \mu_{25} & \theta_0-1 & -\theta_0+6 \\ \mu_{26} & \theta_0+1 & -\theta_0+4 \\ \mu_{27} & \theta_0+2 & -\theta_0+3 \\ \mu_{28} & \theta_0+3 & -\theta_0+2 \\ \mu_{29} & \theta_0+4 & -\theta_0+1 \\ \mu_{30} & 2\theta_0+2 & -2\theta_0+3 \\ \mu_{31} & 2\theta_0 & -2\theta_0+5 \end{pmatrix} \begin{pmatrix} \mu_{15} & 0 & 5 \\ \mu_{16} & 1 & 4 \\ \mu_{17} & 2 & 3 \\ \mu_{18} & 3 & 2 \\ \mu_{19} & 4 & 1 \\ \mu_{20} & -\theta_0+1 & \theta_0+4 \\ \mu_{21} & -\theta_0+2 & \theta_0+3 \\ \mu_{22} & -\theta_0+3 & \theta_0+2 \\ \mu_{23} & -\theta_0+4 & \theta_0+1 \\ \mu_{24} & \theta_0 & -\theta_0+5 \\ \mu_{25} & \theta_0-1 & -\theta_0+6 \\ \mu_{26} & \theta_0+1 & -\theta_0+4 \\ \mu_{27} & \theta_0+2 & -\theta_0+3 \\ \mu_{28} & \theta_0+3 & -\theta_0+2 \\ \mu_{29} & \theta_0+4 & -\theta_0+1 \\ \mu_{30} & 2\,\theta_0+2 & -2\,\theta_0+3 \\ \mu_{31} & 2\,\theta_0 & -2\,\theta_0+5 \end{pmatrix} \tag{4.4.1}$$

where

$$\mu_{15}=\nu_3=-\frac{1}{24\,(\theta_0^5+10\,\theta_0^4+35\,\theta_0^3+50\,\theta_0^2+24\,\theta_0)}\Big\{96\,\big(\theta_0^4+7\,\theta_0^3+14\,\theta_0^2+8\,\theta_0\big)A_0|A_1|^2\overline{A_3}$$
$$+2\,\big(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\big)|A_1|^2\overline{A_0C_2}-192\,\Big(\big(\theta_0^4\overline{\alpha_3}+10\,\theta_0^3\overline{\alpha_3}+35\,\theta_0^2\overline{\alpha_3}+50\,\theta_0\overline{\alpha_3}+24\,\overline{\alpha_3}\big)A_0\overline{A_0}$$
$$+\big(\theta_0^4\overline{\alpha_1}+9\,\theta_0^3\overline{\alpha_1}+26\,\theta_0^2\overline{\alpha_1}+24\,\theta_0\overline{\alpha_1}\big)A_0\overline{A_1}\Big)|A_1|^2-48\,\Big((2\,\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0^4+10\,(2\,\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0^3$$
$$+35\,(2\,\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0^2+50\,(2\,\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0+48\,\overline{\alpha_1\alpha_5}-24\,\alpha_{15}\Big)A_0\overline{A_0}$$
$$-48\,\big((\overline{\alpha_1}^2-\overline{\alpha_4})\theta_0^4+10\,(\overline{\alpha_1}^2-\overline{\alpha_4})\theta_0^3+35\,(\overline{\alpha_1}^2-\overline{\alpha_4})\theta_0^2+50\,(\overline{\alpha_1}^2-\overline{\alpha_4})\theta_0+24\,\overline{\alpha_1}^2-24\,\overline{\alpha_4}\big)A_1\overline{A_0}$$
$$-48\,\big(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0\big)A_1\overline{A_4}-4\,\big(\theta_0^4+9\,\theta_0^3+26\,\theta_0^2+24\,\theta_0\big)\overline{A_1B_3}$$
$$-3\,\big(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\big)\overline{A_0B_6}+48\,\Big(\big(\theta_0^4\overline{\alpha_6}+9\,\theta_0^3\overline{\alpha_6}+26\,\theta_0^2\overline{\alpha_6}+24\,\theta_0\overline{\alpha_6}\big)A_0$$
$$+\big(\theta_0^4\overline{\alpha_3}+9\,\theta_0^3\overline{\alpha_3}+26\,\theta_0^2\overline{\alpha_3}+24\,\theta_0\overline{\alpha_3}\big)A_1\Big)\overline{A_1}+48\,\Big(\big(\theta_0^4\overline{\alpha_5}+8\,\theta_0^3\overline{\alpha_5}+19\,\theta_0^2\overline{\alpha_5}+12\,\theta_0\overline{\alpha_5}\big)A_0$$
$$+\big(\theta_0^4\overline{\alpha_1}+8\,\theta_0^3\overline{\alpha_1}+19\,\theta_0^2\overline{\alpha_1}+12\,\theta_0\overline{\alpha_1}\big)A_1\Big)\overline{A_2}+3\,\Big(\big(\theta_0^4\overline{\alpha_1}+10\,\theta_0^3\overline{\alpha_1}+35\,\theta_0^2\overline{\alpha_1}+50\,\theta_0\overline{\alpha_1}+24\,\overline{\alpha_1}\big)\overline{A_0}$$

$$- 2\left(\theta_0^4 + 8\,\theta_0^3 + 19\,\theta_0^2 + 12\,\theta_0\right)\overline{A_2}\Bigg)\overline{B_1}$$

$$+ \Bigg(4\left(\theta_0^4 + 9\,\theta_0^3 + 26\,\theta_0^2 + 24\,\theta_0\right)|A_1|^2\overline{A_1} + 3\left(\theta_0^4\overline{\alpha_5} + 10\,\theta_0^3\overline{\alpha_5} + 35\,\theta_0^2\overline{\alpha_5} + 50\,\theta_0\overline{\alpha_5} + 24\,\overline{\alpha_5}\right)\overline{A_0}\Bigg)\overline{C_1}\Bigg\}$$

We will not need the precise expression of the other μ_j coefficients for $j \geq 16$, they are just written for completeness.

4.5 Final development of the tensors related to the invariance by inversions

$$
|\vec{\Phi}|^2\vec{\alpha} - 2\langle\vec{\alpha},\vec{\Phi}\rangle\vec{\Phi} = \begin{pmatrix}
-\frac{2\left(\theta_0\overline{\alpha_2}\overline{\zeta_0} + \left(\theta_0^2 + 3\,\theta_0 + 2\right)\zeta_{13}\right)}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & \overline{A_0} & 0 & \theta_0 + 4 \\
-\frac{\theta_0 - 2}{2\,\theta_0^2} & C_1 & 1 & \theta_0 \\
-\frac{\theta_0 - 2}{2\,\theta_0^2} & B_1 & 1 & \theta_0 + 1 \\
-\frac{2\,(\alpha_2\theta_0 - 2\,\alpha_2)}{\theta_0} & A_0 & 1 & \theta_0 + 1 \\
-\frac{\theta_0 - 2}{2\,\theta_0^2} & B_2 & 1 & \theta_0 + 2 \\
-\frac{(\theta_0 - 2)|C_1|^2}{2\,\theta_0^3} & \overline{A_0} & 1 & \theta_0 + 2 \\
-\frac{\theta_0\overline{\alpha_1} - 2\,\overline{\alpha_1}}{2\,(\theta_0^2 + 2\,\theta_0)} & C_1 & 1 & \theta_0 + 2 \\
-\frac{2\,(\theta_0 - 2)\zeta_4}{\theta_0^3 + 2\,\theta_0^2} & A_0 & 1 & \theta_0 + 2 \\
\frac{1}{\theta_0^2} & B_{10} & 1 & \theta_0 + 3 \\
-\frac{2\,\zeta_{18}}{\theta_0} & A_0 & 1 & \theta_0 + 3 \\
-\frac{2\,\zeta_{14}}{\theta_0} & \overline{A_0} & 1 & \theta_0 + 3 \\
-\frac{\alpha_2\theta_0 - 2\,\alpha_2}{4\,\theta_0} & \overline{C_1} & 1 & \theta_0 + 3 \\
-\frac{\theta_0\overline{\alpha_1} - 2\,\overline{\alpha_1}}{2\,(\theta_0^2 + 2\,\theta_0)} & B_1 & 1 & \theta_0 + 3 \\
-\frac{2\,\theta_0^3\overline{\alpha_3} - 2\,\theta_0^2\overline{\alpha_3} - 4\,\theta_0\overline{\alpha_3} - (\theta_0 - 2)\overline{\zeta_2}}{4\,(\theta_0^4 + 4\,\theta_0^3 + 3\,\theta_0^2)} & C_1 & 1 & \theta_0 + 3 \\
-\frac{\theta_0 - 3}{2\,\theta_0^2} & C_2 & 2 & \theta_0 \\
\frac{3\,\zeta_2}{\theta_0^3 + \theta_0^2} & \overline{A_0} & 2 & \theta_0 \\
-\frac{\theta_0 - 3}{2\,\theta_0^2} & B_3 & 2 & \theta_0 + 1 \\
-\frac{2\,(\alpha_2\theta_0 - 2\,\alpha_2)}{\theta_0 + 1} & A_1 & 2 & \theta_0 + 1 \\
-\frac{(\theta_0 - 3)\zeta_3}{\theta_0^3 + \theta_0^2} & A_0 & 2 & \theta_0 + 1 \\
\frac{(5\,\theta_0 + 2)\zeta_2}{\theta_0^4 + 2\,\theta_0^3 + \theta_0^2} & \overline{A_1} & 2 & \theta_0 + 1 \\
-\frac{(\theta_0 - 2)|A_1|^2}{\theta_0^2 + 2\,\theta_0 + 1} & C_1 & 2 & \theta_0 + 1
\end{pmatrix}
$$

$$\begin{pmatrix}
\frac{1}{\theta_0^2} & B_{11} & 2 & \theta_0+2 \\
-\frac{2\,\zeta_{19}}{\theta_0} & A_0 & 2 & \theta_0+2 \\
-\frac{\theta_0\overline{\alpha_5}-2\,\overline{\alpha_5}}{2\,(\theta_0^2+3\,\theta_0+2)} & C_1 & 2 & \theta_0+2 \\
-\frac{\theta_0\overline{\alpha_1}-3\,\overline{\alpha_1}}{2\,(\theta_0^2+2\,\theta_0)} & C_2 & 2 & \theta_0+2 \\
-\frac{2\,((\theta_0+2)\zeta_{15}-\zeta_2\overline{\alpha_1})}{\theta_0^2+2\,\theta_0} & \overline{A_0} & 2 & \theta_0+2 \\
-\frac{2\,(\theta_0-2)\zeta_4}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & A_1 & 2 & \theta_0+2 \\
-\frac{(2\,\theta_0-1)\zeta_2}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & \overline{A_2} & 2 & \theta_0+2 \\
-\frac{(\theta_0-2)|A_1|^2}{\theta_0^2+2\,\theta_0+1} & B_1 & 2 & \theta_0+2 \\
-\frac{\theta_0-4}{2\,\theta_0^2} & C_3 & 3 & \theta_0 \\
-\frac{2\,(\theta_0\zeta_7-\zeta_5)}{\theta_0^2} & \overline{A_0} & 3 & \theta_0 \\
\frac{2\,(\alpha_7\theta_0-4\,\alpha_7)}{\theta_0} & A_0 & 3 & \theta_0 \\
-\frac{\alpha_1\theta_0-2\,\alpha_1}{2\,(\theta_0^2+2\,\theta_0)} & C_1 & 3 & \theta_0 \\
\frac{1}{\theta_0^2} & B_{12} & 3 & \theta_0+1 \\
-\frac{2\,\zeta_7}{\theta_0+1} & \overline{A_1} & 3 & \theta_0+1 \\
-\frac{2\,\zeta_{20}}{\theta_0} & A_0 & 3 & \theta_0+1 \\
-\frac{2\,(\alpha_2\theta_0-2\,\alpha_2)}{\theta_0+2} & A_2 & 3 & \theta_0+1 \\
-\frac{\alpha_1\theta_0-2\,\alpha_1}{2\,(\theta_0^2+2\,\theta_0)} & B_1 & 3 & \theta_0+1 \\
-\frac{\alpha_5\theta_0-2\,\alpha_5}{2\,(\theta_0^2+3\,\theta_0+2)} & C_1 & 3 & \theta_0+1 \\
\frac{2\left(2\,\theta_0\zeta_2|A_1|^2-\left(\theta_0^2+2\,\theta_0+1\right)\zeta_{16}\right)}{\theta_0^3+2\,\theta_0^2+\theta_0} & \overline{A_0} & 3 & \theta_0+1 \\
-\frac{(\theta_0-3)\zeta_3}{\theta_0^3+2\,\theta_0^2+\theta_0} & A_1 & 3 & \theta_0+1 \\
-\frac{(\theta_0-3)|A_1|^2}{\theta_0^2+2\,\theta_0+1} & C_2 & 3 & \theta_0+1 \\
\frac{1}{\theta_0^2} & B_{13} & 4 & \theta_0 \\
-\frac{2\,\zeta_{21}}{\theta_0} & A_0 & 4 & \theta_0 \\
-\frac{2\,((\theta_0+2)\zeta_{17}-\alpha_1\zeta_2)}{\theta_0^2+2\,\theta_0} & \overline{A_0} & 4 & \theta_0 \\
-\frac{\alpha_1\theta_0-3\,\alpha_1}{2\,(\theta_0^2+2\,\theta_0)} & C_2 & 4 & \theta_0 \\
-\frac{4\,\alpha_3\theta_0^2-8\,\alpha_3\theta_0+(2\,\theta_0+1)\zeta_2}{8\,(\theta_0^3+3\,\theta_0^2)} & C_1 & 4 & \theta_0
\end{pmatrix}$$

$$
\begin{pmatrix}
-\frac{2\,(\alpha_2\theta_0 - 2\,\alpha_2)}{\theta_0} & \overline{A_0} & -\theta_0 + 1 & 2\,\theta_0 + 1 \\
-\frac{2\,(\alpha_2\theta_0 - 2\,\alpha_2)}{\theta_0 + 1} & \overline{A_1} & -\theta_0 + 1 & 2\,\theta_0 + 2 \\
\frac{(\theta_0 - 2)\overline{\zeta_0}}{2\,(\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0)} & C_1 & -\theta_0 + 1 & 2\,\theta_0 + 2 \\
-\frac{2\,(\theta_0 - 2)\zeta_4}{\theta_0^3 + 2\,\theta_0^2} & \overline{A_0} & -\theta_0 + 1 & 2\,\theta_0 + 2 \\
-\frac{2\,\zeta_{18}}{\theta_0} & \overline{A_0} & -\theta_0 + 1 & 2\,\theta_0 + 3 \\
-\frac{2\,(\alpha_2\theta_0 - 2\,\alpha_2)}{\theta_0 + 2} & \overline{A_2} & -\theta_0 + 1 & 2\,\theta_0 + 3 \\
\frac{2\,(\theta_0 - 2)\overline{\zeta_1}}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0} & C_1 & -\theta_0 + 1 & 2\,\theta_0 + 3 \\
\frac{(\theta_0 - 2)\overline{\zeta_0}}{2\,(\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0)} & B_1 & -\theta_0 + 1 & 2\,\theta_0 + 3 \\
-\frac{2\,(\theta_0 - 2)\zeta_4}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & \overline{A_1} & -\theta_0 + 1 & 2\,\theta_0 + 3 \\
-\frac{(\theta_0 - 3)\zeta_3}{\theta_0^3 + \theta_0^2} & \overline{A_0} & -\theta_0 + 2 & 2\,\theta_0 + 1 \\
-\frac{2\,\left((\theta_0^3 + 4\,\theta_0^2 + 5\,\theta_0 + 2)\zeta_{19} + \zeta_2\overline{\zeta_0}\right)}{\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0} & \overline{A_0} & -\theta_0 + 2 & 2\,\theta_0 + 2 \\
\frac{(\theta_0 - 3)\overline{\zeta_0}}{2\,(\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0)} & C_2 & -\theta_0 + 2 & 2\,\theta_0 + 2 \\
-\frac{(\theta_0 - 3)\zeta_3}{\theta_0^3 + 2\,\theta_0^2 + \theta_0} & \overline{A_1} & -\theta_0 + 2 & 2\,\theta_0 + 2 \\
-\frac{\alpha_2\theta_0^2 - 6\,\alpha_2\theta_0 + 8\,\alpha_2}{4\,\theta_0^2} & C_1 & -\theta_0 + 3 & 2\,\theta_0 + 1 \\
-\frac{2\,\zeta_{20}}{\theta_0} & \overline{A_0} & -\theta_0 + 3 & 2\,\theta_0 + 1 \\
-\frac{2\,\left(\alpha_7\theta_0^2 - 6\,\alpha_7\theta_0 + 8\,\alpha_7\right)}{\theta_0^3 + \theta_0^2} & \overline{A_1} & -\theta_0 + 3 & 2\,\theta_0 + 1 \\
-\frac{2\,\zeta_{21}}{\theta_0} & \overline{A_0} & -\theta_0 + 4 & 2\,\theta_0 \\
\frac{\overline{\zeta_2}}{\theta_0^3 - 3\,\theta_0^2} & A_1 & \theta_0 & 3 \\
\frac{2\,\overline{\alpha_2}}{\theta_0} & \overline{A_1} & \theta_0 & 3 \\
\frac{1}{\theta_0^2} & B_7 & \theta_0 & 4 \\
-\frac{\overline{\zeta_3}}{\theta_0^2 + \theta_0} & \overline{A_1} & \theta_0 & 4 \\
-\frac{2\,\zeta_{13}}{\theta_0} & A_0 & \theta_0 & 4 \\
-\frac{2\,(\theta_0\overline{\alpha_2} + \overline{\alpha_2})}{\theta_0 + 2} & \overline{A_2} & \theta_0 & 4 \\
\frac{2\,\left(\theta_0^2\overline{\alpha_1\alpha_2} + \theta_0\overline{\alpha_1\alpha_2} - (\theta_0 + 2)\zeta_{22}\right)}{\theta_0^2 + 2\,\theta_0} & \overline{A_0} & \theta_0 & 4
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{(\theta_0-2)|C_1|^2}{4\,\theta_0^3} & A_0 & \theta_0+1 & 2 \\
\frac{1}{\theta_0^2} & B_8 & \theta_0+1 & 3 \\
-\frac{2\,\overline{\zeta_4}}{\theta_0^2+\theta_0} & \overline{A_1} & \theta_0+1 & 3 \\
-\frac{2\,\zeta_{14}}{\theta_0} & A_0 & \theta_0+1 & 3 \\
\frac{2\left(2\,\theta_0^2|A_1|^2\overline{\alpha_2}-(\theta_0+1)\zeta_{23}\right)}{\theta_0^2+\theta_0} & \overline{A_0} & \theta_0+1 & 3 \\
-\frac{(2\,\theta_0-1)\zeta_2}{\theta_0^3+\theta_0^2} & A_0 & \theta_0+2 & 0 \\
\frac{1}{\theta_0^2} & B_9 & \theta_0+2 & 2 \\
-\frac{2\,\zeta_{15}}{\theta_0} & A_0 & \theta_0+2 & 2 \\
-\frac{\theta_0\overline{\alpha_2}+\overline{\alpha_2}}{4\,\theta_0} & C_1 & \theta_0+2 & 2 \\
\frac{2\left(\alpha_1\theta_0^2\overline{\alpha_2}+\alpha_1\theta_0\overline{\alpha_2}-(\theta_0+2)\zeta_{24}\right)}{\theta_0^2+2\,\theta_0} & \overline{A_0} & \theta_0+2 & 2 \\
-\frac{(2\,\theta_0-1)\zeta_2}{8\,(\theta_0^3+\theta_0^2)} & \overline{C_1} & \theta_0+2 & 2 \\
-\frac{2\,\zeta_7}{\theta_0} & A_0 & \theta_0+3 & 0 \\
-\frac{(2\,\theta_0-1)\zeta_2}{\theta_0^3+2\,\theta_0^2+\theta_0} & A_1 & \theta_0+3 & 0 \\
\frac{(\theta_0-2)\zeta_0}{2\,(\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0)} & C_1 & \theta_0+3 & 0 \\
-\frac{2\,\zeta_{16}}{\theta_0} & A_0 & \theta_0+3 & 1 \\
\frac{(\theta_0-2)\zeta_0}{2\,(\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0)} & B_1 & \theta_0+3 & 1 \\
-\frac{2\,\zeta_7}{\theta_0+1} & A_1 & \theta_0+4 & 0 \\
-\frac{2\,\zeta_{17}}{\theta_0} & A_0 & \theta_0+4 & 0 \\
-\frac{2\,\zeta_0\zeta_2}{\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0} & \overline{A_0} & \theta_0+4 & 0 \\
-\frac{(2\,\theta_0-1)\zeta_2}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & A_2 & \theta_0+4 & 0 \\
\frac{2\,(\theta_0-2)\zeta_1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & C_1 & \theta_0+4 & 0 \\
\frac{(\theta_0-3)\zeta_0}{2\,(\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0)} & C_2 & \theta_0+4 & 0
\end{pmatrix}
$$

$$\begin{pmatrix}
-\frac{2\,\overline{\zeta_4}}{\theta_0^2} & A_0 & 2\,\theta_0+1 & -\theta_0+2 \\
-2\,\overline{\alpha_2} & A_1 & 2\,\theta_0+1 & -\theta_0+2 \\
-\frac{\overline{\zeta_3}}{\theta_0^2+\theta_0} & A_1 & 2\,\theta_0+1 & -\theta_0+3 \\
-\frac{2\,\zeta_{23}}{\theta_0} & A_0 & 2\,\theta_0+1 & -\theta_0+3 \\
-\frac{2\,\overline{\zeta_4}}{\theta_0^2+\theta_0} & A_1 & 2\,\theta_0+2 & -\theta_0+2 \\
-\frac{2\,\zeta_{24}}{\theta_0} & A_0 & 2\,\theta_0+2 & -\theta_0+2 \\
-\frac{2\,(\theta_0\overline{\alpha_2}+\overline{\alpha_2})}{\theta_0+2} & A_2 & 2\,\theta_0+2 & -\theta_0+2 \\
-\frac{2\,\zeta_0\overline{\alpha_2}}{\theta_0^2+3\,\theta_0+2} & \overline{A_0} & 2\,\theta_0+2 & -\theta_0+2 \\
-\frac{2\,(\theta_0\overline{\alpha_2}+\overline{\alpha_2})}{\theta_0} & A_0 & 2\,\theta_0 & -\theta_0+2 \\
-\frac{\overline{\zeta_3}}{\theta_0^2} & A_0 & 2\,\theta_0 & -\theta_0+3 \\
-\frac{\theta_0\overline{\alpha_7}-4\,\overline{\alpha_7}}{\theta_0^2} & A_1 & 2\,\theta_0 & -\theta_0+4 \\
-\frac{\theta_0^2\overline{\alpha_2}-\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2}}{4\,\theta_0^2} & \overline{C_1} & 2\,\theta_0 & -\theta_0+4 \\
-\frac{2\,\zeta_{22}}{\theta_0} & A_0 & 2\,\theta_0 & -\theta_0+4
\end{pmatrix}$$

while

$$
-\,g^{-1}\otimes\left(\overline{\partial}|\vec{\Phi}|^2\otimes\vec{h}_0-2\langle\vec{\Phi},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}\right)=
\begin{pmatrix}
-\frac{4\,|A_1|^2}{\theta_0^2+\theta_0} & \overline{A_0} & 0 & \theta_0+1\\
\frac{4\,\overline{\zeta_0}}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & A_1 & 0 & \theta_0+2\\
-\frac{4\,|A_1|^2}{\theta_0^2+\theta_0} & \overline{A_1} & 0 & \theta_0+2\\
2\,\mu_1 & \overline{A_0} & 0 & \theta_0+2\\
\frac{8\,(2\,\theta_0+3)\overline{\zeta_1}}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & A_1 & 0 & \theta_0+3\\
-\frac{4\,|A_1|^2}{\theta_0^2+\theta_0} & \overline{A_2} & 0 & \theta_0+3\\
2\,\mu_2 & \overline{A_0} & 0 & \theta_0+3\\
2\,\mu_1 & \overline{A_1} & 0 & \theta_0+3\\
-\frac{8\,|A_1|^2\overline{\zeta_0}}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & A_0 & 0 & \theta_0+3\\
\frac{\overline{\zeta_0}}{2\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)} & \overline{B_1} & 0 & \theta_0+4\\
-\frac{4\left(\overline{\alpha_1}\overline{\zeta_0}+\left(\theta_0^4+5\,\theta_0^3+8\,\theta_0^2+4\,\theta_0\right)\overline{\zeta_9}\right)}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & A_1 & 0 & \theta_0+4\\
-\frac{4\left(4\,(2\,\theta_0+3)|A_1|^2\overline{\zeta_1}+(\theta_0\overline{\alpha_5}+3\,\overline{\alpha_5})\overline{\zeta_0}\right)}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & A_0 & 0 & \theta_0+4\\
-\frac{4\,|A_1|^2}{\theta_0^2+\theta_0} & \overline{A_3} & 0 & \theta_0+4\\
2\,\mu_2 & \overline{A_1} & 0 & \theta_0+4\\
2\,\mu_{15} & \overline{A_0} & 0 & \theta_0+4\\
2\,\mu_1 & \overline{A_2} & 0 & \theta_0+4\\
\frac{\theta_0-2}{2\,\theta_0^2} & C_1 & 1 & \theta_0\\
-\frac{2\,(\alpha_2\theta_0-2\,\alpha_2)}{\theta_0} & A_0 & 1 & \theta_0+1\\
\frac{\theta_0-2}{2\left(\theta_0^2+\theta_0\right)} & B_1 & 1 & \theta_0+1\\
2\,\mu_3 & \overline{A_0} & 1 & \theta_0+1\\
\frac{8\,\overline{\zeta_0}}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & A_2 & 1 & \theta_0+2\\
\frac{\theta_0-2}{2\left(\theta_0^2+2\,\theta_0\right)} & B_2 & 1 & \theta_0+2\\
\frac{\theta_0\overline{\alpha_1}-2\,\overline{\alpha_1}}{2\left(\theta_0^2+2\,\theta_0\right)} & C_1 & 1 & \theta_0+2\\
-\frac{2\left(\alpha_9\theta_0^3+\alpha_9\theta_0^2-4\,\alpha_9\theta_0+4\,\alpha_1\overline{\zeta_0}-4\,\alpha_9\right)}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & A_0 & 1 & \theta_0+2\\
2\,\mu_4 & \overline{A_0} & 1 & \theta_0+2\\
2\,\mu_3 & \overline{A_1} & 1 & \theta_0+2\\
\frac{\theta_0-2}{2\left(\theta_0^2+3\,\theta_0\right)} & B_4 & 1 & \theta_0+3\\
\frac{\theta_0\overline{\alpha_1}-2\,\overline{\alpha_1}}{2\left(\theta_0^2+3\,\theta_0\right)} & B_1 & 1 & \theta_0+3\\
-\frac{\alpha_2\theta_0^3-3\,\alpha_2\theta_0^2-4\,\alpha_2\theta_0+12\,\alpha_2}{4\left(\theta_0^3+3\,\theta_0^2\right)} & \overline{C_1} & 1 & \theta_0+3
\end{pmatrix}
$$

$$
\begin{pmatrix}
\pi_1 & A_0 & 1 & \theta_0 + 3 \\
\frac{2\,\theta_0^4\overline{\alpha_3} - 2\,\theta_0^3\overline{\alpha_3} - 4\,\theta_0^2\overline{\alpha_3} - \left(\theta_0^2 + \theta_0 - 6\right)\overline{\zeta_2}}{4\left(\theta_0^5 + 4\,\theta_0^4 + 3\,\theta_0^3\right)} & C_1 & 1 & \theta_0 + 3 \\
\frac{16\,(2\,\theta_0 + 3)\overline{\zeta_1}}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0} & A_2 & 1 & \theta_0 + 3 \\
2\,\mu_4 & \overline{A_1} & 1 & \theta_0 + 3 \\
2\,\mu_3 & \overline{A_2} & 1 & \theta_0 + 3 \\
2\,\mu_{16} & \overline{A_0} & 1 & \theta_0 + 3 \\
-\frac{16\,|A_1|^2\overline{\zeta_0}}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_1 & 1 & \theta_0 + 3 \\
\frac{\theta_0 - 3}{2\,\theta_0^2} & C_2 & 2 & \theta_0 \\
2\,\mu_5 & \overline{A_0} & 2 & \theta_0 \\
-\frac{2\,(\alpha_8\theta_0 - 3\,\alpha_8)}{\theta_0} & A_0 & 2 & \theta_0 + 1 \\
-\frac{2\,(\alpha_2\theta_0 - 3\,\alpha_2)}{\theta_0} & A_1 & 2 & \theta_0 + 1 \\
\frac{\theta_0 - 3}{2\left(\theta_0^2 + \theta_0\right)} & B_3 & 2 & \theta_0 + 1 \\
2\,\mu_6 & \overline{A_0} & 2 & \theta_0 + 1 \\
2\,\mu_5 & \overline{A_1} & 2 & \theta_0 + 1 \\
\frac{\left(2\,\theta_0^2 - 7\,\theta_0 + 6\right)|A_1|^2}{2\left(\theta_0^3 + \theta_0^2\right)} & C_1 & 2 & \theta_0 + 1 \\
\frac{12\,\overline{\zeta_0}}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_3 & 2 & \theta_0 + 2 \\
\frac{\theta_0 - 3}{2\left(\theta_0^2 + 2\,\theta_0\right)} & B_5 & 2 & \theta_0 + 2 \\
\frac{\theta_0\overline{\alpha_1} - 3\,\overline{\alpha_1}}{2\left(\theta_0^2 + 2\,\theta_0\right)} & C_2 & 2 & \theta_0 + 2 \\
-\frac{2\left(\alpha_9\theta_0^3 - 7\,\alpha_9\theta_0 + 6\,\alpha_1\overline{\zeta_0} - 6\,\alpha_9\right)}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_1 & 2 & \theta_0 + 2 \\
-\frac{2\left(\alpha_{11}\theta_0^3 - 2\left(\alpha_2\theta_0^3 - 10\,\alpha_2\theta_0 - 12\,\alpha_2\right)|A_1|^2 - 7\,\alpha_{11}\theta_0 + 6\,\alpha_3\overline{\zeta_0} - 6\,\alpha_{11}\right)}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_0 & 2 & \theta_0 + 2 \\
\frac{\mu_1\theta_0^4 + (3\,\mu_1 + 2\,\overline{\alpha_5})\theta_0^3 + 2\,(\mu_1 - 2\,\overline{\alpha_5})\theta_0^2 + 12\,\overline{\alpha_5}}{4\left(\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2\right)} & C_1 & 2 & \theta_0 + 2 \\
2\,\mu_6 & \overline{A_1} & 2 & \theta_0 + 2 \\
2\,\mu_5 & \overline{A_2} & 2 & \theta_0 + 2 \\
2\,\mu_{17} & \overline{A_0} & 2 & \theta_0 + 2 \\
\frac{\left(2\,\theta_0^3 - 3\,\theta_0^2 - 7\,\theta_0 + 4\right)|A_1|^2}{2\left(\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0\right)} & B_1 & 2 & \theta_0 + 2 \\
\frac{\theta_0 - 4}{2\,\theta_0^2} & C_3 & 3 & \theta_0 \\
-\frac{2\,(\alpha_7\theta_0 - 4\,\alpha_7)}{\theta_0} & A_0 & 3 & \theta_0 \\
\frac{\alpha_1\theta_0 - 2\,\alpha_1}{2\left(\theta_0^2 + 2\,\theta_0\right)} & C_1 & 3 & \theta_0 \\
2\,\mu_7 & \overline{A_0} & 3 & \theta_0
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{2\,(\alpha_8\theta_0 - 4\,\alpha_8)}{\theta_0} & A_1 & 3 & \theta_0 + 1 \\
-\frac{2\,(\alpha_2\theta_0 - 4\,\alpha_2)}{\theta_0} & A_2 & 3 & \theta_0 + 1 \\
\frac{\theta_0 - 4}{2\left(\theta_0^2 + \theta_0\right)} & B_6 & 3 & \theta_0 + 1 \\
\frac{2\left((\alpha_1\alpha_2 - \alpha_{12})\theta_0^2 - 16\,\alpha_1\alpha_2 - 2\,(\alpha_1\alpha_2 - \alpha_{12})\theta_0 + 8\,\alpha_{12}\right)}{\theta_0^2 + 2\,\theta_0} & A_0 & 3 & \theta_0 + 1 \\
\frac{\alpha_1\theta_0 - 2\,\alpha_1}{2\left(\theta_0^2 + 3\,\theta_0 + 2\right)} & B_1 & 3 & \theta_0 + 1 \\
\frac{\mu_3\theta_0^4 + (2\,\alpha_5 + 3\,\mu_3)\theta_0^3 - 2\,(2\,\alpha_5 - \mu_3)\theta_0^2 + 16\,\alpha_5}{4\left(\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2\right)} & C_1 & 3 & \theta_0 + 1 \\
2\,\mu_7 & \overline{A_1} & 3 & \theta_0 + 1 \\
2\,\mu_{18} & \overline{A_0} & 3 & \theta_0 + 1 \\
\frac{\left(3\,\theta_0^2 - 13\,\theta_0 + 12\right)|A_1|^2}{3\left(\theta_0^3 + \theta_0^2\right)} & C_2 & 3 & \theta_0 + 1 \\
\frac{\theta_0 - 5}{2\,\theta_0^2} & C_4 & 4 & \theta_0 \\
-\frac{2\,(\alpha_7\theta_0 - 4\,\alpha_7)}{\theta_0} & A_1 & 4 & \theta_0 \\
-\frac{2\,(\alpha_{13}\theta_0 - 5\,\alpha_{13})}{\theta_0} & A_0 & 4 & \theta_0 \\
\frac{\alpha_1\theta_0 - 3\,\alpha_1}{2\left(\theta_0^2 + 2\,\theta_0\right)} & C_2 & 4 & \theta_0 \\
\frac{\mu_5\theta_0^4 + 2\,(\alpha_3 + 2\,\mu_5)\theta_0^3 - (2\,\alpha_3 - 3\,\mu_5)\theta_0^2 - 4\,\alpha_3\theta_0 - (\theta_0 - 2)\zeta_2}{4\left(\theta_0^4 + 4\,\theta_0^3 + 3\,\theta_0^2\right)} & C_1 & 4 & \theta_0 \\
2\,\mu_{19} & \overline{A_0} & 4 & \theta_0 \\
\frac{2\,(\alpha_2\theta_0 - 2\,\alpha_2)}{\theta_0} & \overline{A_0} & -\theta_0 + 1 & 2\,\theta_0 + 1 \\
\frac{2\,(\alpha_2\theta_0 - 2\,\alpha_2)}{\theta_0} & \overline{A_1} & -\theta_0 + 1 & 2\,\theta_0 + 2 \\
-\frac{(\theta_0 - 2)\overline{\zeta_0}}{\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2} & C_1 & -\theta_0 + 1 & 2\,\theta_0 + 2 \\
2\,\mu_8 & \overline{A_0} & -\theta_0 + 1 & 2\,\theta_0 + 2 \\
\frac{2\,(\alpha_2\theta_0 - 2\,\alpha_2)}{\theta_0} & \overline{A_2} & -\theta_0 + 1 & 2\,\theta_0 + 3 \\
-\frac{2\left(2\,\theta_0^2 - \theta_0 - 6\right)\overline{\zeta_1}}{\theta_0^5 + 6\,\theta_0^4 + 11\,\theta_0^3 + 6\,\theta_0^2} & C_1 & -\theta_0 + 1 & 2\,\theta_0 + 3 \\
-\frac{(\theta_0 - 2)\overline{\zeta_0}}{\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0} & B_1 & -\theta_0 + 1 & 2\,\theta_0 + 3 \\
\frac{4\,(\alpha_2\theta_0 - 2\,\alpha_2)\overline{\zeta_0}}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_0 & -\theta_0 + 1 & 2\,\theta_0 + 3 \\
2\,\mu_8 & \overline{A_1} & -\theta_0 + 1 & 2\,\theta_0 + 3 \\
2\,\mu_{20} & \overline{A_0} & -\theta_0 + 1 & 2\,\theta_0 + 3 \\
2\,\mu_9 & \overline{A_0} & -\theta_0 + 2 & 2\,\theta_0 + 1 \\
-\frac{(\theta_0 - 3)\overline{\zeta_0}}{\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2} & C_2 & -\theta_0 + 2 & 2\,\theta_0 + 2 \\
2\,\mu_9 & \overline{A_1} & -\theta_0 + 2 & 2\,\theta_0 + 2 \\
2\,\mu_{21} & \overline{A_0} & -\theta_0 + 2 & 2\,\theta_0 + 2
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{2\left(\alpha_7\theta_0^2 - 6\,\alpha_7\theta_0 + 8\,\alpha_7\right)}{\theta_0^2} & \overline{A_1} & -\theta_0 + 3 & 2\,\theta_0 + 1 \\
\frac{\theta_0 - 2}{2\,\theta_0^2 + \theta_0} & E_2 & -\theta_0 + 3 & 2\,\theta_0 + 1 \\
\frac{2\,\alpha_2\theta_0^3 - 7\,\alpha_2\theta_0^2 + 2\,\alpha_2\theta_0 + 8\,\alpha_2}{4\left(2\,\theta_0^3 + \theta_0^2\right)} & C_1 & -\theta_0 + 3 & 2\,\theta_0 + 1 \\
2\,\mu_{22} & \overline{A_0} & -\theta_0 + 3 & 2\,\theta_0 + 1 \\
\frac{\theta_0 - 2}{2\,\theta_0^2} & E_1 & -\theta_0 + 3 & 2\,\theta_0 \\
\frac{2\left(\alpha_7\theta_0^2 - 6\,\alpha_7\theta_0 + 8\,\alpha_7\right)}{\theta_0^2} & \overline{A_0} & -\theta_0 + 3 & 2\,\theta_0 \\
\frac{2\,\theta_0 - 5}{4\,\theta_0^2} & E_3 & -\theta_0 + 4 & 2\,\theta_0 \\
2\,\mu_{23} & \overline{A_0} & -\theta_0 + 4 & 2\,\theta_0 \\
-\frac{2}{\theta_0} & A_1 & \theta_0 & 0 \\
\frac{4\,|A_1|^2}{\theta_0} & A_0 & \theta_0 & 1 \\
-\frac{1}{4\,\theta_0} & \overline{B_1} & \theta_0 & 2 \\
\frac{2\,\overline{\alpha_5}}{\theta_0} & A_0 & \theta_0 & 2 \\
-\frac{2\left(\theta_0\overline{\alpha_2} + \overline{\alpha_2}\right)}{\theta_0} & \overline{A_0} & \theta_0 & 2 \\
-\frac{1}{6\,\theta_0} & \overline{B_3} & \theta_0 & 3 \\
\frac{\zeta_2}{\theta_0^3 + \theta_0^2} & A_1 & \theta_0 & 3 \\
-\frac{2\left(\theta_0\overline{\alpha_2} + \overline{\alpha_2}\right)}{\theta_0} & \overline{A_1} & \theta_0 & 3 \\
-\frac{2\left(2\,|A_1|^2\overline{\alpha_1} - \overline{\alpha_6}\right)}{\theta_0} & A_0 & \theta_0 & 3 \\
2\,\mu_{10} & \overline{A_0} & \theta_0 & 3 \\
\frac{\left(\theta_0^2 + \theta_0 - 6\right)|A_1|^2}{6\left(\theta_0^3 + \theta_0^2\right)} & \overline{C_1} & \theta_0 & 3 \\
-\frac{1}{8\,\theta_0} & \overline{B_6} & \theta_0 & 4 \\
-\frac{\overline{\alpha_1}}{8\,\theta_0} & \overline{B_1} & \theta_0 & 4 \\
\frac{4\,\mu_1 + \overline{\alpha_5}}{8\,\theta_0} & \overline{C_1} & \theta_0 & 4 \\
-\frac{2\left(\left(\theta_0^2 + 4\,\theta_0\right)\overline{\zeta_8} - \overline{\alpha_4}\right)}{\theta_0} & A_1 & \theta_0 & 4 \\
-\frac{2\left(\theta_0\overline{\alpha_2} + \overline{\alpha_2}\right)}{\theta_0} & \overline{A_2} & \theta_0 & 4 \\
\pi_2 & A_0 & \theta_0 & 4 \\
2\,\mu_{24} & \overline{A_0} & \theta_0 & 4 \\
2\,\mu_{10} & \overline{A_1} & \theta_0 & 4 \\
\frac{\left(\theta_0^2 + \theta_0 - 12\right)|A_1|^2}{12\left(\theta_0^3 + \theta_0^2\right)} & \overline{C_2} & \theta_0 & 4 \\
2\,\mu_{25} & \overline{A_0} & \theta_0 - 1 & 5
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{4}{\theta_0} & A_2 & \theta_0 + 1 & 0 \\
-\frac{4\,\zeta_0}{\theta_0^2 + \theta_0} & \overline{A_0} & \theta_0 + 1 & 0 \\
\frac{4\,\alpha_1}{\theta_0} & A_0 & \theta_0 + 1 & 0 \\
-\frac{4\,\zeta_0}{\theta_0^2 + \theta_0} & \overline{A_1} & \theta_0 + 1 & 1 \\
\frac{4\,\alpha_5}{\theta_0} & A_0 & \theta_0 + 1 & 1 \\
\frac{4\,(\theta_0 + 2)|A_1|^2}{\theta_0^2 + \theta_0} & A_1 & \theta_0 + 1 & 1 \\
-\frac{1}{2\,\theta_0} & \overline{B_2} & \theta_0 + 1 & 2 \\
-\frac{4\,\zeta_0}{\theta_0^2 + \theta_0} & \overline{A_2} & \theta_0 + 1 & 2 \\
\frac{2\,(\theta_0\overline{\alpha_5} + 2\,\overline{\alpha_5})}{\theta_0^2 + \theta_0} & A_1 & \theta_0 + 1 & 2 \\
-\frac{4\,\left(2\,(\theta_0 + 2)|A_1|^4 + (\alpha_1\overline{\alpha_1} - \beta)\theta_0 + \alpha_1\overline{\alpha_1} - \beta\right)}{\theta_0^2 + \theta_0} & A_0 & \theta_0 + 1 & 2 \\
2\,\mu_{11} & \overline{A_0} & \theta_0 + 1 & 2 \\
-\frac{1}{3\,\theta_0} & \overline{B_5} & \theta_0 + 1 & 3 \\
\frac{2\,\zeta_2}{\theta_0^3 + \theta_0^2} & A_2 & \theta_0 + 1 & 3 \\
-\frac{4\,\zeta_0}{\theta_0^2 + \theta_0} & \overline{A_3} & \theta_0 + 1 & 3 \\
\frac{\alpha_5 + 3\,\mu_3}{6\,\theta_0} & \overline{C_1} & \theta_0 + 1 & 3 \\
-\frac{4\,\left((\theta_0\overline{\alpha_1} + 2\,\overline{\alpha_1})|A_1|^2 - \theta_0\overline{\alpha_6} + \theta_0\overline{\zeta_{11}} - \overline{\alpha_6}\right)}{\theta_0^2 + \theta_0} & A_1 & \theta_0 + 1 & 3 \\
-\frac{2\,\left(2\,(\alpha_5\overline{\alpha_1} + \alpha_1\overline{\alpha_3} - \alpha_{16})\theta_0^2 + 4\,\left(\theta_0^2\overline{\alpha_5} + 2\,\theta_0\overline{\alpha_5}\right)|A_1|^2 + 2\,(\alpha_5\overline{\alpha_1} + \alpha_1\overline{\alpha_3} - \alpha_{16})\theta_0 + \alpha_1\overline{\zeta_2}\right)}{\theta_0^3 + \theta_0^2} & A_0 & \theta_0 + 1 & 3 \\
2\,\mu_{26} & \overline{A_0} & \theta_0 + 1 & 3 \\
2\,\mu_{11} & \overline{A_1} & \theta_0 + 1 & 3 \\
-\frac{\left(\theta_0^2 - \theta_0 + 4\right)|A_1|^2}{6\,\left(\theta_0^3 + 2\,\theta_0^2 + \theta_0\right)} & \overline{B_1} & \theta_0 + 1 & 3 \\
-\frac{6}{\theta_0} & A_3 & \theta_0 + 2 & 0 \\
\frac{6\,\alpha_3}{\theta_0} & A_0 & \theta_0 + 2 & 0 \\
\frac{4\,(\alpha_1\theta_0 + 3\,\alpha_1)}{\theta_0^2 + 2\,\theta_0} & A_1 & \theta_0 + 2 & 0 \\
2\,\mu_{12} & \overline{A_0} & \theta_0 + 2 & 0 \\
\frac{4\,(\alpha_5\theta_0 + 3\,\alpha_5)}{\theta_0^2 + 2\,\theta_0} & A_1 & \theta_0 + 2 & 1 \\
\frac{2\,\left(3\,\alpha_6\theta_0^2 - 2\,\left(3\,\alpha_1\theta_0^2 + 13\,\alpha_1\theta_0 + 12\,\alpha_1\right)|A_1|^2 + 9\,\alpha_6\theta_0 + 6\,\alpha_6\right)}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_0 & \theta_0 + 2 & 1 \\
2\,\mu_{13} & \overline{A_0} & \theta_0 + 2 & 1 \\
2\,\mu_{12} & \overline{A_1} & \theta_0 + 2 & 1 \\
\frac{4\,(\theta_0 + 3)|A_1|^2}{\theta_0^2 + \theta_0} & A_2 & \theta_0 + 2 & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{3}{4\,\theta_0} & \overline{B_4} & \theta_0+2 & 2 \\
-\frac{\theta_0^2\overline{\alpha_2}+4\,\theta_0\overline{\alpha_2}-6\,\overline{\alpha_2}}{4\,\theta_0^2} & C_1 & \theta_0+2 & 2 \\
\frac{\mu_5}{2\,\theta_0} & \overline{C_1} & \theta_0+2 & 2 \\
-\frac{\alpha_1}{4\,(\theta_0+2)} & \overline{B_1} & \theta_0+2 & 2 \\
\frac{2\,(\theta_0\overline{\alpha_5}+3\,\overline{\alpha_5})}{\theta_0^2+\theta_0} & A_2 & \theta_0+2 & 2 \\
\pi_3 & A_0 & \theta_0+2 & 2 \\
\pi_4 & A_1 & \theta_0+2 & 2 \\
2\,\mu_{27} & \overline{A_0} & \theta_0+2 & 2 \\
2\,\mu_{13} & \overline{A_1} & \theta_0+2 & 2 \\
2\,\mu_{12} & \overline{A_2} & \theta_0+2 & 2 \\
-\frac{8}{\theta_0} & A_4 & \theta_0+3 & 0 \\
-\frac{\zeta_0}{2\left(\theta_0^2+\theta_0\right)} & C_1 & \theta_0+3 & 0 \\
\frac{4\,(\alpha_1\theta_0+4\,\alpha_1)}{\theta_0^2+2\,\theta_0} & A_2 & \theta_0+3 & 0 \\
-\frac{4\left(4\,\alpha_1^2+\left(\alpha_1^2-2\,\alpha_4\right)\theta_0-4\,\alpha_4\right)}{\theta_0^2+2\,\theta_0} & A_0 & \theta_0+3 & 0 \\
\frac{6\,\alpha_3\theta_0^2+30\,\alpha_3\theta_0+24\,\alpha_3+\zeta_2}{\theta_0^3+4\,\theta_0^2+3\,\theta_0} & A_1 & \theta_0+3 & 0 \\
2\,\mu_{14} & \overline{A_0} & \theta_0+3 & 0 \\
-\frac{\zeta_0}{2\left(\theta_0^2+\theta_0\right)} & B_1 & \theta_0+3 & 1 \\
\frac{4\,(\alpha_5\theta_0+4\,\alpha_5)}{\theta_0^2+2\,\theta_0} & A_2 & \theta_0+3 & 1 \\
\pi_5 & A_1 & \theta_0+3 & 1 \\
\pi_6 & A_0 & \theta_0+3 & 1 \\
2\,\mu_{28} & \overline{A_0} & \theta_0+3 & 1 \\
2\,\mu_{14} & \overline{A_1} & \theta_0+3 & 1 \\
\frac{4\,(\theta_0+4)|A_1|^2}{\theta_0^2+\theta_0} & A_3 & \theta_0+3 & 1 \\
-\frac{10}{\theta_0} & A_5 & \theta_0+4 & 0 \\
-\frac{\zeta_0}{3\left(\theta_0^2+\theta_0\right)} & C_2 & \theta_0+4 & 0 \\
\frac{4\,(\alpha_1\theta_0+5\,\alpha_1)}{\theta_0^2+2\,\theta_0} & A_3 & \theta_0+4 & 0 \\
-\frac{2\left(10\,\alpha_1^2+\left(2\,\alpha_1^2-5\,\alpha_4\right)\theta_0+\left(\theta_0^3+2\,\theta_0^2\right)\zeta_8-10\,\alpha_4\right)}{\theta_0^2+2\,\theta_0} & A_1 & \theta_0+4 & 0 \\
\frac{2\left(3\,\alpha_3\theta_0^2+18\,\alpha_3\theta_0+15\,\alpha_3+\zeta_2\right)}{\theta_0^3+4\,\theta_0^2+3\,\theta_0} & A_2 & \theta_0+4 & 0 \\
-\frac{2\left(5\,(\alpha_1\alpha_3-\overline{\alpha_{14}})\theta_0^3+6\,(7\,\alpha_1\alpha_3-5\,\overline{\alpha_{14}})\theta_0^2+60\,\alpha_1\alpha_3+(97\,\alpha_1\alpha_3-55\,\overline{\alpha_{14}})\theta_0+(\alpha_1\theta_0+2\,\alpha_1)\zeta_2-30\,\overline{\alpha_{14}}\right)}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & A_0 & \theta_0+4 & 0 \\
2\,\mu_{29} & \overline{A_0} & \theta_0+4 & 0 \\
\frac{1}{4}\,\mu_{12} & C_1 & \theta_0+4 & 0
\end{pmatrix}
$$

$$\begin{pmatrix}
\frac{1}{2\,(\theta_0-4)} & \overline{E_1} & 2\,\theta_0-1 & -\theta_0+4 \\
2\,\overline{\alpha_7} & A_0 & 2\,\theta_0-1 & -\theta_0+4 \\
\frac{1}{2\,(\theta_0-5)} & \overline{E_3} & 2\,\theta_0-1 & -\theta_0+5 \\
2\,\overline{\alpha_{13}} & A_0 & 2\,\theta_0-1 & -\theta_0+5 \\
-\frac{\zeta_0}{\theta_0^3+\theta_0^2} & \overline{C_1} & 2\,\theta_0+1 & -\theta_0+2 \\
\frac{2\,(\theta_0\overline{\alpha_9}+2\,\overline{\alpha_9})}{\theta_0} & A_0 & 2\,\theta_0+1 & -\theta_0+2 \\
\frac{2\,(\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2})}{\theta_0} & A_1 & 2\,\theta_0+1 & -\theta_0+2 \\
-\frac{\zeta_0}{\theta_0^3+\theta_0^2} & \overline{C_2} & 2\,\theta_0+1 & -\theta_0+3 \\
\frac{2\,(\theta_0\overline{\alpha_8}+2\,\overline{\alpha_8})}{\theta_0} & A_1 & 2\,\theta_0+1 & -\theta_0+3 \\
-\frac{2\left(2\,(\theta_0\overline{\alpha_2}+4\,\overline{\alpha_2})|A_1|^2-\theta_0\overline{\alpha_{11}}-2\,\overline{\alpha_{11}}\right)}{\theta_0} & A_0 & 2\,\theta_0+1 & -\theta_0+3 \\
-\frac{\zeta_0}{\theta_0^3+2\,\theta_0^2+\theta_0} & \overline{B_1} & 2\,\theta_0+2 & -\theta_0+2 \\
\frac{\mu_{12}}{2\,\theta_0} & \overline{C_1} & 2\,\theta_0+2 & -\theta_0+2 \\
\frac{2\,(\theta_0\overline{\alpha_9}+3\,\overline{\alpha_9})}{\theta_0} & A_1 & 2\,\theta_0+2 & -\theta_0+2 \\
\frac{2\,(\theta_0\overline{\alpha_2}+3\,\overline{\alpha_2})}{\theta_0} & A_2 & 2\,\theta_0+2 & -\theta_0+2 \\
-\frac{2\left((\alpha_1\overline{\alpha_2}-\overline{\alpha_{10}})\theta_0^2+(9\,\alpha_1\overline{\alpha_2}-5\,\overline{\alpha_{10}})\theta_0+12\,\alpha_1\overline{\alpha_2}-6\,\overline{\alpha_{10}}\right)}{\theta_0^2+2\,\theta_0} & A_0 & 2\,\theta_0+2 & -\theta_0+2 \\
2\,\mu_{30} & \overline{A_0} & 2\,\theta_0+2 & -\theta_0+2 \\
\frac{2\,(\theta_0\overline{\alpha_2}+\overline{\alpha_2})}{\theta_0} & A_0 & 2\,\theta_0 & -\theta_0+2 \\
\frac{2\,(\theta_0\overline{\alpha_8}+\overline{\alpha_8})}{\theta_0} & A_0 & 2\,\theta_0 & -\theta_0+3 \\
\frac{2\left(\theta_0^2\overline{\alpha_7}+\theta_0\overline{\alpha_7}-2\,\overline{\alpha_7}\right)}{\theta_0^2} & A_1 & 2\,\theta_0 & -\theta_0+4 \\
-\frac{2\,((\overline{\alpha_1\alpha_2}-\overline{\alpha_{12}})\theta_0+\overline{\alpha_1\alpha_2}-\overline{\alpha_{12}})}{\theta_0} & A_0 & 2\,\theta_0 & -\theta_0+4 \\
\frac{\theta_0+1}{2\left(\theta_0^2-4\,\theta_0\right)} & \overline{E_2} & 2\,\theta_0 & -\theta_0+4 \\
\frac{\theta_0^3\overline{\alpha_2}-3\,\theta_0^2\overline{\alpha_2}+4\,\theta_0\overline{\alpha_2}+8\,\overline{\alpha_2}}{4\left(\theta_0^3-4\,\theta_0^2\right)} & \overline{C_1} & 2\,\theta_0 & -\theta_0+4 \\
2\,\mu_{31} & \overline{A_0} & 2\,\theta_0 & -\theta_0+4
\end{pmatrix}$$

where

$$\pi_1=\frac{2}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0}\Big\{(\alpha_2\overline{\alpha_1}-\alpha_{10})\theta_0^4+4\,(\alpha_2\overline{\alpha_1}-\alpha_{10})\theta_0^3-(\alpha_2\overline{\alpha_1}-\alpha_{10})\theta_0^2-16\,(\alpha_2\overline{\alpha_1}-\alpha_{10})\theta_0-12\,\alpha_2\overline{\alpha_1}$$
$$-4\,(\alpha_5\theta_0+3\,\alpha_5)\overline{\zeta_0}-8\,(2\,\alpha_1\theta_0+3\,\alpha_1)\overline{\zeta_1}+12\,\alpha_{10}\Big\}$$

$$\pi_2=-\frac{2}{\theta_0^4+3\,\theta_0^3+2\,\theta_0^2}\Big\{(\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0^3+3\,(\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0^2+\left(2\,\theta_0^3\overline{\alpha_3}+6\,\theta_0^2\overline{\alpha_3}+4\,\theta_0\overline{\alpha_3}+(\theta_0+2)\overline{\zeta_2}\right)|A_1|^2$$
$$+2\,(\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0+2\left(\theta_0^2\overline{\alpha_2}+\theta_0\overline{\alpha_2}\right)\overline{\zeta_0}\Big\}$$

$$\pi_3=-\frac{2}{\theta_0^3+3\,\theta_0^2+2\,\theta_0}\Big\{3\,(\alpha_3\overline{\alpha_1}+\alpha_1\overline{\alpha_5}-\overline{\alpha_{16}})\theta_0^2+2\left(3\,\alpha_5\theta_0^2+13\,\alpha_5\theta_0+12\,\alpha_5\right)|A_1|^2$$
$$+(9\,\alpha_3\overline{\alpha_1}+13\,\alpha_1\overline{\alpha_5}-9\,\overline{\alpha_{16}})\theta_0+6\,\alpha_3\overline{\alpha_1}+12\,\alpha_1\overline{\alpha_5}-6\,\overline{\alpha_{16}}\Big\}$$

$$\pi_4 = -\frac{2\left(4\left(\theta_0^2 + 5\,\theta_0 + 6\right)|A_1|^4 + (2\,\alpha_1\overline{\alpha_1} - 3\,\beta)\theta_0^2 + (8\,\alpha_1\overline{\alpha_1} - 9\,\beta)\theta_0 + \left(\theta_0^4 + 5\,\theta_0^3 + 8\,\theta_0^2 + 4\,\theta_0\right)\zeta_{10} + 6\,\alpha_1\overline{\alpha_1} - 6\,\beta\right)}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0}$$

$$\pi_5 = \frac{4\left(2\,\alpha_6\theta_0^3 + 12\,\alpha_6\theta_0^2 - \left(3\,\alpha_1\theta_0^3 + 25\,\alpha_1\theta_0^2 + 64\,\alpha_1\theta_0 + 48\,\alpha_1\right)|A_1|^2 + 22\,\alpha_6\theta_0 - \left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)\zeta_{11} + 12\,\alpha_6\right)}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0}$$

$$\pi_6 = -\frac{2}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0}\left\{4\,(\alpha_1\alpha_5 - \overline{\alpha_{15}})\theta_0^3\right.$$
$$+ 8\,(4\,\alpha_1\alpha_5 - 3\,\overline{\alpha_{15}})\theta_0^2 + \left(8\,\alpha_3\theta_0^3 + 60\,\alpha_3\theta_0^2 + 136\,\alpha_3\theta_0 + (\theta_0 + 2)\zeta_2 + 96\,\alpha_3\right)|A_1|^2 + 48\,\alpha_1\alpha_5$$
$$\left. + 4\,(19\,\alpha_1\alpha_5 - 11\,\overline{\alpha_{15}})\theta_0 - 24\,\overline{\alpha_{15}}\right\}$$

Finally,

$\mathrm{Re}\,(\partial_{\overline{z}}\vec{F}(z)) =$

$$\begin{pmatrix}
\frac{\mu_1\theta_0^2 + 2\,\mu_1\theta_0 + 2\,\overline{\alpha_5}}{\theta_0} & \overline{A_0} & 0 & \theta_0 + 1 \\
\frac{\mu_1\theta_0^3 + 5\,\mu_1\theta_0^2 + 2\,(3\,\mu_1 + \overline{\alpha_5})\theta_0 + 6\,\overline{\alpha_5}}{\theta_0^2 + 2\,\theta_0} & \overline{A_1} & 0 & \theta_0 + 2 \\
\frac{\theta_0^3\overline{\mu_{13}} - 4\,(\theta_0 + 3)|A_1|^2\overline{\zeta_0} + 3\,\theta_0^2\overline{\mu_{13}} + 2\,\theta_0\overline{\mu_{13}}}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_0 & 0 & \theta_0 + 2 \\
\frac{\theta_0^3\overline{\mu_{12}} + 3\,\theta_0^2\overline{\mu_{12}} + 2\,\theta_0\overline{\mu_{12}} + 4\,(2\,\theta_0 + 3)\overline{\zeta_1}}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_1 & 0 & \theta_0 + 2 \\
\frac{\mu_2\theta_0^4 + 6\,\mu_2\theta_0^3 + (11\,\mu_2 + 3\,\overline{\alpha_6})\theta_0^2 - 2\left(3\,\theta_0^2\overline{\alpha_1} + 13\,\theta_0\overline{\alpha_1} + 12\,\overline{\alpha_1}\right)|A_1|^2 + 3\,(2\,\mu_2 + 3\,\overline{\alpha_6})\theta_0 + 6\,\overline{\alpha_6}}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & \overline{A_0} & 0 & \theta_0 + 2 \\
\frac{3\,\overline{\zeta_0}}{4\,(\theta_0^3 + 4\,\theta_0^2 + 5\,\theta_0 + 2)} & \overline{B_1} & 0 & \theta_0 + 3 \\
\frac{\mu_1\theta_0^3 + 6\,\mu_1\theta_0^2 + 2\,(4\,\mu_1 + \overline{\alpha_5})\theta_0 + 8\,\overline{\alpha_5}}{\theta_0^2 + 2\,\theta_0} & \overline{A_2} & 0 & \theta_0 + 3 \\
\frac{\theta_0^3\overline{\mu_{14}} + 3\,\theta_0^2\overline{\mu_{14}} + 2\,\theta_0\overline{\mu_{14}} - 2\,(\theta_0\overline{\alpha_1} + 4\,\overline{\alpha_1})\overline{\zeta_0} - 2\left(\theta_0^5 + 9\,\theta_0^4 + 28\,\theta_0^3 + 36\,\theta_0^2 + 16\,\theta_0\right)\overline{\zeta_9}}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_1 & 0 & \theta_0 + 3 \\
\frac{\theta_0^4\overline{\mu_{28}} + 6\,\theta_0^3\overline{\mu_{28}} - 8\left(2\,\theta_0^2 + 11\,\theta_0 + 12\right)|A_1|^2\overline{\zeta_1} + 11\,\theta_0^2\overline{\mu_{28}} + 6\,\theta_0\overline{\mu_{28}} - 2\left(\theta_0^2\overline{\alpha_5} + 7\,\theta_0\overline{\alpha_5} + 12\,\overline{\alpha_5}\right)\overline{\zeta_0}}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0} & A_0 & 0 & \theta_0 + 3 \\
\lambda_1 & \overline{A_1} & 0 & \theta_0 + 3 \\
\lambda_2 & \overline{A_0} & 0 & \theta_0 + 3 \\
-\frac{\theta_0 - 1}{2\,\theta_0^2} & B_1 & 1 & \theta_0 \\
\frac{\mu_3\theta_0^2 + \mu_3\theta_0 + 2\,\alpha_5}{\theta_0} & \overline{A_0} & 1 & \theta_0 \\
-\frac{2\left(\alpha_2\theta_0^2 - \alpha_2\right)}{\theta_0} & A_0 & 1 & \theta_0 \\
-\frac{\theta_0 - 1}{\theta_0^2} & B_2 & 1 & \theta_0 + 1 \\
\frac{\mu_3\theta_0^3 + 3\,\mu_3\theta_0^2 + 2\,(\alpha_5 + \mu_3)\theta_0 + 4\,\alpha_5}{\theta_0^2 + \theta_0} & \overline{A_1} & 1 & \theta_0 + 1 \\
-\frac{\alpha_9\theta_0^4 + (\alpha_9 - 2\,\overline{\mu_{11}})\theta_0^3 - 2\,(2\,\alpha_9 + \overline{\mu_{11}})\theta_0^2 + 4\,\alpha_1\theta_0\overline{\zeta_0} - 4\,\alpha_9\theta_0 + \left(\theta_0^2 - \theta_0 - 2\right)\zeta_4}{\theta_0^3 + \theta_0^2} & A_0 & 1 & \theta_0 + 1 \\
\frac{4\,\mu_4\theta_0^5 + 12\,\mu_4\theta_0^4 - 32\left(\theta_0^3 + 2\,\theta_0^2\right)|A_1|^4 - 8\,(2\,\alpha_1\overline{\alpha_1} - 2\,\beta - \mu_4)\theta_0^3 - 16\,(\alpha_1\overline{\alpha_1} - \beta)\theta_0^2 - \left(\theta_0^3 - 3\,\theta_0 - 2\right)|C_1|^2}{4\,(\theta_0^4 + \theta_0^3)} & \overline{A_0} & 1 & \theta_0 + 1 \\
\frac{1}{\theta_0^2} & \overline{B_9} & 1 & \theta_0 + 2 \\
\frac{\theta_0 + 3}{2\,\theta_0^2} & B_{10} & 1 & \theta_0 + 2 \\
\frac{\theta_0 - 5}{4\,\theta_0} & B_4 & 1 & \theta_0 + 2
\end{pmatrix}$$

$$
\begin{pmatrix}
-\frac{\theta_0\overline{\alpha_1} - \overline{\alpha_1}}{2\,(\theta_0^2 + 2\,\theta_0)} & B_1 & 1 & \theta_0 + 2 \\
\frac{\mu_3\theta_0^3 + 4\,\mu_3\theta_0^2 + (2\,\alpha_5 + 3\,\mu_3)\theta_0 + 6\,\alpha_5}{\theta_0^2 + \theta_0} & \overline{A_2} & 1 & \theta_0 + 2 \\
\frac{2\left(\theta_0^3\overline{\mu_{13}} - 4\,(\theta_0 + 3)|A_1|^2\overline{\zeta_0} + 3\,\theta_0^2\overline{\mu_{13}} + 2\,\theta_0\overline{\mu_{13}}\right)}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_1 & 1 & \theta_0 + 2 \\
\frac{2\left(\theta_0^3\overline{\mu_{12}} + 3\,\theta_0^2\overline{\mu_{12}} + 2\,\theta_0\overline{\mu_{12}} + 4\,(2\,\theta_0 + 3)\overline{\zeta_1}\right)}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_2 & 1 & \theta_0 + 2 \\
\lambda_3 & \overline{A_1} & 1 & \theta_0 + 2 \\
\lambda_4 & \overline{A_0} & 1 & \theta_0 + 2 \\
\lambda_5 & A_0 & 1 & \theta_0 + 2 \\
\frac{2\,\theta_0^3\overline{\mu_5} + 2\,\theta_0^2\overline{\mu_5} - \left(\theta_0^2 + \theta_0 - 3\right)\overline{\zeta_2}}{4\,(\theta_0^4 + \theta_0^3)} & C_1 & 1 & \theta_0 + 2 \\
-\frac{1}{4}\,\alpha_2\theta_0 - \frac{1}{4}\,\alpha_2 & \overline{C_1} & 1 & \theta_0 + 2 \\
-\frac{2\,\theta_0 - 3}{4\,\theta_0^2} & B_3 & 2 & \theta_0 \\
-\frac{2\,\alpha_8\theta_0^3 - 2\,(2\,\alpha_8 + 3\,\overline{\mu_{10}})\theta_0^2 - 6\,\alpha_8\theta_0 + (\theta_0 - 3)\zeta_3}{2\,\theta_0^2} & A_0 & 2 & \theta_0 \\
\frac{\mu_6\theta_0^2 - 6\,\alpha_1|A_1|^2 + \mu_6\theta_0 + 3\,\alpha_6}{\theta_0} & \overline{A_0} & 2 & \theta_0 \\
-\frac{2\,\alpha_2\theta_0^2 - \alpha_2\theta_0 - 3\,\alpha_2}{\theta_0} & A_1 & 2 & \theta_0 \\
\frac{2\,\mu_5\theta_0^4 - 4\,\mu_5\theta_0^3 - 6\,\mu_5\theta_0^2 + (5\,\theta_0 - 12)\zeta_2}{2\,(\theta_0^3 - 3\,\theta_0^2)} & \overline{A_1} & 2 & \theta_0 \\
\frac{2\,\mu_5\theta_0^3 + 2\,\mu_5\theta_0^2 + 3\,\zeta_2}{2\,(\theta_0^2 + \theta_0)} & \overline{A_0} & 2 & \theta_0 - 1 \\
\frac{3}{2\,\theta_0^2} & \overline{B_8} & 2 & \theta_0 + 1 \\
\frac{\theta_0 + 2}{2\,\theta_0^2} & B_{11} & 2 & \theta_0 + 1 \\
\frac{\theta_0 - 5}{4\,\theta_0} & B_5 & 2 & \theta_0 + 1 \\
\frac{\mu_6\theta_0^3 + 3\,\mu_6\theta_0^2 - 6\,(\alpha_1\theta_0 + 2\,\alpha_1)|A_1|^2 + 2\,(3\,\alpha_6 + \mu_6)\theta_0 - 6\,\theta_0\zeta_{11} + 6\,\alpha_6}{\theta_0^2 + \theta_0} & \overline{A_1} & 2 & \theta_0 + 1 \\
-\frac{\alpha_9\theta_0^3 - 3\,\theta_0^2\overline{\mu_{11}} - (7\,\alpha_9 + 3\,\overline{\mu_{11}})\theta_0 + (\theta_0 + 1)\zeta_4 + 6\,\alpha_1\overline{\zeta_0} - 6\,\alpha_9}{\theta_0^2 + \theta_0} & A_1 & 2 & \theta_0 + 1 \\
\lambda_6 & A_0 & 2 & \theta_0 + 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{2\,\mu_5\theta_0^4 + 6\,\mu_5\theta_0^3 + 4\,\mu_5\theta_0^2 - \left(2\,\theta_0^2 - \theta_0 - 6\right)\zeta_2}{2\,(\theta_0^3 + \theta_0^2)} & \overline{A_2} & 2 & \theta_0 + 1 \\
\lambda_7 & \overline{A_0} & 2 & \theta_0 + 1 \\
\frac{\mu_1\theta_0^4 + 3\,\mu_1\theta_0^3 + 2\,(\mu_1 + \overline{\alpha_5} + 3\,\overline{\mu_3})\theta_0^2 + 2\,\theta_0(\overline{\alpha_5} + 3\,\overline{\mu_3}) + 12\,\overline{\alpha_5}}{8\,(\theta_0^3 + \theta_0^2)} & C_1 & 2 & \theta_0 + 1 \\
-\frac{(2\,\theta_0 - 1)|A_1|^2}{2\,(\theta_0^2 + 2\,\theta_0 + 1)} & B_1 & 2 & \theta_0 + 1 \\
\frac{2}{\theta_0^2} & \overline{B_7} & 3 & \theta_0 \\
\frac{\theta_0 + 1}{2\,\theta_0^2} & B_{12} & 3 & \theta_0 \\
\frac{\theta_0 - 5}{4\,\theta_0} & B_6 & 3 & \theta_0 \\
\frac{\mu_7\theta_0^2 + \mu_7\theta_0 - \theta_0\zeta_7 - 4\left(\theta_0^2 + 4\,\theta_0\right)\zeta_8 + 4\,\alpha_4}{\theta_0} & \overline{A_1} & 3 & \theta_0 \\
-\frac{2\,\alpha_8\theta_0^2 - 2\,(3\,\alpha_8 + 4\,\overline{\mu_{10}})\theta_0 - 8\,\alpha_8 + \zeta_3}{2\,\theta_0} & A_1 & 3 & \theta_0 \\
-\frac{\alpha_1}{2\,(\theta_0 + 2)} & B_1 & 3 & \theta_0 \\
\lambda_8 & A_0 & 3 & \theta_0 \\
\frac{\mu_3\theta_0^4 + 3\,\mu_3\theta_0^3 + 2\,(\alpha_5 + \mu_3 + 4\,\overline{\mu_1})\theta_0^2 + 4\,(\alpha_5 + 4\,\overline{\mu_1})\theta_0 + 16\,\alpha_5}{8\,(\theta_0^3 + 2\,\theta_0^2)} & C_1 & 3 & \theta_0 \\
\lambda_9 & \overline{A_0} & 3 & \theta_0 \\
-2\,\alpha_2\theta_0 - 2\,\alpha_2 & A_2 & 3 & \theta_0 \\
\frac{\mu_7\theta_0^2 - \theta_0\zeta_7 + \zeta_5}{\theta_0} & \overline{A_0} & 3 & \theta_0 - 1 \\
\frac{1}{2\,\theta_0} & B_{13} & 4 & \theta_0 - 1 \\
\frac{\theta_0 - 5}{4\,\theta_0} & C_4 & 4 & \theta_0 - 1 \\
\frac{\mu_{19}\theta_0^2 + 2\,\mu_{19}\theta_0 - (\theta_0 + 2)\zeta_{17} + \alpha_1\zeta_2}{\theta_0 + 2} & \overline{A_0} & 4 & \theta_0 - 1 \\
\frac{2\,\mu_5\theta_0^3 + 2\,\mu_5\theta_0^2 - (2\,\theta_0 - 1)\zeta_2}{16\,(\theta_0^2 + \theta_0)} & C_1 & 4 & \theta_0 - 1 \\
-\alpha_7\theta_0 + 4\,\alpha_7 & A_1 & 4 & \theta_0 - 1 \\
-\alpha_{13}\theta_0 + 5\,\alpha_{13} - \zeta_{21} + 5\,\overline{\mu_{25}} & A_0 & 4 & \theta_0 - 1 \\
-\frac{(\alpha_9 - 2\,\mu_8)\theta_0^3 + 2\,(\alpha_9 - 3\,\mu_8)\theta_0^2 - 4\,(\alpha_9 + \mu_8)\theta_0 + (\theta_0 - 2)\zeta_4 - 8\,\alpha_9}{\theta_0^2 + 2\,\theta_0} & \overline{A_0} & -\theta_0 + 1 & 2\,\theta_0 + 1 \\
-\frac{(\alpha_9 - 2\,\mu_8)\theta_0^3 + (3\,\alpha_9 - 7\,\mu_8)\theta_0^2 - 2\,(2\,\alpha_9 + 3\,\mu_8)\theta_0 + (\theta_0 - 2)\zeta_4 - 12\,\alpha_9}{\theta_0^2 + 2\,\theta_0} & \overline{A_1} & -\theta_0 + 1 & 2\,\theta_0 + 2 \\
\lambda_{10} & \overline{A_0} & -\theta_0 + 1 & 2\,\theta_0 + 2 \\
-\frac{\theta_0^4\overline{\mu_{30}} + \theta_0^3\overline{\mu_{30}} - 4\,\theta_0^2\overline{\mu_{30}} - 4\,\theta_0\overline{\mu_{30}} - \left(5\,\alpha_2\theta_0^2 - 4\,\alpha_2\theta_0 - 12\,\alpha_2\right)\overline{\zeta_0}}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0} & A_0 & -\theta_0 + 1 & 2\,\theta_0 + 2
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{\theta_0^4\overline{\mu_{12}} + \theta_0^3\overline{\mu_{12}} - 4\,\theta_0^2\overline{\mu_{12}} - 4\,\theta_0\overline{\mu_{12}} + 4\left(2\,\theta_0^2 - \theta_0 - 6\right)\overline{\zeta_1}}{4\,(\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2)} & C_1 & -\theta_0 + 1 & 2\,\theta_0 + 2 \\
\frac{(\theta_0 - 2)\overline{\zeta_0}}{4\,(\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0)} & B_1 & -\theta_0 + 1 & 2\,\theta_0 + 2 \\
-\frac{2\,(\alpha_8 - 2\,\mu_9)\theta_0^3 - 8\,\mu_9\theta_0^2 - 2\,(7\,\alpha_8 + 2\,\mu_9)\theta_0 + (\theta_0 - 3)\zeta_3 - 12\,\alpha_8}{2\,(\theta_0^2 + \theta_0)} & \overline{A_1} & -\theta_0 + 2 & 2\,\theta_0 + 1 \\
\lambda_{11} & \overline{A_0} & -\theta_0 + 2 & 2\,\theta_0 + 1 \\
-\frac{2\,(\alpha_8 - 2\,\mu_9)\theta_0^3 - 2\,(\alpha_8 + 3\,\mu_9)\theta_0^2 - 2\,(5\,\alpha_8 + \mu_9)\theta_0 + (\theta_0 - 3)\zeta_3 - 6\,\alpha_8}{2\,(\theta_0^2 + \theta_0)} & \overline{A_0} & -\theta_0 + 2 & 2\,\theta_0 \\
\frac{\theta_0 - 4}{4\,\theta_0} & E_1 & -\theta_0 + 3 & 2\,\theta_0 - 1 \\
\frac{\alpha_7\theta_0^2 - 8\,\alpha_7\theta_0 + 16\,\alpha_7}{\theta_0} & \overline{A_0} & -\theta_0 + 3 & 2\,\theta_0 - 1 \\
\frac{\theta_0 - 5}{4\,\theta_0} & E_2 & -\theta_0 + 3 & 2\,\theta_0 \\
\frac{\alpha_2\theta_0 - 5\,\alpha_2}{4\,\theta_0} & C_1 & -\theta_0 + 3 & 2\,\theta_0 \\
\frac{(\alpha_1\alpha_2 - \alpha_{12} + 2\,\mu_{22})\theta_0^2 - 4\,\alpha_1\alpha_2 - (3\,\alpha_1\alpha_2 - 3\,\alpha_{12} - \mu_{22})\theta_0 - (2\,\theta_0 + 1)\zeta_{20} + (\theta_0 - 4)\overline{\zeta_{22}} + 4\,\alpha_{12}}{\theta_0} & \overline{A_0} & -\theta_0 + 3 & 2\,\theta_0 \\
\frac{2\,\alpha_7\theta_0^3 - 17\,\alpha_7\theta_0^2 + 31\,\alpha_7\theta_0 + 20\,\alpha_7}{2\,(\theta_0^2 + \theta_0)} & \overline{A_1} & -\theta_0 + 3 & 2\,\theta_0 \\
-\theta_0\overline{\mu_{31}} + 4\,\overline{\mu_{31}} & A_0 & -\theta_0 + 3 & 2\,\theta_0 \\
\frac{\theta_0 - 5}{4\,\theta_0} & E_3 & -\theta_0 + 4 & 2\,\theta_0 - 1 \\
-(\alpha_{13} - 2\,\mu_{23})\theta_0 + 5\,\alpha_{13} - 2\,\zeta_{21} & \overline{A_0} & -\theta_0 + 4 & 2\,\theta_0 - 1 \\
-\frac{\theta_0 - 1}{2\,\theta_0^2} & \overline{B_1} & \theta_0 & 1 \\
\frac{\theta_0^2\overline{\mu_3} + \theta_0\overline{\mu_3} + 2\,\overline{\alpha_5}}{\theta_0} & A_0 & \theta_0 & 1 \\
-\frac{2\left(\theta_0^2\overline{\alpha_2} - \overline{\alpha_2}\right)}{\theta_0} & \overline{A_0} & \theta_0 & 1 \\
-\frac{2\,\theta_0 - 3}{4\,\theta_0^2} & \overline{B_3} & \theta_0 & 2 \\
-\frac{2\,\theta_0^3\overline{\alpha_8} - 2\,(3\,\mu_{10} + 2\,\overline{\alpha_8})\theta_0^2 - 6\,\theta_0\overline{\alpha_8} + (\theta_0 - 3)\overline{\zeta_3}}{2\,\theta_0^2} & \overline{A_0} & \theta_0 & 2 \\
-\frac{2\,\theta_0^2\overline{\alpha_2} - \theta_0\overline{\alpha_2} - 3\,\overline{\alpha_2}}{\theta_0} & \overline{A_1} & \theta_0 & 2 \\
-\frac{6\,|A_1|^2\overline{\alpha_1} - \theta_0^2\overline{\mu_6} - \theta_0\overline{\mu_6} - 3\,\overline{\alpha_6}}{\theta_0} & A_0 & \theta_0 & 2 \\
\frac{2\,\theta_0^4\overline{\mu_5} - 4\,\theta_0^3\overline{\mu_5} - 6\,\theta_0^2\overline{\mu_5} + (5\,\theta_0 - 12)\overline{\zeta_2}}{2\,(\theta_0^3 - 3\,\theta_0^2)} & A_1 & \theta_0 & 2 \\
\frac{2}{\theta_0^2} & B_7 & \theta_0 & 3 \\
\frac{\theta_0 + 1}{2\,\theta_0^2} & \overline{B_{12}} & \theta_0 & 3
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{\overline{\alpha_1}}{2\,(\theta_0 + 2)} & \overline{B_1} & \theta_0 & 3 \\
\frac{\theta_0 - 5}{4\,\theta_0} & \overline{B_6} & \theta_0 & 3 \\
\frac{\theta_0^2\overline{\mu_7} + \theta_0\overline{\mu_7} - \theta_0\overline{\zeta_7} - 4\left(\theta_0^2 + 4\,\theta_0\right)\overline{\zeta_8} + 4\,\overline{\alpha_4}}{\theta_0} & A_1 & \theta_0 & 3 \\
-\frac{2\,\theta_0^2\overline{\alpha_8} - 2\,(4\,\mu_{10} + 3\,\overline{\alpha_8})\theta_0 - 8\,\overline{\alpha_8} + \overline{\zeta_3}}{2\,\theta_0} & \overline{A_1} & \theta_0 & 3 \\
\lambda_{12} & \overline{A_0} & \theta_0 & 3 \\
\frac{\theta_0^4\overline{\mu_3} + 3\,\theta_0^3\overline{\mu_3} + 2\,(4\,\mu_1 + \overline{\alpha_5} + \overline{\mu_3})\theta_0^2 + 4\,(4\,\mu_1 + \overline{\alpha_5})\theta_0 + 16\,\overline{\alpha_5}}{8\,(\theta_0^3 + 2\,\theta_0^2)} & \overline{C_1} & \theta_0 & 3 \\
\lambda_{13} & A_0 & \theta_0 & 3 \\
-2\,\theta_0\overline{\alpha_2} - 2\,\overline{\alpha_2} & \overline{A_2} & \theta_0 & 3 \\
\frac{2\,\theta_0^3\overline{\mu_5} + 2\,\theta_0^2\overline{\mu_5} + 3\,\overline{\zeta_2}}{2\,(\theta_0^2 + \theta_0)} & A_0 & \theta_0 - 1 & 2 \\
\frac{\theta_0^2\overline{\mu_7} - \theta_0\overline{\zeta_7} + \overline{\zeta_5}}{\theta_0} & A_0 & \theta_0 - 1 & 3 \\
\frac{1}{2\,\theta_0} & \overline{B_{13}} & \theta_0 - 1 & 4 \\
\frac{\theta_0 - 5}{4\,\theta_0} & \overline{C_4} & \theta_0 - 1 & 4 \\
\frac{\theta_0^2\overline{\mu_{19}} + 2\,\theta_0\overline{\mu_{19}} - (\theta_0 + 2)\overline{\zeta_{17}} + \overline{\alpha_1}\overline{\zeta_2}}{\theta_0 + 2} & A_0 & \theta_0 - 1 & 4 \\
\frac{2\,\theta_0^3\overline{\mu_5} + 2\,\theta_0^2\overline{\mu_5} - (2\,\theta_0 - 1)\overline{\zeta_2}}{16\,(\theta_0^2 + \theta_0)} & \overline{C_1} & \theta_0 - 1 & 4 \\
-\theta_0\overline{\alpha_7} + 4\,\overline{\alpha_7} & \overline{A_1} & \theta_0 - 1 & 4 \\
-\theta_0\overline{\alpha_{13}} + 5\,\mu_{25} + 5\,\overline{\alpha_{13}} - \overline{\zeta_{21}} & \overline{A_0} & \theta_0 - 1 & 4 \\
\frac{\theta_0^2\overline{\mu_1} + 2\,\theta_0\overline{\mu_1} + 2\,\alpha_5}{\theta_0} & A_0 & \theta_0 + 1 & 0 \\
-\frac{\theta_0 - 1}{\theta_0^2} & \overline{B_2} & \theta_0 + 1 & 1 \\
\frac{\theta_0^3\overline{\mu_3} + 3\,\theta_0^2\overline{\mu_3} + 2\,\theta_0(\overline{\alpha_5} + \overline{\mu_3}) + 4\,\overline{\alpha_5}}{\theta_0^2 + \theta_0} & A_1 & \theta_0 + 1 & 1 \\
-\frac{\theta_0^4\overline{\alpha_9} - (2\,\mu_{11} - \overline{\alpha_9})\theta_0^3 - 2\,(\mu_{11} + 2\,\overline{\alpha_9})\theta_0^2 + 4\,\theta_0\zeta_0\overline{\alpha_1} - 4\,\theta_0\overline{\alpha_9} + \left(\theta_0^2 - \theta_0 - 2\right)\overline{\zeta_4}}{\theta_0^3 + \theta_0^2} & \overline{A_0} & \theta_0 + 1 & 1 \\
\frac{4\,\theta_0^5\overline{\mu_4} - 32\left(\theta_0^3 + 2\,\theta_0^2\right)|A_1|^4 + 12\,\theta_0^4\overline{\mu_4} - 8\,(2\,\alpha_1\overline{\alpha_1} - 2\,\beta - \overline{\mu_4})\theta_0^3 - 16\,(\alpha_1\overline{\alpha_1} - \beta)\theta_0^2 - \left(\theta_0^3 - 3\,\theta_0 - 2\right)|C_1|^2}{4\,(\theta_0^4 + \theta_0^3)} & A_0 & \theta_0 + 1 & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{3}{2\,\theta_0^2} & B_8 & \theta_0+1 & 2 \\
\frac{\theta_0+2}{2\,\theta_0^2} & \overline{B_{11}} & \theta_0+1 & 2 \\
\frac{\theta_0-5}{4\,\theta_0} & \overline{B_5} & \theta_0+1 & 2 \\
\frac{\theta_0^3\overline{\mu_6} - 6\,(\theta_0\overline{\alpha_1} + 2\,\overline{\alpha_1})|A_1|^2 + 3\,\theta_0^2\overline{\mu_6} + 2\,\theta_0(3\,\overline{\alpha_6} + \overline{\mu_6}) - 6\,\theta_0\overline{\zeta_{11}} + 6\,\overline{\alpha_6}}{\theta_0^2+\theta_0} & A_1 & \theta_0+1 & 2 \\
-\frac{\theta_0^3\overline{\alpha_9} - 3\,\mu_{11}\theta_0^2 - (3\,\mu_{11} + 7\,\overline{\alpha_9})\theta_0 + 6\,\zeta_0\overline{\alpha_1} + (\theta_0+1)\overline{\zeta_4} - 6\,\overline{\alpha_9}}{\theta_0^2+\theta_0} & \overline{A_1} & \theta_0+1 & 2 \\
\lambda_{14} & \overline{A_0} & \theta_0+1 & 2 \\
\frac{2\,\theta_0^4\overline{\mu_5} + 6\,\theta_0^3\overline{\mu_5} + 4\,\theta_0^2\overline{\mu_5} - \left(2\,\theta_0^2 - \theta_0 - 6\right)\overline{\zeta_2}}{2\,(\theta_0^3+\theta_0^2)} & A_2 & \theta_0+1 & 2 \\
\lambda_{15} & A_0 & \theta_0+1 & 2 \\
\frac{\theta_0^4\overline{\mu_1} + 3\,\theta_0^3\overline{\mu_1} + 2\,(\alpha_5 + 3\,\mu_3 + \overline{\mu_1})\theta_0^2 + 2\,(\alpha_5 + 3\,\mu_3)\theta_0 + 12\,\alpha_5}{8\,(\theta_0^3+\theta_0^2)} & \overline{C_1} & \theta_0+1 & 2 \\
-\frac{(2\,\theta_0-1)|A_1|^2}{2\,(\theta_0^2+2\,\theta_0+1)} & \overline{B_1} & \theta_0+1 & 2 \\
\frac{\theta_0^3\overline{\mu_1} + 5\,\theta_0^2\overline{\mu_1} + 2\,(\alpha_5 + 3\,\overline{\mu_1})\theta_0 + 6\,\alpha_5}{\theta_0^2+2\,\theta_0} & A_1 & \theta_0+2 & 0 \\
\frac{\mu_{13}\theta_0^3 - 4\,(\theta_0+3)\zeta_0|A_1|^2 + 3\,\mu_{13}\theta_0^2 + 2\,\mu_{13}\theta_0}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & \overline{A_0} & \theta_0+2 & 0 \\
\frac{\mu_{12}\theta_0^3 + 3\,\mu_{12}\theta_0^2 + 2\,\mu_{12}\theta_0 + 4\,(2\,\theta_0+3)\zeta_1}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & \overline{A_1} & \theta_0+2 & 0 \\
\frac{\theta_0^4\overline{\mu_2} + 6\,\theta_0^3\overline{\mu_2} + (3\,\alpha_6 + 11\,\overline{\mu_2})\theta_0^2 - 2\left(3\,\alpha_1\theta_0^2 + 13\,\alpha_1\theta_0 + 12\,\alpha_1\right)|A_1|^2 + 3\,(3\,\alpha_6 + 2\,\overline{\mu_2})\theta_0 + 6\,\alpha_6}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & A_0 & \theta_0+2 & 0 \\
\frac{1}{\theta_0^2} & B_9 & \theta_0+2 & 1 \\
\frac{\theta_0+3}{2\,\theta_0^2} & \overline{B_{10}} & \theta_0+2 & 1 \\
\frac{\theta_0-5}{4\,\theta_0} & \overline{B_4} & \theta_0+2 & 1 \\
\frac{\theta_0^3\overline{\mu_3} + 4\,\theta_0^2\overline{\mu_3} + \theta_0(2\,\overline{\alpha_5} + 3\,\overline{\mu_3}) + 6\,\overline{\alpha_5}}{\theta_0^2+\theta_0} & A_2 & \theta_0+2 & 1 \\
-\frac{\alpha_1\theta_0 - \alpha_1}{2\,(\theta_0^2+2\,\theta_0)} & \overline{B_1} & \theta_0+2 & 1 \\
\frac{2\left(\mu_{13}\theta_0^3 - 4\,(\theta_0+3)\zeta_0|A_1|^2 + 3\,\mu_{13}\theta_0^2 + 2\,\mu_{13}\theta_0\right)}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & \overline{A_1} & \theta_0+2 & 1 \\
\frac{2\left(\mu_{12}\theta_0^3 + 3\,\mu_{12}\theta_0^2 + 2\,\mu_{12}\theta_0 + 4\,(2\,\theta_0+3)\zeta_1\right)}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & \overline{A_2} & \theta_0+2 & 1 \\
\lambda_{16} & A_0 & \theta_0+2 & 1 \\
\lambda_{17} & A_1 & \theta_0+2 & 1 \\
\lambda_{18} & \overline{A_0} & \theta_0+2 & 1 \\
\frac{2\,\mu_5\theta_0^3 + 2\,\mu_5\theta_0^2 - \left(\theta_0^2 + \theta_0 - 3\right)\zeta_2}{4\,(\theta_0^4+\theta_0^3)} & \overline{C_1} & \theta_0+2 & 1 \\
-\frac{1}{4}\,\theta_0\overline{\alpha_2} - \frac{1}{4}\,\overline{\alpha_2} & C_1 & \theta_0+2 & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{3\,\zeta_0}{4\,(\theta_0^3+4\,\theta_0^2+5\,\theta_0+2)} & B_1 & \theta_0+3 & 0 \\
\frac{\theta_0^3\overline{\mu_1}+6\,\theta_0^2\overline{\mu_1}+2\,(\alpha_5+4\,\overline{\mu_1})\theta_0+8\,\alpha_5}{\theta_0^2+2\,\theta_0} & A_2 & \theta_0+3 & 0 \\
\frac{\mu_{14}\theta_0^3+3\,\mu_{14}\theta_0^2+2\,\mu_{14}\theta_0-2\,(\alpha_1\theta_0+4\,\alpha_1)\zeta_0-2\,\left(\theta_0^5+9\,\theta_0^4+28\,\theta_0^3+36\,\theta_0^2+16\,\theta_0\right)\zeta_9}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & \overline{A_1} & \theta_0+3 & 0 \\
\frac{\mu_{28}\theta_0^4+6\,\mu_{28}\theta_0^3-8\,\left(2\,\theta_0^2+11\,\theta_0+12\right)\zeta_1|A_1|^2+11\,\mu_{28}\theta_0^2+6\,\mu_{28}\theta_0-2\,\left(\alpha_5\theta_0^2+7\,\alpha_5\theta_0+12\,\alpha_5\right)\zeta_0}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & \overline{A_0} & \theta_0+3 & 0 \\
\lambda_{19} & A_1 & \theta_0+3 & 0 \\
\lambda_{20} & A_0 & \theta_0+3 & 0 \\
\frac{\theta_0-4}{4\,\theta_0} & \overline{E_1} & 2\,\theta_0-1 & -\theta_0+3 \\
\frac{\theta_0^2\overline{\alpha_7}-8\,\theta_0\overline{\alpha_7}+16\,\overline{\alpha_7}}{\theta_0} & A_0 & 2\,\theta_0-1 & -\theta_0+3 \\
\frac{\theta_0-5}{4\,\theta_0} & \overline{E_3} & 2\,\theta_0-1 & -\theta_0+4 \\
-\theta_0(\overline{\alpha_{13}}-2\,\overline{\mu_{23}})+5\,\overline{\alpha_{13}}-2\,\overline{\zeta_{21}} & A_0 & 2\,\theta_0-1 & -\theta_0+4 \\
-\frac{\theta_0^3(\overline{\alpha_9}-2\,\overline{\mu_8})+2\,\theta_0^2(\overline{\alpha_9}-3\,\overline{\mu_8})-4\,\theta_0(\overline{\alpha_9}+\overline{\mu_8})+(\theta_0-2)\overline{\zeta_4}-8\,\overline{\alpha_9}}{\theta_0^2+2\,\theta_0} & A_0 & 2\,\theta_0+1 & -\theta_0+1 \\
-\frac{2\,\theta_0^3(\overline{\alpha_8}-2\,\overline{\mu_9})-8\,\theta_0^2\overline{\mu_9}-2\,\theta_0(7\,\overline{\alpha_8}+2\,\overline{\mu_9})+(\theta_0-3)\overline{\zeta_3}-12\,\overline{\alpha_8}}{2\,(\theta_0^2+\theta_0)} & A_1 & 2\,\theta_0+1 & -\theta_0+2 \\
\lambda_{21} & A_0 & 2\,\theta_0+1 & -\theta_0+2 \\
-\frac{\theta_0^3(\overline{\alpha_9}-2\,\overline{\mu_8})+\theta_0^2(3\,\overline{\alpha_9}-7\,\overline{\mu_8})-2\,\theta_0(2\,\overline{\alpha_9}+3\,\overline{\mu_8})+(\theta_0-2)\overline{\zeta_4}-12\,\overline{\alpha_9}}{\theta_0^2+2\,\theta_0} & A_1 & 2\,\theta_0+2 & -\theta_0+1 \\
\lambda_{22} & A_0 & 2\,\theta_0+2 & -\theta_0+1 \\
-\frac{\mu_{30}\theta_0^4+\mu_{30}\theta_0^3-4\,\mu_{30}\theta_0^2-4\,\mu_{30}\theta_0-\left(5\,\theta_0^2\overline{\alpha_2}-4\,\theta_0\overline{\alpha_2}-12\,\overline{\alpha_2}\right)\zeta_0}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & \overline{A_0} & 2\,\theta_0+2 & -\theta_0+1 \\
-\frac{\mu_{12}\theta_0^4+\mu_{12}\theta_0^3-4\,\mu_{12}\theta_0^2-4\,\mu_{12}\theta_0+4\,\left(2\,\theta_0^2-\theta_0-6\right)\zeta_1}{4\,(\theta_0^4+3\,\theta_0^3+2\,\theta_0^2)} & \overline{C_1} & 2\,\theta_0+2 & -\theta_0+1 \\
\frac{(\theta_0-2)\zeta_0}{4\,(\theta_0^4+4\,\theta_0^3+5\,\theta_0^2+2\,\theta_0)} & \overline{B_1} & 2\,\theta_0+2 & -\theta_0+1 \\
-\frac{2\,\theta_0^3(\overline{\alpha_8}-2\,\overline{\mu_9})-2\,\theta_0^2(\overline{\alpha_8}+3\,\overline{\mu_9})-2\,\theta_0(5\,\overline{\alpha_8}+\overline{\mu_9})+(\theta_0-3)\overline{\zeta_3}-6\,\overline{\alpha_8}}{2\,(\theta_0^2+\theta_0)} & A_0 & 2\,\theta_0 & -\theta_0+2 \\
\frac{\theta_0-5}{4\,\theta_0} & \overline{E_2} & 2\,\theta_0 & -\theta_0+3 \\
\frac{\theta_0\overline{\alpha_2}-5\,\overline{\alpha_2}}{4\,\theta_0} & \overline{C_1} & 2\,\theta_0 & -\theta_0+3 \\
\frac{(\overline{\alpha_1\alpha_2}-\overline{\alpha_{12}}+2\,\overline{\mu_{22}})\theta_0^2-(3\,\overline{\alpha_1\alpha_2}-3\,\overline{\alpha_{12}}-\overline{\mu_{22}})\theta_0+(\theta_0-4)\zeta_{22}-4\,\overline{\alpha_1\alpha_2}-(2\,\theta_0+1)\overline{\zeta_{20}}+4\,\overline{\alpha_{12}}}{\theta_0} & A_0 & 2\,\theta_0 & -\theta_0+3 \\
\frac{2\,\theta_0^3\overline{\alpha_7}-17\,\theta_0^2\overline{\alpha_7}+31\,\theta_0\overline{\alpha_7}+20\,\overline{\alpha_7}}{2\,(\theta_0^2+\theta_0)} & A_1 & 2\,\theta_0 & -\theta_0+3 \\
-\mu_{31}\theta_0+4\,\mu_{31} & \overline{A_0} & 2\,\theta_0 & -\theta_0+3
\end{pmatrix}
$$

where the λ coefficients are given as follows.

$$
\lambda_1=\frac{1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0}\Big\{\mu_2\theta_0^5+10\,\mu_2\theta_0^4+(35\,\mu_2+4\,\overline{\alpha_6})\theta_0^3+2\,(25\,\mu_2+12\,\overline{\alpha_6})\theta_0^2
$$
$$
-2\left(3\,\theta_0^3\overline{\alpha_1}+25\,\theta_0^2\overline{\alpha_1}+64\,\theta_0\overline{\alpha_1}+48\,\overline{\alpha_1}\right)|A_1|^2+4\,(6\,\mu_2+11\,\overline{\alpha_6})\theta_0-2\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)\overline{\zeta_{11}}+24\,\overline{\alpha_6}\Big\}
$$

$$
\lambda_2=\frac{1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0}\Big\{\mu_{15}\theta_0^5+10\,\mu_{15}\theta_0^4-(4\,\overline{\alpha_1\alpha_5}-4\,\alpha_{15}-35\,\mu_{15})\theta_0^3-2\,(16\,\overline{\alpha_1\alpha_5}-12\,\alpha_{15}-25\,\mu_{15})\theta_0^2
$$
$$
-\left(8\,\theta_0^3\overline{\alpha_3}+60\,\theta_0^2\overline{\alpha_3}+136\,\theta_0\overline{\alpha_3}+(\theta_0+2)\overline{\zeta_2}+96\,\overline{\alpha_3}\right)|A_1|^2-4\,(19\,\overline{\alpha_1\alpha_5}-11\,\alpha_{15}-6\,\mu_{15})\theta_0
$$

$$- \left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)\zeta_{13} - 48\,\overline{\alpha_1\alpha_5} - \left(\theta_0^3\overline{\alpha_2} + 7\,\theta_0^2\overline{\alpha_2} + 12\,\theta_0\overline{\alpha_2}\right)\overline{\zeta_0} - \left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)\overline{\zeta_{16}} + 24\,\alpha_{15}\Bigg\}$$

$$\lambda_3 = \frac{1}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0}\Bigg\{\mu_4\theta_0^4 - 8\left(\theta_0^2 + 5\,\theta_0 + 6\right)|A_1|^4 + 6\,\mu_4\theta_0^3 - (4\,\alpha_1\overline{\alpha_1} - 6\,\beta - 11\,\mu_4)\theta_0^2 - 2\,(8\,\alpha_1\overline{\alpha_1} - 9\,\beta - 3\,\mu_4)\theta_0$$
$$- 12\,\alpha_1\overline{\alpha_1} - 2\left(\theta_0^4 + 5\,\theta_0^3 + 8\,\theta_0^2 + 4\,\theta_0\right)\overline{\zeta_{10}} + 12\,\beta\Bigg\}$$

$$\lambda_4 = \frac{1}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0}\Bigg\{\mu_{16}\theta_0^4 + 6\,\mu_{16}\theta_0^3 - (6\,\alpha_5\overline{\alpha_1} + 6\,\alpha_1\overline{\alpha_3} - 6\,\alpha_{16} - 11\,\mu_{16})\theta_0^2 - 4\left(3\,\theta_0^2\overline{\alpha_5} + 13\,\theta_0\overline{\alpha_5} + 12\,\overline{\alpha_5}\right)|A_1|^2$$
$$- 2\,(13\,\alpha_5\overline{\alpha_1} + 9\,\alpha_1\overline{\alpha_3} - 9\,\alpha_{16} - 3\,\mu_{16})\theta_0 - \left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)\zeta_{14} - 24\,\alpha_5\overline{\alpha_1} - 12\,\alpha_1\overline{\alpha_3}$$
$$- 2\left(\theta_0^2 + 3\,\theta_0 + 2\right)\overline{\zeta_{15}} + 12\,\alpha_{16}\Bigg\}$$

$$\lambda_5 = \frac{1}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0}\Bigg\{(\alpha_2\overline{\alpha_1} - \alpha_{10})\theta_0^4 + 2\,(3\,\alpha_2\overline{\alpha_1} - 2\,\alpha_{10} + \overline{\mu_{27}})\theta_0^3 + (3\,\alpha_2\overline{\alpha_1} + \alpha_{10} + 6\,\overline{\mu_{27}})\theta_0^2$$
$$- 2\,(7\,\alpha_2\overline{\alpha_1} - 8\,\alpha_{10} - 2\,\overline{\mu_{27}})\theta_0 - \left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)\zeta_{18} - 12\,\alpha_2\overline{\alpha_1} - 4\,(\alpha_5\theta_0 + 3\,\alpha_5)\overline{\zeta_0} - 8\,(2\,\alpha_1\theta_0 + 3\,\alpha_1)\overline{\zeta_1}$$
$$- 2\left(\theta_0^2 + 3\,\theta_0 + 2\right)\overline{\zeta_{24}} + 12\,\alpha_{10}\Bigg\}$$

$$\lambda_6 = -\frac{1}{\theta_0^2 + \theta_0}\Bigg\{\alpha_{11}\theta_0^3 - 2\left(\alpha_2\theta_0^3 + 3\,\alpha_2\theta_0^2 - 10\,\alpha_2\theta_0 - 12\,\alpha_2\right)|A_1|^2 - 3\,\theta_0^2\overline{\mu_{26}} - (7\,\alpha_{11} + 3\,\overline{\mu_{26}})\theta_0 + \left(\theta_0^2 + 3\,\theta_0 + 2\right)\zeta_{19}$$
$$+ 6\,\alpha_3\overline{\zeta_0} + 3\,(\theta_0 + 1)\overline{\zeta_{23}} - 6\,\alpha_{11}\Bigg\}$$

$$\lambda_7 = \frac{1}{\theta_0^3 + \theta_0^2}\Bigg\{\mu_{17}\theta_0^4 + 3\,\mu_{17}\theta_0^3 - 2\,(3\,\alpha_3\overline{\alpha_1} + 3\,\alpha_1\overline{\alpha_5} - \mu_{17} - 3\,\overline{\alpha_{16}})\theta_0^2 - 12\left(\alpha_5\theta_0^2 + 2\,\alpha_5\theta_0\right)|A_1|^2$$
$$- 6\,(\alpha_3\overline{\alpha_1} + \alpha_1\overline{\alpha_5} - \overline{\alpha_{16}})\theta_0 - \left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)\zeta_{15} + \left(\theta_0^2\overline{\alpha_1} + \theta_0\overline{\alpha_1} - 3\,\overline{\alpha_1}\right)\zeta_2 - 3\left(\theta_0^2 + \theta_0\right)\overline{\zeta_{14}}\Bigg\}$$

$$\lambda_8 = \frac{1}{\theta_0^2 + 2\,\theta_0}\Bigg\{(\alpha_1\alpha_2 - \alpha_{12})\theta_0^3 + (3\,\alpha_1\alpha_2 + \alpha_{12} + 4\,\overline{\mu_{24}})\theta_0^2 - 16\,\alpha_1\alpha_2 - 2\,(7\,\alpha_1\alpha_2 - 5\,\alpha_{12} - 4\,\overline{\mu_{24}})\theta_0$$
$$- \left(\theta_0^2 + 3\,\theta_0 + 2\right)\zeta_{20} - 4\,(\theta_0 + 2)\overline{\zeta_{22}} + 8\,\alpha_{12}\Bigg\}$$

$$\lambda_9 = \frac{1}{\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2}\Bigg\{\mu_{18}\theta_0^5 + 4\,\mu_{18}\theta_0^4 - (4\,\alpha_1\alpha_5 - 5\,\mu_{18} - 4\,\overline{\alpha_{15}})\theta_0^3 - 2\,(6\,\alpha_1\alpha_5 - \mu_{18} - 6\,\overline{\alpha_{15}})\theta_0^2$$
$$- 2\left(4\,\alpha_3\theta_0^3 + 12\,\alpha_3\theta_0^2 + 8\,\alpha_3\theta_0 - \left(\theta_0^3 + 2\,\theta_0^2 - 2\,\theta_0 - 4\right)\zeta_2\right)|A_1|^2 - 8\,(\alpha_1\alpha_5 - \overline{\alpha_{15}})\theta_0 - 8\left(\alpha_2\theta_0^2 + \alpha_2\theta_0\right)\zeta_0$$
$$- \left(\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0\right)\zeta_{16} - 4\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)\overline{\zeta_{13}}\Bigg\}$$

$$\lambda_{10} = \frac{1}{\theta_0^2 + 2\,\theta_0}\Bigg\{(\alpha_2\overline{\alpha_1} - \alpha_{10} + 2\,\mu_{20})\theta_0^3 + (7\,\alpha_2\overline{\alpha_1} - 3\,\alpha_{10} + 7\,\mu_{20})\theta_0^2 - 2\,(3\,\alpha_2\overline{\alpha_1} - 2\,\alpha_{10} - 3\,\mu_{20})\theta_0$$
$$- \left(2\,\theta_0^2 + 7\,\theta_0 + 6\right)\zeta_{18} - 24\,\alpha_2\overline{\alpha_1} + \left(\theta_0^2 - 4\right)\overline{\zeta_{24}} + 12\,\alpha_{10}\Bigg\}$$

$$\lambda_{11} = -\frac{1}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0}\Bigg\{(\alpha_{11} - 2\,\mu_{21})\theta_0^4 + 2\,(\alpha_{11} - 4\,\mu_{21})\theta_0^3 - (7\,\alpha_{11} + 10\,\mu_{21})\theta_0^2$$
$$- 2\left(\alpha_2\theta_0^4 + 4\,\alpha_2\theta_0^3 - 7\,\alpha_2\theta_0^2 - 34\,\alpha_2\theta_0 - 24\,\alpha_2\right)|A_1|^2 - 4\,(5\,\alpha_{11} + \mu_{21})\theta_0 + 2\left(\theta_0^3 + 4\,\theta_0^2 + 5\,\theta_0 + 2\right)\zeta_{19} + 2\,\zeta_2\overline{\zeta_0}$$
$$- \left(\theta_0^3 - 7\,\theta_0 - 6\right)\overline{\zeta_{23}} - 12\,\alpha_{11}\Bigg\}$$

$$\lambda_{12} = \frac{1}{\theta_0^2 + 2\,\theta_0}\Bigg\{(\overline{\alpha_1\alpha_2} - \overline{\alpha_{12}})\theta_0^3 + (3\,\overline{\alpha_1\alpha_2} + 4\,\mu_{24} + \overline{\alpha_{12}})\theta_0^2 - 2\,(7\,\overline{\alpha_1\alpha_2} - 4\,\mu_{24} - 5\,\overline{\alpha_{12}})\theta_0 - 4\,(\theta_0 + 2)\zeta_{22}$$
$$- 16\,\overline{\alpha_1\alpha_2} - \left(\theta_0^2 + 3\,\theta_0 + 2\right)\overline{\zeta_{20}} + 8\,\overline{\alpha_{12}}\Bigg\}$$

$$\lambda_{13} = \frac{1}{\theta_0^4 + 3\,\theta_0^3 + 2\,\theta_0^2}\Big\{\theta_0^5\overline{\mu_{18}} + 4\,\theta_0^4\overline{\mu_{18}} - (4\,\overline{\alpha_1\alpha_5} - 4\,\alpha_{15} - 5\,\overline{\mu_{18}})\theta_0^3 - 2\,(6\,\overline{\alpha_1\alpha_5} - 6\,\alpha_{15} - \overline{\mu_{18}})\theta_0^2$$
$$- 2\left(4\,\theta_0^3\overline{\alpha_3} + 12\,\theta_0^2\overline{\alpha_3} + 8\,\theta_0\overline{\alpha_3} - \left(\theta_0^3 + 2\,\theta_0^2 - 2\,\theta_0 - 4\right)\overline{\zeta_2}\right)|A_1|^2 - 8\,(\overline{\alpha_1\alpha_5} - \alpha_{15})\theta_0 - 4\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)\zeta_{13}$$
$$- 8\left(\theta_0^2\overline{\alpha_2} + \theta_0\overline{\alpha_2}\right)\overline{\zeta_0} - \left(\theta_0^4 + 4\,\theta_0^3 + 5\,\theta_0^2 + 2\,\theta_0\right)\overline{\zeta_{16}}\Big\}$$
$$\lambda_{14} = -\frac{1}{\theta_0^2 + \theta_0}\Big\{\theta_0^3\overline{\alpha_{11}} - 3\,\mu_{26}\theta_0^2 - 2\left(\theta_0^3\overline{\alpha_2} + 3\,\theta_0^2\overline{\alpha_2} - 10\,\theta_0\overline{\alpha_2} - 12\,\overline{\alpha_2}\right)|A_1|^2 - (3\,\mu_{26} + 7\,\overline{\alpha_{11}})\theta_0 + 3\,(\theta_0 + 1)\zeta_{23}$$
$$+ 6\,\zeta_0\overline{\alpha_3} + \left(\theta_0^2 + 3\,\theta_0 + 2\right)\overline{\zeta_{19}} - 6\,\overline{\alpha_{11}}\Big\}$$
$$\lambda_{15} = \frac{1}{\theta_0^3 + \theta_0^2}\Big\{\theta_0^4\overline{\mu_{17}} + 3\,\theta_0^3\overline{\mu_{17}} - 2\,(3\,\alpha_5\overline{\alpha_1} + 3\,\alpha_1\overline{\alpha_3} - 3\,\alpha_{16} - \overline{\mu_{17}})\theta_0^2 - 12\left(\theta_0^2\overline{\alpha_5} + 2\,\theta_0\overline{\alpha_5}\right)|A_1|^2$$
$$- 6\,(\alpha_5\overline{\alpha_1} + \alpha_1\overline{\alpha_3} - \alpha_{16})\theta_0 - 3\left(\theta_0^2 + \theta_0\right)\zeta_{14} - \left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)\overline{\zeta_{15}} + \left(\alpha_1\theta_0^2 + \alpha_1\theta_0 - 3\,\alpha_1\right)\overline{\zeta_2}\Big\}$$
$$\lambda_{16} = \frac{1}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0}\Big\{\theta_0^4\overline{\mu_{16}} + 6\,\theta_0^3\overline{\mu_{16}} - (6\,\alpha_3\overline{\alpha_1} + 6\,\alpha_1\overline{\alpha_5} - 6\,\overline{\alpha_{16}} - 11\,\overline{\mu_{16}})\theta_0^2 - 4\left(3\,\alpha_5\theta_0^2 + 13\,\alpha_5\theta_0 + 12\,\alpha_5\right)|A_1|^2$$
$$- 2\,(9\,\alpha_3\overline{\alpha_1} + 13\,\alpha_1\overline{\alpha_5} - 9\,\overline{\alpha_{16}} - 3\,\overline{\mu_{16}})\theta_0 - 2\left(\theta_0^2 + 3\,\theta_0 + 2\right)\zeta_{15} - 12\,\alpha_3\overline{\alpha_1} - 24\,\alpha_1\overline{\alpha_5}$$
$$- \left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)\overline{\zeta_{14}} + 12\,\overline{\alpha_{16}}\Big\}$$
$$\lambda_{17} = -\frac{1}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0}\Big\{8\left(\theta_0^2 + 5\,\theta_0 + 6\right)|A_1|^4 - \theta_0^4\overline{\mu_4} - 6\,\theta_0^3\overline{\mu_4} + (4\,\alpha_1\overline{\alpha_1} - 6\,\beta - 11\,\overline{\mu_4})\theta_0^2 + 2\,(8\,\alpha_1\overline{\alpha_1} - 9\,\beta - 3\,\overline{\mu_4})\theta_0$$
$$+ 2\left(\theta_0^4 + 5\,\theta_0^3 + 8\,\theta_0^2 + 4\,\theta_0\right)\zeta_{10} + 12\,\alpha_1\overline{\alpha_1} - 12\,\beta\Big\}$$
$$\lambda_{18} = \frac{1}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0}\Big\{(\alpha_1\overline{\alpha_2} - \overline{\alpha_{10}})\theta_0^4 + 2\,(3\,\alpha_1\overline{\alpha_2} + \mu_{27} - 2\,\overline{\alpha_{10}})\theta_0^3 + (3\,\alpha_1\overline{\alpha_2} + 6\,\mu_{27} + \overline{\alpha_{10}})\theta_0^2$$
$$- 2\,(7\,\alpha_1\overline{\alpha_2} - 2\,\mu_{27} - 8\,\overline{\alpha_{10}})\theta_0 - 4\,(\theta_0\overline{\alpha_5} + 3\,\overline{\alpha_5})\zeta_0 - 8\,(2\,\theta_0\overline{\alpha_1} + 3\,\overline{\alpha_1})\zeta_1 - 2\left(\theta_0^2 + 3\,\theta_0 + 2\right)\zeta_{24} - 12\,\alpha_1\overline{\alpha_2}$$
$$- \left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)\overline{\zeta_{18}} + 12\,\overline{\alpha_{10}}\Big\}$$
$$\lambda_{19} = \frac{1}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0}\Big\{\theta_0^5\overline{\mu_2} + 10\,\theta_0^4\overline{\mu_2} + (4\,\alpha_6 + 35\,\overline{\mu_2})\theta_0^3 + 2\,(12\,\alpha_6 + 25\,\overline{\mu_2})\theta_0^2$$
$$- 2\left(3\,\alpha_1\theta_0^3 + 25\,\alpha_1\theta_0^2 + 64\,\alpha_1\theta_0 + 48\,\alpha_1\right)|A_1|^2 + 4\,(11\,\alpha_6 + 6\,\overline{\mu_2})\theta_0 - 2\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)\zeta_{11} + 24\,\alpha_6\Big\}$$
$$\lambda_{20} = \frac{1}{\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0}\Big\{\theta_0^5\overline{\mu_{15}} + 10\,\theta_0^4\overline{\mu_{15}} - (4\,\alpha_1\alpha_5 - 4\,\overline{\alpha_{15}} - 35\,\overline{\mu_{15}})\theta_0^3 - 2\,(16\,\alpha_1\alpha_5 - 12\,\overline{\alpha_{15}} - 25\,\overline{\mu_{15}})\theta_0^2$$
$$- \left(8\,\alpha_3\theta_0^3 + 60\,\alpha_3\theta_0^2 + 136\,\alpha_3\theta_0 + (\theta_0 + 2)\zeta_2 + 96\,\alpha_3\right)|A_1|^2 - 48\,\alpha_1\alpha_5 - 4\,(19\,\alpha_1\alpha_5 - 11\,\overline{\alpha_{15}} - 6\,\overline{\mu_{15}})\theta_0$$
$$- \left(\alpha_2\theta_0^3 + 7\,\alpha_2\theta_0^2 + 12\,\alpha_2\theta_0\right)\zeta_0 - \left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)\zeta_{16} - \left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)\overline{\zeta_{13}} + 24\,\overline{\alpha_{15}}\Big\}$$
$$\lambda_{21} = -\frac{1}{\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0}\Big\{\theta_0^4(\overline{\alpha_{11}} - 2\,\overline{\mu_{21}}) + 2\,\theta_0^3(\overline{\alpha_{11}} - 4\,\overline{\mu_{21}}) - \theta_0^2(7\,\overline{\alpha_{11}} + 10\,\overline{\mu_{21}})$$
$$- 2\left(\theta_0^4\overline{\alpha_2} + 4\,\theta_0^3\overline{\alpha_2} - 7\,\theta_0^2\overline{\alpha_2} - 34\,\theta_0\overline{\alpha_2} - 24\,\overline{\alpha_2}\right)|A_1|^2 - \left(\theta_0^3 - 7\,\theta_0 - 6\right)\zeta_{23} - 4\,\theta_0(5\,\overline{\alpha_{11}} + \overline{\mu_{21}})$$
$$+ 2\left(\theta_0^3 + 4\,\theta_0^2 + 5\,\theta_0 + 2\right)\overline{\zeta_{19}} + 2\,\zeta_0\overline{\zeta_2} - 12\,\overline{\alpha_{11}}\Big\}$$
$$\lambda_{22} = \frac{1}{\theta_0^2 + 2\,\theta_0}\Big\{(\alpha_1\overline{\alpha_2} - \overline{\alpha_{10}} + 2\,\overline{\mu_{20}})\theta_0^3 + (7\,\alpha_1\overline{\alpha_2} - 3\,\overline{\alpha_{10}} + 7\,\overline{\mu_{20}})\theta_0^2 - 2\,(3\,\alpha_1\overline{\alpha_2} - 2\,\overline{\alpha_{10}} - 3\,\overline{\mu_{20}})\theta_0 + \left(\theta_0^2 - 4\right)\zeta_{24}$$
$$- 24\,\alpha_1\overline{\alpha_2} - \left(2\,\theta_0^2 + 7\,\theta_0 + 6\right)\overline{\zeta_{18}} + 12\,\overline{\alpha_{10}}\Big\}$$

We shall only need the precise expression of λ_2.

Also, recall that

$$\zeta_{13}=\frac{1}{\theta_0^4-7\,\theta_0^2-6\,\theta_0}\Big\{\big(\theta_0^3-7\,\theta_0-6\big)B_7\overline{A_0}+2\left(\theta_0^5\overline{\alpha_2}-7\,\theta_0^3\overline{\alpha_2}-6\,\theta_0^2\overline{\alpha_2}\right)\overline{A_1}^2+2\left(\theta_0^5\overline{\alpha_2}-\theta_0^4\overline{\alpha_2}-5\,\theta_0^3\overline{\alpha_2}-3\,\theta_0^2\overline{\alpha_2}\right)\overline{A_0A_2}$$
$$+\Big(\big(\theta_0^2+2\,\theta_0\big)A_1\overline{\zeta_2}+\big(\theta_0^3-\theta_0^2-6\,\theta_0\big)\overline{A_0\zeta_3}\Big)\overline{A_1}\Big\}$$

$$\zeta_{14}=\frac{1}{48\,(\theta_0^2+\theta_0)}\Big\{12\,A_0(\theta_0-2)|C_1|^2\overline{A_1}+96\,\theta_0\overline{A_0A_1\zeta_4}+48\,A_0B_{10}(\theta_0+1)+48\,B_8(\theta_0+1)\overline{A_0}$$
$$-3\left(\theta_0^2-\theta_0-2\right)B_1\overline{C_1}-2\left(\theta_0^2-\theta_0-2\right)C_1\overline{C_2}\Big\}$$

$$\zeta_{15}=-\frac{1}{16\,(\theta_0^2+\theta_0)}\Big\{8\,A_1(\theta_0-2)|C_1|^2\overline{A_0}-4\,(\theta_0+1)\zeta_2\overline{A_0C_1}+8\left(\theta_0^2-2\,\theta_0\right)A_1B_2-16\,A_0B_{11}(\theta_0+1)$$
$$-16\,B_9(\theta_0+1)\overline{A_0}+\left(\theta_0^2-2\,\theta_0-3\right)C_2\overline{C_1}-\Big(4\left(\theta_0^3\overline{\alpha_2}+2\,\theta_0^2\overline{\alpha_2}+\theta_0\overline{\alpha_2}\right)\overline{A_0}-\left(\theta_0^2-2\,\theta_0\right)\overline{B_1}\Big)C_1\Big\}$$

$$\zeta_{16}=\frac{1}{2\,(\theta_0^3+3\,\theta_0^2+2\,\theta_0)}\Big\{4\left(\theta_0^2+2\,\theta_0\right)A_1\zeta_2\overline{A_0}+4\left(\theta_0^2+3\,\theta_0+2\right)A_0\zeta_5\overline{A_0}+4\left(\alpha_7\theta_0^4-\alpha_7\theta_0^3-10\,\alpha_7\theta_0^2-8\,\alpha_7\theta_0\right)A_0^2$$
$$-\left(\theta_0^3-\theta_0^2-2\,\theta_0\right)A_2C_1-\left(\theta_0^3-\theta_0^2-6\,\theta_0\right)A_1C_2-\left(\theta_0^3-\theta_0^2-10\,\theta_0-8\right)A_0C_3\Big\}$$

$$\zeta_{17}=\frac{1}{2\,(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0)}\Big\{4\left(\theta_0^3+4\,\theta_0^2+3\,\theta_0\right)A_2\zeta_2\overline{A_0}+2\left(\theta_0^3+6\,\theta_0^2+11\,\theta_0+6\right)A_0B_{13}$$
$$-\left(\theta_0^4+\theta_0^3-4\,\theta_0^2-4\,\theta_0\right)A_3C_1-\left(\theta_0^4+\theta_0^3-9\,\theta_0^2-9\,\theta_0\right)A_2C_2-\left(\theta_0^4+\theta_0^3-14\,\theta_0^2-24\,\theta_0\right)A_1C_3$$
$$+4\Big(\left(\theta_0^3+5\,\theta_0^2+6\,\theta_0\right)\zeta_5\overline{A_0}+\left(\alpha_7\theta_0^5+\alpha_7\theta_0^4-14\,\alpha_7\theta_0^3-24\,\alpha_7\theta_0^2\right)A_0\Big)A_1\Big\}$$

$$\zeta_{18}=-\frac{\left(\theta_0^3+\theta_0^2-4\,\theta_0-4\right)|C_1|^2\overline{A_0}^2+\left(\theta_0^4+\theta_0^3-4\,\theta_0^2-4\,\theta_0\right)B_2\overline{A_0}+\left(\theta_0^4-4\,\theta_0^2\right)B_1\overline{A_1}+\left(\theta_0^4-\theta_0^3-2\,\theta_0^2\right)C_1\overline{A_2}}{2\,(\theta_0^4+3\,\theta_0^3+2\,\theta_0^2)}$$

$$\zeta_{19}=-\frac{1}{2\,(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0)}\Big\{\left(\theta_0^3+3\,\theta_0^2-4\,\theta_0-12\right)|C_1|^2\overline{A_0A_1}-2\left(\theta_0^3+6\,\theta_0^2+11\,\theta_0+6\right)B_{10}\overline{A_0}$$
$$+\left(\theta_0^4+3\,\theta_0^3-4\,\theta_0^2-12\,\theta_0\right)B_2\overline{A_1}+\left(\theta_0^4+2\,\theta_0^3-5\,\theta_0^2-6\,\theta_0\right)B_1\overline{A_2}+\left(\theta_0^4+\theta_0^3-4\,\theta_0^2-4\,\theta_0\right)C_1\overline{A_3}\Big\}$$

$$\zeta_{20}=\frac{4\left(\theta_0^2+2\,\theta_0\right)\zeta_2\overline{A_1}^2+4\left(\theta_0^2+\theta_0\right)\zeta_2\overline{A_0A_2}+2\left(\theta_0^2+3\,\theta_0+2\right)B_{11}\overline{A_0}-\left(\theta_0^3-\theta_0^2-6\,\theta_0\right)B_3\overline{A_1}-\left(\theta_0^3-2\,\theta_0^2-3\,\theta_0\right)C_2\overline{A_2}}{2\,(\theta_0^3+3\,\theta_0^2+2\,\theta_0)}$$

$$\zeta_{21}=\frac{1}{16\,(\theta_0^2+\theta_0)}\Big\{16\,B_{12}(\theta_0+1)\overline{A_0}-8\left(\theta_0^2-4\,\theta_0\right)C_3\overline{A_1}+\Big(16\left(\alpha_2\theta_0^2-\alpha_2\theta_0-2\,\alpha_2\right)A_0-\left(2\,\theta_0^2-3\,\theta_0-2\right)B_1\Big)C_1$$
$$+32\Big(\theta_0\zeta_5\overline{A_0}+\left(\alpha_7\theta_0^3-5\,\alpha_7\theta_0^2+6\,\alpha_7\theta_0-8\,\alpha_7\right)A_0\Big)\overline{A_1}\Big\}$$

Here, we will only need ζ_{13} and ζ_{16}.

4.6 The coefficient in $(0, \theta_0+3)$ in the Taylor expansion of $\mathrm{Re}\,(\partial_{\overline{z}}\vec{F}(z)) = 0$

The coefficient in $\overline{z}^{\theta_0+3}$ in the Taylor expansion of $\mathrm{Re}\,(\partial_{\overline{z}}\vec{F}(z)) = 0$ is

$$\begin{pmatrix} \dfrac{3\,\overline{\zeta_0}}{4\,(\theta_0^3+4\,\theta_0^2+5\,\theta_0+2)} & \overline{B_1} & 0 & \theta_0+3 & \mathbf{(1)} \\ \dfrac{\mu_1\theta_0^3+6\,\mu_1\theta_0^2+2\,(4\,\mu_1+\overline{\alpha_5})\theta_0+8\,\overline{\alpha_5}}{\theta_0^2+2\,\theta_0} & \overline{A_2} & 0 & \theta_0+3 & \mathbf{(2)} \\ \dfrac{\theta_0^3\overline{\mu_{14}}+3\,\theta_0^2\overline{\mu_{14}}+2\,\theta_0\overline{\mu_{14}}-2\,(\theta_0\overline{\alpha_1}+4\,\overline{\alpha_1})\overline{\zeta_0}-2\left(\theta_0^5+9\,\theta_0^4+28\,\theta_0^3+36\,\theta_0^2+16\,\theta_0\right)\overline{\zeta_9}}{\theta_0^3+3\,\theta_0^2+2\,\theta_0} & A_1 & 0 & \theta_0+3 & \mathbf{(3)} \\ \dfrac{\theta_0^4\overline{\mu_{28}}+6\,\theta_0^3\overline{\mu_{28}}-8\left(2\,\theta_0^2+11\,\theta_0+12\right)|A_1|^2\overline{\zeta_1}+11\,\theta_0^2\overline{\mu_{28}}+6\,\theta_0\overline{\mu_{28}}-2\left(\theta_0^2\overline{\alpha_5}+7\,\theta_0\overline{\alpha_5}+12\,\overline{\alpha_5}\right)\overline{\zeta_0}}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0} & A_0 & 0 & \theta_0+3 & \mathbf{(4)} \\ \lambda_1 & \overline{A_1} & 0 & \theta_0+3 & \mathbf{(5)} \\ \lambda_2 & \overline{A_0} & 0 & \theta_0+3 & \mathbf{(6)} \end{pmatrix} \tag{4.6.1}$$

where

$$\begin{aligned}\lambda_1 &= \frac{1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0}\Big\{\mu_2\theta_0^5+10\,\mu_2\theta_0^4+(35\,\mu_2+4\,\overline{\alpha_6})\theta_0^3+2\,(25\,\mu_2+12\,\overline{\alpha_6})\theta_0^2 \\ &\quad -2\left(3\,\theta_0^3\overline{\alpha_1}+25\,\theta_0^2\overline{\alpha_1}+64\,\theta_0\overline{\alpha_1}+48\,\overline{\alpha_1}\right)|A_1|^2+4\,(6\,\mu_2+11\,\overline{\alpha_6})\theta_0-2\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)\overline{\zeta_{11}}+24\,\overline{\alpha_6}\Big\}\end{aligned}$$

$$\begin{aligned}\lambda_2 &= \frac{1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0}\Big\{\mu_{15}\theta_0^5+10\,\mu_{15}\theta_0^4-(4\,\overline{\alpha_1\alpha_5}-4\,\alpha_{15}-35\,\mu_{15})\theta_0^3-2\,(16\,\overline{\alpha_1\alpha_5}-12\,\alpha_{15}-25\,\mu_{15})\theta_0^2 \\ &\quad -\left(8\,\theta_0^3\overline{\alpha_3}+60\,\theta_0^2\overline{\alpha_3}+136\,\theta_0\overline{\alpha_3}+(\theta_0+2)\overline{\zeta_2}+96\,\overline{\alpha_3}\right)|A_1|^2-4\,(19\,\overline{\alpha_1\alpha_5}-11\,\alpha_{15}-6\,\mu_{15})\theta_0 \\ &\quad -\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)\zeta_{13}-48\,\overline{\alpha_1\alpha_5}-\left(\theta_0^3\overline{\alpha_2}+7\,\theta_0^2\overline{\alpha_2}+12\,\theta_0\overline{\alpha_2}\right)\overline{\zeta_0}-\left(\theta_0^3+6\,\theta_0^2+11\,\theta_0+6\right)\overline{\zeta_{16}}+24\,\alpha_{15}\Big\}\end{aligned}$$

Now, recall that by (3.5.1) and as

$$|\vec{A}_0|^2 = \frac{1}{2}, \quad \alpha_5 = 2\langle\overline{\vec{A}_1}, \vec{A}_2\rangle,$$

we obtain

$$\begin{aligned}\mu_1 &= -\frac{1}{4\,(\theta_0^3+3\,\theta_0^2+2\,\theta_0)}\Big\{16\left(\theta_0^2+2\,\theta_0\right)A_0|A_1|^2\overline{A_1}+8\left(\theta_0^2\overline{\alpha_5}+3\,\theta_0\overline{\alpha_5}+2\,\overline{\alpha_5}\right)A_0\overline{A_0}+8\left(\theta_0^2\overline{\alpha_1}+3\,\theta_0\overline{\alpha_1}+2\,\overline{\alpha_1}\right)A_1\overline{A_0} \\ &\quad -8\left(\theta_0^2+\theta_0\right)A_1\overline{A_2}-\left(\theta_0^2+3\,\theta_0+2\right)\overline{A_0B_1}\Big\} \\ &= -\frac{1}{4\,(\theta_0^3+3\,\theta_0^2+2\,\theta_0)}\Big\{4(\theta_0^2+3\theta_0+2)\overline{\alpha_5}-4(\theta_0^2+\theta_0)\overline{\alpha_5}\Big\} \\ &= -\frac{1}{4\,(\theta_0^3+3\,\theta_0^2+2\,\theta_0)}\Big\{4(2\theta_0+2)\overline{\alpha_5}\} \\ &= -\frac{2(\theta_0+1)}{(\theta_0^3+3\,\theta_0^2+2\,\theta_0)}\overline{\alpha_5} \\ &= -\frac{2(\theta_0+1)}{\theta_0(\theta_0+1)(\theta_0+2)}\overline{\alpha_5} \\ &= -\frac{2}{\theta_0(\theta_0+2)}\overline{\alpha_5}\end{aligned}$$

Then, we have

$$\theta_0^3+6\theta_0^4+8\theta_0 = \theta_0(\theta_0+2)(\theta_0+4),$$

so

$$
\begin{aligned}
\frac{\mu_1\theta_0^3+6\,\mu_1\theta_0^2+2\,(4\,\mu_1+\overline{\alpha_5})\theta_0+8\,\overline{\alpha_5}}{\theta_0^2+2\,\theta_0} &= \frac{1}{\theta_0(\theta_0+2)}\left\{\left(\theta_0^3+6\theta_0^2+8\right)\mu_1+2(\theta_0+4)\overline{\alpha_5}\right\}\\
&= \frac{1}{\theta_0(\theta_0+2)}\left\{\theta_0(\theta_0+2)(\theta_0+4)\left(-\frac{2}{\theta_0(\theta_0+2)}\overline{\alpha_5}\right)+2(\theta_0+4)\overline{\alpha_5}\right\}\\
&=0
\end{aligned}
$$

and in (4.6.1), we have

$$
\mathbf{(2)}=0 \tag{4.6.2}
$$

Therefore, we obtain by (4.6.1)

$$
\begin{aligned}
&\frac{3\,\overline{\zeta_0}}{4\,(\theta_0^3+4\,\theta_0^2+5\,\theta_0+2)}\overline{B_1}+\frac{\theta_0^3\overline{\mu_{14}}+3\,\theta_0^2\overline{\mu_{14}}+2\,\theta_0\overline{\mu_{14}}-2\,(\theta_0\overline{\alpha_1}+4\,\overline{\alpha_1})\overline{\zeta_0}-2\left(\theta_0^5+9\,\theta_0^4+28\,\theta_0^3+36\,\theta_0^2+16\,\theta_0\right)\overline{\zeta_9}}{\theta_0^3+3\,\theta_0^2+2\,\theta_0}A_1\\
&+\frac{\theta_0^4\overline{\mu_{28}}+6\,\theta_0^3\overline{\mu_{28}}-8\left(2\,\theta_0^2+11\,\theta_0+12\right)|A_1|^2\overline{\zeta_1}+11\,\theta_0^2\overline{\mu_{28}}+6\,\theta_0\overline{\mu_{28}}-2\left(\theta_0^2\overline{\alpha_5}+7\,\theta_0\overline{\alpha_5}+12\,\overline{\alpha_5}\right)\overline{\zeta_0}}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0}A_0\\
&+\lambda_1\overline{A_1}+\lambda_2\overline{A_0}=0
\end{aligned} \tag{4.6.3}
$$

Furthermore, as

$$
\langle\vec{A}_0,\vec{A}_0\rangle=\langle\vec{A}_0,\vec{A}_1\rangle=\langle\vec{A}_0,\overline{\vec{A}_1}\rangle=0,
$$

we deduce that $\overline{\vec{A}_0}$ is linearly independent with $\vec{A}_0,\vec{A}_1$ and $\overline{\vec{A}_1}$, and as we also have

$$
\vec{B}_1=-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0\in\mathrm{Span}(\vec{A}_0)
$$

the linear relation in $\overline{\vec{A}_0}$ in (4.6.3) must be trivial, so that

$$
\frac{3\,\overline{\zeta_0}}{4\,(\theta_0^3+4\,\theta_0^2+5\,\theta_0+2)}\overline{B_1}+\lambda_2\overline{A_0}=0. \tag{4.6.4}
$$

The rest of the proof will amount at computing this coefficient in front of $\overline{\vec{A}_0}$ in (4.6.4), which must cancel as $\vec{A}_0\neq 0$ by the very definition of a branch point of multiplicity $\theta_0\in\mathbb{N}$.

We recall that

$$
\alpha_{15}=\frac{1}{4}\,|A_1|^2\overline{A_0C_2}+2\,A_1\overline{A_4}+\frac{1}{8}\left(\overline{A_0}\overline{\alpha_1}+2\,\overline{A_2}\right)\overline{B_1}+\frac{1}{6}\,\overline{A_1B_3}+\frac{1}{8}\,\overline{A_0B_6}+\frac{1}{24}\left(8\,|A_1|^2\overline{A_1}+3\,\overline{A_0}\overline{\alpha_5}\right)\overline{C_1}
$$

while

$$
\begin{cases}
\vec{B}_1=-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0\\
\vec{B}_2=-\dfrac{(\theta_0+2)}{4\theta_0}|\vec{C}_1|^2\overline{\vec{A}_0}+\left(\overline{\alpha_0}\langle\overline{\vec{A}_1},\vec{C}_1\rangle-2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\right)\vec{A}_0\\
\vec{B}_3=-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_1+\dfrac{2}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_1}+\dfrac{2\overline{\alpha_0}}{\theta_0-3}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_0}+2\left(\alpha_0\langle\overline{\vec{A}_1},\vec{C}_1\rangle-\langle\overline{\vec{A}_1},\vec{C}_2\rangle\right)\vec{A}_0\\
\vec{B}_4=-\dfrac{(\theta_0+3)}{6\theta_0}\langle\vec{C}_1,\overline{\vec{C}_2}\rangle\overline{\vec{A}_0}-\dfrac{\overline{\zeta_2}}{6\theta_0}\vec{C}_1-\dfrac{(\theta_0+3)}{6\theta_0}|\vec{C}_1|^2\overline{\vec{A}_1}-\dfrac{\theta_0(\theta_0+1)}{6}\alpha_2\overline{\vec{C}_1}+2\left(\overline{\alpha_1}\langle\overline{\vec{A}_1},\vec{C}_1\rangle-\langle\overline{\vec{A}_3},\vec{C}_1\rangle\right)\vec{A}_0.\\
\vec{B}_5=-\left(\dfrac{(\theta_0+2)}{4}\langle\overline{\vec{C}_1},\vec{C}_2\rangle+\dfrac{2}{\theta_0-3}\overline{\alpha_1}\zeta_2\right)\overline{\vec{A}_0}+\dfrac{2\,\zeta_2}{\theta_0-3}\overline{\vec{A}_2}-2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\vec{A}_1\\
\qquad+\left(8\theta_0(\theta_0+1)|\vec{A}_1|^2\alpha_2-2\langle\overline{\vec{A}_2},\vec{C}_2\rangle-\dfrac{2}{\theta_0-3}\overline{\zeta_0}\zeta_2\right)\vec{A}_0\\
\vec{B}_6=\dfrac{2\,\zeta_5}{\theta_0-4}\overline{\vec{A}_1}-\dfrac{4}{\theta_0-3}|\vec{A}_1|^2\zeta_2\overline{\vec{A}_0}+\left(-2\langle\overline{\vec{A}_1},\vec{C}_3\rangle+4\theta_0(\theta_0+1)\alpha_1\alpha_2\right)\vec{A}_0-4\theta_0(\theta_0+1)\alpha_2\vec{A}_2-2\langle\overline{\vec{A}_1},C_2\rangle\vec{A}_1\\
\vec{E}_1=-\dfrac{1}{2\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_0}\\
\vec{E}_2=-\alpha_2\,\vec{C}_1-\dfrac{2(2\theta_0+1)(\theta_0-4)}{\theta_0+1}\alpha_7\overline{\vec{A}_1}\\
\vec{E}_3=-\dfrac{1}{\theta_0}\langle\vec{C}_1,\vec{C}_2\rangle\overline{\vec{A}_0}
\end{cases}
$$

From now on, we assume that

$$|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle=\overline{\langle\vec{A}_1,\vec{C}_1\rangle}\langle\vec{A}_1,\vec{A}_1\rangle \tag{4.6.5}$$

thanks of the meromorphy of the quartic form (see (2.5.1)). As

$$\begin{cases}\alpha_2=\dfrac{1}{2\theta_0(\theta_0+1)}\overline{\langle\vec{A}_1,\vec{C}_1\rangle},\\ \zeta_0=\langle\vec{A}_1,\vec{A}_1\rangle\\ \zeta_2=\langle\vec{A}_1,\vec{C}_1\rangle\end{cases} \tag{4.6.6}$$

this also implies that

$$|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle=|\vec{A}_1|^2\zeta_2=2\theta_0(\theta_0+1)\zeta_0\alpha_2=2\theta_0(\theta_0+1)\langle\vec{A}_1,\vec{A}_1\rangle\alpha_2=\langle\vec{A}_1,\vec{A}_1\rangle\overline{\langle\vec{A}_1,\vec{C}_1\rangle} \tag{4.6.7}$$

In particular, we have as $\langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_0,\vec{A}_2\rangle=0$

$$\langle\vec{A}_2,\vec{B}_1\rangle=\langle\vec{A}_2,-2\overline{\langle\vec{A}_1,\vec{C}_1\rangle}\vec{A}_0\rangle=-2\overline{\langle\vec{A}_1,\vec{C}_1\rangle}\langle\vec{A}_0,\vec{A}_2\rangle=\overline{\langle\vec{A}_1,\vec{C}_1\rangle}\langle\vec{A}_1,\vec{A}_1\rangle=|\vec{A}_1|^2\zeta_2. \tag{4.6.8}$$

Then, we have

$$\begin{aligned}\langle\vec{A}_1,\vec{B}_3\rangle&=-2\overline{\langle\vec{A}_1,\vec{C}_1\rangle}\langle\vec{A}_1,\vec{A}_1\rangle+\frac{2}{\theta_0-3}|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle\\&=\left(-2+\frac{2}{\theta_0-3}\right)|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle\\&=-2\frac{(\theta_0-4)}{\theta_0-3}|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle\\&=-\frac{2(\theta_0-4)}{\theta_0-3}|\vec{A}_1|^2\zeta_2\end{aligned} \tag{4.6.9}$$

as

$$\zeta_2=\langle\vec{A}_1,\vec{C}_1\rangle.$$

Now, we compute as $|\vec{A}_0|^2=\dfrac{1}{2}$ and $\langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_0,\vec{A}_2\rangle=0$

$$\begin{aligned}\langle\vec{A}_0,\vec{B}_6\rangle&=\left\langle\vec{A}_0,\frac{2\,\zeta_5}{\theta_0-4}\overline{\vec{A}_1}-\frac{4}{\theta_0-3}|\vec{A}_1|^2\zeta_2\overline{\vec{A}_0}+\left(-2\overline{\langle\vec{A}_1,\vec{C}_3\rangle}+4\theta_0(\theta_0+1)\alpha_1\alpha_2\right)\vec{A}_0-4\theta_0(\theta_0+1)\alpha_2\vec{A}_2-2\overline{\langle\vec{A}_1,C_2\rangle}\vec{A}_1\right\rangle\\&=-\frac{2}{\theta_0-3}|\vec{A}_1|^2\zeta_2-4\theta_0(\theta_0+1)\alpha_2\langle\vec{A}_0,\vec{A}_2\rangle\\&=-\frac{2}{\theta_0-3}|\vec{A}_1|^2\zeta_2+2\theta_0(\theta_0+1)\alpha_2\langle\vec{A}_1,\vec{A}_1\rangle\\&=\left(-\frac{2}{\theta_0-3}+1\right)|\vec{A}_1|^2\zeta_2\\&=\frac{(\theta_0-5)}{\theta_0-3}|\vec{A}_1|^2\zeta_2.\end{aligned} \tag{4.6.10}$$

Finally, we have as $\langle\vec{A}_1,\vec{C}_1\rangle+\langle\vec{A}_0,\vec{C}_2\rangle=0$ and thanks of- (4.6.8), (4.6.9) and (4.6.10)

$$\begin{aligned}\alpha_{15}&=\frac{1}{4}\,|A_1|^2\overline{A_0C_2}+2\,A_1\overline{A_4}+\frac{1}{8}\left(\overline{A_0\alpha_1}+2\,A_2\right)\overline{B_1}+\frac{1}{6}\,\overline{A_1B_3}+\frac{1}{8}\,\overline{A_0B_6}+\frac{1}{24}\left(8\,|A_1|^2\overline{A_1}+3\,\overline{A_0\alpha_5}\right)\overline{C_1}\\&=-\frac{1}{4}|\vec{A}_1|^2\overline{\zeta_2}+2\langle\vec{A}_1,\overline{\vec{A}_4}\rangle+\frac{1}{4}\overline{\langle\vec{A}_2,\vec{B}_1\rangle}+\frac{1}{6}\overline{\langle\vec{A}_1,\vec{B}_3\rangle}+\frac{1}{8}\overline{\langle\vec{A}_0,\vec{B}_6\rangle}+\frac{1}{3}|\vec{A}_1|^2\overline{\zeta_2}\\&=2\langle\vec{A}_1,\overline{\vec{A}_4}\rangle+\left(-\frac{1}{4}+\frac{1}{4}+\frac{1}{6}\times\left(-\frac{2(\theta_0-4)}{\theta_0-3}\right)+\frac{1}{8}\times\left(\frac{(\theta_0-5)}{\theta_0-3}\right)+\frac{1}{3}\right)|\vec{A}_1|^2\overline{\zeta_2}\end{aligned}$$

$$= 2\langle \vec{A}_1, \overline{\vec{A}_4}\rangle + \frac{3\theta_0 - 7}{24(\theta_0 - 3)}|\vec{A}_1|^2\overline{\zeta_2}. \tag{4.6.11}$$

Then, we have

$$\zeta_{13} = \frac{1}{\theta_0^4 - 7\,\theta_0^2 - 6\,\theta_0}\Big\{\left(\theta_0^3 - 7\,\theta_0 - 6\right)B_7\overline{A_0} + 2\left(\theta_0^5\overline{\alpha_2} - 7\,\theta_0^3\overline{\alpha_2} - 6\,\theta_0^2\overline{\alpha_2}\right)\overline{A_1}^2$$
$$+ 2\left(\theta_0^5\overline{\alpha_2} - \theta_0^4\overline{\alpha_2} - 5\,\theta_0^3\overline{\alpha_2} - 3\,\theta_0^2\overline{\alpha_2}\right)\overline{A_0 A_2} + \left(\left(\theta_0^2 + 2\,\theta_0\right)A_1\overline{\zeta_2} + \left(\theta_0^3 - \theta_0^2 - 6\,\theta_0\right)\overline{A_0\zeta_3}\right)\overline{A_1}\Big\}$$

and

$$\zeta_{16} = \frac{4\left(\theta_0^2 + 2\,\theta_0\right)A_1\zeta_2\overline{A_1} - \left(\theta_0^3 - \theta_0^2 - 2\,\theta_0\right)A_2B_1 + 2\left(\theta_0^2 + 3\,\theta_0 + 2\right)A_0B_{12} - \left(\theta_0^3 - \theta_0^2 - 6\,\theta_0\right)A_1B_3}{2\left(\theta_0^3 + 3\,\theta_0^2 + 2\,\theta_0\right)} \tag{4.6.12}$$

$$\begin{cases} \zeta_0 = \langle \vec{A}_1, \vec{A}_1\rangle \\ \zeta_1 = \langle \vec{A}_1, \vec{A}_2\rangle \\ \zeta_2 = \langle \vec{A}_1, \vec{C}_1\rangle \\ \zeta_3 = \langle \overline{\vec{A}_1}, \vec{C}_2\rangle \\ \zeta_4 = \langle \overline{\vec{A}_2}, \vec{C}_1\rangle \\ \zeta_5 = 2\langle \vec{A}_2, \vec{C}_1\rangle + \langle \vec{A}_1, \vec{C}_2\rangle \end{cases} \tag{4.6.13}$$

Then, we have

$$\vec{B}_7 = \begin{pmatrix} \frac{1}{2} & \overline{B_6} & 0 & -\theta_0 + 4 & \mathbf{(6)} \\ 2\,A_1\overline{C_1} & \overline{A_2} & 0 & -\theta_0 + 4 & \mathbf{(7)} \\ -4\,A_0|A_1|^2\overline{C_2} - 2\,A_1\overline{C_1}\overline{\alpha_1} - 2\,A_0\overline{C_1}\overline{\alpha_5} + \frac{1}{4}\,\overline{B_1C_1} + 2\,A_1\overline{C_3} & \overline{A_0} & 0 & -\theta_0 + 4 & \mathbf{(8)} \\ -4\,A_0|A_1|^2\overline{C_1} + 2\,A_1\overline{C_2} & \overline{A_1} & 0 & -\theta_0 + 4 & \mathbf{(9)} \end{pmatrix}$$

so

$$\begin{aligned} \langle \overline{\vec{A}_0}, \vec{B}_7\rangle &= \frac{1}{2}\overline{\langle \vec{A}_0, \vec{B}_6\rangle} + 2\langle \vec{A}_1, \overline{\vec{C}_1}\rangle\overline{\langle \vec{A}_0, \vec{A}_2\rangle} \\ &= \frac{1}{2}\overline{\langle \vec{A}_0, \vec{B}_6\rangle} - \langle \vec{A}_1, \overline{\vec{C}_1}\rangle\overline{\langle \vec{A}_1, \vec{A}_1\rangle} \\ &= \frac{1}{2}\overline{\langle \vec{A}_0, \vec{B}_6\rangle} - |\vec{A}_1|^2\overline{\zeta_2}. \end{aligned}$$

Then, we obtain by (4.6.10)

$$\langle \vec{A}_0, \vec{B}_6\rangle = \frac{(\theta_0 - 5)}{\theta_0 - 3}|\vec{A}_1|^2\zeta_2$$

so

$$\begin{aligned} \langle \overline{\vec{A}_0}, \vec{B}_7\rangle &= \frac{(\theta_0 - 5)}{2(\theta_0 - 3)}|\vec{A}_1|^2\overline{\zeta_2} - |\vec{A}_1|^2\overline{\zeta_2} \\ &= -\frac{(\theta_0 - 1)}{2(\theta_0 - 3)}|\vec{A}_1|^2\overline{\zeta_2} \end{aligned} \tag{4.6.14}$$

Finally, we obtain as $2\langle \vec{A}_0, \vec{A}_2\rangle = -\langle \vec{A}_1, \vec{A}_1\rangle$

$$
\begin{aligned}
\zeta_{13} &= \frac{1}{\theta_0^4 - 7\,\theta_0^2 - 6\,\theta_0}\bigg\{\left(\theta_0^3 - 7\,\theta_0 - 6\right)B_7\overline{A_0} + 2\left(\theta_0^5\overline{\alpha_2} - 7\,\theta_0^3\overline{\alpha_2} - 6\,\theta_0^2\overline{\alpha_2}\right)\overline{A_1}^2 \\
&\quad + 2\left(\theta_0^5\overline{\alpha_2} - \theta_0^4\overline{\alpha_2} - 5\,\theta_0^3\overline{\alpha_2} - 3\,\theta_0^2\overline{\alpha_2}\right)\overline{A_0A_2} + \Big(\left(\theta_0^2 + 2\,\theta_0\right)A_1\overline{\zeta_2} + \left(\theta_0^3 - \theta_0^2 - 6\,\theta_0\right)\overline{A_0\zeta_3}\Big)\overline{A_1}\bigg\} \\
&= \frac{1}{\theta_0(\theta_0+1)(\theta_0+2)(\theta_0-3)}\bigg\{(\theta_0+1)(\theta_0+2)(\theta_0-3)\left(-\frac{(\theta_0-1)}{2(\theta_0-3)}|\vec{A}_1|^2\overline{\zeta_2}\right) \\
&\quad + \Big(2(\theta_0^5 - 7\theta_0^3 - 6\theta_0^2) - (\theta_0^5 - \theta_0^4 - 5\theta_0^3 - 3\theta_0^2)\Big)\overline{\zeta_0}\overline{\alpha_2} + \theta_0(\theta_0+2)|\vec{A}_1|^2\overline{\zeta_2}\bigg\} \\
&= \frac{1}{\theta_0(\theta_0+1)(\theta_0+2)(\theta_0-3)}\bigg\{-\frac{1}{2}(\theta_0-1)(\theta_0+1)(\theta_0+2)|\vec{A}_1|^2\overline{\zeta_2} + \theta_0^2(\theta_0+1)(\theta_0+3)(\theta_0-3)\overline{\zeta_0}\overline{\alpha_2} \\
&\quad + \theta_0(\theta_0+2)|\vec{A}_1|^2\overline{\zeta_2}\bigg\} \\
&= \frac{1}{\theta_0(\theta_0+1)(\theta_0+2)(\theta_0-3)}\bigg\{-\frac{1}{2}(\theta_0-1)(\theta_0+1)(\theta_0+2) + \frac{1}{2}\theta_0(\theta_0+3)(\theta_0-3) + \theta_0(\theta_0+2)\bigg\}|\vec{A}_1|^2\overline{\zeta_2} \\
&= \frac{-(2\theta_0-1)}{\theta_0(\theta_0+1)(\theta_0+2)(\theta_0-3)}|\vec{A}_1|^2\overline{\zeta_2} \qquad (4.6.15)
\end{aligned}
$$

as

$$
\begin{aligned}
&2(\theta_0^5 - 7\theta_0^3 - 6\theta_0^2) - (\theta_0^5 - \theta_0^4 - 5\theta_0^3 - 3\theta_0^2) = \theta_0^2(\theta_0+1)(\theta_0+3)(\theta_0-3) \\
&\theta_0(\theta_0+1)\overline{\zeta_0}\overline{\alpha_2} = \frac{1}{2}\overline{\langle \vec{A}_1, \vec{A}_1\rangle}\langle \vec{A}_1, \vec{C}_1\rangle = \frac{1}{2}|\vec{A}_1|^2\overline{\zeta_2} \\
&-\frac{1}{2}(\theta_0-1)(\theta_0+1)(\theta_0+2) + \frac{1}{2}\theta_0(\theta_0+3)(\theta_0-3) + \theta_0(\theta_0+2) = -2\theta_0 + 1
\end{aligned}
$$

Now we recall that

$$
\vec{B}_{12} = \begin{pmatrix}
-\frac{1}{2}\,\theta_0 + 2 & B_6 & -\theta_0+3 & 1 & \textbf{(49)} \\
\frac{1}{2}\,B_1C_1 & A_0 & -\theta_0+3 & 1 & \textbf{(50)} \\
-4\,A_0C_1\alpha_1 + 4\,A_2C_1 + 2\,A_1C_2 & \overline{A_1} & -\theta_0+3 & 1 & \textbf{(51)} \\
-8\,A_1C_1|A_1|^2 - 4\,A_0C_2|A_1|^2 - 4\,A_0B_1\alpha_1 - 4\,A_0C_1\alpha_5 + 4\,A_2B_1 + 2\,A_1B_3 & \overline{A_0} & -\theta_0+3 & 1 & \textbf{(52)}
\end{pmatrix}
$$

Furthermore, remark that $\langle \vec{A}_2, -2\vec{A}_0\rangle = \langle \vec{A}_1, \vec{A}_1\rangle$ so by (4.6.5)

$$
\begin{aligned}
&-8\,A_1C_1|A_1|^2 - 4\,A_0C_2|A_1|^2 - 4\,A_0B_1\alpha_1 - 4\,A_0C_1\alpha_5 + 4\,A_2B_1 + 2\,A_1B_3 \\
&= -4|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle + 4\langle \vec{A}_2, -2\overline{\langle \vec{A}_1, \vec{C}_1\rangle}\vec{A}_0\rangle \\
&= -4|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle + 4\overline{\langle \vec{A}_1, \vec{C}_1\rangle}\langle \vec{A}_1, \vec{A}_1\rangle \\
&= 0
\end{aligned}
$$

so

$$
(\textbf{52}) = 0 \qquad (4.6.16)
$$

and by (4.6.10), we obtain

$$
\langle \vec{A}_0, \vec{B}_{12}\rangle = -\frac{(\theta_0-4)}{2}\langle \vec{A}_0, \vec{B}_6\rangle = -\frac{(\theta_0-4)}{2}\left(\frac{(\theta_0-5)}{\theta_0-3}|\vec{A}_1|^2\zeta_2\right) = -\frac{(\theta_0-4)(\theta_0-5)}{2(\theta_0-3)}|\vec{A}_1|^2\zeta_2 \qquad (4.6.17)
$$

We deduce by (4.6.8), (4.6.9), (4.6.10) and (4.6.17) that

$$\begin{aligned}
\zeta_{16} &= \frac{4\left(\theta_0^2+2\,\theta_0\right)A_1\zeta_2\overline{A_1}-\left(\theta_0^3-\theta_0^2-2\,\theta_0\right)A_2B_1+2\left(\theta_0^2+3\,\theta_0+2\right)A_0B_{12}-\left(\theta_0^3-\theta_0^2-6\,\theta_0\right)A_1B_3}{2\left(\theta_0^3+3\,\theta_0^2+2\,\theta_0\right)}\\
&=\frac{1}{2\theta_0(\theta_0+1)(\theta_0+2)}\Bigg\{4\theta_0(\theta_0+2)-\theta_0(\theta_0+1)(\theta_0-2)\times 1+2(\theta_0+1)(\theta_0+2)\times\left(-\frac{(\theta_0-4)(\theta_0-5)}{2(\theta_0-3)}\right)\\
&-\theta_0(\theta_0+2)(\theta_0-3)\times\left(-\frac{2(\theta_0-4)}{\theta_0-3}\right)\Bigg\}|\vec{A}_1|^2\zeta_2\\
&=\frac{2(\theta_0^3-\theta_0^2-6\theta_0-10)}{\theta_0(\theta_0+1)(\theta_0+2)(\theta_0-3)}|\vec{A}_1|^2\zeta_2
\end{aligned}\tag{4.6.18}$$

as

$$\begin{aligned}
&4\theta_0(\theta_0+2)-\theta_0(\theta_0+1)(\theta_0-2)\times 1+2(\theta_0+1)(\theta_0+2)\times\left(-\frac{(\theta_0-4)(\theta_0-5)}{2(\theta_0-3)}\right)\\
&-\theta_0(\theta_0+2)(\theta_0-3)\times\left(-\frac{2(\theta_0-4)}{\theta_0-3}\right)=4\theta_0(\theta_0+2)-\frac{40}{\theta_0-3}=\frac{4\left(\theta_0^3-\theta_0^2-6\theta_0-10\right)}{\theta_0-3}
\end{aligned}$$

$$\begin{aligned}
\mu_{15} &= -\frac{1}{24\left(\theta_0^5+10\,\theta_0^4+35\,\theta_0^3+50\,\theta_0^2+24\,\theta_0\right)}\Bigg\{96\left(\theta_0^4+7\,\theta_0^3+14\,\theta_0^2+8\,\theta_0\right)A_0|A_1|^2\overline{A_3}\\
&+2\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)|A_1|^2\overline{A_0C_2}-192\left\{\left(\theta_0^4\overline{\alpha_3}+10\,\theta_0^3\overline{\alpha_3}+35\,\theta_0^2\overline{\alpha_3}+50\,\theta_0\overline{\alpha_3}+24\,\overline{\alpha_3}\right)A_0\overline{A_0}\right.\\
&\left.+\left(\theta_0^4\overline{\alpha_1}+9\,\theta_0^3\overline{\alpha_1}+26\,\theta_0^2\overline{\alpha_1}+24\,\theta_0\overline{\alpha_1}\right)A_0\overline{A_1}\right\}|A_1|^2-48\left\{(2\,\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0^4+10\,(2\,\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0^3+35\,(2\,\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0^2\right.\\
&\left.+50\,(2\,\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0+48\,\overline{\alpha_1\alpha_5}-24\,\alpha_{15}\right\}A_0\overline{A_0}-48\left\{\left(\overline{\alpha_1}^2-\overline{\alpha_4}\right)\theta_0^4+10\left(\overline{\alpha_1}^2-\overline{\alpha_4}\right)\theta_0^3+35\left(\overline{\alpha_1}^2-\overline{\alpha_4}\right)\theta_0^2\right.\\
&\left.+50\left(\overline{\alpha_1}^2-\overline{\alpha_4}\right)\theta_0+24\,\overline{\alpha_1}^2-24\,\overline{\alpha_4}\right\}A_1\overline{A_0}-48\left(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0\right)A_1\overline{A_4}-4\left(\theta_0^4+9\,\theta_0^3+26\,\theta_0^2+24\,\theta_0\right)\overline{A_1B_3}\\
&-3\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)\overline{A_0B_6}\\
&+48\left(\left(\theta_0^4\overline{\alpha_6}+9\,\theta_0^3\overline{\alpha_6}+26\,\theta_0^2\overline{\alpha_6}+24\,\theta_0\overline{\alpha_6}\right)A_0+\left(\theta_0^4\overline{\alpha_3}+9\,\theta_0^3\overline{\alpha_3}+26\,\theta_0^2\overline{\alpha_3}+24\,\theta_0\overline{\alpha_3}\right)A_1\right)\overline{A_1}\\
&+48\left(\left(\theta_0^4\overline{\alpha_5}+8\,\theta_0^3\overline{\alpha_5}+19\,\theta_0^2\overline{\alpha_5}+12\,\theta_0\overline{\alpha_5}\right)A_0+\left(\theta_0^4\overline{\alpha_1}+8\,\theta_0^3\overline{\alpha_1}+19\,\theta_0^2\overline{\alpha_1}+12\,\theta_0\overline{\alpha_1}\right)A_1\right)\overline{A_2}\\
&+3\left(\left(\theta_0^4\overline{\alpha_1}+10\,\theta_0^3\overline{\alpha_1}+35\,\theta_0^2\overline{\alpha_1}+50\,\theta_0\overline{\alpha_1}+24\,\overline{\alpha_1}\right)\overline{A_0}-2\left(\theta_0^4+8\,\theta_0^3+19\,\theta_0^2+12\,\theta_0\right)\overline{A_2}\right)\overline{B_1}\\
&+\left(4\left(\theta_0^4+9\,\theta_0^3+26\,\theta_0^2+24\,\theta_0\right)|A_1|^2\overline{A_1}+3\left(\theta_0^4\overline{\alpha_5}+10\,\theta_0^3\overline{\alpha_5}+35\,\theta_0^2\overline{\alpha_5}+50\,\theta_0\overline{\alpha_5}+24\,\overline{\alpha_5}\right)\overline{A_0}\right)\overline{C_1}\Bigg\}\\
&=-\frac{1}{24\left(\theta_0^5+10\,\theta_0^4+35\,\theta_0^3+50\,\theta_0^2+24\,\theta_0\right)}\Bigg\{96\left(\theta_0^4+7\,\theta_0^3+14\,\theta_0^2+8\,\theta_0\right)A_0|A_1|^2\overline{A_3}\\
&+2\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)|A_1|^2\overline{A_0C_2}-192\left\{\left(\theta_0^4\overline{\alpha_3}+10\,\theta_0^3\overline{\alpha_3}+35\,\theta_0^2\overline{\alpha_3}+50\,\theta_0\overline{\alpha_3}+24\,\overline{\alpha_3}\right)A_0\overline{A_0}\right\}|A_1|^2\\
&-48\left\{(2\,\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0^4+10\,(2\,\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0^3+35\,(2\,\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0^2+50\,(2\,\overline{\alpha_1\alpha_5}-\alpha_{15})\theta_0+48\,\overline{\alpha_1\alpha_5}-24\,\alpha_{15}\right\}A_0\overline{A_0}\\
&-48\left(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0\right)A_1\overline{A_4}-4\left(\theta_0^4+9\,\theta_0^3+26\,\theta_0^2+24\,\theta_0\right)\overline{A_1B_3}\\
&-3\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)\overline{A_0B_6}\\
&+48\left(\theta_0^4\overline{\alpha_3}+9\,\theta_0^3\overline{\alpha_3}+26\,\theta_0^2\overline{\alpha_3}+24\,\theta_0\overline{\alpha_3}\right)A_1\overline{A_1}\\
&+48\left(\left(\theta_0^4\overline{\alpha_5}+8\,\theta_0^3\overline{\alpha_5}+19\,\theta_0^2\overline{\alpha_5}+12\,\theta_0\overline{\alpha_5}\right)A_0+\left(\theta_0^4\overline{\alpha_1}+8\,\theta_0^3\overline{\alpha_1}+19\,\theta_0^2\overline{\alpha_1}+12\,\theta_0\overline{\alpha_1}\right)A_1\right)\overline{A_2}
\end{aligned}\tag{4.6.19}$$

$$- 6\left(\theta_0^4 + 8\,\theta_0^3 + 19\,\theta_0^2 + 12\,\theta_0\right)\overline{A_2 B_1}$$
$$\left. + 4\left(\theta_0^4 + 9\,\theta_0^3 + 26\,\theta_0^2 + 24\,\theta_0\right)|A_1|^2\overline{A_1 C_1}\right\} \tag{4.6.20}$$

Now, recall that

$$\begin{cases} \alpha_1 = 2\langle\overline{\vec{A}_0}, \vec{A}_2\rangle \\ \alpha_3 = 2\langle\overline{\vec{A}_0}, \vec{A}_3\rangle + \dfrac{1}{12}\langle\vec{A}_1, \vec{C}_1\rangle \\ \alpha_5 = 2\langle\overline{\vec{A}_1}, \vec{A}_2\rangle \\ \alpha_{15} = 2\langle\vec{A}_1, \overline{\vec{A}_4}\rangle + \dfrac{3\theta_0 - 7}{24(\theta_0 - 3)}|\vec{A}_1|^2\overline{\zeta_2}. \end{cases}$$

so we have

$$\begin{aligned} \langle\overline{\vec{A}_0}, \vec{A}_3\rangle &= \frac{1}{2}\alpha_3 - \frac{1}{24}\zeta_2 \\ \langle\vec{A}_1, \overline{\vec{A}_4}\rangle &= \frac{1}{2}\alpha_{15} - \frac{3\theta_0 - 7}{48(\theta_0 - 3)}|\vec{A}_1|^2\overline{\zeta_2}. \end{aligned}$$

We will compute separately each term in (4.6.19). We first have

$$\begin{aligned} \text{(I)} &= 96\left(\theta_0^4 + 7\,\theta_0^3 + 14\,\theta_0^2 + 8\,\theta_0\right)A_0|A_1|^2\overline{A_3} = 96|\vec{A}_1|^2\left(\theta_0^4 + 7\,\theta_0^3 + 14\,\theta_0^2 + 8\,\theta_0\right)\left(\frac{1}{2}\overline{\alpha_3} - \frac{1}{24}\overline{\zeta_2}\right) \\ &= 48\left(\theta_0^4 + 7\,\theta_0^3 + 14\,\theta_0^2 + 8\,\theta_0\right)|\vec{A}_1|^2\overline{\alpha_3} - 4\left(\theta_0^4 + 7\,\theta_0^3 + 14\,\theta_0^2 + 8\,\theta_0\right)|\vec{A}_1|^2\overline{\zeta_2}. \end{aligned} \tag{4.6.21}$$

Now, as $\langle\vec{A}_1, \vec{C}_1\rangle + \langle\vec{A}_0, \vec{C}_2\rangle = 0$, we deduce that

$$\begin{aligned} \text{(II)} &= 2\left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)|A_1|^2\overline{A_0 C_2} \quad = -2\left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)|A_1|^2\overline{A_1 C_1} \\ &= -2\left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)|A_1|^2\overline{\zeta_2}. \end{aligned} \tag{4.6.22}$$

Then, we have as $|\vec{A}_0|^2 = \dfrac{1}{2}$ the identity

$$\begin{aligned} \text{(III)} &= -192\left\{\left(\theta_0^4\overline{\alpha_3} + 10\,\theta_0^3\overline{\alpha_3} + 35\,\theta_0^2\overline{\alpha_3} + 50\,\theta_0\overline{\alpha_3} + 24\,\overline{\alpha_3}\right)A_0\overline{A_0}\right\}|A_1|^2 \\ &= -96\left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)|\vec{A}_1|^2\overline{\alpha_3} \end{aligned} \tag{4.6.23}$$

Similarly, we have

$$\begin{aligned} \text{(IV)} &= -48\Big\{(2\,\overline{\alpha_1\alpha_5} - \alpha_{15})\theta_0^4 + 10\,(2\,\overline{\alpha_1\alpha_5} - \alpha_{15})\theta_0^3 + 35\,(2\,\overline{\alpha_1\alpha_5} - \alpha_{15})\theta_0^2 + 50\,(2\,\overline{\alpha_1\alpha_5} - \alpha_{15})\theta_0 \\ &\quad + 48\,\overline{\alpha_1\alpha_5} - 24\,\alpha_{15}\Big\}A_0\overline{A_0} \\ &\quad = -24\left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)\left(2\overline{\alpha_1\alpha_5} - \alpha_{15}\right) \end{aligned} \tag{4.6.24}$$

Then, we have

$$\begin{aligned} \text{(V)} &= -48\left(\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0\right)A_1\overline{A_4} = -48\left(\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0\right)\left(\frac{1}{2}\alpha_{15} - \frac{3\theta_0 - 7}{48(\theta_0 - 3)}|\vec{A}_1|^2\overline{\zeta_2}\right) \\ &= -24\left(\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0\right)\alpha_{15} + \frac{\left(\theta_0^4 + 6\,\theta_0^3 + 11\,\theta_0^2 + 6\,\theta_0\right)(3\theta_0 - 7)}{\theta_0 - 3}|\vec{A}_1|^2\overline{\zeta_2} \end{aligned} \tag{4.6.25}$$

Now, recalling by (4.6.9) that

$$\langle\vec{A}_1, \vec{B}_3\rangle = -\frac{2(\theta_0 - 4)}{\theta_0 - 3}|\vec{A}_1|^2\overline{\zeta_2},$$

we obtain

$$(\mathrm{VI}) = -4\left(\theta_0^4 + 9\,\theta_0^3 + 26\,\theta_0^2 + 24\,\theta_0\right)\overline{A_1 B_3} = \frac{8\left(\theta_0^4 + 9\,\theta_0^3 + 26\,\theta_0^2 + 24\,\theta_0\right)(\theta_0 - 4)}{\theta_0 - 3}|\vec{A}_1|^2\overline{\zeta_2}. \tag{4.6.26}$$

Now, we have as by (4.6.10)

$$\langle \vec{A}_0, \vec{B}_6\rangle = \frac{(\theta_0 - 5)}{\theta_0 - 3}|\vec{A}_1|^2\zeta_2,$$

the identity

$$(\mathrm{VII}) = -3\left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)\overline{A_0 B_6} = \frac{-3\left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)(\theta_0 - 5)}{\theta_0 - 3}|\vec{A}_1|^2\overline{\zeta_2} \tag{4.6.27}$$

Then, we trivially have

$$(\mathrm{VIII}) = 48\left(\theta_0^4\overline{\alpha_3} + 9\,\theta_0^3\overline{\alpha_3} + 26\,\theta_0^2\overline{\alpha_3} + 24\,\theta_0\overline{\alpha_3}\right)A_1\overline{A_1} = 48\left(\theta_0^4 + 9\,\theta_0^3 + 26\,\theta_0^2 + 24\,\theta_0\right)|\vec{A}_1|^2\overline{\alpha_3}. \tag{4.6.28}$$

The next coefficient is as

$$\overline{\alpha_1}\langle \vec{A}_1, \overline{\vec{A}_2}\rangle = \overline{\alpha_5}\langle \vec{A}_0, \overline{\vec{A}_2}\rangle = \frac{1}{2}\overline{\alpha_1 \alpha_5}$$

word

$$\begin{aligned}(\mathrm{IX}) &= 48\left(\left(\theta_0^4\overline{\alpha_5} + 8\,\theta_0^3\overline{\alpha_5} + 19\,\theta_0^2\overline{\alpha_5} + 12\,\theta_0\overline{\alpha_5}\right)A_0 + \left(\theta_0^4\overline{\alpha_1} + 8\,\theta_0^3\overline{\alpha_1} + 19\,\theta_0^2\overline{\alpha_1} + 12\,\theta_0\overline{\alpha_1}\right)A_1\right)\overline{A_2} \\ &= 48\left(\theta_0^4 + 8\,\theta_0^3 + 19\,\theta_0^2 + 12\,\theta_0\right)\overline{\alpha_1\alpha_5}\end{aligned} \tag{4.6.29}$$

Then, we have by (4.6.8)

$$(\mathrm{X}) = -6\left(\theta_0^4 + 8\,\theta_0^3 + 19\,\theta_0^2 + 12\,\theta_0\right)\overline{A_2 B_1} = -6\left(\theta_0^4 + 8\,\theta_0^3 + 19\,\theta_0^2 + 12\,\theta_0\right)|\vec{A}_1|^2\overline{\zeta_2} \tag{4.6.30}$$

and finally

$$(\mathrm{XI}) = 4\left(\theta_0^4 + 9\,\theta_0^3 + 26\,\theta_0^2 + 24\,\theta_0\right)|A_1|^2\overline{A_1 C_1} = 4\left(\theta_0^4 + 9\,\theta_0^3 + 26\,\theta_0^2 + 24\,\theta_0\right)|A_1|^2\overline{\zeta_2} \tag{4.6.31}$$

And by the very definition of μ_{15}, we have

$$\begin{aligned}\mu_{15} = -\frac{1}{24\left(\theta_0^5 + 10\,\theta_0^4 + 35\,\theta_0^3 + 50\,\theta_0^2 + 24\,\theta_0\right)}\Big\{&(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}) + (\mathrm{VI}) + (\mathrm{VII}) + (\mathrm{VIII}) \\ &+ (\mathrm{IX}) + (\mathrm{X}) + (\mathrm{XI})\Big\}\end{aligned}$$

Now, remark that

$$\frac{3\overline{\zeta_0}}{4(\theta_0^3 + 4\theta_0^2 + 5\theta_0 + 2)}\overline{\vec{B}_1} = -\frac{3\overline{\langle \vec{A}_1, \vec{A}_1\rangle}\langle \vec{A}_1, \overline{\vec{C}_1}\rangle}{2(\theta_0^3 + 4\theta_0^2 + 5\theta_0 + 2)}\overline{\vec{A}_0} = -\frac{3|\vec{A}_1|^2\overline{\zeta_2}}{2(\theta_0^3 + 4\theta_0^2 + 5\theta_0 + 2)}\overline{\vec{A}_0}$$

As the coefficient in $\overline{\vec{A}_0}\overline{z}^{\theta_0+3}$ in the Taylor development of

$$\mathrm{Re}\left(\partial_{\overline{z}}\vec{F}(z)\right) = 0$$

is

$$\frac{3\overline{\zeta_0}}{2(\theta_0^3 + 4\theta_0^2 + 5\theta_0 + 2)}\overline{\vec{B}_1} + \lambda_2\overline{\vec{A}_0} = \left(-\frac{3|\vec{A}_1|^2\overline{\zeta_2}}{2(\theta_0^3 + 4\theta_0^2 + 5\theta_0 + 2)} + \lambda_2\right)\overline{\vec{A}_0}$$

while the coefficient in $\overline{\vec{A}_2}$ vanishes, and the vectors are

$$\vec{A}_0, \vec{A}_1, \overline{\vec{A}_1}$$

while

$$\langle \vec{A}_0, \vec{A}_0\rangle = \langle \vec{A}_0, \vec{A}_1\rangle = \langle \vec{A}_0, \overline{\vec{A}_1}\rangle = 0$$

we deduce that

$$\Omega = -\frac{3|\vec{A}_1|^2\overline{\zeta_2}}{2(\theta_0^3+4\theta_0^2+5\theta_0+2)} + \lambda_2 = 0 \tag{4.6.32}$$

Now, we only need to compute

$$\begin{aligned}
\lambda_2 &= \frac{1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0}\Big\{\mu_{15}\theta_0^5 + 10\,\mu_{15}\theta_0^4 - (4\,\overline{\alpha_1\alpha_5} - 4\,\alpha_{15} - 35\,\mu_{15})\theta_0^3 - 2\,(16\,\overline{\alpha_1\alpha_5} - 12\,\alpha_{15} - 25\,\mu_{15})\theta_0^2 \\
&- \left(8\,\theta_0^3\overline{\alpha_3} + 60\,\theta_0^2\overline{\alpha_3} + 136\,\theta_0\overline{\alpha_3} + (\theta_0+2)\overline{\zeta_2} + 96\,\overline{\alpha_3}\right)|A_1|^2 - 4\,(19\,\overline{\alpha_1\alpha_5} - 11\,\alpha_{15} - 6\,\mu_{15})\theta_0 \\
&- \left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)\zeta_{13} - 48\,\overline{\alpha_1\alpha_5} - \left(\theta_0^3\overline{\alpha_2} + 7\,\theta_0^2\overline{\alpha_2} + 12\,\theta_0\overline{\alpha_2}\right)\overline{\zeta_0} - \left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)\overline{\zeta_{16}} + 24\,\alpha_{15}\Big\} \\
&= \frac{1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0}\Big\{\mu_{15}\theta_0^5 + 10\,\mu_{15}\theta_0^4 - (4\,\overline{\alpha_1\alpha_5} - 4\,\alpha_{15} - 35\,\mu_{15})\theta_0^3 - 2\,(16\,\overline{\alpha_1\alpha_5} - 12\,\alpha_{15} - 25\,\mu_{15})\theta_0^2 \\
&- \left(8\,\theta_0^3\overline{\alpha_3} + 60\,\theta_0^2\overline{\alpha_3} + 136\,\theta_0\overline{\alpha_3} + (\theta_0+2)\overline{\zeta_2} + 96\,\overline{\alpha_3}\right)|A_1|^2 - 4\,(19\,\overline{\alpha_1\alpha_5} - 11\,\alpha_{15} - 6\,\mu_{15})\theta_0 \\
&- \left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)\zeta_{13} - 48\,\overline{\alpha_1\alpha_5} - \frac{\left(\theta_0^2 + 7\,\theta_0 + 12\,\right)}{2(\theta_0+1)}|A_1|^2\overline{\zeta_2} - \left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)\overline{\zeta_{16}} + 24\,\alpha_{15}\Big\}
\end{aligned}$$

Here the only change between the two lines is to write

$$\left(\theta_0^3\overline{\alpha_2} + 7\,\theta_0^2\overline{\alpha_2} + 12\,\theta_0\overline{\alpha_2}\right)\overline{\zeta_0} = \frac{\left(\theta_0^2 + 7\,\theta_0\overline{\alpha_2} + 12\,\overline{\alpha_2}\right)}{2(\theta_0+1)}|A_1|^2\overline{\zeta_2}$$

as

$$2\theta_0(\theta_0+1)\alpha_2\zeta_0 = \langle\overline{\vec{A}_1}, \vec{C}_1\rangle\langle\vec{A}_1, \vec{A}_1\rangle = |\vec{A}_1|^2\langle\vec{A}_1, \vec{C}_1\rangle = |\vec{A}_1|^2\zeta_2$$

We emphasize the new development of λ_2 here

$$\begin{aligned}
\lambda_2 &= \frac{1}{\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0}\Big\{\mu_{15}\theta_0^5 + 10\,\mu_{15}\theta_0^4 - (4\,\overline{\alpha_1\alpha_5} - 4\,\alpha_{15} - 35\,\mu_{15})\theta_0^3 - 2\,(16\,\overline{\alpha_1\alpha_5} - 12\,\alpha_{15} - 25\,\mu_{15})\theta_0^2 \\
&- \left(8\,\theta_0^3\overline{\alpha_3} + 60\,\theta_0^2\overline{\alpha_3} + 136\,\theta_0\overline{\alpha_3} + (\theta_0+2)\overline{\zeta_2} + 96\,\overline{\alpha_3}\right)|A_1|^2 - 4\,(19\,\overline{\alpha_1\alpha_5} - 11\,\alpha_{15} - 6\,\mu_{15})\theta_0 \\
&- \left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)\zeta_{13} - 48\,\overline{\alpha_1\alpha_5} - \frac{\left(\theta_0^2 + 7\,\theta_0 + 12\,\right)}{2(\theta_0+1)}|A_1|^2\overline{\zeta_2} - \left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)\overline{\zeta_{16}} + 24\,\alpha_{15}\Big\}
\end{aligned} \tag{4.6.33}$$

Now, we will code each term and compare with the expression with obtained, to finally obtain the expression of the coefficient which we shall not name Ω, as it did not bring us any luck so far. **Sage version**

$$\begin{aligned}
&\lambda_2 = \frac{1}{2\,(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0)}\Big\{2\,\mu_{15}\theta_0^5 + 20\,\mu_{15}\theta_0^4 - 2\,(4\,\overline{\alpha_1\alpha_5} - 4\,\alpha_{15} - 35\,\mu_{15})\theta_0^3 \\
&- 4\,(16\,\overline{\alpha_1\alpha_5} - 12\,\alpha_{15} - 25\,\mu_{15})\theta_0^2 - 2\left(8\,\theta_0^3\overline{\alpha_3} + 60\,\theta_0^2\overline{\alpha_3} + 136\,\theta_0\overline{\alpha_3} + (\theta_0+2)\overline{\zeta_2} + 96\,\overline{\alpha_3}\right)|A_1|^2 \\
&- \frac{\left(\theta_0^2 + 7\,\theta_0 + 12\right)|A_1|^2\overline{\zeta_2}}{\theta_0+1} - 8\,(19\,\overline{\alpha_1\alpha_5} - 11\,\alpha_{15} - 6\,\mu_{15})\theta_0 - 2\left(\theta_0^4 + 10\,\theta_0^3 + 35\,\theta_0^2 + 50\,\theta_0 + 24\right)\zeta_{13} \\
&- 96\,\overline{\alpha_1\alpha_5} - 2\left(\theta_0^3 + 6\,\theta_0^2 + 11\,\theta_0 + 6\right)\overline{\zeta_{16}} + 48\,\alpha_{15}\Big\}
\end{aligned}$$

Sage decided to factor the $\frac{1}{2}$ coming from the last line and changes the order of some factors, but one can nevertheless check that both expressions coincide. We also check that (the left -hand side is our expression and the right-hand side is the Sage expression - the attentive reader will check that Sage always puts spaces between numbers and symbolic characters)

$$-\frac{3|\vec{A}_1|^2\overline{\zeta_2}}{2(\theta_0^3+4\theta_0^2+5\theta_0+2)}=\Omega-\lambda_2=-\frac{3\,{\left|A_1\right|}^2\overline{\zeta_2}}{2\,{\left(\theta_0^3+4\,\theta_0^2+5\,\theta_0+2\right)}}.$$

Then, we have

$$\begin{aligned}\mu_{15}&=-\frac{1}{24\,{\left(\theta_0^5+10\,\theta_0^4+35\,\theta_0^3+50\,\theta_0^2+24\,\theta_0\right)}}\Big\{\text{(I)}+\text{(II)}+\text{(III)}+\text{(IV)}+\text{(V)}+\text{(VI)}+\text{(VII)}+\text{(VIII)}\\&\quad+\text{(IX)}+\text{(X)}+\text{(XI)}\Big\}\\&=-\frac{1}{24\,{\left(\theta_0^5+10\,\theta_0^4+35\,\theta_0^3+50\,\theta_0^2+24\,\theta_0\right)}}\Big\{\mathit{roman}_1+\mathit{roman}_{10}+\mathit{roman}_{11}+\mathit{roman}_2+\mathit{roman}_3+\mathit{roman}_4\\&\quad+\mathit{roman}_5+\mathit{roman}_6+\mathit{roman}_7+\mathit{roman}_8+\mathit{roman}_9\Big\}\end{aligned}$$

where for all $1\leq j\leq n$, we have roman_j is the corresponding Roman numeral to the Arabic numeral j. Now, we have

$$\begin{aligned}\zeta_{13}&=\frac{-(2\theta_0-1)}{\theta_0(\theta_0+1)(\theta_0+2)(\theta_0-3)}|\vec{A}_1|^2\overline{\zeta_2}\\&=-\frac{(2\,\theta_0-1){\left|A_1\right|}^2\overline{\zeta_2}}{(\theta_0+2)(\theta_0+1)(\theta_0-3)\theta_0}\end{aligned}$$

and

$$\begin{aligned}\overline{\zeta_{16}}&=\frac{2(\theta_0^3-\theta_0^2-6\theta_0-10)}{\theta_0(\theta_0+1)(\theta_0+2)(\theta_0-3)}|\vec{A}_1|^2\overline{\zeta_2}\\&=\frac{2\,{\left(\theta_0^3-\theta_0^2-6\,\theta_0-10\right)}{\left|A_1\right|}^2\overline{\zeta_2}}{(\theta_0+2)(\theta_0+1)(\theta_0-3)\theta_0}\end{aligned}$$

Finally, we have (the first line is the $\TeX$, and the second one the Sage version)

$$\begin{aligned}\text{(I)}&=48\big(\theta_0^4+7\,\theta_0^3+14\,\theta_0^2+8\,\theta_0\big)|\vec{A}_1|^2\overline{\alpha_3}-4\big(\theta_0^4+7\,\theta_0^3+14\,\theta_0^2+8\,\theta_0\big)|\vec{A}_1|^2\overline{\zeta_2}\\&=48\,{\left(\theta_0^4+7\,\theta_0^3+14\,\theta_0^2+8\,\theta_0\right)}{\left|A_1\right|}^2\overline{\alpha_3}-4\,{\left(\theta_0^4+7\,\theta_0^3+14\,\theta_0^2+8\,\theta_0\right)}{\left|A_1\right|}^2\overline{\zeta_2}\\\text{(II)}&=-2\,{\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)}{\left|A_1\right|}^2\overline{\zeta_2}\\&=-2\,{\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)}{\left|A_1\right|}^2\overline{\zeta_2}\\\text{(III)}&=-96\big(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\,\big)|\vec{A}_1|^2\overline{\alpha_3}\\&=-96\,{\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)}{\left|A_1\right|}^2\overline{\alpha_3}\\\text{(IV)}&=-24\,{\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)}\,{\left(2\overline{\alpha_1\alpha_5}-\alpha_{15}\right)}\\&=-24\,{\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)}{\left(2\,\overline{\alpha_1\alpha_5}-\alpha_{15}\right)}\\\text{(V)}&=-24\,{\left(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0\right)}\alpha_{15}+\frac{\big(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0\big)(3\theta_0-7)}{\theta_0-3}|\vec{A}_1|^2\overline{\zeta_2}\\&=\frac{{\left(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0\right)}(3\,\theta_0-7){\left|A_1\right|}^2\overline{\zeta_2}}{\theta_0-3}-24\,{\left(\theta_0^4+6\,\theta_0^3+11\,\theta_0^2+6\,\theta_0\right)}\alpha_{15}\\\text{(VI)}&=\frac{8\big(\theta_0^4+9\,\theta_0^3+26\,\theta_0^2+24\,\theta_0\big)(\theta_0-4)}{\theta_0-3}|\vec{A}_1|^2\overline{\zeta_2}\\&=\frac{8\,{\left(\theta_0^4+9\,\theta_0^3+26\,\theta_0^2+24\,\theta_0\right)}(\theta_0-4){\left|A_1\right|}^2\overline{\zeta_2}}{\theta_0-3}\end{aligned}$$

$$
\begin{aligned}
\text{(VII)} &= \frac{-3\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)(\theta_0-5)}{\theta_0-3}|\vec{A}_1|^2\overline{\zeta_2}\\
&= -\frac{3\left(\theta_0^4+10\,\theta_0^3+35\,\theta_0^2+50\,\theta_0+24\right)(\theta_0-5)|A_1|^2\overline{\zeta_2}}{\theta_0-3}\\
\text{(VIII)} &= 48\big(\theta_0^4+9\,\theta_0^3+26\,\theta_0^2+24\,\theta_0\big)|\vec{A}_1|^2\overline{\alpha_3}\\
&= 48\left(\theta_0^4+9\,\theta_0^3+26\,\theta_0^2+24\,\theta_0\right)|A_1|^2\overline{\alpha_3}\\
\text{(IX)} &= 48\big(\theta_0^4+8\,\theta_0^3+19\,\theta_0^2+12\,\theta_0\big)\overline{\alpha_1\alpha_5}\\
&= 48\left(\theta_0^4+8\,\theta_0^3+19\,\theta_0^2+12\,\theta_0\right)\overline{\alpha_1\alpha_5}\\
\text{(X)} &= -6\left(\theta_0^4+8\,\theta_0^3+19\,\theta_0^2+12\,\theta_0\right)|\vec{A}_1|^2\overline{\zeta_2}\\
&= -6\left(\theta_0^4+8\,\theta_0^3+19\,\theta_0^2+12\,\theta_0\right)|A_1|^2\overline{\zeta_2}\\
\text{(XI)} &= 4\left(\theta_0^4+9\,\theta_0^3+26\,\theta_0^2+24\,\theta_0\right)|A_1|^2\overline{\zeta_2}\\
&= 4\left(\theta_0^4+9\,\theta_0^3+26\,\theta_0^2+24\,\theta_0\right)|A_1|^2\overline{\zeta_2}
\end{aligned}
$$

And finally

$$
\Omega = -\frac{2\,(\theta_0-4)|A_1|^2\overline{\zeta_2}}{\theta_0^3-3\,\theta_0^2} = 0. \tag{4.6.34}
$$

where $\zeta_2 = \langle \vec{A}_1, \vec{C}_1\rangle$. This works for $\theta_0 \geq 5$... Therefore, we obtain the relation

$$
\overline{\Omega} = -\frac{2(\theta_0-4)}{\theta_0^2(\theta_0-3)}|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle = 0. \tag{4.6.35}
$$

To get this relation, we have used the meromorphy of the quartic form to obtain

$$
|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle = \langle \overline{\vec{A}_1}, \vec{C}_1\rangle\langle \vec{A}_1, \vec{A}_1\rangle.
$$

4.7 Conclusion

Thanks of (4.6.35), we obtain

$$
\langle \vec{A}_1, \vec{C}_1\rangle = 0 \tag{4.7.1}
$$

and the holomorphy of the quartic form, as

$$
\mathscr{Q}_{\vec{\Phi}} = (\theta_0-1)(\theta_0-2)\langle \vec{A}_1, \vec{C}_1\rangle\frac{dz^4}{z} + O(1).
$$

Furthermore, notice that as

$$
\begin{aligned}
|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle &= \langle \overline{\vec{A}_1}, \vec{C}_1\rangle\langle \vec{A}_1, \vec{C}_1\rangle\\
|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle &= \langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle = 0,
\end{aligned}
$$

we have $\vec{C}_1 = 0$ or $\langle \vec{A}_1, \vec{A}_1\rangle = 0$, and $\langle \vec{A}_1, \vec{A}_1\rangle = 0$ or $\langle \overline{\vec{A}_1}, \vec{C}_1\rangle = 0$.

Chapter 5

Removability of the poles of the octic form

As the holomorphy of the octic form $\mathscr{O}_{\vec{\Phi}}$ does not follow trivially from the asymptotic expansion of $\vec{h}_0$, we will need a normal derivative free expression, which is the content of the following proposition.

Proposition A. *Let $\vec{\Phi}:\Sigma^2\to S^4$ a smooth immersion. Then we have*

$$\begin{aligned}
&\mathscr{O}_{\vec{\Phi}} = -\frac{1}{4}\,g^{-3}\otimes\partial\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\\
&+g^{-2}\otimes\left\{\frac{1}{4}\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\overline{\partial}\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{4}\left(\partial\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\right.\\
&-\frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-\frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\\
&-\frac{1}{4}\Big\{2\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\otimes\left(\overline{\partial}\vec{h}_0\otimes\vec{h}_0\right)+\langle\vec{H},\partial\vec{h}_0\rangle\otimes\partial\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)+\langle\vec{H},\vec{h}_0\rangle\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\Big\}\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\\
&\left.+\frac{1}{4}\left\langle\vec{H},\overline{\partial}\left(\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\vec{h}_0\right)\right\rangle\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{4}\left\langle\vec{H},\partial\left(\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\vec{h}_0\right)\right\rangle\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\right\}\\
&+\frac{1}{4}\,g^{-1}\otimes\left\{\left(\langle\vec{H},\partial\overline{\partial}\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle-\langle\vec{H},\partial\vec{h}_0\rangle\otimes\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\right.\\
&\left.-\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0-\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle^2\right)\right\}\\
&+\frac{1}{4}|\vec{h}_0|^2_{WP}\,g^{-1}\otimes\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-\frac{1}{4}\,|\vec{h}_0|^2_{WP}\left\{\left(\partial\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0\right)\otimes\langle\vec{H},\vec{h}_0\rangle-\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\langle\vec{H},\vec{h}_0\rangle^2\right\}\\
&+\frac{1}{16}\left\{\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)^2+|\vec{H}|^2\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\langle\vec{H},\vec{h}_0\rangle+2|\vec{h}_0|^2_{WP}\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\langle\vec{H},\vec{h}_0\rangle^2\right\}\\
&+\frac{1}{4}\left(1+|\vec{H}|^2\right)\,g^{-1}\otimes\left\{\frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{2}\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\right\}\\
&+\frac{1}{64}\left(1+|\vec{H}|^2\right)^2\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-\frac{1}{8}\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle^2\right)^2 \qquad (5.0.1)
\end{aligned}$$

Proof. We first remark that for any two tensors $\vec{\alpha},\vec{\beta}$, and any two differential operators $\vec{D}_1,\vec{D}_2\in\{\partial,\overline{\partial}\}$, we have

$$\begin{aligned}
\vec{D}_1^{\perp}\vec{\alpha}\,\dot{\otimes}\,\vec{D}_2^{N}\vec{\beta}&=\left(\vec{D}_1-\vec{D}_1^{\top}\right)\vec{\alpha}\,\dot{\otimes}\,\left(\vec{D}_2-\vec{D}_2^{\top}\right)\vec{\beta}\\
&=\vec{D}_1\vec{\alpha}\,\dot{\otimes}\,\vec{D}_2\vec{\beta}-\vec{D}_1\vec{\alpha}\,\dot{\otimes}\,\vec{D}_2^{\top}\vec{\beta}-\vec{D}_1^{\top}\vec{\alpha}\,\dot{\otimes}\,\vec{D}_2\vec{\beta}+\vec{D}_1^{\top}\vec{\alpha}\,\dot{\otimes}\,\vec{D}_2^{\top}\vec{\beta}\\
&=\vec{D}_1\vec{\alpha}\,\dot{\otimes}\,\vec{D}_2\vec{\beta}-\vec{D}_1^{\top}\vec{\alpha}\,\dot{\otimes}\,\vec{D}_2^{\top}\vec{\beta} \qquad (5.0.2)
\end{aligned}$$

as $\vec{D}_1\vec{\alpha}\,\dot{\otimes}\,\vec{D}_2^\top\vec{\beta}=\vec{D}_1^\top\vec{\alpha}\,\dot{\otimes}\,\vec{D}_2^\top\vec{\beta}$. Now, recall that

$$\begin{cases}\partial^\top\vec{h}_0=-\langle\vec{H},\vec{h}_0\rangle\otimes\partial\vec{\Phi}-g^{-1}\otimes(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0)\otimes\overline{\partial}\vec{\Phi}\\ \overline{\partial}^\top\vec{h}_0=-|\vec{h}_0|^2_{WP}\,g\otimes\partial\vec{\Phi}-\langle\vec{H},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}\end{cases}\tag{5.0.3}$$

As $\vec{\Phi}$ is conformal, $g=\partial\,\dot{\otimes}\,\overline{\partial}\vec{\Phi}$ and $|\vec{h}_0|^2_{WP}=g^{-2}\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)$, we obtain

$$\begin{cases}\partial^\top\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^\top\vec{h}_0=\dfrac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle^2\right)\\ \partial^\top\vec{h}_0\,\dot{\otimes}\,\partial^\top\vec{h}_0=\langle\vec{H},\vec{h}_0\rangle\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\\ \overline{\partial}^\top\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^\top\vec{h}_0=|\vec{h}_0|^2_{WP}\,g^2\otimes\langle\vec{H},\vec{h}_0\rangle=\langle\vec{H},\vec{h}_0\rangle\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right).\end{cases}\tag{5.0.4}$$

Then we compute thanks to (5.0.3)

$$\begin{aligned}\partial^N\overline{\partial}^N\vec{h}_0&=\partial^N\left(\overline{\partial}\vec{h}_0+|\vec{h}_0|^2_{WP}\,g\otimes\partial\vec{\Phi}+\langle\vec{H},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}\right)\\&=\partial^N\overline{\partial}\vec{h}_0+|\vec{h}_0|^2_{WP}\,g\otimes\partial^N\partial\vec{\Phi}+\langle\vec{H},\vec{h}_0\rangle\otimes\partial^N\overline{\partial}\vec{\Phi}\\&=\partial^N\overline{\partial}\vec{h}_0+\frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle\vec{H}\right)\end{aligned}\tag{5.0.5}$$

In particular, as $\vec{h}_0$ is normal, we have

$$\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\vec{h}_0=\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle^2\right).\tag{5.0.6}$$

Then, by (5.0.4) and (5.0.2), we have

$$\partial^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0=\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0-\partial^\top\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^\top\vec{h}_0=\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0-\frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle^2\right).\tag{5.0.7}$$

Therefore, we finally obtain by (5.0.6) and (5.0.7)

$$\begin{aligned}&\left(\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\left(\partial^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\\&=\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right).\end{aligned}$$

which shows by normality of $\vec{h}_0$ that

$$\begin{aligned}&g^{-1}\otimes\left\{\frac{1}{2}\left(\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-\left(\partial^N\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{2}\left(\partial^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\right\}\\&=g^{-1}\otimes\left\{\frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{2}\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\right\}\end{aligned}\tag{5.0.8}$$

so we can simply remove the normal derivatives. Then, we have almost by definition

$$\partial^\top\overline{\partial}\vec{h}_0=2\,g^{-1}\otimes\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{\Phi}\right)\partial\vec{\Phi}+2\,g^{-1}\otimes\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\vec{\Phi}\right)\overline{\partial}\vec{\Phi}.\tag{5.0.9}$$

Recall now that

$$\vec{h}_0=2\,\mathbb{I}(\partial_z\vec{\Phi},\partial_z\vec{\Phi})dz^2=2\left(\partial_z^2\vec{\Phi}-2(\partial_z\lambda)\partial_z\vec{\Phi}\right)dz^2.$$

We have as $\vec{h}_0$ is normal the identity

$$\langle\overline{\partial}\vec{h}_0,\partial\vec{\Phi}\rangle=-\langle\vec{h}_0,\partial\overline{\partial}\vec{\Phi}\rangle=-\frac{1}{2}\,g\otimes\langle\vec{H},\vec{h}_0\rangle$$

so by Codazzi identity, we have

$$
\begin{aligned}
\langle \partial\overline{\partial}\vec{h}_0, \partial\vec{\Phi}\rangle &= -\frac{1}{2}\partial\left(g\otimes\langle \vec{H},\vec{h}_0\rangle\right) - \langle \overline{\partial}\vec{h}_0, \partial_z^2\vec{\Phi}\rangle\otimes dz^2\\
&= -\frac{1}{2}\left((\partial g)\otimes\langle \vec{H},\vec{h}_0\rangle + g\otimes\langle \partial\vec{H},\vec{h}_0\rangle + g\otimes\langle \vec{H},\partial\vec{h}_0\rangle\right) - \langle \overline{\partial}\vec{h}_0, \partial_z^2\vec{\Phi}\rangle\otimes dz^2\\
&= -\frac{1}{2}\left((\partial g)\otimes\langle \vec{H},\vec{h}_0\rangle + \langle \overline{\partial}\vec{h}_0,\vec{h}_0\rangle + g\otimes\langle \vec{H},\partial\vec{h}_0\rangle\right) - \langle \overline{\partial}\vec{h}_0, \partial_z^2\vec{\Phi}\rangle\otimes dz^2 \qquad (5.0.10)
\end{aligned}
$$

but

$$
\begin{aligned}
\langle \overline{\partial}\vec{h}_0, \partial_z^2\vec{\Phi}\rangle\otimes dz^2 &= \frac{1}{2}\langle \overline{\partial}\vec{h}_0,\vec{h}_0\rangle + 2(\partial_z\lambda)\langle \overline{\partial}\vec{h}_0,\partial_z\vec{\Phi}\rangle\otimes dz^2 = \frac{1}{2}\langle \overline{\partial}\vec{h}_0,\vec{h}_0\rangle - (\partial_z\lambda)e^{2\lambda}\langle \vec{H},\vec{h}_0\rangle\otimes dz^2\otimes d\overline{z}\\
&= \frac{1}{2}\langle \overline{\partial}\vec{h}_0,\vec{h}_0\rangle - \frac{1}{2}\partial_z(e^{2\lambda})dz^2\otimes d\overline{z}\otimes\langle \vec{H},\vec{h}_0\rangle\\
&= \frac{1}{2}\langle \overline{\partial}\vec{h}_0,\vec{h}_0\rangle - \frac{1}{2}(\partial g)\otimes\langle \vec{H},\vec{h}_0\rangle. \qquad (5.0.11)
\end{aligned}
$$

Therefore, by (5.0.50) and (5.0.67), we have

$$
\begin{aligned}
\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\vec{\Phi} &= -\frac{1}{2}\left(\cancel{(\partial g)\otimes\langle \vec{H},\vec{h}_0\rangle} + \langle \overline{\partial}\vec{h}_0,\vec{h}_0\rangle + g\otimes\langle \vec{H},\partial\vec{h}_0\rangle\right) - \left(\frac{1}{2}\langle \overline{\partial}\vec{h}_0,\vec{h}_0\rangle - \frac{1}{2}\cancel{(\partial g)\otimes\langle \vec{H},\vec{h}_0\rangle}\right)\\
&= -\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0 - \frac{1}{2}\,g\otimes\langle \vec{H},\partial\vec{h}_0\rangle. \qquad (5.0.12)
\end{aligned}
$$

Now, we have again by normality of $\vec{h}_0$ the identity

$$
\langle \overline{\partial}\vec{h}_0, \overline{\partial}\vec{\Phi}\rangle = -\langle \vec{h}_0, \overline{\partial}^N\overline{\partial}\vec{\Phi}\rangle = -\frac{1}{2}\,\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0} \qquad (5.0.13)
$$

so (as for any tensor $\partial\overline{\alpha} = \overline{\overline{\partial}\alpha}$)

$$
\begin{aligned}
\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{\Phi} &= -\frac{1}{2}\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0} + \vec{h}_0\,\dot{\otimes}\,\partial\overline{\vec{h}_0}\right) - \langle \overline{\partial}\vec{h}_0, \partial\overline{\partial}\vec{\Phi}\rangle]\\
&= -\frac{1}{2}\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0} + \vec{h}_0\,\dot{\otimes}\,\overline{\overline{\partial}\vec{h}_0}\right) - \frac{1}{2}\,g\otimes\langle \vec{H},\overline{\partial}\vec{h}_0\rangle. \qquad (5.0.14)
\end{aligned}
$$

Putting together (5.0.9), (5.0.12) and (5.0.14), we obtain

$$
\partial^\top\overline{\partial}\vec{h}_0 = -\left(g^{-1}\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0} + \vec{h}_0\,\dot{\otimes}\,\overline{\overline{\partial}\vec{h}_0}\right) + \langle \vec{H},\overline{\partial}\vec{h}_0\rangle\right)\partial\vec{\Phi} - \left(2\,g^{-1}\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right) + \langle \vec{H},\partial\vec{h}_0\rangle\right)\overline{\partial}\vec{\Phi} \qquad (5.0.15)
$$

This relation implies immediately that

$$
\partial^\top\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial^\top\overline{\partial}\vec{h}_0 = g\otimes\left(g^{-1}\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0} + \vec{h}_0\,\dot{\otimes}\,\overline{\overline{\partial}\vec{h}_0}\right) + \langle \vec{H},\overline{\partial}\vec{h}_0\rangle\right)\otimes\left(2\,g^{-1}\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right) + \langle \vec{H},\partial\vec{h}_0\rangle\right). \qquad (5.0.16)
$$

By (5.0.2) and (5.0.16), we have

$$
\begin{aligned}
\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\partial^N\overline{\partial}^N\vec{h}_0 &= \left(\partial^N\overline{\partial}\vec{h}_0 + \frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0 + \langle \vec{H},\vec{h}_0\rangle\vec{H}\right)\right)\dot{\otimes}\left(\partial^N\overline{\partial}\vec{h}_0 + \frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0 + \langle \vec{H},\vec{h}_0\rangle\vec{H}\right)\right)\\
&= \partial^N\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial^N\overline{\partial}\vec{h}_0 + g\otimes\left(|\vec{h}_0|^2_{WP}\,\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0 + \langle \vec{H},\partial\overline{\partial}\vec{h}_0\rangle\,\dot{\otimes}\,\langle \vec{H},\vec{h}_0\rangle\right)\\
&+ \frac{1}{4}\,g^2\otimes\left(|\vec{h}_0|^4_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0 + |\vec{H}|^2\langle \vec{H},\vec{h}_0\rangle + 2|\vec{h}_0|^2_{WP}\langle \vec{H},\vec{h}_0\rangle^2\right)\\
&= \partial\overline{\partial}\vec{h}_0\otimes\partial\overline{\partial}\vec{h}_0 - \partial^\top\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial^\top\overline{\partial}\vec{h}_0 + g\otimes\left(|\vec{h}_0|^2_{WP}\,\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0 + \langle \vec{H},\partial\overline{\partial}\vec{h}_0\rangle\,\dot{\otimes}\,\langle \vec{H},\vec{h}_0\rangle\right)\\
&+ \frac{1}{4}\,g^2\otimes\left(|\vec{h}_0|^4_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0 + |\vec{H}|^2\langle \vec{H},\vec{h}_0\rangle + 2|\vec{h}_0|^2_{WP}\langle \vec{H},\vec{h}_0\rangle^2\right)
\end{aligned}
$$

$$= \partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\overline{\partial}\vec{h}_0 - g\otimes\left(g^{-1}\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}+\vec{h}_0\,\dot{\otimes}\,\partial\overline{\vec{h}_0}\right)+\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\right)\otimes\left(2\,g^{-1}\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\langle\vec{H},\partial\vec{h}_0\rangle\right)$$
$$+\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\partial\overline{\partial}\vec{h}_0\rangle\,\dot{\otimes}\,\langle\vec{H},\vec{h}_0\rangle\right)+\frac{1}{4}\,g^2\otimes\left(|\vec{h}_0|^4_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+|\vec{H}|^2\langle\vec{H},\vec{h}_0\rangle+2|\vec{h}_0|^2_{WP}\langle\vec{H},\vec{h}_0\rangle^2\right) \tag{5.0.17}$$

Now, we have

$$g\otimes\left(g^{-1}\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}+\vec{h}_0\,\dot{\otimes}\,\partial\overline{\vec{h}_0}\right)+\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\right)\otimes\left(2\,g^{-1}\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\langle\vec{H},\partial\vec{h}_0\rangle\right)$$
$$=2\,g^{-1}\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+2\,g^{-1}\otimes\left(\partial\overline{\vec{h}_0}\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+g\otimes\langle\vec{H},\partial\vec{h}_0\rangle\,\dot{\otimes}\,\langle\vec{H},\overline{\partial}\vec{h}_0\rangle$$
$$+\,2\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\langle\vec{H},\partial\vec{h}_0\rangle\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)+\langle\vec{H},\partial\vec{h}_0\rangle\otimes\left(\partial\overline{\vec{h}_0}\,\dot{\otimes}\,\vec{h}_0\right) \tag{5.0.18}$$

As

$$g\otimes\left(|\vec{h}_0|^2_{WP}\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)=g\otimes\left(g^{-2}\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\,\dot{\otimes}\,\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\right)=g^{-1}\otimes\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\otimes\overline{\vec{h}_0}\right), \tag{5.0.19}$$

we finally obtain by (5.0.17), (5.0.18) and (5.0.19)

$$\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\partial^N\overline{\partial}^N\vec{h}_0=\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\overline{\partial}\vec{h}_0$$
$$+\,g^{-1}\otimes\left(\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\otimes\overline{\vec{h}_0}\right)-2\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-2\left(\partial\overline{\vec{h}_0}\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\right)$$
$$-\left(2\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\langle\vec{H},\partial\vec{h}_0\rangle\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)+\langle\vec{H},\partial\vec{h}_0\rangle\otimes\left(\partial\overline{\vec{h}_0}\,\dot{\otimes}\,\vec{h}_0\right)\right)$$
$$+\,g\otimes\left(\langle\vec{H},\partial\overline{\partial}\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle-\langle\vec{H},\partial\vec{h}_0\rangle\otimes\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\right)$$
$$+\,\frac{1}{4}\,g^2\otimes\left(|\vec{h}_0|^4_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+|\vec{H}|^2\langle\vec{H},\vec{h}_0\rangle+2|\vec{h}_0|^2_{WP}\langle\vec{H},\vec{h}_0\rangle^2\right). \tag{5.0.20}$$

Remains only to compute

$$\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\partial^N\vec{h}_0,\quad\text{and }\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0$$

Thanks to (5.0.2), we have

$$\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\partial^N\vec{h}_0=\left(\partial^N\overline{\partial}\vec{h}_0+\frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle\vec{H}\right)\right)\otimes\partial^N\vec{h}_0$$
$$=\partial^N\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial^N\vec{h}_0+\frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\partial\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle\right)$$
$$=\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0-\partial^\top\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial^\top\vec{h}_0+\frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\partial\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle\right). \tag{5.0.21}$$

Now, by (5.0.3) and (5.0.15), as $g=2\,\partial\vec{\Phi}\,\dot{\otimes}\,\overline{\partial}\vec{\Phi}$, we have

$$\partial^\top\overline{\partial}\vec{h}_0\vec{h}_0\,\dot{\otimes}\,\partial^\top\vec{h}_0=\Bigg\{-\left(g^{-1}\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}+\vec{h}_0\,\dot{\otimes}\,\overline{\overline{\partial}\vec{h}_0}\right)+\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\right)\partial\vec{\Phi}$$
$$-\left(2\,g^{-1}\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\langle\vec{H},\partial\vec{h}_0\rangle\right)\overline{\partial}\vec{\Phi}\Bigg\}\,\dot{\otimes}\,\Bigg\{-\langle\vec{H},\vec{h}_0\rangle\otimes\partial\vec{\Phi}-g^{-1}\otimes(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0)\otimes\overline{\partial}\vec{\Phi}\Bigg\}$$
$$=\frac{1}{2}\,\left(g^{-1}\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}+\vec{h}_0\,\dot{\otimes}\,\overline{\overline{\partial}\vec{h}_0}\right)+\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)$$
$$+\,\langle\vec{H},\vec{h}_0\rangle\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{2}\,g\otimes\langle\vec{H},\partial\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle. \tag{5.0.22}$$

Putting together (5.0.21) and (5.0.22), we have

$$\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\partial^N\vec{h}_0=\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0-\left(\frac{1}{2}\,\left(g^{-1}\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}+\vec{h}_0\,\dot{\otimes}\,\overline{\overline{\partial}\vec{h}_0}\right)+\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\right.$$

$$+\langle\vec{H},\vec{h}_0\rangle\,\dot{\otimes}\,\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{2}\,g\otimes\langle\vec{H},\partial\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle\Big)+\frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\partial\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle\right)$$
$$=\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0+\frac{1}{2}\,g^{-1}\otimes\left\{\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)-\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-\left(\partial\overline{\vec{h}_0}\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\right\}$$
$$-\frac{1}{2}\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-\langle\vec{H},\vec{h}_0\rangle\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)$$
$$=\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0+\frac{1}{2}\,g^{-1}\otimes\left\{\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)-\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-\left(\partial\overline{\vec{h}_0}\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\right\}$$
$$-\frac{1}{2}\left\langle\vec{H},\overline{\partial}\left(\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\vec{h}_0\right)\right\rangle. \tag{5.0.23}$$

Finally, we have by (5.0.21) (indeed, we just need to virtually replace $\partial\vec{h}_0$ by $\overline{\partial}\vec{h}_0$)

$$\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0=\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0-\partial^\top\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^\top\vec{h}_0+\frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle\right). \tag{5.0.24}$$

Now, by (5.0.3) and (5.0.15), we have

$$\partial^\top\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^\top\vec{h}_0=\Big\{-\left(g^{-1}\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}+\vec{h}_0\,\dot{\otimes}\,\overline{\partial\vec{h}_0}\right)+\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\right)\partial\vec{\Phi}$$
$$-\left(2\,g^{-1}\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\langle\vec{H},\partial\vec{h}_0\rangle\right)\overline{\partial}\vec{\Phi}\Big\}\,\dot{\otimes}\,\Big\{-|\vec{h}_0|^2_{WP}\,g\otimes\partial\vec{\Phi}-\langle\vec{H},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}\Big\}$$
$$=\frac{1}{2}\,g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}+\partial\overline{\vec{h}_0}\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{2}\,g\otimes\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle$$
$$+g^{-1}\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)+\frac{1}{2}\langle\vec{H},\partial\vec{h}_0\rangle\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right) \tag{5.0.25}$$

as $|\vec{h}_0|^2_{WP}\,g^2=\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}$. Finally, by (5.0.24) and (5.0.25), we have

$$\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0=\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0-\left\{\frac{1}{2}\,g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}+\partial\overline{\vec{h}_0}\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{2}\,g\otimes\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle\right.$$
$$\left.+g^{-1}\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)+\frac{1}{2}\langle\vec{H},\partial\vec{h}_0\rangle\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\right\}+\frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle\right)$$
$$=\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0-\frac{1}{2}\,g^{-1}\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)-\frac{1}{2}\left\langle\vec{H},\partial\left(\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\vec{h}_0\right)\right\rangle \tag{5.0.26}$$

as

$$g\otimes\left(|\vec{h}_0|^2_{WP}\,\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)=g^{-1}\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)$$
$$\frac{1}{2}\,g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}+\partial\overline{\vec{h}_0}\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{2}\langle\vec{H},\partial\vec{h}_0\rangle\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)=\frac{1}{2}\left\langle\vec{H},\partial\left(\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\vec{h}_0\right)\right\rangle$$

We see that in $\mathscr{O}_{\vec{\Phi}}$, the part that we need to express without normal derivatives is

$$\widetilde{\mathscr{O}}_{\vec{\Phi}}=g^{-2}\otimes\Big\{\frac{1}{4}(\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\partial^N\overline{\partial}^N\vec{h}_0)\otimes(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0)+\frac{1}{4}(\partial^N\vec{h}_0\,\dot{\otimes}\,\partial^N\vec{h}_0)\otimes(\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0)$$
$$-\frac{1}{2}(\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\partial^N\vec{h}_0)\otimes(\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\vec{h}_0)-\frac{1}{2}(\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0)\otimes(\partial^N\vec{h}_0\,\dot{\otimes}\,\vec{h}_0)$$
$$+\frac{1}{2}(\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\vec{h}_0)\otimes(\partial^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0)\Big\}$$
$$=g^{-2}\otimes\Big\{\frac{1}{4}(\mathrm{I})+\frac{1}{4}(\mathrm{II})-\frac{1}{2}(\mathrm{III})-\frac{1}{2}(\mathrm{IV})+\frac{1}{2}(\mathrm{V})\Big\}. \tag{5.0.27}$$

We first have by (5.0.20)

$$(\mathrm{I})=\left(\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\partial^N\overline{\partial}^N\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)=\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\overline{\partial}\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+g^{-1}\otimes\Big(\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\otimes\overline{\vec{h}_0}\right)$$

$$- 2\left(\partial\vec{h}_0\dot{\otimes}\overline{\vec{h}_0}\right)\otimes\left(\overline{\partial}\vec{h}_0\dot{\otimes}\vec{h}_0\right) - 2\left(\partial\overline{\vec{h}_0}\dot{\otimes}\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\dot{\otimes}\vec{h}_0\right)\Bigg)\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)$$

$$- \left(2\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\otimes\left(\overline{\partial}\vec{h}_0\dot{\otimes}\vec{h}_0\right) + \langle\vec{H},\partial\vec{h}_0\rangle\otimes\left(\partial\vec{h}_0\dot{\otimes}\overline{\vec{h}_0}\right) + \langle\vec{H},\partial\vec{h}_0\rangle\otimes\left(\partial\overline{\vec{h}_0}\dot{\otimes}\vec{h}_0\right)\right)\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)$$

$$+ g\otimes\left(\langle\vec{H},\partial\overline{\partial}\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle - \langle\vec{H},\partial\vec{h}_0\rangle\otimes\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\right)\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)$$

$$+ \frac{1}{4}\,g^2\otimes\left(|\vec{h}_0|^4_{WP}\,\vec{h}_0\dot{\otimes}\vec{h}_0 + |\vec{H}|^2\langle\vec{H},\vec{h}_0\rangle + 2|\vec{h}_0|^2_{WP}\langle\vec{H},\vec{h}_0\rangle^2\right)\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right). \tag{5.0.28}$$

Therefore, we have

$$\frac{1}{4}\,g^{-2}\otimes(\mathrm{I}) = \frac{1}{4}\,g^{-2}\otimes\left(\partial\overline{\partial}\vec{h}_0\otimes\partial\overline{\partial}\vec{h}_0\right)\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right) + \frac{1}{4}|\vec{h}_0|^2_{WP}\,g^{-1}\otimes\left(\partial\overline{\partial}\vec{h}_0\otimes\vec{h}_0\right)\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)$$
$$- \frac{1}{2}\,g^{-3}\otimes\partial\left(\vec{h}_0\dot{\otimes}\overline{\vec{h}_0}\right)\otimes\left(\overline{\partial}\vec{h}_0\dot{\otimes}\vec{h}_0\right)\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)$$
$$- \frac{1}{4}\,g^{-2}\otimes\left\{2\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\otimes\left(\overline{\partial}\vec{h}_0\otimes\vec{h}_0\right) + \langle\vec{H},\partial\vec{h}_0\rangle\otimes\partial\left(\vec{h}_0\dot{\otimes}\overline{\vec{h}_0}\right)\right\}\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)$$
$$+ \frac{1}{4}\,g^{-1}\otimes\left(\langle\vec{H},\partial\overline{\partial}\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle - \langle\vec{H},\partial\vec{h}_0\rangle\otimes\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\right)\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)$$
$$+ \frac{1}{16}\left\{\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\dot{\otimes}\vec{h}_0\right)^2 + |\vec{H}|^2\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)\otimes\langle\vec{H},\vec{h}_0\rangle + 2|\vec{h}_0|^2_{WP}\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)\otimes\langle\vec{H},\vec{h}_0\rangle^2\right\} \tag{5.0.29}$$

Then, by (5.0.4), we have

$$\begin{cases}\partial^N\vec{h}_0\dot{\otimes}\overline{\partial}^N\vec{h}_0 = \partial\vec{h}_0\dot{\otimes}\overline{\partial}\vec{h}_0 - \dfrac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\dot{\otimes}\vec{h}_0 + \langle\vec{H},\vec{h}_0\rangle^2\right)\\ \partial^N\vec{h}_0\dot{\otimes}\partial^N\vec{h}_0 = \partial\vec{h}_0\dot{\otimes}\partial\vec{h}_0 - \langle\vec{H},\vec{h}_0\rangle\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)\\ \overline{\partial}^N\vec{h}_0\dot{\otimes}\overline{\partial}^N\vec{h}_0 = \overline{\partial}\vec{h}_0\dot{\otimes}\overline{\partial}\vec{h}_0 - \langle\vec{H},\vec{h}_0\rangle\otimes\left(\vec{h}_0\dot{\otimes}\overline{\vec{h}_0}\right).\end{cases} \tag{5.0.30}$$

Therefore, we have

$$(\mathrm{II}) = \left(\partial^N\vec{h}_0\dot{\otimes}\partial^N\vec{h}_0\right)\otimes\left(\overline{\partial}^N\vec{h}_0\dot{\otimes}\overline{\partial}^N\vec{h}_0\right) = \left(\partial\vec{h}_0\dot{\otimes}\partial\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\dot{\otimes}\overline{\partial}\vec{h}_0\right)$$
$$- \langle\vec{H},\vec{h}_0\rangle\otimes\left\{\left(\partial\vec{h}_0\dot{\otimes}\partial\vec{h}_0\right)\otimes\left(\vec{h}_0\dot{\otimes}\overline{\vec{h}_0}\right) + \left(\overline{\partial}\vec{h}_0\dot{\otimes}\overline{\partial}\vec{h}_0\right)\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)\right\} + \langle\vec{H},\vec{h}_0\rangle^2\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)\otimes\left(\vec{h}_0\dot{\otimes}\overline{\vec{h}_0}\right) \tag{5.0.31}$$

and

$$\frac{1}{4}\,g^{-2}\otimes(\mathrm{II}) = \frac{1}{4}\,g^{-2}\otimes\left(\partial\vec{h}_0\dot{\otimes}\partial\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\dot{\otimes}\overline{\partial}\vec{h}_0\right) - \frac{1}{4}|\vec{h}_0|^2_{WP}\langle\vec{H},\vec{h}_0\rangle\otimes\left(\partial\vec{h}_0\dot{\otimes}\partial\vec{h}_0\right)$$
$$- \frac{1}{4}\,g^{-2}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\left(\overline{\partial}\vec{h}_0\dot{\otimes}\overline{\partial}\vec{h}_0\right)\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right) + \frac{1}{4}|\vec{h}_0|^2_{WP}\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)\otimes\langle\vec{H},\vec{h}_0\rangle^2 \tag{5.0.32}$$

Then, we have by (5.0.23)

$$(\mathrm{III}) = (\partial^N\overline{\partial}^N\vec{h}_0\dot{\otimes}\partial^N\vec{h}_0)\otimes(\overline{\partial}^N\vec{h}_0\dot{\otimes}\vec{h}_0) = \left(\partial\overline{\partial}\vec{h}_0\dot{\otimes}\partial\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\dot{\otimes}\vec{h}_0\right)$$
$$+ \frac{1}{2}\,g^{-1}\otimes\left\{\left(\partial\vec{h}_0\dot{\otimes}\vec{h}_0\right)\otimes\left(\vec{h}_0\dot{\otimes}\overline{\vec{h}_0}\right) - \left(\partial\vec{h}_0\dot{\otimes}\overline{\vec{h}_0}\right)\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)\right.$$
$$\left. - \left(\partial\overline{\vec{h}_0}\dot{\otimes}\vec{h}_0\right)\otimes\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)\right\}\otimes\left(\overline{\partial}\vec{h}_0\dot{\otimes}\vec{h}_0\right) - \frac{1}{2}\left\langle\vec{H},\overline{\partial}\left(\left(\vec{h}_0\dot{\otimes}\vec{h}_0\right)\vec{h}_0\right)\right\rangle\otimes\left(\overline{\partial}\vec{h}_0\dot{\otimes}\vec{h}_0\right) \tag{5.0.33}$$

so

$$- \frac{1}{2}\,g^{-2}\otimes(\mathrm{III}) = -\frac{1}{2}\,g^{-2}\otimes\left(\partial\overline{\partial}\vec{h}_0\otimes\partial\vec{h}_0\right)\dot{\otimes}\left(\overline{\partial}\vec{h}_0\dot{\otimes}\vec{h}_0\right) - \frac{1}{4}|\vec{h}_0|^2_{WP}\,g^{-1}\otimes\left(\partial\vec{h}_0\dot{\otimes}\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\dot{\otimes}\vec{h}_0\right)$$

$$+\frac{1}{4}\,g^{-3}\otimes\partial\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{4}\,g^{-2}\otimes\left\langle\vec{H},\overline{\partial}\left(\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\vec{h}_0\right)\right\rangle\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right). \tag{5.0.34}$$

Now, we easily have by (5.0.26)

$$\begin{aligned}(\mathrm{IV})&=\left(\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0\right)\,\dot{\otimes}\,\left(\partial^N\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)=\left(\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0\right)\,\dot{\otimes}\,\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\\&=\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-\frac{1}{2}\,g^{-1}\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\\&-\frac{1}{2}\left\langle\vec{H},\partial\left(\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\vec{h}_0\right)\right\rangle\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\end{aligned} \tag{5.0.35}$$

so

$$\begin{aligned}-\frac{1}{2}\,g^{-2}\otimes(\mathrm{IV})&=-\frac{1}{2}\,g^{-2}\otimes\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{4}|\vec{h}_0|^2_{WP}\,g^{-1}\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\\&+\frac{1}{4}\,g^{-2}\otimes\left\langle\vec{H},\partial\left(\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\vec{h}_0\right)\right\rangle\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\end{aligned} \tag{5.0.36}$$

As by (5.0.30), we have

$$\partial^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0=\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0-\frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle^2\right)$$

and by (5.0.6)

$$\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\vec{h}_0=\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\frac{1}{2}\,g\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle^2\right),$$

we readily obtain that

$$\begin{aligned}&(\mathrm{V})=\left(\partial^N\overline{\partial}^N\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\partial^N\vec{h}_0\,\dot{\otimes}\,\overline{\partial}^N\vec{h}_0\right)=\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\partial\vec{h}_0\otimes\overline{\partial}\vec{h}_0\right)\\&-\frac{1}{2}\,g\otimes\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0-\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle^2\right)-\frac{1}{4}\,g^2\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle^2\right)^2.\end{aligned} \tag{5.0.37}$$

and

$$\begin{aligned}&\frac{1}{2}\,g^{-2}\otimes(\mathrm{V})=\frac{1}{2}\,g^{-2}\otimes\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\\&-\frac{1}{4}\,g^{-1}\otimes\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0-\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\otimes\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle^2\right)-\frac{1}{8}\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0+\langle\vec{H},\vec{h}_0\rangle^2\right)^2\end{aligned} \tag{5.0.38}$$

and we see that we recover the main term of the quartic form $\mathscr{Q}_{\vec{\Phi}}$. Putting together (5.0.29), (5.0.32), (5.0.34), (5.0.36), (5.0.38), we obtain

$$\begin{aligned}&\widetilde{\mathscr{O}}_{\vec{\Phi}}=g^{-2}\otimes\left\{\frac{1}{4}(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\overline{\partial}\vec{h}_0)\otimes(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0)+\frac{1}{4}(\partial\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0)\otimes(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0)\right.\\&\left.-\frac{1}{2}(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0)\otimes(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0)-\frac{1}{2}(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0)\otimes(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0)+\frac{1}{2}(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0)\otimes(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0)\right\}\\&+\frac{1}{4}|\vec{h}_0|^2_{WP}\,g^{-1}\otimes\left(\partial\overline{\partial}\vec{h}_0\otimes\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-\frac{1}{2}\,g^{-3}\otimes\partial\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\\&-\frac{1}{4}\,g^{-2}\otimes\left\{2\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\otimes\left(\overline{\partial}\vec{h}_0\otimes\vec{h}_0\right)+\langle\vec{H},\partial\vec{h}_0\rangle\otimes\partial\left(\vec{h}_0\,\dot{\otimes}\,\overline{\vec{h}_0}\right)\right\}\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\\&+\frac{1}{4}\,g^{-1}\otimes\left(\langle\vec{H},\partial\overline{\partial}\vec{h}_0\rangle\otimes\langle\vec{H},\vec{h}_0\rangle-\langle\vec{H},\partial\vec{h}_0\rangle\otimes\langle\vec{H},\overline{\partial}\vec{h}_0\rangle\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\\&+\frac{1}{16}\left\{\left(|\vec{h}_0|^2_{WP}\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)^2+|\vec{H}|^2\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\langle\vec{H},\vec{h}_0\rangle+2|\vec{h}_0|^2_{WP}\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\langle\vec{H},\vec{h}_0\rangle^2\right\}\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{4}|\vec{h}_0|^2_{WP}\langle\vec{H},\vec{h}_0\rangle \otimes \left(\partial\vec{h}_0 \dot{\otimes} \partial\vec{h}_0\right) - \frac{1}{4}\, g^{-2} \otimes \langle\vec{H},\vec{h}_0\rangle \otimes \left(\overline{\partial}\vec{h}_0 \dot{\otimes} \overline{\partial}\vec{h}_0\right) \otimes \left(\vec{h}_0 \dot{\otimes} \vec{h}_0\right)\\
&+ \frac{1}{4}|\vec{h}_0|^2_{WP} \left(\vec{h}_0 \dot{\otimes} \vec{h}_0\right) \otimes \langle\vec{H},\vec{h}_0\rangle^2 - \frac{1}{4}|\vec{h}_0|^2_{WP}\, g^{-1} \otimes \left(\partial\vec{h}_0 \dot{\otimes} \vec{h}_0\right) \otimes \left(\overline{\partial}\vec{h}_0 \dot{\otimes} \vec{h}_0\right)\\
&+ \frac{1}{4}\, g^{-3} \otimes \partial \left(\vec{h}_0 \dot{\otimes} \overline{\vec{h}_0}\right) \otimes \left(\overline{\partial}\vec{h}_0 \dot{\otimes} \vec{h}_0\right) \otimes \left(\vec{h}_0 \dot{\otimes} \vec{h}_0\right) + \frac{1}{4}\, g^{-2} \otimes \left\langle \vec{H}, \overline{\partial}\left(\left(\vec{h}_0 \dot{\otimes} \vec{h}_0\right) \vec{h}_0\right)\right\rangle \otimes \left(\overline{\partial}\vec{h}_0 \dot{\otimes} \vec{h}_0\right)\\
&+ \frac{1}{4}|\vec{h}_0|^2_{WP}\, g^{-1} \otimes \left(\partial\vec{h}_0 \dot{\otimes} \vec{h}_0\right) \otimes \left(\overline{\partial}\vec{h}_0 \dot{\otimes} \vec{h}_0\right) + \frac{1}{4}\, g^{-2} \otimes \left\langle \vec{H}, \partial\left(\left(\vec{h}_0 \dot{\otimes} \overline{\vec{h}_0}\right) \vec{h}_0\right)\right\rangle \otimes \left(\partial\vec{h}_0 \dot{\otimes} \vec{h}_0\right)\\
&- \frac{1}{4}\, g^{-1} \otimes \left(\partial\overline{\partial}\vec{h}_0 \dot{\otimes} \vec{h}_0 - \partial\vec{h}_0 \dot{\otimes} \overline{\partial}\vec{h}_0\right) \otimes \left(|\vec{h}_0|^2_{WP}\, \vec{h}_0 \dot{\otimes} \vec{h}_0 + \langle\vec{H},\vec{h}_0\rangle^2\right) - \frac{1}{8}\left(|\vec{h}_0|^2_{WP}\, \vec{h}_0 \dot{\otimes} \vec{h}_0 + \langle\vec{H},\vec{h}_0\rangle^2\right)^2
\end{aligned}
\tag{5.0.39}
$$

Finally, we obtain by (**??**), (5.0.8), and (5.0.39)

$$
\begin{aligned}
\mathscr{O}_{\vec{\Phi}} = \widetilde{\mathscr{O}}_{\vec{\Phi}} &+ \frac{1}{4}\left(1+|\vec{H}|^2\right) g^{-1} \otimes \left\{\frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0 \dot{\otimes} \vec{h}_0\right) \otimes \left(\vec{h}_0 \dot{\otimes} \vec{h}_0\right) - \left(\partial\vec{h}_0 \dot{\otimes} \vec{h}_0\right) \otimes \left(\overline{\partial}\vec{h}_0 \dot{\otimes} \vec{h}_0\right)\right.\\
&\left.+ \frac{1}{2}\left(\partial\vec{h}_0 \dot{\otimes} \overline{\partial}\vec{h}_0\right) \otimes \left(\vec{h}_0 \dot{\otimes} \vec{h}_0\right)\right\} + \frac{1}{64}\left(1+|\vec{H}|^2\right)^2 \left(\vec{h}_0 \dot{\otimes} \vec{h}_0\right)^2
\end{aligned}
$$

which concludes the proof of the proposition. □

We recall that

$$
\vec{h}_0 = \begin{pmatrix}
-\dfrac{\theta_0 - 2}{2\,\theta_0} & C_1 & 0 & \theta_0\\
-\dfrac{\theta_0 - 2}{2\,(\theta_0 + 1)} & B_1 & 0 & \theta_0 + 1\\
2\,\alpha_2\theta_0 - 4\,\alpha_2 & A_0 & 0 & \theta_0 + 1\\
-\dfrac{\theta_0 - 3}{2\,\theta_0} & C_2 & 1 & \theta_0\\
4 & A_2 & \theta_0 & 0\\
-4\,\alpha_1 & A_0 & \theta_0 & 0\\
-4\,\alpha_5 & A_0 & \theta_0 & 1
\end{pmatrix}
\begin{pmatrix}
-4\,|A_1|^2 & A_1 & \theta_0 & 1\\
2 & A_1 & \theta_0 - 1 & 0\\
-4\,|A_1|^2 & A_0 & \theta_0 - 1 & 1\\
\dfrac{1}{4} & \overline{B_1} & \theta_0 - 1 & 2\\
-2\,\overline{\alpha_5} & A_0 & \theta_0 - 1 & 2\\
6 & A_3 & \theta_0 + 1 & 0\\
-6\,\alpha_3 & A_0 & \theta_0 + 1 & 0\\
-4\,\alpha_1 & A_1 & \theta_0 + 1 & 0\\
-2\,\theta_0\overline{\alpha_2} - 2\,\overline{\alpha_2} & A_0 & 2\,\theta_0 - 1 & -\theta_0 + 2
\end{pmatrix}
$$

Now, we compute

$$\vec{h}_0 \dot{\otimes} \vec{h}_0 = \begin{pmatrix} \frac{\left(\theta_0^2 - 4\,\theta_0 + 4\right) C_1^2}{4\,\theta_0^2} & 0 & 2\,\theta_0 \\ -\frac{2\,(A_1 C_2 (\theta_0 - 3) - 2\,((\alpha_1\theta_0 - 2\,\alpha_1) A_0 - A_2(\theta_0 - 2)) C_1)}{\theta_0} & \theta_0 & \theta_0 \\ -\frac{2\,A_1 C_1 (\theta_0 - 2)}{\theta_0} & \theta_0 - 1 & \theta_0 \\ \frac{2\left(2\left(\theta_0^2 - \theta_0 - 2\right) A_0 C_1 |A_1|^2 + 4\left(\alpha_2\theta_0^3 - \alpha_2\theta_0^2 - 2\,\alpha_2\theta_0\right) A_0 A_1 - \left(\theta_0^2 - 2\,\theta_0\right) A_1 B_1\right)}{\theta_0^2 + \theta_0} & \theta_0 - 1 & \theta_0 + 1 \\ 4\,A_1^2 & 2\,\theta_0 - 2 & 0 \\ -16\,A_0 A_1 |A_1|^2 & 2\,\theta_0 - 2 & 1 \\ 16\,A_0^2 |A_1|^4 - 8\,A_0 A_1 \overline{\alpha_5} + A_1 \overline{B_1} & 2\,\theta_0 - 2 & 2 \\ -16\,A_0 A_1 \alpha_1 + 16\,A_1 A_2 & 2\,\theta_0 - 1 & 0 \\ -32\,A_0 A_2 |A_1|^2 - 16\,A_0 A_1 \alpha_5 + 16\left(2\,A_0^2\alpha_1 - A_1^2\right)|A_1|^2 & 2\,\theta_0 - 1 & 1 \\ -8\,(\theta_0\overline{\alpha_2} + \overline{\alpha_2}) A_0 A_1 & 3\,\theta_0 - 2 & -\theta_0 + 2 \\ 16\,A_0^2\alpha_1^2 - 16\,A_1^2\alpha_1 - 32\,A_0 A_2 \alpha_1 - 24\,A_0 A_1 \alpha_3 + 16\,A_2^2 + 24\,A_1 A_3 & 2\,\theta_0 & 0 \end{pmatrix}$$

$\partial\vec{h}_0 \dot{\otimes} \vec{h}_0 =$

$$\begin{pmatrix} -\frac{\left(\theta_0^2 - 3\,\theta_0 + 2\right) A_1 C_1}{\theta_0} & \theta_0 - 2 & \theta_0 \\ \frac{2\left(\theta_0^3 - 2\,\theta_0^2 - \theta_0 + 2\right) A_0 C_1 |A_1|^2 + 4\left(\alpha_2\theta_0^4 - 2\,\alpha_2\theta_0^3 - \alpha_2\theta_0^2 + 2\,\alpha_2\theta_0\right) A_0 A_1 - \left(\theta_0^3 - 3\,\theta_0^2 + 2\,\theta_0\right) A_1 B_1}{\theta_0^2 + \theta_0} & \theta_0 - 2 & \theta_0 + 1 \\ -A_1 C_2(\theta_0 - 3) + 2\,((\alpha_1\theta_0 - 2\,\alpha_1) A_0 - A_2(\theta_0 - 2)) C_1 & \theta_0 - 1 & \theta_0 \\ 4\,A_1^2(\theta_0 - 1) & 2\,\theta_0 - 3 & 0 \\ -16\,A_0 A_1 (\theta_0 - 1)|A_1|^2 & 2\,\theta_0 - 3 & 1 \\ 16\,A_0^2(\theta_0 - 1)|A_1|^4 - 8\,(\theta_0\overline{\alpha_5} - \overline{\alpha_5}) A_0 A_1 + A_1(\theta_0 - 1)\overline{B_1} & 2\,\theta_0 - 3 & 2 \\ -8\,(2\,\alpha_1\theta_0 - \alpha_1) A_0 A_1 + 8\,A_1 A_2 (2\,\theta_0 - 1) & 2\,\theta_0 - 2 & 0 \\ -16\,A_0 A_2 (2\,\theta_0 - 1)|A_1|^2 - 8\,(2\,\alpha_5\theta_0 - \alpha_5) A_0 A_1 + 8\left(2\,(2\,\alpha_1\theta_0 - \alpha_1) A_0^2 - A_1^2(2\,\theta_0 - 1)\right)|A_1|^2 & 2\,\theta_0 - 2 & 1 \\ 16\,A_0^2\alpha_1^2\theta_0 - 16\,A_1^2\alpha_1\theta_0 - 32\,A_0 A_2 \alpha_1\theta_0 - 24\,A_0 A_1 \alpha_3\theta_0 + 16\,A_2^2\theta_0 + 24\,A_1 A_3 \theta_0 & 2\,\theta_0 - 1 & 0 \\ -4\left(3\,\theta_0^2\overline{\alpha_2} + \theta_0\overline{\alpha_2} - 2\,\overline{\alpha_2}\right) A_0 A_1 & 3\,\theta_0 - 3 & -\theta_0 + 2 \end{pmatrix}$$

$\overline{\partial}\vec{h}_0 \dot{\otimes} \vec{h}_0 =$

$$\begin{pmatrix} \frac{\left(\theta_0^2 - 4\,\theta_0 + 4\right) C_1^2}{4\,\theta_0} & 0 & 2\,\theta_0 - 1 \\ -A_1 C_2(\theta_0 - 3) + 2\,((\alpha_1\theta_0 - 2\,\alpha_1) A_0 - A_2(\theta_0 - 2)) C_1 & \theta_0 & \theta_0 - 1 \\ \frac{2\left(\theta_0^2 - \theta_0 - 2\right) A_0 C_1 |A_1|^2 + 4\left(\alpha_2\theta_0^3 - \alpha_2\theta_0^2 - 2\,\alpha_2\theta_0\right) A_0 A_1 - \left(\theta_0^2 - 2\,\theta_0\right) A_1 B_1}{\theta_0} & \theta_0 - 1 & \theta_0 \\ -A_1 C_1 (\theta_0 - 2) & \theta_0 - 1 & \theta_0 - 1 \\ -8\,A_0 A_1 |A_1|^2 & 2\,\theta_0 - 2 & 0 \\ 16\,A_0^2 |A_1|^4 - 8\,A_0 A_1 \overline{\alpha_5} + A_1 \overline{B_1} & 2\,\theta_0 - 2 & 1 \\ -16\,A_0 A_2 |A_1|^2 - 8\,A_0 A_1 \alpha_5 + 8\left(2\,A_0^2\alpha_1 - A_1^2\right)|A_1|^2 & 2\,\theta_0 - 1 & 0 \\ 4\left(\theta_0^2\overline{\alpha_2} - \theta_0\overline{\alpha_2} - 2\,\overline{\alpha_2}\right) A_0 A_1 & 3\,\theta_0 - 2 & -\theta_0 + 1 \end{pmatrix}$$

$\partial\vec{h}_0 \dot{\otimes} \partial\vec{h}_0 =$

$$
\begin{pmatrix}
-\frac{2\left(\theta_0^2-4\,\theta_0+3\right)A_1C_2}{\theta_0} & \theta_0-2 & \theta_0 \\
4\left(\theta_0^2-2\,\theta_0+1\right)A_1^2 & 2\,\theta_0-4 & 0 \\
-16\left(\theta_0^2-2\,\theta_0+1\right)A_0A_1|A_1|^2 & 2\,\theta_0-4 & 1 \\
16\left(\theta_0^2-2\,\theta_0+1\right)A_0^2|A_1|^4-8\left(\theta_0^2\overline{\alpha_5}-2\,\theta_0\overline{\alpha_5}+\overline{\alpha_5}\right)A_0A_1+\left(\theta_0^2-2\,\theta_0+1\right)A_1\overline{B_1} & 2\,\theta_0-4 & 2 \\
-16\left(\alpha_1\theta_0^2-\alpha_1\theta_0\right)A_0A_1+16\left(\theta_0^2-\theta_0\right)A_1A_2 & 2\,\theta_0-3 & 0 \\
-32\left(\theta_0^2-\theta_0\right)A_0A_2|A_1|^2-16\left(\alpha_5\theta_0^2-\alpha_5\theta_0\right)A_0A_1+16\left(2\left(\alpha_1\theta_0^2-\alpha_1\theta_0\right)A_0^2-\left(\theta_0^2-\theta_0\right)A_1^2\right)|A_1|^2 & 2\,\theta_0-3 & 1 \\
16\,A_0^2\alpha_1^2\theta_0^2-32\,A_0A_2\alpha_1\theta_0^2+16\,A_2^2\theta_0^2-24\left(\alpha_3\theta_0^2-\alpha_3\right)A_0A_1-16\left(\alpha_1\theta_0^2-\alpha_1\right)A_1^2+24\left(\theta_0^2-1\right)A_1A_3 & 2\,\theta_0-2 & 0 \\
-8\left(2\,\theta_0^3\overline{\alpha_2}-\theta_0^2\overline{\alpha_2}-2\,\theta_0\overline{\alpha_2}+\overline{\alpha_2}\right)A_0A_1 & 3\,\theta_0-4 & -\theta_0+2
\end{pmatrix}
$$

$\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}\vec{h}_0 =$

$$
\begin{pmatrix}
\frac{1}{4}\left(\theta_0^2-4\,\theta_0+4\right)C_1^2 & 0 & 2\,\theta_0-2 \\
-\frac{1}{2}\left(4\left(\alpha_2\theta_0^3-3\,\alpha_2\theta_0^2+4\,\alpha_2\right)A_0-\left(\theta_0^2-4\,\theta_0+4\right)B_1\right)C_1 & 0 & 2\,\theta_0-1 \\
\frac{1}{2}\left(\theta_0^2-5\,\theta_0+6\right)C_1C_2 & 1 & 2\,\theta_0-2 \\
4\,A_0C_2(\theta_0-3)|A_1|^2+4\left(A_1(\theta_0-2)|A_1|^2+(\alpha_5\theta_0-2\,\alpha_5)A_0\right)C_1 & \theta_0 & \theta_0-1 \\
-16\left(\alpha_2\theta_0^2-\alpha_2\theta_0-2\,\alpha_2\right)A_0^2|A_1|^2+4\,A_0B_1(\theta_0-2)|A_1|^2+\frac{1}{2}\left(8\,(\theta_0\overline{\alpha_5}-2\,\overline{\alpha_5})A_0-(\theta_0-2)\overline{B_1}\right)C_1 & \theta_0-1 & \theta_0 \\
4\,A_0C_1(\theta_0-2)|A_1|^2 & \theta_0-1 & \theta_0-1 \\
16\,A_0^2|A_1|^4 & 2\,\theta_0-2 & 0 \\
32\,A_0^2|A_1|^2\overline{\alpha_5}-4\,A_0|A_1|^2\overline{B_1} & 2\,\theta_0-2 & 1 \\
32\,A_0A_1|A_1|^4+32\,A_0^2\alpha_5|A_1|^2-2\left(\theta_0^3\overline{\alpha_2}-3\,\theta_0^2\overline{\alpha_2}+4\,\overline{\alpha_2}\right)A_0C_1 & 2\,\theta_0-1 & 0 \\
-16\left(\theta_0^2\overline{\alpha_2}-\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2}\right)A_0^2|A_1|^2 & 3\,\theta_0-2 & -\theta_0+1
\end{pmatrix}
$$

$\partial\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}\vec{h}_0 =$

$$
\begin{pmatrix}
2\left(\theta_0^2-3\,\theta_0+2\right)A_0C_1|A_1|^2+4\left(\alpha_2\theta_0^3-2\,\alpha_2\theta_0^2-\alpha_2\theta_0+2\,\alpha_2\right)A_0A_1-\left(\theta_0^2-3\,\theta_0+2\right)A_1B_1 & \theta_0-2 & \theta_0 \\
-\left(\theta_0^2-3\,\theta_0+2\right)A_1C_1 & \theta_0-2 & \theta_0-1 \\
-\left(\theta_0^2-4\,\theta_0+3\right)A_1C_2+2\left(\left(\alpha_1\theta_0^2-2\,\alpha_1\theta_0\right)A_0-\left(\theta_0^2-2\,\theta_0\right)A_2\right)C_1 & \theta_0-1 & \theta_0-1 \\
-8\,A_0A_1(\theta_0-1)|A_1|^2 & 2\,\theta_0-3 & 0 \\
16\,A_0^2(\theta_0-1)|A_1|^4-8\,(\theta_0\overline{\alpha_5}-\overline{\alpha_5})A_0A_1+A_1(\theta_0-1)\overline{B_1} & 2\,\theta_0-3 & 1 \\
-16\,A_0A_2\theta_0|A_1|^2-8\,(\alpha_5\theta_0-\alpha_5)A_0A_1+8\left(2\,A_0^2\alpha_1\theta_0-A_1^2(\theta_0-1)\right)|A_1|^2 & 2\,\theta_0-2 & 0 \\
4\left(\theta_0^3\overline{\alpha_2}-2\,\theta_0^2\overline{\alpha_2}-\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2}\right)A_0A_1 & 3\,\theta_0-3 & -\theta_0+1
\end{pmatrix}
$$

$\partial\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0 =$

$$
\begin{pmatrix}
\frac{2\left(\theta_0^2-3\,\theta_0+2\right)A_0C_1|A_1|^2}{\theta_0} & \theta_0-2 & \theta_0 \\
-A_1C_2(\theta_0-3) & \theta_0-1 & \theta_0-1 \\
-8\,A_0A_1(\theta_0-1)|A_1|^2 & 2\,\theta_0-3 & 0 \\
16\,A_0^2(\theta_0-1)|A_1|^4-8\,(\theta_0\overline{\alpha_5}-\overline{\alpha_5})A_0A_1+A_1(\theta_0-1)\overline{B_1} & 2\,\theta_0-3 & 1 \\
-16\,A_0A_2(\theta_0-1)|A_1|^2-8\,A_0A_1\alpha_5\theta_0+8\left(2\,(\alpha_1\theta_0-\alpha_1)A_0^2-A_1^2\theta_0\right)|A_1|^2 & 2\,\theta_0-2 & 0 \\
4\left(2\,\theta_0^3\overline{\alpha_2}-3\,\theta_0^2\overline{\alpha_2}-3\,\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2}\right)A_0A_1 & 3\,\theta_0-3 & -\theta_0+1
\end{pmatrix}
$$

$\partial\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \partial\vec{h}_0 =$

$$\begin{pmatrix} -\left(\theta_0^2-4\,\theta_0+3\right)A_1C_2 & \theta_0-2 & \theta_0-1 \\ -8\left(\theta_0^2-2\,\theta_0+1\right)A_0A_1|A_1|^2 & 2\,\theta_0-4 & 0 \\ 16\left(\theta_0^2-2\,\theta_0+1\right)A_0^2|A_1|^4-8\left(\theta_0^2\overline{\alpha_5}-2\,\theta_0\overline{\alpha_5}+\overline{\alpha_5}\right)A_0A_1+\left(\theta_0^2-2\,\theta_0+1\right)A_1\overline{B_1} & 2\,\theta_0-4 & 1 \\ -16\left(\theta_0^2-\theta_0\right)A_0A_2|A_1|^2-8\left(\alpha_5\theta_0^2-\alpha_5\theta_0\right)A_0A_1+8\left(2\left(\alpha_1\theta_0^2-\alpha_1\theta_0\right)A_0^2-\left(\theta_0^2-\theta_0\right)A_1^2\right)|A_1|^2 & 2\,\theta_0-3 & 0 \\ 4\left(2\,\theta_0^4\overline{\alpha_2}-5\,\theta_0^3\overline{\alpha_2}+5\,\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2}\right)A_0A_1 & 3\,\theta_0-4 & -\theta_0+1 \end{pmatrix}$$

$\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0 =$

$$\begin{pmatrix} \frac{1}{4}\left(\theta_0^2-5\,\theta_0+6\right)C_1C_2 & 0 & 2\,\theta_0-2 \\ \lambda_1 & \theta_0-2 & \theta_0 \\ 2\left(\theta_0^2-3\,\theta_0+2\right)A_0C_1|A_1|^2 & \theta_0-2 & \theta_0-1 \\ 2\left(\theta_0^2-3\,\theta_0\right)A_0C_2|A_1|^2+2\left(\left(\theta_0^2-2\,\theta_0\right)A_1|A_1|^2+\left(\alpha_5\theta_0^2-2\,\alpha_5\theta_0\right)A_0\right)C_1 & \theta_0-1 & \theta_0-1 \\ 16\,A_0^2(\theta_0-1)|A_1|^4 & 2\,\theta_0-3 & 0 \\ 32\left(\theta_0\overline{\alpha_5}-\overline{\alpha_5}\right)A_0^2|A_1|^2-4\,A_0(\theta_0-1)|A_1|^2\overline{B_1} & 2\,\theta_0-3 & 1 \\ 16\,A_0A_1(2\,\theta_0-1)|A_1|^4+16\left(2\,\alpha_5\theta_0-\alpha_5\right)A_0^2|A_1|^2-\left(2\,\theta_0^4\overline{\alpha_2}-7\,\theta_0^3\overline{\alpha_2}+3\,\theta_0^2\overline{\alpha_2}+8\,\theta_0\overline{\alpha_2}-4\,\overline{\alpha_2}\right)A_0C_1 & 2\,\theta_0-2 & 0 \\ -8\left(3\,\theta_0^3\overline{\alpha_2}-5\,\theta_0^2\overline{\alpha_2}-4\,\theta_0\overline{\alpha_2}+4\,\overline{\alpha_2}\right)A_0^2|A_1|^2 & 3\,\theta_0-3 & -\theta_0+1 \end{pmatrix}$$

$$\lambda_1=-8\left(\alpha_2\theta_0^3-2\,\alpha_2\theta_0^2-\alpha_2\theta_0+2\,\alpha_2\right)A_0^2|A_1|^2+2\left(\theta_0^2-3\,\theta_0+2\right)A_0B_1|A_1|^2$$
$$+\frac{1}{4}\left(8\left(\theta_0^2\overline{\alpha_5}-3\,\theta_0\overline{\alpha_5}+2\,\overline{\alpha_5}\right)A_0-\left(\theta_0^2-3\,\theta_0+2\right)\overline{B_1}\right)C_1$$

$$\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\overline{\partial}\vec{h}_0=\begin{pmatrix} 4\left(\theta_0^2-4\,\theta_0+3\right)A_0C_2|A_1|^2 & \theta_0-2 & \theta_0-1 \\ 16\left(\theta_0^2-2\,\theta_0+1\right)A_0^2|A_1|^4 & 2\,\theta_0-4 & 0 \\ 32\left(\theta_0^2\overline{\alpha_5}-2\,\theta_0\overline{\alpha_5}+\overline{\alpha_5}\right)A_0^2|A_1|^2-4\left(\theta_0^2-2\,\theta_0+1\right)A_0|A_1|^2\overline{B_1} & 2\,\theta_0-4 & 1 \\ 32\left(\theta_0^2-\theta_0\right)A_0A_1|A_1|^4+32\left(\alpha_5\theta_0^2-\alpha_5\theta_0\right)A_0^2|A_1|^2 & 2\,\theta_0-3 & 0 \\ -16\left(2\,\theta_0^4\overline{\alpha_2}-5\,\theta_0^3\overline{\alpha_2}+5\,\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2}\right)A_0^2|A_1|^2 & 3\,\theta_0-4 & -\theta_0+1 \end{pmatrix}$$

We have for some constants $\alpha_1\in\mathbb{C}$ and $\vec{A}_0,\vec{A}_1,\vec{A}_2,\vec{C}_1\in\mathbb{C}^n$ the development

$$\begin{cases} g=|z|^{2\theta_0-2}|dz|^2 \\ \vec{H}=\mathrm{Re}\left(\dfrac{\vec{C}_1}{z^{\theta_0-2}}\right)+O(|z|^{3-\theta_0}) \\ \vec{h}_0=2\left(\vec{A}_1-2|\vec{A}_1|^2\vec{A}_0\overline{z}\right)z^{\theta_0-1}+4\left(\vec{A}_2-\alpha_1\vec{A}_0\right)z^{\theta_0}dz^2-\dfrac{(\theta_0-2)}{2\theta_0}\vec{C}_1\overline{z}^{\theta_0}dz^2+O(|z|^{\theta_0+1}) \end{cases} \tag{5.0.40}$$

In particular, we have

$$\begin{cases} \partial\vec{h}_0=\left(2(\theta_0-1)\left(\vec{A}_1-2|\vec{A}_1|^2\vec{A}_0\overline{z}\right)z^{\theta_0-2}+4\left(\vec{A}_2-\alpha_1\vec{A}_0\right)z^{\theta_0-1}\right)dz^3+O(|z|^{\theta_0})=O(|z|^{\theta_0-2}) \\ \overline{\partial}\vec{h}_0=\left(-4|\vec{A}_1|^2\vec{A}_0z^{\theta_0-1}-\dfrac{(\theta_0-2)}{2}\vec{C}_1\overline{z}^{\theta_0-1}\right)dz^2\,\dot{\otimes}\,d\overline{z}+O(|z|^{\theta_0})=O(|z|^{\theta_0-1}) \\ \partial\overline{\partial}\vec{h}_0=-4(\theta_0-1)|\vec{A}_1|^2\vec{A}_0z^{\theta_0-2}+O(|z|^{\theta_0-1}) \end{cases} \tag{5.0.41}$$

Now, recall that

$$\begin{cases} \langle\vec{A}_0,\vec{A}_0\rangle=\langle\vec{A}_0,\vec{A}_1\rangle=\langle\vec{A}_0,\overline{\vec{A}_1}\rangle=\langle\vec{A}_0,\vec{C}_1\rangle=\langle\vec{A}_0,\overline{\vec{C}_1}\rangle=0,\quad \langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_0,\vec{A}_2\rangle=0, \\ \langle\vec{A}_1,\vec{C}_1\rangle=\langle\vec{A}_0,\vec{C}_2\rangle=0,\quad |\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle=0 \end{cases}$$

we obtain

$$
\vec{h}_0 \dot{\otimes} \vec{h}_0 = \begin{pmatrix}
\frac{\left(\theta_0^2 - 4\,\theta_0 + 4\right) C_1^2}{4\,\theta_0^2} & 0 & 2\,\theta_0 \\
-\frac{2\,\left(A_1 C_2(\theta_0 - 3) - 2\,\left((\alpha_1\theta_0 - 2\,\alpha_1)A_0 - A_2(\theta_0 - 2)\right)C_1\right)}{\theta_0} & \theta_0 & \theta_0 \\
-\frac{2\,A_1 C_1(\theta_0 - 2)}{\theta_0} & \theta_0 - 1 & \theta_0 \\
\frac{2\left(2\left(\theta_0^2 - \theta_0 - 2\right)A_0 C_1 |A_1|^2 + 4\left(\alpha_2\theta_0^3 - \alpha_2\theta_0^2 - 2\,\alpha_2\theta_0\right)A_0 A_1 - \left(\theta_0^2 - 2\,\theta_0\right)A_1 B_1\right)}{\theta_0^2 + \theta_0} & \theta_0 - 1 & \theta_0 + 1 \\
4\,A_1^2 & 2\,\theta_0 - 2 & 0 \\
-16\,A_0 A_1 |A_1|^2 & 2\,\theta_0 - 2 & 1 \\
16\,A_0^2 |A_1|^4 - 8\,A_0 A_1 \overline{\alpha_5} + A_1 \overline{B_1} & 2\,\theta_0 - 2 & 2 \\
-16\,A_0 A_1 \alpha_1 + 16\,A_1 A_2 & 2\,\theta_0 - 1 & 0 \\
-32\,A_0 A_2 |A_1|^2 - 16\,A_0 A_1 \alpha_5 + 16\left(2\,A_0^2\alpha_1 - A_1^2\right)|A_1|^2 & 2\,\theta_0 - 1 & 1 \\
-8\,(\theta_0\overline{\alpha_2} + \overline{\alpha_2})A_0 A_1 & 3\,\theta_0 - 2 & -\theta_0 + 2 \\
16\,A_0^2\alpha_1^2 - 16\,A_1^2\alpha_1 - 32\,A_0 A_2 \alpha_1 - 24\,A_0 A_1 \alpha_3 + 16\,A_2^2 + 24\,A_1 A_3 & 2\,\theta_0 & 0
\end{pmatrix} \tag{5.0.42}
$$

$$
= \begin{pmatrix}
\frac{\left(\theta_0^2 - 4\,\theta_0 + 4\right) C_1^2}{4\,\theta_0^2} & 0 & 2\,\theta_0 & \mathbf{(1)} \\
-\frac{2\,(A_1 C_2(\theta_0 - 3) + 2\,(\theta_0 - 2)A_2 C_1)}{\theta_0} & \theta_0 & \theta_0 & \mathbf{(2)} \\
4\,A_1^2 & 2\,\theta_0 - 2 & 0 & \mathbf{(3)} \\
16\,A_1 A_2 & 2\,\theta_0 - 1 & 0 & \mathbf{(4)} \\
16\,A_2^2 + 24\,A_1 A_3 & 2\,\theta_0 & 0 & \mathbf{(5)}
\end{pmatrix} \tag{5.0.43}
$$

Then, we have

$$
\partial\vec{h}_0 \dot{\otimes} \vec{h}_0 = \tag{5.0.44}
$$

$$
\begin{pmatrix}
-\frac{\left(\theta_0^2 - 3\,\theta_0 + 2\right)A_1 C_1 \theta_0}{\ } & \theta_0 - 2 & \theta_0 \\
\frac{2\left(\theta_0^3 - 2\,\theta_0^2 - \theta_0 + 2\right)A_0 C_1 |A_1|^2 + 4\left(\alpha_2\theta_0^4 - 2\,\alpha_2\theta_0^3 - \alpha_2\theta_0^2 + 2\,\alpha_2\theta_0\right)A_0 A_1 - \left(\theta_0^3 - 3\,\theta_0^2 + 2\,\theta_0\right)A_1 B_1}{\theta_0^2 + \theta_0} & \theta_0 - 2 & \theta_0 + 1 \\
-A_1 C_2(\theta_0 - 3) + 2\left((\alpha_1\theta_0 - 2\,\alpha_1)A_0 - A_2(\theta_0 - 2)\right)C_1 & \theta_0 - 1 & \theta_0 \\
4\,A_1^2(\theta_0 - 1) & 2\,\theta_0 - 3 & 0 \\
-16\,A_0 A_1(\theta_0 - 1)|A_1|^2 & 2\,\theta_0 - 3 & 1 \\
16\,A_0^2(\theta_0 - 1)|A_1|^4 - 8\,(\theta_0\overline{\alpha_5} - \overline{\alpha_5})A_0 A_1 + A_1(\theta_0 - 1)\overline{B_1} & 2\,\theta_0 - 3 & 2 \\
-8\,(2\,\alpha_1\theta_0 - \alpha_1)A_0 A_1 + 8\,A_1 A_2(2\,\theta_0 - 1) & 2\,\theta_0 - 2 & 0 \\
-16\,A_0 A_2(2\,\theta_0 - 1)|A_1|^2 - 8\,(2\,\alpha_5\theta_0 - \alpha_5)A_0 A_1 + 8\left(2\,(2\,\alpha_1\theta_0 - \alpha_1)A_0^2 - A_1^2(2\,\theta_0 - 1)\right)|A_1|^2 & 2\,\theta_0 - 2 & 1 \\
16\,A_0^2\alpha_1^2\theta_0 - 16\,A_1^2\alpha_1\theta_0 - 32\,A_0 A_2 \alpha_1\theta_0 - 24\,A_0 A_1 \alpha_3\theta_0 + 16\,A_2^2\theta_0 + 24\,A_1 A_3\theta_0 & 2\,\theta_0 - 1 & 0 \\
-4\left(3\,\theta_0^2\overline{\alpha_2} + \theta_0\overline{\alpha_2} - 2\,\overline{\alpha_2}\right)A_0 A_1 & 3\,\theta_0 - 3 & -\theta_0 + 2
\end{pmatrix}
$$

$$= \begin{pmatrix} -A_1C_2(\theta_0-3) - 2\,(\theta_0-2)A_2C_1 & \theta_0-1 & \theta_0 & \mathbf{(1)} \\ 4\,A_1^2(\theta_0-1) & 2\,\theta_0-3 & 0 & \mathbf{(2)} \\ 8\,A_1A_2(2\,\theta_0-1) & 2\,\theta_0-2 & 0 & \mathbf{(3)} \\ 16\,A_2^2\theta_0+24\,A_1A_3\theta_0 & 2\,\theta_0-1 & 0 & \mathbf{(4)} \end{pmatrix} \tag{5.0.45}$$

Now,

$$\overline{\partial}\vec{h}_0 \dot{\otimes} \vec{h}_0 = \tag{5.0.46}$$

$$\begin{pmatrix} \frac{\left(\theta_0^2-4\,\theta_0+4\right)C_1^2}{4\,\theta_0} & 0 & 2\,\theta_0-1 \\ -A_1C_2(\theta_0-3)+2\left((\alpha_1\theta_0-2\,\alpha_1)A_0-A_2(\theta_0-2)\right)C_1 & \theta_0 & \theta_0-1 \\ \frac{2\left(\theta_0^2-\theta_0-2\right)A_0C_1|A_1|^2+4\left(\alpha_2\theta_0^3-\alpha_2\theta_0^2-2\,\alpha_2\theta_0\right)A_0A_1-\left(\theta_0^2-2\,\theta_0\right)A_1B_1}{\theta_0} & \theta_0-1 & \theta_0 \\ -A_1C_1(\theta_0-2) & \theta_0-1 & \theta_0-1 \\ -8\,A_0A_1|A_1|^2 & 2\,\theta_0-2 & 0 \\ 16\,A_0^2|A_1|^4-8\,A_0A_1\overline{\alpha_5}+A_1\overline{B_1} & 2\,\theta_0-2 & 1 \\ -16\,A_0A_2|A_1|^2-8\,A_0A_1\alpha_5+8\left(2\,A_0^2\alpha_1-A_1^2\right)|A_1|^2 & 2\,\theta_0-1 & 0 \\ 4\left(\theta_0^2\overline{\alpha_2}-\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2}\right)A_0A_1 & 3\,\theta_0-2 & -\theta_0+1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{(\theta_0-2)^2C_1^2}{4\,\theta_0} & 0 & 2\,\theta_0-1 & \mathbf{(1)} \\ -A_1C_2(\theta_0-3)-2\,(\theta_0-2)A_2C_1 & \theta_0 & \theta_0-1 & \mathbf{(2)} \end{pmatrix} \tag{5.0.47}$$

Then

$$\partial\vec{h}_0 \dot{\otimes} \partial\vec{h}_0 =$$

$$\begin{pmatrix} -\frac{2\left(\theta_0^2-4\,\theta_0+3\right)A_1C_2}{\theta_0} & \theta_0-2 & \theta_0 \\ 4\left(\theta_0^2-2\,\theta_0+1\right)A_1^2 & 2\,\theta_0-4 & 0 \\ -16\left(\theta_0^2-2\,\theta_0+1\right)A_0A_1|A_1|^2 & 2\,\theta_0-4 & 1 \\ 16\left(\theta_0^2-2\,\theta_0+1\right)A_0^2|A_1|^4-8\left(\theta_0^2\overline{\alpha_5}-2\,\theta_0\overline{\alpha_5}+\overline{\alpha_5}\right)A_0A_1+\left(\theta_0^2-2\,\theta_0+1\right)A_1\overline{B_1} & 2\,\theta_0-4 & 2 \\ -16\left(\alpha_1\theta_0^2-\alpha_1\theta_0\right)A_0A_1+16\left(\theta_0^2-\theta_0\right)A_1A_2 & 2\,\theta_0-3 & 0 \\ -32\left(\theta_0^2-\theta_0\right)A_0A_2|A_1|^2-16\left(\alpha_5\theta_0^2-\alpha_5\theta_0\right)A_0A_1+16\left(2\left(\alpha_1\theta_0^2-\alpha_1\theta_0\right)A_0^2-\left(\theta_0^2-\theta_0\right)A_1^2\right)|A_1|^2 & 2\,\theta_0-3 & 1 \\ 16\,A_0^2\alpha_1^2\theta_0^2-32\,A_0A_2\alpha_1\theta_0^2+16\,A_2^2\theta_0^2-24\left(\alpha_3\theta_0^2-\alpha_3\right)A_0A_1-16\left(\alpha_1\theta_0^2-\alpha_1\right)A_1^2+24\left(\theta_0^2-1\right)A_1A_3 & 2\,\theta_0-2 & 0 \\ -8\left(2\,\theta_0^3\overline{\alpha_2}-\theta_0^2\overline{\alpha_2}-2\,\theta_0\overline{\alpha_2}+\overline{\alpha_2}\right)A_0A_1 & 3\,\theta_0-4 & -\theta_0+2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2\left(\theta_0^2-4\,\theta_0+3\right)A_1C_2}{\theta_0} & \theta_0-2 & \theta_0 & \mathbf{(1)} \\ 4\left(\theta_0^2-2\,\theta_0+1\right)A_1^2 & 2\,\theta_0-4 & 0 & \mathbf{(2)} \\ 16\,A_2^2\theta_0^2+24\left(\theta_0^2-1\right)A_1A_3 & 2\,\theta_0-2 & 0 & \mathbf{(3)} \end{pmatrix} \tag{5.0.48}$$

Now

$$\overline{\partial}\vec{h}_0 \dot{\otimes} \overline{\partial}\vec{h}_0 =$$

$$\begin{pmatrix} \frac{1}{4}\left(\theta_0^2-4\,\theta_0+4\right)C_1^2 & 0 & 2\,\theta_0-2 \\ -\frac{1}{2}\left(4\left(\alpha_2\theta_0^3-3\,\alpha_2\theta_0^2+4\,\alpha_2\right)A_0-\left(\theta_0^2-4\,\theta_0+4\right)B_1\right)C_1 & 0 & 2\,\theta_0-1 \\ \frac{1}{2}\left(\theta_0^2-5\,\theta_0+6\right)C_1C_2 & 1 & 2\,\theta_0-2 \\ 4\,A_0C_2(\theta_0-3)|A_1|^2+4\left(A_1(\theta_0-2)|A_1|^2+(\alpha_5\theta_0-2\,\alpha_5)A_0\right)C_1 & \theta_0 & \theta_0-1 \\ -16\left(\alpha_2\theta_0^2-\alpha_2\theta_0-2\,\alpha_2\right)A_0^2|A_1|^2+4\,A_0B_1(\theta_0-2)|A_1|^2+\frac{1}{2}\left(8\,(\theta_0\overline{\alpha_5}-2\,\overline{\alpha_5})A_0-(\theta_0-2)\overline{B_1}\right)C_1 & \theta_0-1 & \theta_0 \\ 4\,A_0C_1(\theta_0-2)|A_1|^2 & \theta_0-1 & \theta_0-1 \\ 16\,A_0^2|A_1|^4 & 2\,\theta_0-2 & 0 \\ 32\,A_0^2|A_1|^2\overline{\alpha_5}-4\,A_0|A_1|^2\overline{B_1} & 2\,\theta_0-2 & 1 \\ 32\,A_0A_1|A_1|^4+32\,A_0^2\alpha_5|A_1|^2-2\left(\theta_0^3\overline{\alpha_2}-3\,\theta_0^2\overline{\alpha_2}+4\,\overline{\alpha_2}\right)A_0C_1 & 2\,\theta_0-1 & 0 \\ -16\left(\theta_0^2\overline{\alpha_2}-\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2}\right)A_0^2|A_1|^2 & 3\,\theta_0-2 & -\theta_0+1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}\left(\theta_0^2-4\,\theta_0+4\right)C_1^2 & 0 & 2\,\theta_0-2 & \textbf{(1)} \\ \frac{1}{2}\left(\theta_0^2-5\,\theta_0+6\right)C_1C_2 & 1 & 2\,\theta_0-2 & \textbf{(2)} \\ 8\theta_0(\theta_0+1)|\vec{A}_1|^2\overline{\alpha_2} & 2\,\theta_0-2 & 1 & \textbf{(3)} \end{pmatrix} \tag{5.0.49}$$

as

$$\begin{aligned} &\alpha_2 = \frac{1}{2\theta_0(\theta_0+1)}\langle\overline{\vec{A}_1},\vec{C}_1\rangle \\ &\langle\vec{A}_0,\overline{\vec{B}_1}\rangle = \langle\vec{A}_0,-2\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\overline{\vec{A}_0}\rangle = -\langle\vec{A}_1,\overline{\vec{C}_1}\rangle = -2\theta_0(\theta_0+1)\overline{\alpha_2}. \end{aligned} \tag{5.0.50}$$

Now, we have

$\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0 =$

$$\begin{pmatrix} 2\left(\theta_0^2-3\,\theta_0+2\right)A_0C_1|A_1|^2+4\left(\alpha_2\theta_0^3-2\,\alpha_2\theta_0^2-\alpha_2\theta_0+2\,\alpha_2\right)A_0A_1-\left(\theta_0^2-3\,\theta_0+2\right)A_1B_1 & \theta_0-2 & \theta_0 \\ -\left(\theta_0^2-3\,\theta_0+2\right)A_1C_1 & \theta_0-2 & \theta_0-1 \\ -\left(\theta_0^2-4\,\theta_0+3\right)A_1C_2+2\left(\left(\alpha_1\theta_0^2-2\,\alpha_1\theta_0\right)A_0-\left(\theta_0^2-2\,\theta_0\right)A_2\right)C_1 & \theta_0-1 & \theta_0-1 \\ -8\,A_0A_1(\theta_0-1)|A_1|^2 & 2\,\theta_0-3 & 0 \\ 16\,A_0^2(\theta_0-1)|A_1|^4-8\left(\theta_0\overline{\alpha_5}-\overline{\alpha_5}\right)A_0A_1+A_1(\theta_0-1)\overline{B_1} & 2\,\theta_0-3 & 1 \\ -16\,A_0A_2\theta_0|A_1|^2-8\left(\alpha_5\theta_0-\alpha_5\right)A_0A_1+8\left(2\,A_0^2\alpha_1\theta_0-A_1^2(\theta_0-1)\right)|A_1|^2 & 2\,\theta_0-2 & 0 \\ 4\left(\theta_0^3\overline{\alpha_2}-2\,\theta_0^2\overline{\alpha_2}-\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2}\right)A_0A_1 & 3\,\theta_0-3 & -\theta_0+1 \end{pmatrix}$$

$$= \begin{pmatrix} -\left(\theta_0^2-4\,\theta_0+3\right)A_1C_2-2\left(\theta_0^2-2\,\theta_0\right)A_2C_1 & \theta_0-1 & \theta_0-1 \end{pmatrix} \tag{5.0.51}$$

Then

$\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0 =$

$$
\begin{pmatrix}
\frac{2\left(\theta_0^2-3\,\theta_0+2\right)A_0C_1|A_1|^2}{\theta_0} & \theta_0-2 & \theta_0\\
-A_1C_2(\theta_0-3) & \theta_0-1 & \theta_0-1\\
-8\,A_0A_1(\theta_0-1)|A_1|^2 & 2\,\theta_0-3 & 0\\
16\,A_0^2(\theta_0-1)|A_1|^4-8\,(\theta_0\overline{\alpha_5}-\overline{\alpha_5})A_0A_1+A_1(\theta_0-1)\overline{B_1} & 2\,\theta_0-3 & 1\\
-16\,A_0A_2(\theta_0-1)|A_1|^2-8\,A_0A_1\alpha_5\theta_0+8\left(2\,(\alpha_1\theta_0-\alpha_1)A_0^2-A_1^2\theta_0\right)|A_1|^2 & 2\,\theta_0-2 & 0\\
4\left(2\,\theta_0^3\overline{\alpha_2}-3\,\theta_0^2\overline{\alpha_2}-3\,\theta_0\overline{\alpha_2}+2\,\overline{\alpha_2}\right)A_0A_1 & 3\,\theta_0-3 & -\theta_0+1
\end{pmatrix}
$$

$$
=\begin{pmatrix}-A_1C_2(\theta_0-3) & \theta_0-1 & \theta_0-1\end{pmatrix} \tag{5.0.52}
$$

while

$\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0 =$

$$
\begin{pmatrix}
-\left(\theta_0^2-4\,\theta_0+3\right)A_1C_2 & \theta_0-2 & \theta_0-1\\
-8\left(\theta_0^2-2\,\theta_0+1\right)A_0A_1|A_1|^2 & 2\,\theta_0-4 & 0\\
16\left(\theta_0^2-2\,\theta_0+1\right)A_0^2|A_1|^4-8\left(\theta_0^2\overline{\alpha_5}-2\,\theta_0\overline{\alpha_5}+\overline{\alpha_5}\right)A_0A_1+\left(\theta_0^2-2\,\theta_0+1\right)A_1\overline{B_1} & 2\,\theta_0-4 & 1\\
-16\left(\theta_0^2-\theta_0\right)A_0A_2|A_1|^2-8\left(\alpha_5\theta_0^2-\alpha_5\theta_0\right)A_0A_1+8\left(2\left(\alpha_1\theta_0^2-\alpha_1\theta_0\right)A_0^2-\left(\theta_0^2-\theta_0\right)A_1^2\right)|A_1|^2 & 2\,\theta_0-3 & 0\\
4\left(2\,\theta_0^4\overline{\alpha_2}-5\,\theta_0^3\overline{\alpha_2}+5\,\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2}\right)A_0A_1 & 3\,\theta_0-4 & -\theta_0+1
\end{pmatrix}
$$

$$
=\begin{pmatrix}-\left(\theta_0^2-4\,\theta_0+3\right)A_1C_2 & \theta_0-2 & \theta_0-1\end{pmatrix} \tag{5.0.53}
$$

Now

$\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0 =$

$$
\begin{pmatrix}
\frac{1}{4}\left(\theta_0^2-5\,\theta_0+6\right)C_1C_2 & 0 & 2\,\theta_0-2\\
\lambda_1 & \theta_0-2 & \theta_0\\
2\left(\theta_0^2-3\,\theta_0+2\right)A_0C_1|A_1|^2 & \theta_0-2 & \theta_0-1\\
2\left(\theta_0^2-3\,\theta_0\right)A_0C_2|A_1|^2+2\left(\left(\theta_0^2-2\,\theta_0\right)A_1|A_1|^2+\left(\alpha_5\theta_0^2-2\,\alpha_5\theta_0\right)A_0\right)C_1 & \theta_0-1 & \theta_0-1\\
16\,A_0^2(\theta_0-1)|A_1|^4 & 2\,\theta_0-3 & 0\\
32\,(\theta_0\overline{\alpha_5}-\overline{\alpha_5})A_0^2|A_1|^2-4\,A_0(\theta_0-1)|A_1|^2\overline{B_1} & 2\,\theta_0-3 & 1\\
16\,A_0A_1(2\,\theta_0-1)|A_1|^4+16\,(2\,\alpha_5\theta_0-\alpha_5)A_0^2|A_1|^2-\left(2\,\theta_0^4\overline{\alpha_2}-7\,\theta_0^3\overline{\alpha_2}+3\,\theta_0^2\overline{\alpha_2}+8\,\theta_0\overline{\alpha_2}-4\,\overline{\alpha_2}\right)A_0C_1 & 2\,\theta_0-2 & 0\\
-8\left(3\,\theta_0^3\overline{\alpha_2}-5\,\theta_0^2\overline{\alpha_2}-4\,\theta_0\overline{\alpha_2}+4\,\overline{\alpha_2}\right)A_0^2|A_1|^2 & 3\,\theta_0-3 & -\theta_0+1
\end{pmatrix}
$$

$$
=\begin{pmatrix}
\frac{1}{4}\left(\theta_0^2-5\,\theta_0+6\right)C_1C_2 & 0 & 2\,\theta_0-2\\
8\theta_0(\theta_0+1)|\vec{A}_1|^2\overline{\alpha_2} & 2\,\theta_0-3 & 1
\end{pmatrix} \tag{5.0.54}
$$

as

$$
\begin{aligned}
\lambda_1 &= -8\left(\alpha_2\theta_0^3-2\,\alpha_2\theta_0^2-\alpha_2\theta_0+2\,\alpha_2\right)A_0^2|A_1|^2+2\left(\theta_0^2-3\,\theta_0+2\right)A_0B_1|A_1|^2\\
&+\frac{1}{4}\left(8\left(\theta_0^2\overline{\alpha_5}-3\,\theta_0\overline{\alpha_5}+2\,\overline{\alpha_5}\right)A_0-\left(\theta_0^2-3\,\theta_0+2\right)\overline{B_1}\right)C_1=0
\end{aligned}
$$

Finally,

$$\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\overline{\partial}\vec{h}_0=\begin{pmatrix} 4\left(\theta_0^2-4\,\theta_0+3\right)A_0C_2|A_1|^2 & \theta_0-2 & \theta_0-1\\ 16\left(\theta_0^2-2\,\theta_0+1\right)A_0^2|A_1|^4 & 2\,\theta_0-4 & 0\\ 32\left(\theta_0^2\overline{\alpha_5}-2\,\theta_0\overline{\alpha_5}+\overline{\alpha_5}\right)A_0^2|A_1|^2-4\left(\theta_0^2-2\,\theta_0+1\right)A_0|A_1|^2\overline{B_1} & 2\,\theta_0-4 & 1\\ 32\left(\theta_0^2-\theta_0\right)A_0A_1|A_1|^4+32\left(\alpha_5\theta_0^2-\alpha_5\theta_0\right)A_0^2|A_1|^2 & 2\,\theta_0-3 & 0\\ -16\left(2\,\theta_0^4\overline{\alpha_2}-5\,\theta_0^3\overline{\alpha_2}+5\,\theta_0\overline{\alpha_2}-2\,\overline{\alpha_2}\right)A_0^2|A_1|^2 & 3\,\theta_0-4 & -\theta_0+1 \end{pmatrix}$$

$$=\begin{pmatrix}8\theta_0(\theta_0-1)^2(\theta_0+1)|\vec{A}_1|^2\overline{\alpha}_2 & 2\,\theta_0-4 & 1\end{pmatrix}\tag{5.0.55}$$

thanks of (5.0.50). Finally, thanks of (5.0.42), (5.0.44), (5.0.46), (5.0.48), (5.0.49), (5.0.51), (5.0.52), (5.0.53), (5.0.54), (5.0.55), we have (omitting the dz, $d\overline{z}$ to simplify notations)

$$\begin{cases} \vec{h}_0\,\dot{\otimes}\,\vec{h}_0=4\langle\vec{A}_1,\vec{A}_1\rangle z^{2\theta_0-2}+16\langle\vec{A}_1,\vec{A}_2\rangle z^{2\theta_0-1}+\left(16\langle\vec{A}_2,\vec{A}_2\rangle+24\langle\vec{A}_1,\vec{A}_3\rangle\right)z^{2\theta_0}\\ \quad-\dfrac{2}{\theta_0}\left((\theta_0-3)\langle\vec{A}_1,\vec{C}_2\rangle+2(\theta_0-2)\langle\vec{A}_2,\vec{C}_1\rangle\right)|z|^{2\theta_0}+\dfrac{(\theta_0-2)^2}{4\theta_0^2}\langle\vec{C}_1,\vec{C}_1\rangle\overline{z}^{2\theta_0}+O(|z|^{2\theta_0+1})\\ \partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0=4(\theta_0-1)\langle\vec{A}_1,\vec{A}_1\rangle z^{2\theta_0-3}+8(2\theta_0-1)\langle\vec{A}_1,\vec{A}_2\rangle z^{2\theta_0-2}+\theta_0\left(16\langle\vec{A}_2,\vec{A}_2\rangle+24\langle\vec{A}_1,\vec{A}_2\rangle\right)z^{2\theta_0-1}\\ \quad-\left((\theta_0-3)\langle\vec{A}_1,\vec{C}_2\rangle+2(\theta_0-2)\langle\vec{A}_2,\vec{C}_1\rangle\right)z^{\theta_0-1}\overline{z}^{\theta_0}+O(|z|^{2\theta_0})\\ \overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0=-\left((\theta_0-3)\langle\vec{A}_1,\vec{C}_2\rangle+2(\theta_0-2)\langle\vec{A}_2,\vec{C}_1\rangle\right)z^{\theta_0}\overline{z}^{\theta_0-1}+\dfrac{(\theta_0-2)^2}{4\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle\overline{z}^{2\theta_0-1}+O(|z|^{2\theta_0})\\ \partial\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0=4(\theta_0-1)^2\langle\vec{A}_1,\vec{A}_1\rangle z^{2\theta_0-4}+\left(16\theta_0^2\langle\vec{A}_2,\vec{A}_2\rangle+24(\theta_0-1)(\theta_0+1)\langle\vec{A}_1,\vec{A}_3\rangle\right)z^{2\theta_0-2}\\ \quad-\dfrac{2(\theta_0-1)(\theta_0-3)}{\theta_0}\langle\vec{A}_1,\vec{C}_2\rangle z^{\theta_0-2}\overline{z}^{\theta_0}+O(|z|^{2\theta_0-1})\\ \overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0=\dfrac{1}{4}(\theta_0-2)^2\langle\vec{C}_1,\vec{C}_1\rangle\overline{z}^{2\theta_0-2}+\dfrac{1}{2}(\theta_0-2)(\theta_0-3)\langle\vec{C}_1,\vec{C}_2\rangle z\overline{z}^{2\theta_0-2}+8\theta_0(\theta_0+1)|\vec{A}_1|^2\overline{\alpha_2}z^{2\theta_0-2}\overline{z}\\ \quad+O(|z|^{2\theta_0})\\ \partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0=-\left((\theta_0-1)(\theta_0-3)\langle\vec{A}_1,\vec{C}_2\rangle+2\theta_0(\theta_0-2)\langle\vec{A}_2,\vec{C}_1\rangle\right)|z|^{2\theta_0-2}+O(|z|^{2\theta_0-1})\\ \partial\overline{\partial}\vec{h}_0\otimes\vec{h}_0=-(\theta_0-3)\langle\vec{A}_1,\vec{C}_2\rangle|z|^{2\theta_0-2}+O(|z|^{2\theta_0-1})\\ \partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0=-(\theta_0-1)(\theta_0-3)\langle\vec{A}_1,\vec{C}_2\rangle z^{\theta_0-2}\overline{z}^{\theta_0-1}+O(|z|^{2\theta_0-2})\\ \partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0=\dfrac{1}{4}(\theta_0-2)(\theta_0-3)\langle\vec{C}_1,\vec{C}_2\rangle\overline{z}^{2\theta_0-2}+8\theta_0(\theta_0+1)|\vec{A}_1|^2\overline{\alpha_2}z^{2\theta_0-3}\overline{z}+O(|z|^{2\theta_0-1})\\ \partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\overline{\partial}\vec{h}_0=8\theta_0(\theta_0-1)^2(\theta_0+1)|\vec{A}_1|^2\overline{\alpha_2}z^{2\theta_0-4}\overline{z}+O(|z|^{2\theta_0-2}). \end{cases}\tag{5.0.56}$$

We can easily check that the only potential singular term in $\mathscr{O}_{\vec{\Phi}}$ is

$$O(\vec{h}_0)=g^{-2}\otimes\left\{\frac{1}{4}\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\overline{\partial}\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{4}\left(\partial\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\right.$$
$$\left.-\frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)-\frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)+\frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\right\}$$

We have by (5.0.56)

$$\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\overline{\partial}\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)$$
$$=\left(8\theta_0(\theta_0-1)^2(\theta_0+1)|\vec{A}_1|^2\overline{\alpha_2}z^{2\theta_0-4}\overline{z}+O(|z|^{2\theta_0-2})\right)\cdot\left(4\langle\vec{A}_1,\vec{A}_1\rangle z^{2\theta_0-2}+O(|z|^{2\theta_0-1})\right)$$

$$= 32\theta_0(\theta_0-1)^2(\theta_0+1)|\vec{A}_1|^2\overline{\alpha_2}\langle\vec{A}_1,\vec{A}_1\rangle z^{4\theta_0-6}\overline{z}+O(|z|^{4\theta_0-4})$$
$$= 16(\theta_0-1)^2|\vec{A}_1|^2\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\langle\vec{A}_1,\vec{A}_1\rangle z^{4\theta_0-6}\overline{z}+O(|z|^{4\theta_0-4}) \tag{5.0.57}$$

but

$$|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle = 0 \tag{5.0.58}$$

so either $\vec{C}_1 = 0$, or $\langle\vec{A}_1,\vec{A}_1\rangle = 0$, so that in all cases

$$\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\langle\vec{A}_1,\vec{A}_1\rangle = 0, \tag{5.0.59}$$

and by (5.0.57) and (5.0.59)

$$\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\overline{\partial}\vec{h}_0\right)\otimes\left(\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right) = O(|z|^{4\theta_0-4}). \tag{5.0.60}$$

Now, we have

$$\left(\partial\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right) = \left(4(\theta_0-1)^2\langle\vec{A}_1,\vec{A}_1\rangle z^{2\theta_0-4}+O(|z|^{2\theta_0-2})\right)\times$$
$$\left(\frac{1}{4}(\theta_0-2)^2\langle\vec{C}_1,\vec{C}_1\rangle\overline{z}^{2\theta_0-2}+\frac{1}{2}(\theta_0-2)(\theta_0-3)\langle\vec{C}_1,\vec{C}_2\rangle z\overline{z}^{2\theta_0-2}+8\theta_0(\theta_0+1)|\vec{A}_1|^2\overline{\alpha_2}z^{2\theta_0-2}\overline{z}+O(|z|^{2\theta_0})\right)$$
$$= (\theta_0-1)^2(\theta_0-2)^2\langle\vec{A}_1,\vec{A}_1\rangle\langle\vec{C}_1,\vec{C}_1\rangle z^{2\theta_0-4}\overline{z}^{2\theta_0-2}+2(\theta_0-1)^2(\theta_0-2)(\theta_0-3)\langle\vec{A}_1,\vec{A}_1\rangle\langle\vec{C}_1,\vec{C}_2\rangle z^{2\theta_0-5}\overline{z}^{2\theta_0-2}$$
$$+16(\theta_0-1)^2|\vec{A}_1|^2\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\langle\vec{A}_1,\vec{A}_1\rangle z^{4\theta_0-6}\overline{z}+O(|z|^{4\theta_0-4})$$
$$= O(|z|^{4\theta_0-4}) \tag{5.0.61}$$

thanks of (5.0.59) and as by (5.0.58), we have

$$\langle\vec{A}_1,\vec{A}_1\rangle\vec{C}_1 = 0,$$

so that

$$\langle\vec{A}_1,\vec{A}_1\rangle\langle\vec{C}_1,\vec{C}_1\rangle = \langle\vec{A}_1,\vec{A}_1\rangle\langle\vec{C}_1,\vec{C}_2\rangle = 0. \tag{5.0.62}$$

Then, we compute

$$\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\partial\vec{h}_0\right)\otimes\left(\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right) = \left(-(\theta_0-1)(\theta_0-3)\langle\vec{A}_1,\vec{C}_2\rangle z^{\theta_0-2}\overline{z}^{\theta_0-1}+O(|z|^{2\theta_0-2})\right)\times$$
$$\left(-\left((\theta_0-3)\langle\vec{A}_1,\vec{C}_2\rangle+2(\theta_0-2)\langle\vec{A}_2,\vec{C}_1\rangle\right)z^{\theta_0}\overline{z}^{\theta_0-1}+\frac{(\theta_0-2)^2}{4\theta_0}\langle\vec{C}_1,\vec{C}_1\rangle\overline{z}^{2\theta_0-1}+O(|z|^{2\theta_0})\right)$$
$$= O(|z|^{2\theta_0-3}\cdot|z|^{2\theta_0-1}) = O(|z|^{4\theta_0-4}). \tag{5.0.63}$$

The next term is thanks of (5.0.58) and (5.0.62)

$$\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right)\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right) = \left(\frac{1}{4}(\theta_0-2)(\theta_0-3)\langle\vec{C}_1,\vec{C}_2\rangle\overline{z}^{2\theta_0-2}+8\theta_0(\theta_0+1)|\vec{A}_1|^2\overline{\alpha_2}z^{2\theta_0-3}\overline{z}+O(|z|^{2\theta_0-1})\right)\times$$
$$\left(4(\theta_0-1)\langle\vec{A}_1,\vec{A}_1\rangle z^{2\theta_0-3}+O(|z|^{2\theta_0-2}\right)$$
$$= (\theta_0-1)(\theta_0-2)(\theta_0-3)\langle\vec{A}_1,\vec{A}_1\rangle\langle\vec{C}_1,\vec{C}_2\rangle z^{2\theta_0-3}\overline{z}^{2\theta_0-2}+16(\theta_0-1)|\vec{A}_1|^2\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\langle\vec{A}_1,\vec{A}_1\rangle z^{4\theta_0-6}\overline{z}+O(|z|^{4\theta_0-4})$$
$$= O(|z|^{4\theta_0-4}) \tag{5.0.64}$$

Finally, we have

$$\left(\partial\overline{\partial}\vec{h}_0\,\dot{\otimes}\,\vec{h}_0\right)\otimes\left(\partial\vec{h}_0\,\dot{\otimes}\,\overline{\partial}\vec{h}_0\right) = \left(-(\theta_0-3)\langle\vec{A}_1,\vec{C}_2\rangle|z|^{2\theta_0-2}+O(|z|^{2\theta_0-1}\right)\times$$
$$\left(-\left((\theta_0-1)(\theta_0-3)\langle\vec{A}_1,\vec{C}_2\rangle+2\theta_0(\theta_0-2)\langle\vec{A}_2,\vec{C}_1\rangle|z|^{2\theta_0-2}+O(|z|^{2\theta_0-1})\right)\right)$$

$$= O(|z|^{4\theta_0 - 4}) \tag{5.0.65}$$

Finally, as $g = |z|^{2\theta_0 - 2}(1 + O(|z|))$, we have

$$g^{-2} = \frac{|z|^{4-4\theta_0}}{|dz|^4}(1 + O(|z|^2))$$

and thanks of (5.0.60), (5.0.61), (5.0.63), (5.0.64), (5.0.65), we have

$$\begin{aligned}
O(\vec{h}_0) = g^{-2} \otimes \Big\{ &\frac{1}{4}\left(\partial\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \partial\overline{\partial}\vec{h}_0\right) \otimes \left(\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) + \frac{1}{4}\left(\partial\vec{h}_0 \,\dot{\otimes}\, \partial\vec{h}_0\right) \otimes \left(\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}\vec{h}_0\right) \\
- \frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \partial\vec{h}_0\right) \otimes \left(\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) &- \frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}\vec{h}_0\right) \otimes \left(\partial\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) + \frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \otimes \left(\partial\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}\vec{h}_0\right) \Big\} \in L^\infty(D^2)
\end{aligned} \tag{5.0.66}$$

Now, if we suppose that $\mathscr{O}_{\vec{\Phi}}$ is holomorphic, then

Now, recall that

$$\begin{aligned}
&\frac{1}{4}|\vec{H}|^2\, g^{-1} \otimes \left\{ \frac{1}{2}\left(\partial^N\overline{\partial}^N\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \otimes \left(\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) - \left(\partial^N\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \otimes \left(\overline{\partial}^N\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) + \frac{1}{2}\left(\partial^N\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}^N\vec{h}_0\right) \otimes \left(\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \right\} \\
&= \frac{1}{4}|\vec{H}|^2\, g^{-1} \otimes \left\{ \frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \otimes \left(\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) - \left(\partial\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \otimes \left(\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) + \frac{1}{2}\left(\partial\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}\vec{h}_0\right) \otimes \left(\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \right\}
\end{aligned} \tag{5.0.67}$$

We compute thanks of (5.0.56)

$$\begin{aligned}
&\left(\partial\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \otimes \left(\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) = \left(-(\theta_0 - 3)\langle \vec{A}_1, \vec{C}_2\rangle |z|^{2\theta_0 - 2} + O(|z|^{2\theta_0 - 1})\right) \times \left(4\langle \vec{A}_1, \vec{A}_1\rangle z^{2\theta_0 - 2} + O(|z|^{2\theta_0 - 1})\right) \\
&= O(|z|^{4\theta_0 - 4}). \\
&\left(\partial\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}\vec{h}_0\right) \otimes \left(\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) = \left(4(\theta_0 - 1)\langle \vec{A}_1, \vec{A}_1\rangle z^{2\theta_0 - 3} + O(|z|^{2\theta_0 - 2})\right) \\
&\times \left(-\left((\theta_0 - 3)\langle \vec{A}_1, \vec{C}_2\rangle + 2(\theta_0 - 2)\langle \vec{A}_2, \vec{C}_1\rangle\right) z^{\theta_0}\overline{z}^{\theta_0 - 1} + \frac{(\theta_0 - 2)^2}{4\theta_0}\langle \vec{C}_1, \vec{C}_1\rangle \overline{z}^{2\theta_0 - 1} + O(|z|^{2\theta_0}) \right) = O(|z|^{4\theta_0 - 4}) \\
&\left(\partial\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}\vec{h}_0\right) \otimes \left(\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) = O(|z|^{4\theta_0 - 4})
\end{aligned}$$

and as

$$\vec{H} = \mathrm{Re}\left(\frac{\vec{C}_1}{z^{\theta_0 - 2}}\right) + O(|z|^{3-\theta_0}) = O(|z|^{2-\theta_0})$$

we have $|\vec{H}|^2 = O(|z|^{4-2\theta_0})$, so that

$$\frac{1}{4}|\vec{H}|^2\, g^{-1} \otimes \left\{ \frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \otimes \left(\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) - \left(\partial\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \otimes \left(\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) + \frac{1}{2}\left(\partial\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}\vec{h}_0\right) \otimes \left(\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \right\} = O(|z|^2)$$

and *a fortiori*

$$\frac{1}{4}|\vec{H}|^2\, g^{-1} \otimes \left\{ \frac{1}{2}\left(\partial\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \otimes \left(\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) - \left(\partial\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \otimes \left(\overline{\partial}\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) + \frac{1}{2}\left(\partial\vec{h}_0 \,\dot{\otimes}\, \overline{\partial}\vec{h}_0\right) \otimes \left(\vec{h}_0 \,\dot{\otimes}\, \vec{h}_0\right) \right\} \in L^\infty(D^2). \tag{5.0.68}$$

One can check the other bounds similarly, which proves the holomorphy of $\mathscr{O}_{\vec{\Phi}}$ once it is meromorphic.

Chapter 6

The special cases of low multiplicity $\theta_0 = 2, 3, 4$

6.1 The case where $\theta_0 = 4$

In this case, we already have by the holomorphy of the quartic form the relation

$$|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle = \langle \overline{\vec{A}_1}, \vec{C}_1\rangle\langle \vec{A}_1, \vec{A}_1\rangle \tag{6.1.1}$$

and we can show that the coefficient in $z^{\theta_0}\overline{z}^{2-\theta_0}dz^4 = z^4\overline{z}^{-2}dz^4$ in the quartic form is

$$\Omega_0 = \frac{3}{4}\left(|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle - \langle \vec{A}_1, \overline{\vec{C}_1}\rangle\langle \vec{A}_1, \vec{C}_1\rangle\right) = 0$$

so we obtain

$$|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle = \langle \vec{A}_1, \overline{\vec{C}_1}\rangle\langle \vec{A}_1, \vec{C}_1\rangle. \tag{6.1.2}$$

Coupling this equation with (6.1.1), we have recovered the system

$$\begin{cases} |\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle = \langle \overline{\vec{A}_1}, \vec{C}_1\rangle\langle \vec{A}_1, \vec{A}_1\rangle \\ |\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle = \langle \vec{A}_1, \overline{\vec{C}_1}\rangle\langle \vec{A}_1, \vec{C}_1\rangle, \end{cases} \tag{6.1.3}$$

Remarking that is a linear system in $(\langle \vec{A}_1, \vec{C}_1\rangle, \langle \vec{A}_1, \vec{A}_1\rangle)$, we can recast (6.1.3) as

$$\begin{pmatrix} |\vec{A}_1|^2 & -\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \\ -\langle \vec{A}_1, \overline{\vec{C}_1}\rangle & |\vec{C}_1|^2 \end{pmatrix}\begin{pmatrix} \langle \vec{A}_1, \vec{C}_1\rangle \\ \langle \vec{A}_1, \vec{A}_1\rangle \end{pmatrix} = 0. \tag{6.1.4}$$

Thanks of Cauchy-Schwarz inequality, we obtain

$$\det\begin{pmatrix} |\vec{A}_1|^2 & -\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \\ -\langle \vec{A}_1, \overline{\vec{C}_1}\rangle & |\vec{C}_1|^2 \end{pmatrix} = |\vec{A}_1|^2|\vec{C}_1|^2 - |\langle \vec{A}_1, \overline{\vec{C}_1}\rangle|^2 \geq 0. \tag{6.1.5}$$

and the determinant vanishes if and only if (by the equality case of the Cauchy-Schwarz identity) there exists $\lambda \in \mathbb{C}$ such that

$$\vec{A}_1 = \lambda\,\vec{C}_1 \quad \text{or} \quad \vec{C}_1 = \lambda\vec{A}_1. \tag{6.1.6}$$

If the determinant in (6.1.5) is strictly positive, then we are done. Therefore, suppose now that (**??**) holds. If $\lambda = 0$, we are also done, so we can suppose that $\lambda \neq 0$ and that the relation

$$\vec{C}_1 = \lambda \vec{A}_1 \tag{6.1.7}$$

holds. Then, the conclusion will follow immediately by remarking that the only non-trivial coefficient in $(-1\ 4\ 1)$ or

$$\frac{\overline{z}^4}{z}\log|z|$$

in the Taylor expansion of the quartic form is

$$\Omega_1 = -\frac{3}{8}\langle \vec{A}_1, \vec{C}_1\rangle\overline{\langle \vec{C}_1, \vec{C}_1\rangle} = \frac{3\overline{\lambda}}{8}|\langle \vec{A}_1, \vec{C}_1\rangle|^2 = 0. \tag{6.1.8}$$

As $\lambda \neq 0$, we immediately obtain

$$\langle \vec{A}_1, \vec{C}_1\rangle = 0. \tag{6.1.9}$$

Therefore, in all cases, (6.1.9) is always satisfied, and we are done in the case $\theta_0 = 4$.

Now, let us present the details of this argument.

As we will have to integrate functions of the type

$$z^\alpha \overline{z}^\beta \log^p|z|$$

for $\alpha, b \in \mathbb{Z}$, with $\alpha \neq 1$ and $p \in \mathbb{N}$, let $a_0^p, \cdots, a_p^p \in \mathbb{R}$ and

$$f_p(z) = \frac{1}{\alpha+1} z^{\alpha+1}\overline{z}^\beta\left(\sum_{j=0}^p a_j^p \log^{p-j}|z|\right)$$

such that

$$\partial_z f_p(z) = z^\alpha \overline{z}^\beta \log^p|z|.$$

As

$$\partial_z\left(\frac{1}{\alpha+1}z^{\alpha+1}\overline{z}^\beta\log|z|\right) = z^\alpha\overline{z}^\beta\log|z| + \frac{1}{2(\alpha+1)}z^\alpha\overline{z}^\beta,$$

we deduce that

$$f_1(z) = \frac{1}{\alpha+1}\overline{z}^{\alpha+1}\overline{z}^\beta\left(\log|z| - \frac{1}{2(\alpha+1)}\right),$$

so

$$a_0^1 = 1, \quad a_1^1 = -\frac{1}{2(\alpha+1)}.$$

In the general case, we have

$$\begin{aligned}
\partial_z f_p(z) &= z^\alpha\overline{z}^\beta\left(\sum_{j=0}^p a_j^p \log^{p-j}(z)\right) + \frac{1}{2(\alpha+1)}z^\alpha\overline{z}^\beta\left(\sum_{j=0}^{p-1}(p-j)a_j^p\log^{p-j-1}|z|\right)\\
&= z^\alpha\overline{z}^\beta a_0^p\log^p|z| + z^\alpha\overline{z}^\beta\sum_{j=1}^p\left(a_j^p + \frac{(p-j+1)}{2(\alpha+1)}a_{j-1}^p\right)\log^{p-j}|z|\\
&= z^\alpha\overline{z}^\beta\log^p|z|
\end{aligned}$$

if and only if

$$\begin{cases} a_0^p = 1 \\ a_j^p = -\dfrac{(p-j+1)}{2(\alpha+1)} a_{j-1}^p, \quad j = 1, \cdots, p. \end{cases} \tag{6.1.10}$$

In particular, we obtain

$$a_j^p = (-1)^j \frac{p!}{(p-j)!} \frac{1}{2^j(\alpha+1)^j}, \quad j = 0, \cdots, p. \tag{6.1.11}$$

As for $\alpha = -1$, we trivially have

$$f_{-1}(z) = \frac{2}{p+1} \overline{z}^\beta \log^{p+1}|z|,$$

one can check that the function `intz` precisely gives these formulae. Indeed, fixing some arbitrary any $a_1, a_2, a_4, b_1, b_2, b_3, b_4 \in \mathbb{Z}$ for example

$$h(z) = \lambda_1 z^{a_1} \overline{z}^{b_1} \log^4|z| + \lambda_2 \overline{z}^{b_2} \frac{\log^2|z|}{z} + \lambda_3 z^{a_3} \overline{z}^{b_3} \log|z| + \lambda_4 z^{a_4} z^{b_4}$$

then a primitive f of h is given thanks of (6.1.11) by

$$\begin{aligned} f(z) &= \frac{\lambda_1}{a_1+1} z^{a+1} \overline{z}^{b_1} \left(\log^4|z| - \frac{2}{a_1+1} \log^3|z| + \frac{3}{(a_1+1)^2} \log^2|z| - \frac{3}{(a_1+1)^3} \log|z| + \frac{3}{2(a_1+1)^4} \right) \\ &\quad + \frac{2}{3} \lambda_2 \overline{z}^{b_2} \log^3|z| + \frac{\lambda_3}{a_3+1} z^{a_3+1} \overline{z}^{b_3} \left(\log|z| - \frac{1}{2(a_3+1)} \right) + \frac{\lambda_4}{a_4+1} z^{a_4+1} \overline{z}^{b_4}. \end{aligned} \tag{6.1.12}$$

and using the code

```
var('lambda1,lambda2,lambda3,lambda4',domain='complex'),var('a,b,p', domain='real')

def intz(v):
length=0
for i in range(v.nrows()):
if bool((v[i,v.ncols()-3]+1).is_zero()): # if the coefficient is z^{-1}\z^{b}\log^p|z|,\\
the primitive has only one components
length=length+1
else:
length=length+v[i,v.ncols()-1]+1 # if the coefficient is z^{a}\z^{b}\log^p|z|\\
with a\neq -1, then the primitive has p+1 components
m=matrix(SR,length,v.ncols())
n=0
if v.ncols()==4:
for i in range(v.nrows()):
if (v[i,1]+1).is_zero():        #integration of 1/z
m[i+n,0]=2*v[i,0]/(v[i,3]+1)
m[i+n,1]=0
m[i+n,2]=v[i,2]
m[i+n,3]=v[i,3]+1
else:
if v[i,3].is_zero():
m[i+n,0]=v[i,0]/(v[i,1]+1)
m[i+n,1]=v[i,1]+1
m[i+n,2]=v[i,2]
else:
a0=1/(v[i,1]+1)
```

```
m[i+n,0]=a0*v[i,0]
m[i+n,1]=v[i,1]+1
m[i+n,2]=v[i,2]
m[i+n,3]=v[i,3]
n=n+1
for j in range(v[i,3]):
a0=-(v[i,3]-j)/(2*(v[i,1]+1))*a0
m[i+n,0]=a0*v[i,0]
m[i+n,1]=v[i,1]+1
m[i+n,2]=v[i,2]
m[i+n,3]=v[i,3]-j-1
n=n+1
n=n-1
return m
else:
for i in range(v.nrows()):
if (v[i,2]+1).is_zero():        #integration of 1/z
m[i+n,0]=2*v[i,0]/(v[i,4]+1)
m[i+n,1]=v[i,1]
m[i+n,2]=0
m[i+n,3]=v[i,3]
m[i+n,4]=v[i,4]+1
else:
if v[i,4].is_zero():
m[i+n,0]=v[i,0]/(v[i,2]+1)
m[i+n,1]=v[i,1]
m[i+n,2]=v[i,2]+1
m[i+n,3]=v[i,3]
else:
a0=1/(v[i,2]+1)
m[i+n,0]=a0*v[i,0]
m[i+n,1]=v[i,1]
m[i+n,2]=v[i,2]+1
m[i+n,3]=v[i,3]
m[i+n,4]=v[i,4]
n=n+1
for j in range(v[i,4]):
a0=-(v[i,4]-j)/(2*(v[i,2]+1))*a0
m[i+n,0]=a0*v[i,0]
m[i+n,1]=v[i,1]
m[i+n,2]=v[i,2]+1
m[i+n,3]=v[i,3]
m[i+n,4]=v[i,4]-j-1
n=n+1
n=n-1
return m

m1=matrix([[lambda1,a,b,4],[lambda2,-1,b,2],[lambda3,a,b,1],[lambda4,a,b,0]])

latex(intz(m1))
```

yields

$$f(z) = \begin{pmatrix} \dfrac{\lambda_1}{a_1+1} & a_1+1 & b_1 & 4 \\ -\dfrac{2\,\lambda_1}{(a_1+1)^2} & a_1+1 & b_1 & 3 \\ \dfrac{3\,\lambda_1}{(a_1+1)^3} & a_1+1 & b_1 & 2 \\ -\dfrac{3\,\lambda_1}{(a_1+1)^4} & a_1+1 & b_1 & 1 \\ \dfrac{3\,\lambda_1}{2\,(a_1+1)^5} & a_1+1 & b_1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{2}{3}\,\lambda_2 & 0 & b_2 & 3 \\ \dfrac{\lambda_3}{a_3+1} & a_3+1 & b_3 & 1 \\ -\dfrac{\lambda_3}{2\,(a_3+1)^2} & a_3+1 & b_3 & 0 \\ \dfrac{\lambda_4}{a_4+1} & a_4+1 & b_4 & 0 \end{pmatrix}$$

which coincides with (6.1.12).

From now on, we assume that $\theta_0 = 4$. We remark that the developments (2.1.2), (2.1.4), (2.1.5) and (2.2.5) and the relation $\alpha_0 = 0$ (see (2.4.7)) hold for all $\theta_0 \geq 4$, so that

$$\left\{\begin{aligned} &\partial_z\vec{\Phi} = \begin{pmatrix} 1 & A_0 & \theta_0-1 & 0 \\ 1 & A_1 & \theta_0 & 0 \\ 1 & A_2 & \theta_0+1 & 0 \\ \dfrac{1}{4\,\theta_0} & C_1 & 1 & \theta_0 \\ \dfrac{1}{8} & \overline{C_1} & \theta_0-1 & 2 \end{pmatrix} \\ &\vec{h}_0 = \begin{pmatrix} 2 & A_1 & \theta_0-1 & 0 \\ -4\,|A_1|^2 & A_0 & \theta_0-1 & 1 \\ 4 & A_2 & \theta_0 & 0 \\ -\dfrac{\theta_0-2}{2\,\theta_0} & C_1 & \theta_0-1 & 0 \\ -4\,\alpha_1 & A_0 & \theta_0 & 0\theta_0 \end{pmatrix} \end{aligned}\right. \qquad \left\{\begin{aligned} &g = \begin{pmatrix} 1 & \theta_0-1 & \theta_0-1 \\ 2\,|A_1|^2 & \theta_0 & \theta_0 \\ \alpha_1 & \theta_0 & \theta_0-1 \\ \overline{\alpha_1} & \theta_0-1 & \theta_0 \end{pmatrix} \\ &\vec{H} = \begin{pmatrix} \dfrac{1}{2} & C_1 & -\theta_0+2 & 0 \\ \dfrac{1}{2} & \overline{C_1} & 0 & -\theta_0+2 \\ \dfrac{1}{2} & C_2 & -\theta_0+3 & 0 \\ \dfrac{1}{2} & \overline{C_2} & 0 & -\theta_0+3 \\ \dfrac{1}{2} & B_1 & -\theta_0+2 & 1 \\ \dfrac{1}{2} & \overline{B_1} & 1 & -\theta_0+2 \end{pmatrix} \end{aligned}\right.$$

where by (2.2.4),

$$\begin{cases} \vec{C}_2 = \vec{D}_2 + \dfrac{2}{\theta_0-3}\langle \vec{A}_1, \vec{C}_1\rangle \overline{\vec{A}_0}, \\ \vec{B}_1 = -2\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \vec{A}_0 \end{cases} \tag{6.1.13}$$

Taking $\theta_0 = 4$, we obtain

$$\begin{cases} \vec{C}_2 = \vec{D}_2 + 2\langle \vec{A}_1, \vec{C}_1\rangle \overline{\vec{A}_0}, \\ \vec{B}_1 = -2\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \vec{A}_0 \end{cases} \tag{6.1.14}$$

and

$$\left\{\begin{aligned}
&\partial_z\vec{\Phi} = \begin{pmatrix} 1 & A_0 & 3 & 0 & 0 \\ 1 & A_1 & 4 & 0 & 0 \\ 1 & A_2 & 5 & 0 & 0 \\ \frac{1}{16} & C_1 & 1 & 4 & 0 \\ \frac{1}{8} & \overline{C_1} & 3 & 2 & 0 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 3 & 3 & 0 \\ 2\,|A_1|^2 & 4 & 4 & 0 \\ \alpha_1 & 5 & 3 & 0 \\ \overline{\alpha_1} & 3 & 5 & 0 \end{pmatrix} \\
&H = \begin{pmatrix} \frac{1}{2} & C_1 & -2 & 0 & 0 \\ \frac{1}{2} & \overline{C_1} & 0 & -2 & 0 \\ \frac{1}{2} & C_2 & -1 & 0 & 0 \\ \frac{1}{2} & \overline{C_2} & 0 & -1 & 0 \\ \frac{1}{2} & B_1 & -2 & 1 & 0 \\ \frac{1}{2} & \overline{B_1} & 1 & -2 & 0 \end{pmatrix} \qquad \vec{h}_0 = \begin{pmatrix} 2 & A_1 & 3 & 0 & 0 \\ 4 & A_2 & 4 & 0 & 0 \\ -\frac{1}{4} & C_1 & 0 & 4 & 0 \\ -4\,|A_1|^2 & A_0 & 3 & 1 & 0 \\ -4\,\alpha_1 & A_0 & 4 & 0 & 0 \end{pmatrix}
\end{aligned}\right.$$

In some sense, the result is much more mysterious when we fix the parameter θ_0, as the coefficients allowing the different cancellations which were polynomials in θ_0 become random constants. This explains why the computer is so convenient in this special case.

In the following expressions, the fifth column will indicate the logarithmic power, as indicated in remark A.

If $\vec{Q} \in C^\infty(D^2 \setminus \{0\}, \mathbb{C}^n)$ is as usual the anti-meromorphic free function such that

$$\partial\vec{Q} = -|\vec{H}|^2\partial\vec{\Phi} - 2\,g^{-1} \otimes \langle \vec{H}, \vec{h}_0\rangle \otimes \overline{\partial}\vec{\Phi},$$

then we have

$$\vec{Q} = \begin{pmatrix} -\frac{1}{16}\,C_1^2 & \overline{A_0} & -4 & 4 & 0 \\ -\frac{1}{8}\,C_1\overline{C_1} & \overline{A_0} & -2 & 2 & 0 \\ 2\,A_1C_1 & \overline{A_0} & -1 & 0 & 0 \\ 2\,A_1C_1 & \overline{A_1} & -1 & 1 & 0 \\ -4\,A_0C_1|A_1|^2 + 2\,A_1B_1 & \overline{A_0} & -1 & 1 & 0 \\ -\frac{1}{2}\,C_1^2 & A_0 & 0 & 0 & 1 \\ 8\,A_0C_1\alpha_1 - 8\,A_2C_1 - 4\,A_1C_2 & \overline{A_0} & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2\,A_1\overline{C_1} & \overline{A_0} & 1 & -2 & 0 \\ -2\,A_1\overline{C_1} & \overline{A_1} & 1 & -1 & 0 \\ 4\,A_0|A_1|^2\overline{C_1} - 2\,A_1C_2 & \overline{A_0} & 1 & -1 & 0 \\ -\frac{1}{4}\,C_1\overline{C_1} & A_0 & 2 & -2 & 0 \\ 2\,A_0\alpha_1\overline{C_1} - A_1\overline{B_1} - 2\,A_2\overline{C_1} & \overline{A_0} & 2 & -2 & 0 \\ -\frac{1}{16}\,\overline{C_1}^2 & A_0 & 4 & -4 & 0 \end{pmatrix} \tag{6.1.15}$$

and as

$$\partial\left(\vec{H} - 2i\vec{L} + \vec{\gamma}_0\log|z|\right) = -|\vec{H}|^2\partial\vec{\Phi} - 2\,g^{-1} \otimes \langle \vec{H}, \vec{h}_0\rangle \otimes \overline{\partial}\vec{\Phi},$$

there exists $\vec{D}_2, \vec{D}_3 \in \mathbb{C}^n$ such that

$$\vec{H} - 2i\vec{L} + \vec{\gamma}_0\log|z| = \frac{\overline{\vec{C}_1}}{\overline{z}^2} + \frac{\overline{\vec{D}_2}}{\overline{z}} + \overline{\vec{D}_3} + \vec{Q} + O(|z|^{1-\varepsilon})$$

and

$$\vec{H} + \vec{\gamma}_0 \log|z| = \mathrm{Re}\left(\frac{\vec{C}_1}{z^2} + \frac{\vec{D}_2}{z} + \vec{D}_3 + \vec{Q}\right) + O(|z|^{1-\varepsilon})$$

so we obtain

$$\vec{H} + \vec{\gamma}_0 \log|z| = \mathrm{Re}\left(\frac{\vec{C}_1}{z^2} + \frac{\vec{D}_2}{z} + \vec{D}_3\right) + \tag{6.1.16}$$

$$\begin{pmatrix}
-\frac{1}{16}\,C_1^2 & \overline{A_0} & -4 & 4 & 0\\
-C_1\overline{A_1} & A_0 & -2 & 1 & 0\\
-\frac{3}{16}\,C_1\overline{C_1} & \overline{A_0} & -2 & 2 & 0\\
C_1\overline{A_0}\alpha_1 - \frac{1}{2}B_1\overline{A_1} - C_1\overline{A_2} & A_0 & -2 & 2 & 0\\
A_1C_1 & \overline{A_0} & -1 & 0 & 0\\
A_1C_1 & \overline{A_1} & -1 & 1 & 0\\
-2\,A_0C_1|A_1|^2 + A_1B_1 & \overline{A_0} & -1 & 1 & 0\\
-C_1\overline{A_1} & A_1 & -1 & 1 & 0\\
2\,C_1|A_1|^2\overline{A_0} - C_2\overline{A_1} & A_0 & -1 & 1 & 0\\
\overline{A_1C_1} & A_0 & 0 & -1 & 0\\
4\,\overline{A_0C_1\alpha_1} - \frac{1}{4}\,C_1^2 - 2\,C_2\overline{A_1} - 4\,\overline{A_2C_1} & A_0 & 0 & 0 & 1\\
4\,A_0C_1\alpha_1 - 4\,A_2C_1 - 2\,A_1C_2 - \frac{1}{4}\,\overline{C_1}^2 & \overline{A_0} & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
-A_1\overline{C_1} & \overline{A_0} & 1 & -2 & 0\\
\overline{A_1C_1} & A_1 & 1 & -1 & 0\\
-2\,|A_1|^2\overline{A_0}C_1 + \overline{A_1}B_1 & A_0 & 1 & -1 & 0\\
-A_1\overline{C_1} & \overline{A_1} & 1 & -1 & 0\\
2\,A_0|A_1|^2\overline{C_1} - A_1C_2 & \overline{A_0} & 1 & -1 & 0\\
-\frac{3}{16}\,C_1\overline{C_1} & A_0 & 2 & -2 & 0\\
A_0\alpha_1\overline{C_1} - \frac{1}{2}A_1\overline{B_1} - A_2\overline{C_1} & \overline{A_0} & 2 & -2 & 0\\
-\frac{1}{16}\,\overline{C_1}^2 & A_0 & 4 & -4 & 0
\end{pmatrix}
\tag{6.1.17}$$

There are some cancellations, here, as

$$\langle \vec{A}_0, \vec{A}_1\rangle = \langle \vec{A}_0, \overline{\vec{A}_1}\rangle = 0, \quad \langle \vec{A}_0, \vec{C}_1\rangle = \langle \vec{A}_0, \overline{\vec{C}_1}\rangle = 0$$

and thanks of (6.1.14),

$$\langle \vec{A}_1, \vec{B}_1\rangle = \langle \overline{\vec{A}_1}, \vec{B}_1\rangle = 0$$

so with (we just need to multiply the corresponding coefficients in (6.1.16) by 2)

$$\begin{cases}
\vec{C}_2 = \vec{D}_2 + 2\langle \vec{A}_1, \vec{C}_1\rangle\overline{\vec{A}_0}\\
\vec{C}_3 = \mathrm{Re}\,(\vec{D}_3)\\
\vec{B}_1 = -2\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \vec{A}_0\\
\vec{B}_2 = -\frac{3}{8}|\vec{C}_1|^2\overline{\vec{A}_0} - 2\langle \overline{\vec{A}_2}, \vec{C}_1\rangle \vec{A}_0\\
\vec{B}_3 = -2\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \vec{A}_1 + 2\langle \vec{A}_1, \vec{C}_1\rangle\overline{\vec{A}_1} - 2\langle \overline{\vec{A}_1}, \vec{C}_2\rangle \vec{A}_0\\
\vec{E}_1 = -\frac{1}{8}\langle \vec{C}_1, \vec{C}_1\rangle\overline{\vec{A}_0}\\
\vec{\gamma}_1 = -\vec{\gamma}_0 - \mathrm{Re}\left\{\left(\frac{1}{2}\langle \vec{C}_1, \vec{C}_1\rangle + 8\overline{\langle \vec{A}_2, \vec{C}_1\rangle} + 4\overline{\langle \vec{A}_1, \vec{C}_2\rangle}\right)\vec{A}_0\right\} \in \mathbb{R}^n
\end{cases}\tag{6.1.18}$$

and we obtain

$$\vec{H} = \mathrm{Re}\left(\frac{\vec{C}_1}{z^2} + \frac{\vec{C}_2}{z} + \vec{C}_3 + \vec{B}_1\frac{\overline{z}}{z^2} + \vec{B}_2\frac{\overline{z}^2}{z^2} + \vec{B}_3\frac{\overline{z}}{z} + \vec{E}_1\frac{\overline{z}^4}{z^4}\right) + \vec{\gamma}_1\log|z| + O(|z|^{1-\varepsilon}). \tag{6.1.19}$$

The notations were chosen to be consistent with the higher development of the previous chapters. Translated into code, this is

$$\vec{H}=\begin{pmatrix}\frac{1}{2} & C_1 & -2 & 0 & 0\\ \frac{1}{2} & C_2 & -1 & 0 & 0\\ 1 & C_3 & 0 & 0 & 0\\ \frac{1}{2} & B_1 & -2 & 1 & 0\\ \frac{1}{2} & B_2 & -2 & 2 & 0\\ \frac{1}{2} & B_3 & -1 & 1 & 0\\ \frac{1}{2} & E_1 & -4 & 4 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \overline{C_1} & 0 & -2 & 0\\ \frac{1}{2} & \overline{C_2} & 0 & -1 & 0\\ \frac{1}{2} & \overline{B_1} & 1 & -2 & 0\\ \frac{1}{2} & \overline{B_2} & 2 & -2 & 0\\ \frac{1}{2} & \overline{B_3} & 1 & -1 & 0\\ \frac{1}{2} & \overline{E_1} & 4 & -4 & 0\\ 1 & \gamma_1 & 0 & 0 & 1\end{pmatrix} \tag{6.1.20}$$

Now, as

$$\partial_{\overline{z}}\left(\partial_z\vec{\Phi}\right)=\frac{e^{2\lambda}}{2}\vec{H},$$

we deduce that

$$\partial_z\vec{\Phi}=\begin{pmatrix}1 & A_0 & 3 & 0 & 0\\ 1 & A_1 & 4 & 0 & 0\\ 1 & A_2 & 5 & 0 & 0\\ 1 & A_3 & 6 & 0 & 0\\ 1 & A_4 & 7 & 0 & 0\\ \frac{1}{16} & C_1 & 1 & 4 & 0\\ \frac{1}{16} & C_2 & 2 & 4 & 0\\ \frac{1}{8} & C_3 & 3 & 4 & 0\\ \frac{1}{20} & B_1 & 1 & 5 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{24} & B_2 & 1 & 6 & 0\\ \frac{1}{20} & B_3 & 2 & 5 & 0\\ \frac{1}{32} & E_1 & -1 & 8 & 0\\ \frac{1}{8} & \overline{C_1} & 3 & 2 & 0\\ \frac{1}{12} & \overline{C_2} & 3 & 3 & 0\\ \frac{1}{8} & \overline{B_1} & 4 & 2 & 0\\ \frac{1}{8} & \overline{B_2} & 5 & 2 & 0\\ \frac{1}{12} & \overline{B_3} & 4 & 3 & 0\\ \frac{1}{2} & \overline{E_1} & 7 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{8} & \gamma_1 & 3 & 4 & 1\\ -\frac{1}{64} & \gamma_1 & 3 & 4 & 0\\ \frac{1}{10}\left|A_1\right|^2 & C_1 & 2 & 5 & 0\\ \frac{1}{6}\left|A_1\right|^2 & \overline{C_1} & 4 & 3 & 0\\ \frac{1}{16}\,\alpha_1 & C_1 & 3 & 4 & 0\\ \frac{1}{8}\,\alpha_1 & \overline{C_1} & 5 & 2 & 0\\ \frac{1}{24}\,\overline{\alpha_1} & C_1 & 1 & 6 & 0\\ \frac{1}{16}\,\overline{\alpha_1} & \overline{C_1} & 3 & 4 & 0\end{pmatrix} \tag{6.1.21}$$

and by conformality of $\vec{\Phi}$, we obtain

$$0=\langle\partial_z\vec{\Phi},\partial_z\vec{\Phi}\rangle=\begin{pmatrix} A_0^2 & 6 & 0 & 0\\ 2\,A_0A_1 & 7 & 0 & 0\\ A_1^2+2\,A_0A_2 & 8 & 0 & 0\\ 2\,A_1A_2+2\,A_0A_3 & 9 & 0 & 0\\ A_2^2+2\,A_1A_3+2\,A_0A_4 & 10 & 0 & 0\\ \frac{1}{8}\,A_0C_1 & 4 & 4 & 0\\ \frac{1}{8}\,A_1C_1+\frac{1}{8}\,A_0C_2 & 5 & 4 & 0\\ \frac{1}{8}\,A_0C_1\alpha_1+\frac{1}{8}\,A_0\overline{C_1}\overline{\alpha_1}+\frac{1}{8}\,A_2C_1+\frac{1}{8}\,A_1C_2+\frac{1}{4}\,A_0C_3-\frac{1}{32}\,A_0\gamma_1+\frac{1}{64}\,\overline{C_1}^2 & 6 & 4 & 0\\ \frac{1}{10}\,A_0B_1 & 4 & 5 & 0\\ \frac{1}{12}\,A_0C_1\overline{\alpha_1}+\frac{1}{12}\,A_0B_2+\frac{1}{64}\,C_1\overline{C_1} & 4 & 6 & 0\\ \frac{1}{5}\,A_0C_1|A_1|^2+\frac{1}{10}\,A_1B_1+\frac{1}{10}\,A_0B_3 & 5 & 5 & 0\\ \frac{1}{256}\,C_1^2+\frac{1}{16}\,A_0E_1 & 2 & 8 & 0\\ \frac{1}{4}\,A_0\overline{C_1} & 6 & 2 & 0\\ \frac{1}{6}\,A_0\overline{C_2} & 6 & 3 & 0\\ \frac{1}{4}\,A_0\overline{B_1}+\frac{1}{4}\,A_1\overline{C_1} & 7 & 2 & 0\\ \frac{1}{4}\,A_0\alpha_1\overline{C_1}+\frac{1}{4}\,A_1\overline{B_1}+\frac{1}{4}\,A_0\overline{B_2}+\frac{1}{4}\,A_2\overline{C_1} & 8 & 2 & 0\\ \frac{1}{3}\,A_0|A_1|^2\overline{C_1}+\frac{1}{6}\,A_0\overline{B_3}+\frac{1}{6}\,A_1\overline{C_2} & 7 & 3 & 0\\ A_0\overline{E_1} & 10 & 0 & 1\\ \frac{1}{4}\,A_0\gamma_1 & 6 & 4 & 1 \end{pmatrix} \tag{6.1.22}$$

We already know that the only interesting relations are

$$\begin{cases}\langle\vec{A}_0,\vec{A}_0\rangle=\langle\vec{A}_0,\vec{A}_1\rangle=\langle\vec{A}_0,\overline{\vec{A}_1}\rangle=\langle\vec{A}_0,\vec{\gamma}_1\rangle=0,\\ \langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_0,\vec{A}_2\rangle=0,\quad\langle\vec{A}_1,\vec{A}_2\rangle+\langle\vec{A}_0,\vec{A}_3\rangle=0\\ \langle\vec{A}_0,\vec{C}_1\rangle=\langle\vec{A}_0,\overline{\vec{C}_1}\rangle=\langle\vec{A}_0,\overline{\vec{C}_2}\rangle=0,\quad\langle\vec{A}_1,\vec{C}_1\rangle+\langle\vec{A}_0,\vec{C}_2\rangle=0.\end{cases} \tag{6.1.23}$$

where we added the previous $\langle\vec{A}_0,\overline{\vec{A}_1}\rangle=\langle\vec{A}_0,\overline{\vec{A}_2}\rangle=0$ coming from the upper regularity. The rest of the cancellations in (6.1.22) is either trivial or useless.

and

$$e^{2\lambda}= \tag{6.1.24}$$

$$
\begin{pmatrix}
2\,A_0\overline{A_0} & 3 & 3 & 0\\
2\,A_0\overline{A_1} & 3 & 4 & 0\\
2\,A_0\overline{A_2}+\frac{1}{4}\,\overline{A_0C_1} & 3 & 5 & 0\\
2\,A_0\overline{A_3}+\frac{1}{4}\,\overline{A_1C_1}+\frac{1}{6}\,\overline{A_0C_2} & 3 & 6 & 0\\
\frac{1}{8}\,C_1\alpha_1\overline{A_0}+\frac{1}{8}\,\overline{A_0C_1}\overline{\alpha_1}+\frac{1}{64}\,C_1^2+\frac{1}{4}\,C_3\overline{A_0}-\frac{1}{32}\,\gamma_1\overline{A_0}+2\,A_0\overline{A_4}+\frac{1}{4}\,\overline{A_2C_1}+\frac{1}{6}\,\overline{A_1C_2} & 3 & 7 & 0\\
\frac{1}{8}\,A_0\overline{C_1} & 7 & 1 & 0\\
\frac{1}{8}\,A_0\overline{C_2} & 7 & 2 & 0\\
\frac{1}{8}\,A_0C_1\alpha_1+\frac{1}{8}\,A_0\overline{C_1}\overline{\alpha_1}+\frac{1}{4}\,A_2C_1+\frac{1}{6}\,A_1C_2+\frac{1}{4}\,A_0C_3-\frac{1}{32}\,A_0\gamma_1+2\,A_4\overline{A_0}+\frac{1}{64}\,\overline{C_1}^2 & 7 & 3 & 0\\
\frac{1}{10}\,A_0\overline{B_1}+\frac{1}{8}\,A_1\overline{C_1} & 8 & 1 & 0\\
\frac{1}{12}\,A_0\alpha_1\overline{C_1}+\frac{1}{10}\,A_1\overline{B_1}+\frac{1}{12}\,A_0\overline{B_2}+\frac{1}{8}\,A_2\overline{C_1} & 9 & 1 & 0\\
\frac{1}{5}\,A_0|A_1|^2\overline{C_1}+\frac{1}{10}\,A_0\overline{B_3}+\frac{1}{8}\,A_1\overline{C_2} & 8 & 2 & 0\\
\frac{1}{16}\,A_0\overline{E_1} & 11 & -1 & 0\\
\frac{1}{4}\,A_0C_1+2\,A_2\overline{A_0} & 5 & 3 & 0\\
\frac{1}{4}\,A_1C_1+\frac{1}{6}\,A_0C_2+2\,A_3\overline{A_0} & 6 & 3 & 0\\
\frac{1}{4}\,A_0B_1+2\,A_2\overline{A_1} & 5 & 4 & 0\\
\frac{1}{4}\,\alpha_1\overline{A_0C_1}+\frac{1}{4}\,A_0C_1\overline{\alpha_1}+\frac{1}{4}\,A_0B_2+2\,A_2\overline{A_2}+\frac{1}{4}\,\overline{A_0B_2}+\frac{5}{128}\,C_1\overline{C_1} & 5 & 5 & 0\\
\frac{1}{3}\,A_0C_1|A_1|^2+\frac{1}{4}\,A_1B_1+\frac{1}{6}\,A_0B_3+2\,A_3\overline{A_1} & 6 & 4 & 0\\
A_0E_1+\frac{1}{4}\,\gamma_1\overline{A_0} & 3 & 7 & 1\\
\frac{1}{4}\,A_0\gamma_1+\overline{A_0E_1} & 7 & 3 & 1\\
2\,A_1\overline{A_0} & 4 & 3 & 0\\
2\,A_1\overline{A_1} & 4 & 4 & 0\\
2\,A_1\overline{A_2}+\frac{1}{4}\,\overline{A_0B_1} & 4 & 5 & 0\\
\frac{1}{3}\,|A_1|^2\overline{A_0C_1}+2\,A_1\overline{A_3}+\frac{1}{4}\,\overline{A_1B_1}+\frac{1}{6}\,\overline{A_0B_3} & 4 & 6 & 0\\
\frac{1}{8}\,C_1\overline{A_0} & 1 & 7 & 0\\
\frac{1}{10}\,B_1\overline{A_0}+\frac{1}{8}\,C_1\overline{A_1} & 1 & 8 & 0\\
\frac{1}{12}\,C_1\overline{A_0}\overline{\alpha_1}+\frac{1}{12}\,B_2\overline{A_0}+\frac{1}{10}\,B_1\overline{A_1}+\frac{1}{8}\,C_1\overline{A_2} & 1 & 9 & 0\\
\frac{1}{8}\,C_2\overline{A_0} & 2 & 7 & 0\\
\frac{1}{5}\,C_1|A_1|^2\overline{A_0}+\frac{1}{10}\,B_3\overline{A_0}+\frac{1}{8}\,C_2\overline{A_1} & 2 & 8 & 0\\
\frac{1}{16}\,E_1\overline{A_0} & -1 & 11 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{(1)}\\ \mathbf{(2)}\\ \mathbf{(3)}\\ \mathbf{(4)}\\ \mathbf{(5)}\\ \mathbf{(6)}\\ \mathbf{(7)}\\ \mathbf{(8)}\\ \mathbf{(9)}\\ \mathbf{(10)}\\ \mathbf{(11)}\\ \mathbf{(12)}\\ \mathbf{(13)}\\ \mathbf{(14)}\\ \mathbf{(15)}\\ \mathbf{(16)}\\ \mathbf{(17)}\\ \mathbf{(18)}\\ \mathbf{(19)}\\ \mathbf{(20)}\\ \mathbf{(21)}\\ \mathbf{(22)}\\ \mathbf{(23)}\\ \mathbf{(24)}\\ \mathbf{(25)}\\ \mathbf{(26)}\\ \mathbf{(27)}\\ \mathbf{(28)}\\ \mathbf{(29)}
\end{pmatrix}
\tag{6.1.25}
$$

We see easily that

$$(2),\ (6),\ (7),\ (12),\ (20),\ (24),\ (27),\ (29)$$

all vanish. Therefore, for some

$$\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8,\alpha_9\in\mathbb{C},\quad \zeta_0\in\mathbb{C},\quad \beta\in\mathbb{R},$$

we have $\left(\text{recall that } |\vec{A}_0|^2=\dfrac{1}{2}\right)$, with TEX on the left, and Sage on the right

$$e^{2\lambda}=\begin{pmatrix} 1 & 3 & 3 & 0\\ 2|\vec{A}_1|^2 & 4 & 4 & 0\\ \beta & 5 & 5 & 0\\ \alpha_1 & 5 & 3 & 0\\ \alpha_2 & 1 & 8 & 0\\ \alpha_3 & 6 & 3 & 0\\ \alpha_4 & 5 & 4 & 0\\ \alpha_5 & 6 & 4 & 0\\ \alpha_6 & 8 & 2 & 0\\ \alpha_7 & 9 & 1 & 0\\ \zeta_0 & 7 & 3 & 1 \end{pmatrix}\begin{pmatrix} \overline{\alpha_1} & 3 & 5 & 0\\ \overline{\alpha_2} & 8 & 1 & 0\\ \overline{\alpha_3} & 3 & 6 & 0\\ \overline{\alpha_4} & 4 & 5 & 0\\ \overline{\alpha_5} & 4 & 6 & 0\\ \overline{\alpha_6} & 2 & 8 & 0\\ \overline{\alpha_7} & 1 & 9 & 0\\ \overline{\zeta_0} & 3 & 7 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 3 & 0\\ 2\,|A_1|^2 & 4 & 4 & 0\\ \beta & 5 & 5 & 0\\ \alpha_1 & 5 & 3 & 0\\ \alpha_2 & 1 & 8 & 0\\ \alpha_3 & 6 & 3 & 0\\ \alpha_4 & 5 & 4 & 0\\ \alpha_5 & 6 & 4 & 0\\ \alpha_6 & 8 & 2 & 0\\ \alpha_7 & 9 & 1 & 0\\ \zeta_0 & 7 & 3 & 1 \end{pmatrix}\begin{pmatrix} \overline{\alpha_1} & 3 & 5 & 0\\ \overline{\alpha_2} & 8 & 1 & 0\\ \overline{\alpha_3} & 3 & 6 & 0\\ \overline{\alpha_4} & 4 & 5 & 0\\ \overline{\alpha_5} & 4 & 6 & 0\\ \overline{\alpha_6} & 2 & 8 & 0\\ \overline{\alpha_7} & 1 & 9 & 0\\ \overline{\zeta_0} & 3 & 7 & 1 \end{pmatrix} \tag{6.1.26}$$

At this point, this is easy to see that the first non-trivial logarithm arising in the quartic form will be given by the expression in the beginning of the chapter, so we are done. First, α_3 is the coefficient in (6 3 0) in (6.1.24), which is the coefficient corresponding to the line (14), that is

$$\alpha_3=\frac{1}{4}\,A_1C_1+\frac{1}{6}\,A_0C_2+2\,A_3\overline{A_0}=\frac{1}{12}\langle\vec{A}_1,\vec{C}_1\rangle+2\langle\overline{\vec{A}_0},\vec{A}_3\rangle, \tag{6.1.27}$$

where we have used $\langle\vec{A}_1,\vec{C}_1\rangle+\langle\vec{A}_0,\vec{C}_2\rangle=0$ in (6.1.23). Then, α_4 corresponds to the coefficient in (5, 4, 0), *i.e.* to the line (15), giving

$$\alpha_4=\frac{1}{4}\,A_0B_1+2\,A_2\overline{A_1}=2\langle\overline{\vec{A}_1},\vec{A}_2\rangle \tag{6.1.28}$$

as $\vec{B}_1\in\mathrm{Span}(\vec{A}_0)$. Finally, α_2 corresponds to (1, 8, 0), (25)

$$\alpha_2=\frac{1}{10}\,B_1\overline{A_0}+\frac{1}{8}\,C_1\overline{A_1}=-\frac{1}{10}\langle\overline{\vec{A}_1},\vec{C}_1\rangle+\frac{1}{8}\langle\overline{\vec{A}_1},\vec{C}_1\rangle=\frac{1}{40}\langle\overline{\vec{A}_1},\vec{C}_1\rangle \tag{6.1.29}$$

as

$$\langle\overline{\vec{A}_0},\vec{B}_1\rangle=\left\langle\overline{\vec{A}_0},-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0\right\rangle=-\langle\overline{\vec{A}_1},\vec{C}_1\rangle.$$

Gathering (6.1.27), (6.1.28) and (6.1.29), we obtain

$$\begin{cases}\alpha_2=\dfrac{1}{40}\langle\overline{\vec{A}_1},\vec{C}_1\rangle\\ \alpha_3=\dfrac{1}{12}\langle\vec{A}_1,\vec{C}_1\rangle+2\langle\overline{\vec{A}_0},\vec{A}_3\rangle\\ \alpha_4=2\langle\overline{\vec{A}_1},\vec{A}_2\rangle\end{cases} \tag{6.1.30}$$

We now check that our transcription in Sage was made without mistakes. We have so

$$
\vec{h}_0 = \begin{pmatrix} 2 & A_1 & 3 & 0 & 0 \\ 4 & A_2 & 4 & 0 & 0 \\ 6 & A_3 & 5 & 0 & 0 \\ 8 & A_4 & 6 & 0 & 0 \\ -\frac{1}{4} & C_1 & 0 & 4 & 0 \\ -\frac{1}{8} & C_2 & 1 & 4 & 0 \\ -\frac{1}{5} & B_1 & 0 & 5 & 0 \\ -\frac{1}{6} & B_2 & 0 & 6 & 0 \\ -\frac{1}{10} & B_3 & 1 & 5 & 0 \\ -\frac{1}{4} & E_1 & -2 & 8 & 0 \\ \frac{1}{4} & \overline{B_1} & 3 & 2 & 0 \\ \frac{1}{2} & \overline{B_2} & 4 & 2 & 0 \\ \frac{1}{6} & \overline{B_3} & 3 & 3 & 0 \\ 4 & \overline{E_1} & 6 & 0 & 1 \\ \frac{1}{2} & \overline{E_1} & 6 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{8} & \gamma_1 & 2 & 4 & 0 \\ -\frac{9}{20}\,|A_1|^2 & C_1 & 1 & 5 & 0 \\ -\frac{1}{6}\,|A_1|^2 & \overline{C_1} & 3 & 3 & 0 \\ -\frac{1}{4}\,\alpha_1 & C_1 & 2 & 4 & 0 \\ -\frac{1}{6}\,\overline{\alpha_1} & C_1 & 0 & 6 & 0 \\ -4\,|A_1|^2 & A_0 & 3 & 1 & 0 \\ -4\,|A_1|^2 & A_1 & 4 & 1 & 0 \\ -4\,|A_1|^2 & A_2 & 5 & 1 & 0 \\ 8\,|A_1|^4 + 4\,\alpha_1\overline{\alpha_1} - 4\,\beta & A_0 & 4 & 2 & 0 \\ -4\,\alpha_1 & A_0 & 4 & 0 & 0 \\ -4\,\alpha_1 & A_1 & 5 & 0 & 0 \\ -4\,\alpha_1 & A_2 & 6 & 0 & 0 \\ 4\,\alpha_2 & A_0 & 0 & 5 & 0 \\ 4\,\alpha_2 & A_1 & 1 & 5 & 0 \end{pmatrix} \begin{pmatrix} -6\,\alpha_3 & A_0 & 5 & 0 & 0 \\ -6\,\alpha_3 & A_1 & 6 & 0 & 0 \\ -4\,\alpha_4 & A_0 & 4 & 1 & 0 \\ -4\,\alpha_4 & A_1 & 5 & 1 & 0 \\ 12\,\alpha_1|A_1|^2 - 6\,\alpha_5 & A_0 & 5 & 1 & 0 \\ -10\,\alpha_6 & A_0 & 7 & -1 & 0 \\ -12\,\alpha_7 & A_0 & 8 & -2 & 0 \\ -8\,\zeta_0 & A_0 & 6 & 0 & 1 \\ 4\,\alpha_1^2 - \zeta_0 & A_0 & 6 & 0 & 0 \\ -10\,\overline{\alpha_2} & A_0 & 7 & -2 & 0 \\ -10\,\overline{\alpha_2} & A_1 & 8 & -2 & 0 \\ -2\,\overline{\alpha_4} & A_0 & 3 & 2 & 0 \\ -2\,\overline{\alpha_4} & A_1 & 4 & 2 & 0 \\ 4\,|A_1|^2\overline{\alpha_1} - 2\,\overline{\alpha_5} & A_0 & 3 & 3 & 0 \\ 2\,\overline{\alpha_6} & A_0 & 1 & 5 & 0 \\ 4\,\overline{\alpha_7} & A_0 & 0 & 6 & 0 \\ -\overline{\zeta_0} & A_0 & 2 & 4 & 0 \end{pmatrix} \tag{6.1.31}
$$

Let $Q : \mathscr{QD}(D^2, \mathbb{C}^n) \to \mathbb{C}$ be the function defined on the $\mathbb{C}$-vector space of $\mathbb{C}^n$-valued quadratic differential $\mathscr{QD}(D^2, \mathbb{C}^n)$, such that for all $\vec{\alpha} \in \mathscr{QD}(D^2, \mathbb{C}^n)$, we have

$$
Q(\alpha) = \partial\overline{\partial}\vec{\alpha} \,\dot{\otimes}\, \vec{\alpha} - \partial\vec{\alpha} \,\dot{\otimes}\, \overline{\partial}\vec{\alpha}.
$$

Then we have

$$
\mathscr{Q}_{\vec{\Phi}} = g^{-1} \otimes Q(\vec{h}_0) + \left(\frac{1}{4}|\vec{H}|^2 + |\vec{h}_0|^2_{WP}\right) \vec{h}_0 \,\dot{\otimes}\, \vec{h}_0 + \langle \vec{H}, \vec{h}_0 \rangle^2 \tag{6.1.32}
$$

As for all $\theta_0 \geq 4$, we have

$$
\vec{H} = O(|z|^{2-\theta_0}), \quad \vec{h}_0 = O(|z|^{\theta_0 - 1})
$$

we have

$$
\frac{1}{4}|\vec{H}|^2 \vec{h}_0 \,\dot{\otimes}\, \vec{h}_0 + \langle \vec{H}, \vec{h}_0 \rangle^2 = O(|z|^2)
$$

while

$$
|\vec{h}_0|^2_{WP} \vec{h}_0 \,\dot{\otimes}\, \vec{h}_0 = \left(|\vec{H}|^2 - K_g\right) \vec{h}_0 \,\dot{\otimes}\, \vec{h}_0 = -K_g \vec{h}_0 \,\dot{\otimes}\, \vec{h}_0 + O(|z|^2)
$$

Then we compute

$$
g^{-1} \otimes Q(\vec{h}_0) =
$$

$$\begin{pmatrix} -16\,A_0C_1\alpha_1+16\,A_2C_1+2\,A_1C_2 & 0 & 0 & 0\\ -30\,A_0C_1\alpha_3+8\,A_0A_1\overline{\zeta_0}+30\,A_3C_1+6\,A_2C_2-6\,(4\,A_1C_1+A_0C_2)\alpha_1-A_1\gamma_1 & 1 & 0 & 0\\ 20\,A_1E_1 & -3 & 4 & 0\\ \omega_1 & 0 & 1 & 0\\ \lambda_2 & 3 & -2 & 0\\ 192\,A_0^2\alpha_2|A_1|^2-\frac{48}{5}\,A_0B_1|A_1|^2-3\,A_0C_1\overline{\alpha_4}-144\,A_0A_1\overline{\alpha_7}+6\,A_1B_2+\frac{3}{8}\,C_1\overline{B_1} & -1 & 2 & 0\\ -24\,A_0A_1\zeta_0+12\,A_1\overline{E_1} & 5 & -4 & 0\\ -16\,A_0^2\alpha_1|A_1|^2-8\,A_1^2|A_1|^2+16\,A_0A_2|A_1|^2-8\,A_0A_1\alpha_4 & 3 & -3 & 0\\ 16\,A_0A_1\alpha_1|A_1|^2-48\,A_0^2\alpha_3|A_1|^2-16\,A_1A_2|A_1|^2+48\,A_0A_3|A_1|^2-16\,A_1^2\alpha_4-24\,A_0A_1\alpha_5 & 4 & -3 & 0\\ -9\,A_0C_1|A_1|^2-120\,A_0A_1\alpha_2+6\,A_1B_1 & -1 & 1 & 0\\ -480\,A_0^2|A_1|^2\overline{\alpha_2}+80\,A_0A_1\alpha_6 & 6 & -5 & 0\\ 240\,A_0A_1\alpha_7-40\left(6\,A_0^2\alpha_1-5\,A_1^2-6\,A_0A_2\right)\overline{\alpha_2} & 7 & -6 & 0\\ 160\,A_0A_1\overline{\alpha_2} & 6 & -6 & 0\\ 6\,A_1C_1 & -1 & 0 & 0 \end{pmatrix} \tag{6.1.33}$$

where

$$\begin{aligned}
&\lambda_1=-15\,A_1C_1|A_1|^2-3\,A_0C_2|A_1|^2-16\,A_0B_1\alpha_1-12\,A_0C_1\alpha_4-40\,A_0A_1\overline{\alpha_6}+16\,A_2B_1+2\,A_1B_3\\
&+80\left(4\,A_0^2\alpha_1-A_1^2-4\,A_0A_2\right)\alpha_2\\
&\lambda_2=32\,A_0A_1|A_1|^4+16\,A_0A_1\alpha_1\overline{\alpha_1}-16\,A_0A_1\beta+2\,A_0\alpha_1\overline{B_1}-105\,A_0C_1\overline{\alpha_2}-2\,A_2\overline{B_1}+2\,A_1\overline{B_2}\\
&-8\left(2\,A_0^2\alpha_1+A_1^2-2\,A_0A_2\right)\overline{\alpha_4}
\end{aligned} \tag{6.1.34}$$

Then we have

$$-K_g\vec{h}_0\,\dot{\otimes}\,\vec{h}_0=\begin{pmatrix} 16\,A_1^2|A_1|^2 & 3 & -3 & 0\\ -64\,A_0A_1\alpha_1|A_1|^2+64\,A_1A_2|A_1|^2+16\,A_1^2\alpha_4 & 4 & -3 & 0\\ -4\,A_1C_1|A_1|^2-80\,A_1^2\alpha_2 & 0 & 1 & 0\\ -64\,A_0A_1|A_1|^4+16\,A_1^2\overline{\alpha_4} & 3 & -2 & 0\\ -80\,A_1^2\overline{\alpha_2} & 7 & -6 & 0 \end{pmatrix} \tag{6.1.35}$$

and we finally obtain as $|\vec{H}|^2\vec{h}_0\,\dot{\otimes}\,\vec{h}_0=O(|z|^2)$ and $\langle\vec{H},\vec{h}_0\rangle^2=O(|z|^2)$

$$\mathscr{Q}_{\vec{\Phi}}=$$

$$
\begin{pmatrix}
-16\,A_0C_1\alpha_1+16\,A_2C_1+2\,A_1C_2 & 0 & 0 & 0\\
-30\,A_0C_1\alpha_3+8\,A_0A_1\overline{\zeta_0}+30\,A_3C_1+6\,A_2C_2-6\,(4\,A_1C_1+A_0C_2)\alpha_1-A_1\gamma_1 & 1 & 0 & 0\\
20\,A_1E_1 & -3 & 4 & 0\\
\mu_1 & 0 & 1 & 0\\
\mu_2 & 3 & -2 & 0\\
192\,A_0^2\alpha_2|A_1|^2-\dfrac{48}{5}\,A_0B_1|A_1|^2-3\,A_0C_1\overline{\alpha_4}-144\,A_0A_1\overline{\alpha_7}+6\,A_1B_2+\dfrac{3}{8}\,C_1\overline{B_1} & -1 & 2 & 0\\
-24\,A_0A_1\zeta_0+12\,A_1\overline{E_1} & 5 & -4 & 0\\
-16\,A_0^2\alpha_1|A_1|^2+8\,A_1^2|A_1|^2+16\,A_0A_2|A_1|^2-8\,A_0A_1\alpha_4 & 3 & -3 & 0\\
-48\,A_0A_1\alpha_1|A_1|^2-48\,A_0^2\alpha_3|A_1|^2+48\,A_1A_2|A_1|^2+48\,A_0A_3|A_1|^2-24\,A_0A_1\alpha_5 & 4 & -3 & 0\\
-9\,A_0C_1|A_1|^2-120\,A_0A_1\alpha_2+6\,A_1B_1 & -1 & 1 & 0\\
-480\,A_0^2|A_1|^2\overline{\alpha_2}+80\,A_0A_1\alpha_6 & 6 & -5 & 0\\
240\,A_0A_1\alpha_7-80\,A_1^2\overline{\alpha_2}-40\left(6\,A_0^2\alpha_1-5\,A_1^2-6\,A_0A_2\right)\overline{\alpha_2} & 7 & -6 & 0\\
160\,A_0A_1\overline{\alpha_2} & 6 & -6 & 0\\
6\,A_1C_1 & -1 & 0 & 0
\end{pmatrix}
\tag{6.1.36}
$$

where

$$
\begin{aligned}
\mu_1&=-19\,A_1C_1|A_1|^2-3\,A_0C_2|A_1|^2-16\,A_0B_1\alpha_1-80\,A_1^2\alpha_2-12\,A_0C_1\alpha_4-40\,A_0A_1\overline{\alpha_6}+16\,A_2B_1+2\,A_1B_3\\
&\quad+80\left(4\,A_0^2\alpha_1-A_1^2-4\,A_0A_2\right)\alpha_2\\
\mu_2&=-32\,A_0A_1|A_1|^4+16\,A_0A_1\alpha_1\overline{\alpha_1}-16\,A_0A_1\beta+2\,A_0\alpha_1\overline{B_1}-105\,A_0C_1\overline{\alpha_2}+16\,A_1^2\overline{\alpha_4}-2\,A_2\overline{B_1}+2\,A_1\overline{B_2}\\
&\quad-8\left(2\,A_0^2\alpha_1+A_1^2-2\,A_0A_2\right)\overline{\alpha_4}
\end{aligned}
$$

It is easy so see that all relations coming from the meromorphy of $\mathscr{Q}_{\vec{\Phi}}$ are trivial, besides

$$\mu_1=0.$$

As

$$\langle\vec{A}_0,\vec{A}_1\rangle=\langle\vec{A}_0,\vec{C}_1\rangle=\langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_0,\vec{A}_2\rangle=0,$$

we have

$$
\begin{aligned}
\mu_1&=-19\,A_1C_1|A_1|^2-3\,A_0C_2|A_1|^2-16\,A_0B_1\alpha_1-80\,A_1^2\alpha_2-12\,A_0C_1\alpha_4-40\,A_0A_1\overline{\alpha_6}+16\,A_2B_1+2\,A_1B_3\\
&\quad+80\left(4\,A_0^2\alpha_1-A_1^2-4\,A_0A_2\right)\alpha_2\\
&=-19|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle-3|\vec{A}_1|^2\langle\vec{A}_0,\vec{C}_2\rangle+16\langle\vec{A}_2,\vec{B}_1\rangle+2\langle\vec{A}_1,\vec{B}_3\rangle-15|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle-160\left(A_1^2+2\,A_0A_2\right)\\
&=-19|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle-3|\vec{A}_1|^2\langle\vec{A}_0,\vec{C}_2\rangle+16\langle\vec{A}_2,\vec{B}_1\rangle+2\langle\vec{A}_1,\vec{B}_3\rangle-15|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle\\
&=-16|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle+16\langle\vec{A}_2,\vec{B}_1\rangle+2\langle\vec{A}_1,\vec{B}_3\rangle
\end{aligned}
$$

as

$$\langle\vec{A}_1,\vec{C}_1\rangle+\langle\vec{A}_0,\vec{C}_2\rangle=0.$$

Then, recall that

$$
\begin{cases}\vec{C}_2=\vec{D}_2+2\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_0}\\
\vec{C}_3=\vec{D}_3\\
\vec{B}_1=-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0\\
\vec{B}_2=-\dfrac{3}{8}|\vec{C}_1|^2\overline{\vec{A}_0}-2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\vec{A}_0\\
\vec{B}_3=-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_1+2\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_1}-2\langle\overline{\vec{A}_1},\vec{C}_2\rangle\vec{A}_0\\
\vec{E}_1=-\dfrac{1}{8}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_0}\\
\vec{\gamma}_1=-\vec{\gamma}_0-\mathrm{Re}\left\{\left(\dfrac{1}{2}\langle\vec{C}_1,\vec{C}_1\rangle+8\overline{\langle\vec{A}_2,\vec{C}_1\rangle}+4\overline{\langle\vec{A}_1,\vec{C}_2\rangle}\right)\vec{A}_0\right\}\in\mathbb{R}^n
\end{cases}
\tag{6.1.37}
$$

In particular, we have

$$
\begin{cases}\langle\vec{A}_2,\vec{B}_1\rangle=-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_0,\vec{A}_2\rangle=\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_1,\vec{A}_1\rangle\\
\langle\vec{A}_1,\vec{B}_3\rangle=-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_1,\vec{A}_1\rangle+2|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle
\end{cases}
\tag{6.1.38}
$$

and

$$
\begin{aligned}
\mu_1&=-16|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle+16\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_1,\vec{A}_1\rangle+2\left(-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_1,\vec{A}_1\rangle+2|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle\right)\\
&=-12|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle+12\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_1,\vec{A}_1\rangle=0
\end{aligned}
$$

so we recover as expected

$$
|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle=\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_1,\vec{A}_1\rangle
\tag{6.1.39}
$$

In particular, looking at (6.1.38), we see that

$$
\langle\vec{A}_2,\vec{B}_1\rangle=|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle,\quad\langle\vec{A}_1,\vec{B}_3\rangle=0
\tag{6.1.40}
$$

Then we get for some $\vec{D}_4\in\vec{C}^n$

$$
\vec{H}+\vec{\gamma}_0\log|z|=\mathrm{Re}\left(\frac{\vec{C}_1}{z^2}+\frac{\vec{D}_2}{z}+\vec{D}_3+\vec{D}_4 z\right)+
$$

$$
\begin{pmatrix}
-\frac{1}{16}\,C_1^2 & \overline{A_0} & -4 & 4 & 0 \\
-\frac{9}{160}\,C_1^2 & \overline{A_1} & -4 & 5 & 0 \\
\frac{1}{2}\,A_0C_1\alpha_2-\frac{17}{160}\,B_1C_1 & \overline{A_0} & -4 & 5 & 0 \\
-\frac{1}{80}\,C_1\overline{A_1} & C_1 & -4 & 5 & 0 \\
C_1\alpha_2\overline{A_0}-\frac{1}{5}\,E_1\overline{A_1} & A_0 & -4 & 5 & 0 \\
-\frac{1}{8}\,C_1C_2+\frac{1}{3}\,A_1E_1 & \overline{A_0} & -3 & 4 & 0 \\
-C_1\overline{A_1} & A_0 & -2 & 1 & 0 \\
-\frac{3}{16}\,C_1\overline{C_1} & \overline{A_0} & -2 & 2 & 0 \\
C_1\overline{A_0}\overline{\alpha_1}-\frac{1}{2}\,B_1\overline{A_1}-C_1\overline{A_2} & A_0 & -2 & 2 & 0 \\
-\frac{7}{48}\,C_1\overline{C_1} & \overline{A_1} & -2 & 3 & 0 \\
A_0\alpha_2\overline{C_1}-\frac{2}{15}\,B_1\overline{C_1}-\frac{7}{48}\,C_1\overline{C_2} & \overline{A_0} & -2 & 3 & 0 \\
-\frac{1}{48}\,\overline{A_1C_1} & C_1 & -2 & 3 & 0 \\
\frac{5}{3}\,\alpha_2\overline{A_0C_1}+C_1\overline{A_0}\overline{\alpha_3}-\frac{1}{3}\,B_2\overline{A_1}-\frac{2}{3}\,B_1\overline{A_2}-C_1\overline{A_3}+\frac{1}{3}\left(2\,B_1\overline{A_0}+3\,C_1\overline{A_1}\right)\overline{\alpha_1} & A_0 & -2 & 3 & 0 \\
-\frac{1}{24}\,C_1\overline{A_1} & \overline{C_1} & -2 & 3 & 0 \\
A_1C_1 & \overline{A_0} & -1 & 0 & 0 \\
A_1C_1 & \overline{A_1} & -1 & 1 & 0 \\
-2\,A_0C_1|A_1|^2+A_1B_1 & \overline{A_0} & -1 & 1 & 0 \\
-C_1\overline{A_1} & A_1 & -1 & 1 & 0 \\
2\,C_1|A_1|^2\overline{A_0}-C_2\overline{A_1} & A_0 & -1 & 1 & 0 \\
-2\,A_0B_1|A_1|^2-A_1C_1\overline{\alpha_1}-A_0C_1\overline{\alpha_4}+A_1B_2-\frac{1}{8}\,C_1\overline{B_1}-\frac{3}{16}\,C_2\overline{C_1} & \overline{A_0} & -1 & 2 & 0 \\
A_1C_1 & \overline{A_2} & -1 & 2 & 0 \\
-2\,A_0C_1|A_1|^2+A_1B_1 & \overline{A_1} & -1 & 2 & 0 \\
B_1|A_1|^2\overline{A_0}+2\,C_1|A_1|^2\overline{A_1}+C_2\overline{A_0}\overline{\alpha_1}+C_1\overline{A_0}\overline{\alpha_4}-\frac{1}{2}\,B_3\overline{A_1}-C_2\overline{A_2} & A_0 & -1 & 2 & 0 \\
C_1\overline{A_0}\overline{\alpha_1}-\frac{1}{2}\,B_1\overline{A_1}-C_1\overline{A_2} & A_1 & -1 & 2 & 0 \\
\overline{A_1C_1} & A_0 & 0 & -1 & 0 \\
4\,\overline{A_0C_1}\overline{\alpha_1}-\frac{1}{4}\,C_1^2-4\,\overline{A_2C_1}-2\,\overline{A_1C_2} & A_0 & 0 & 0 & 1 \\
4\,A_0C_1\alpha_1-4\,A_2C_1-2\,A_1C_2-\frac{1}{4}\,\overline{C_1}^2 & \overline{A_0} & 0 & 0 & 1 \\
-\frac{1}{8}\,\overline{C_1}^2 & \overline{A_1} & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{(1)} \\ \mathbf{(2)} \\ \mathbf{(3)} \\ \mathbf{(4)} \\ \mathbf{(5)} \\ \mathbf{(6)} \\ \mathbf{(7)} \\ \mathbf{(8)} \\ \mathbf{(9)} \\ \mathbf{(10)} \\ \mathbf{(11)} \\ \mathbf{(12)} \\ \mathbf{(13)} \\ \mathbf{(14)} \\ \mathbf{(15)} \\ \mathbf{(16)} \\ \mathbf{(17)} \\ \mathbf{(18)} \\ \mathbf{(19)} \\ \mathbf{(20)} \\ \mathbf{(21)} \\ \mathbf{(22)} \\ \mathbf{(23)} \\ \mathbf{(24)} \\ \mathbf{(25)} \\ \mathbf{(26)} \\ \mathbf{(27)} \\ \mathbf{(28)}
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{1}{4}\,\overline{C_1C_2} & \overline{A_0} & 0 & 1 & 0 \\
-\frac{1}{8}\,\overline{A_1C_1} & \overline{C_1} & 0 & 1 & 0 \\
\lambda_1 & A_0 & 0 & 1 & 0 \\
-C_1\overline{A_1} & A_2 & 0 & 1 & 0 \\
2\,C_1|A_1|^2\overline{A_0} - C_2\overline{A_1} & A_1 & 0 & 1 & 0 \\
-\frac{1}{2}\,B_1C_1 - 2\,\gamma_1\overline{A_1} & A_0 & 0 & 1 & 1 \\
8\,A_1C_1|A_1|^2 + 4\,A_0C_2|A_1|^2 + 4\,A_0B_1\alpha_1 + 4\,A_0C_1\alpha_4 - 4\,A_2B_1 - 2\,A_1B_3 & \overline{A_0} & 0 & 1 & 1 \\
4\,A_0C_1\alpha_1 - 4\,A_2C_1 - 2\,A_1C_2 & \overline{A_1} & 0 & 1 & 1 \\
-A_1\overline{C_1} & \overline{A_0} & 1 & -2 & 0 \\
\overline{A_1C_1} & A_1 & 1 & -1 & 0 \\
-2\,|A_1|^2\overline{A_0C_1} + \overline{A_1B_1} & A_0 & 1 & -1 & 0 \\
-A_1\overline{C_1} & \overline{A_1} & 1 & -1 & 0 \\
2\,A_0|A_1|^2\overline{C_1} - A_1\overline{C_2} & \overline{A_0} & 1 & -1 & 0 \\
-\frac{1}{8}\,C_1^2 & A_1 & 1 & 0 & 0 \\
-\frac{1}{4}\,C_1C_2 & A_0 & 1 & 0 & 0 \\
-\frac{1}{8}\,A_1C_1 & C_1 & 1 & 0 & 0 \\
\lambda_2 & \overline{A_0} & 1 & 0 & 0 \\
-A_1\overline{C_1} & \overline{A_2} & 1 & 0 & 0 \\
2\,A_0|A_1|^2\overline{C_1} - A_1\overline{C_2} & \overline{A_1} & 1 & 0 & 0 \\
-2\,A_1\gamma_1 - \frac{1}{2}\,\overline{B_1C_1} & \overline{A_0} & 1 & 0 & 1 \\
8\,|A_1|^2\overline{A_1C_1} + 4\,|A_1|^2\overline{A_0C_2} + 4\,\overline{A_0B_1}\overline{\alpha_1} + 4\,\overline{A_0C_1}\overline{\alpha_4} - 4\,\overline{A_2B_1} - 2\,\overline{A_1B_3} & A_0 & 1 & 0 & 1 \\
4\,\overline{A_0C_1}\overline{\alpha_1} - 4\,\overline{A_2C_1} - 2\,\overline{A_1C_2} & A_1 & 1 & 0 & 1 \\
-\frac{3}{16}\,C_1\overline{C_1} & A_0 & 2 & -2 & 0 \\
A_0\alpha_1\overline{C_1} - \frac{1}{2}\,A_1\overline{B_1} - A_2\overline{C_1} & \overline{A_0} & 2 & -2 & 0 \\
-2\,|A_1|^2\overline{A_0B_1} - \alpha_4\overline{A_0C_1} - \alpha_1\overline{A_1C_1} + \overline{A_1B_2} - \frac{1}{8}\,B_1\overline{C_1} - \frac{3}{16}\,C_1\overline{C_2} & A_0 & 2 & -1 & 0 \\
\overline{A_1C_1} & A_2 & 2 & -1 & 0 \\
-2\,|A_1|^2\overline{A_0C_1} + \overline{A_1B_1} & A_1 & 2 & -1 & 0 \\
A_0|A_1|^2\overline{B_1} + 2\,A_1|A_1|^2\overline{C_1} + A_0\alpha_4\overline{C_1} + A_0\alpha_1\overline{C_2} - \frac{1}{2}\,A_1\overline{B_3} - A_2\overline{C_2} & \overline{A_0} & 2 & -1 & 0 \\
A_0\alpha_1\overline{C_1} - \frac{1}{2}\,A_1\overline{B_1} - A_2\overline{C_1} & \overline{A_1} & 2 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{(29)} \\ \mathbf{(30)} \\ \mathbf{(31)} \\ \mathbf{(32)} \\ \mathbf{(33)} \\ \mathbf{(34)} \\ \mathbf{(35)} \\ \mathbf{(36)} \\ \mathbf{(37)} \\ \mathbf{(38)} \\ \mathbf{(39)} \\ \mathbf{(40)} \\ \mathbf{(41)} \\ \mathbf{(42)} \\ \mathbf{(43)} \\ \mathbf{(44)} \\ \mathbf{(45)} \\ \mathbf{(46)} \\ \mathbf{(47)} \\ \mathbf{(48)} \\ \mathbf{(49)} \\ \mathbf{(50)} \\ \mathbf{(51)} \\ \mathbf{(52)} \\ \mathbf{(53)} \\ \mathbf{(54)} \\ \mathbf{(55)} \\ \mathbf{(56)} \\ \mathbf{(57)}
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{7}{48}\,C_1\overline{C_1} & A_1 & 3 & -2 & 0\\
C_1\overline{A_0}\,\overline{\alpha_2}-\frac{2}{15}\,C_1\overline{B_1}-\frac{7}{48}\,C_2\overline{C_1} & A_0 & 3 & -2 & 0\\
-\frac{1}{48}\,A_1C_1 & \overline{C_1} & 3 & -2 & 0\\
A_0\alpha_3\overline{C_1}+\frac{5}{3}\,A_0C_1\overline{\alpha_2}+\frac{1}{3}\left(2\,A_0\overline{B_1}+3\,A_1\overline{C_1}\right)\alpha_1-\frac{2}{3}\,A_2\overline{B_1}-\frac{1}{3}\,A_1\overline{B_2}-A_3\overline{C_1} & \overline{A_0} & 3 & -2 & 0\\
-\frac{1}{24}\,A_1\overline{C_1} & C_1 & 3 & -2 & 0\\
-\frac{1}{16}\,\overline{C_1}^2 & A_0 & 4 & -4 & 0\\
-\frac{1}{8}\,\overline{C_1C_2}+\frac{1}{3}\,\overline{A_1E_1} & A_0 & 4 & -3 & 0\\
-\frac{9}{160}\,\overline{C_1}^2 & A_1 & 5 & -4 & 0\\
\frac{1}{2}\,\overline{A_0C_1\alpha_2}-\frac{17}{160}\,\overline{B_1C_1} & A_0 & 5 & -4 & 0\\
-\frac{1}{80}\,A_1\overline{C_1} & \overline{C_1} & 5 & -4 & 0\\
A_0\overline{C_1}\,\overline{\alpha_2}-\frac{1}{5}\,A_1\overline{E_1} & \overline{A_0} & 5 & -4 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{(58)}\\ \mathbf{(59)}\\ \mathbf{(60)}\\ \mathbf{(61)}\\ \mathbf{(62)}\\ \mathbf{(63)}\\ \mathbf{(64)}\\ \mathbf{(65)}\\ \mathbf{(66)}\\ \mathbf{(67)}\\ \mathbf{(68)}
\end{pmatrix}
$$

where

$$
\begin{aligned}
\lambda_1 &= 2\,C_2|A_1|^2\overline{A_0}+C_1\alpha_4\overline{A_0}+C_1\alpha_1\overline{A_1}+3\,\overline{A_0C_1}\,\overline{\alpha_3}-\frac{1}{8}\,B_1C_1-2\,C_3\overline{A_1}+\gamma_1\overline{A_1}-3\,\overline{A_3C_1}-2\,\overline{A_2C_2}\\
&\quad+\left(3\,\overline{A_1C_1}+2\,\overline{A_0C_2}\right)\overline{\alpha_1}\\
\lambda_2 &= 2\,A_0|A_1|^2\overline{C_2}+3\,A_0C_1\alpha_3+A_1\overline{C_1}\,\overline{\alpha_1}+A_0\overline{C_1}\,\overline{\alpha_4}-3\,A_3C_1-2\,A_2C_2-2\,A_1C_3\\
&\quad+(3\,A_1C_1+2\,A_0C_2)\alpha_1+A_1\gamma_1-\frac{1}{8}\,\overline{B_1C_1}
\end{aligned}
$$

The new powers arising are

$$
\begin{pmatrix}
-4 & 5 & 0\\
-3 & 4 & 0\\
-2 & 3 & 0
\end{pmatrix}
\begin{pmatrix}
-1 & 2 & 0\\
1 & 0 & 0\\
1 & 0 & 1
\end{pmatrix}
$$

so there exists $\vec{C}_4,\vec{B}_4,\vec{B}_5,\vec{E}_2,\vec{E}_3,\vec{\gamma}_2\in\mathbb{C}^n$ such that

$$
\begin{aligned}
\vec{H} &= \mathrm{Re}\left(\frac{\vec{C}_1}{z^2}+\frac{\vec{C}_2}{z}+\vec{C}_4z+\left(\vec{B}_1\overline{z}+\vec{B}_2\overline{z}^2+\vec{B}_4\overline{z}^3\right)\frac{1}{z^2}+\left(\vec{B}_3\overline{z}+\vec{B}_5\overline{z}^2\right)\frac{1}{z}+\left(\vec{E}_1\overline{z}^4+\vec{E}_2\overline{z}^5\right)\frac{1}{z^4}+\vec{E}_3\frac{\overline{z}^4}{z^3}\right)\\
&\quad+\vec{C}_3+\vec{\gamma}_1\log|z|+\mathrm{Re}\left(\vec{\gamma}_2\,z\right)\log|z|+O(|z|^{2-\varepsilon}). \qquad (6.1.41)
\end{aligned}
$$

First, we check that the mean curvature given in the code coincides with (6.1.41) (recall that $\vec{C}_3 \in \mathbb{R}^n$).

$$\vec{H} = \begin{pmatrix} \frac{1}{2} & C_1 & -2 & 0 & 0 \\ \frac{1}{2} & C_2 & -1 & 0 & 0 \\ 1 & C_3 & 0 & 0 & 0 \\ \frac{1}{2} & C_4 & 1 & 0 & 0 \\ \frac{1}{2} & B_1 & -2 & 1 & 0 \\ \frac{1}{2} & B_2 & -2 & 2 & 0 \\ \frac{1}{2} & B_4 & -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & B_3 & -1 & 1 & 0 \\ \frac{1}{2} & B_5 & -1 & 2 & 0 \\ \frac{1}{2} & E_1 & -4 & 4 & 0 \\ \frac{1}{2} & E_2 & -4 & 5 & 0 \\ \frac{1}{2} & E_3 & -3 & 4 & 0 \\ \frac{1}{2} & \gamma_2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \overline{C_1} & 0 & -2 & 0 \\ \frac{1}{2} & \overline{C_2} & 0 & -1 & 0 \\ \frac{1}{2} & \overline{C_4} & 0 & 1 & 0 \\ \frac{1}{2} & \overline{B_1} & 1 & -2 & 0 \\ \frac{1}{2} & \overline{B_2} & 2 & -2 & 0 \\ \frac{1}{2} & \overline{B_4} & 3 & -2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \overline{B_3} & 1 & -1 & 0 \\ \frac{1}{2} & \overline{B_5} & 2 & -1 & 0 \\ \frac{1}{2} & \overline{E_1} & 4 & -4 & 0 \\ \frac{1}{2} & \overline{E_2} & 5 & -4 & 0 \\ \frac{1}{2} & \overline{E_3} & 4 & -3 & 0 \\ \frac{1}{2} & \overline{\gamma_2} & 0 & 1 & 1 \end{pmatrix} \tag{6.1.42}$$

We see that $\vec{\gamma}_2$ corresponds to *twice* the lines

$$(48),\ (49)\ (50)$$

so

$$\vec{\gamma}_2 = -2\left(4\overline{\langle \vec{A}_2, \vec{C}_1\rangle} + 2\overline{\langle \vec{A}_1, \vec{C}_2\rangle}\right)\vec{A}_1 + 4\langle \vec{A}_1, \vec{\gamma}_0\rangle\overline{\vec{A}_0}$$

as by (6.1.38) and (6.1.39), $\langle \vec{A}_1, \vec{B}_3\rangle = 0$ and by (6.1.23)

$$\begin{aligned} &8\,|A_1|^2\overline{A_1 C_1} + 4\,|A_1|^2\overline{A_0 C_2} + 4\,\overline{A_0 B_1}\overline{\alpha_1} + 4\,\overline{A_0 C_1}\overline{\alpha_4} - 4\,\overline{A_2 B_1} - 2\,\overline{A_1 B_3} \\ &= 4|\vec{A}_1|^2\overline{\langle \vec{A}_1, \vec{C}_1\rangle} - 4\langle \overline{\vec{A}_1}, \vec{C}_1\rangle\langle \vec{A}_1, \vec{A}_1\rangle = 0. \end{aligned}$$

Now we have for some $\vec{D}_5 \in \mathbb{C}^n$

$\vec{H} + \vec{\gamma}_0 \log|z| = \mathrm{Re}\left(\dfrac{\vec{C}_1}{z^2} + \dfrac{\vec{D}_2}{z} + \vec{D}_3 + \vec{D}_4 z + \vec{D}_5 z^2\right) +$

$-\frac{7}{96}\,C_1E_1$	$\overline{A_0}$	-6	8	0	**(1)**
$-\frac{1}{16}\,C_1^2$	$\overline{A_0}$	-4	4	0	**(2)**
$-\frac{9}{160}\,C_1^2$	$\overline{A_1}$	-4	5	0	**(3)**
$\frac{1}{2}\,A_0C_1\alpha_2 - \frac{17}{160}\,B_1C_1$	$\overline{A_0}$	-4	5	0	**(4)**
$-\frac{1}{80}\,C_1\overline{A_1}$	C_1	-4	5	0	**(5)**
$C_1\alpha_2\overline{A_0} - \frac{1}{5}\,E_1\overline{A_1}$	A_0	-4	5	0	**(6)**
$\frac{1}{2}\,A_0B_1\alpha_2 + \frac{1}{96}\,C_1^2\overline{\alpha_1} + \frac{1}{2}\,A_0C_1\overline{\alpha_7} - \frac{11}{240}\,B_1^2 - \frac{3}{32}\,B_2C_1 - \frac{7}{96}\,E_1\overline{C_1}$	$\overline{A_0}$	-4	6	0	**(7)**
$-\frac{5}{96}\,C_1^2$	$\overline{A_2}$	-4	6	0	**(8)**
$\frac{1}{2}\,A_0C_1\alpha_2 - \frac{47}{480}\,B_1C_1$	$\overline{A_1}$	-4	6	0	**(9)**
$-\frac{1}{120}\,C_1\overline{A_1}$	B_1	-4	6	0	**(10)**
$\frac{1}{3}\,E_1\overline{A_0}\,\overline{\alpha_1} + C_1\overline{A_0}\,\overline{\alpha_7} + \frac{1}{6}\left(5\,B_1\overline{A_0} + 6\,C_1\overline{A_1}\right)\alpha_2 - \frac{1}{6}\,E_2\overline{A_1} - \frac{1}{3}\,E_1\overline{A_2}$	A_0	-4	6	0	**(11)**
$\frac{1}{48}\,C_1\overline{A_0}\,\overline{\alpha_1} - \frac{1}{96}\,B_1\overline{A_1} - \frac{1}{48}\,C_1\overline{A_2}$	C_1	-4	6	0	**(12)**
$-\frac{1}{8}\,C_1C_2 + \frac{1}{3}\,A_1E_1$	$\overline{A_0}$	-3	4	0	**(13)**
$-\frac{9}{80}\,C_1C_2 + \frac{1}{3}\,A_1E_1$	$\overline{A_1}$	-3	5	0	**(14)**
$\frac{1}{120}\,C_1^2\lvert A_1\rvert^2 - \frac{2}{3}\,A_0E_1\lvert A_1\rvert^2 + \frac{1}{3}\,A_0C_1\overline{\alpha_6} - \frac{13}{120}\,B_3C_1 - \frac{5}{48}\,B_1C_2 + \frac{1}{3}\,A_1E_2 + \frac{1}{3}\,(A_1C_1 + 2\,A_0C_2)\alpha_2$	$\overline{A_0}$	-3	5	0	**(15)**
$-\frac{1}{80}\,C_1\overline{A_1}$	C_2	-3	5	0	**(16)**
$\frac{1}{40}\,C_1\lvert A_1\rvert^2\overline{A_0} - \frac{1}{80}\,C_2\overline{A_1}$	C_1	-3	5	0	**(17)**
$\frac{2}{5}\,E_1\lvert A_1\rvert^2\overline{A_0} + C_2\alpha_2\overline{A_0} + C_1\overline{A_0}\,\overline{\alpha_6} - \frac{1}{5}\,E_3\overline{A_1}$	A_0	-3	5	0	**(18)**
$C_1\alpha_2\overline{A_0} - \frac{1}{5}\,E_1\overline{A_1}$	A_1	-3	5	0	**(19)**
$-C_1\overline{A_1}$	A_0	-2	1	0	**(20)**
$-\frac{3}{16}\,C_1\overline{C_1}$	$\overline{A_0}$	-2	2	0	**(21)**
$C_1\overline{A_0}\,\overline{\alpha_1} - \frac{1}{2}\,B_1\overline{A_1} - C_1\overline{A_2}$	A_0	-2	2	0	**(22)**
$-\frac{7}{48}\,C_1\overline{C_1}$	$\overline{A_1}$	-2	3	0	**(23)**
$A_0\alpha_2\overline{C_1} - \frac{2}{15}\,B_1\overline{C_1} - \frac{7}{48}\,C_1\overline{C_2}$	$\overline{A_0}$	-2	3	0	**(24)**
$-\frac{1}{48}\,\overline{A_1C_1}$	C_1	-2	3	0	**(25)**
$\frac{5}{3}\,\alpha_2\overline{A_0C_1} + C_1\overline{A_0}\,\overline{\alpha_3} - \frac{1}{3}\,B_2\overline{A_1} - \frac{2}{3}\,B_1\overline{A_2} - C_1\overline{A_3} + \frac{1}{3}\left(2\,B_1\overline{A_0} + 3\,C_1\overline{A_1}\right)\overline{\alpha_1}$	A_0	-2	3	0	**(26)**
$-\frac{1}{24}\,C_1\overline{A_1}$	$\overline{C_1}$	-2	3	0	**(27)**
$\frac{1}{32}\,\overline{A_0C_1}\,\overline{\alpha_1} - \frac{1}{128}\,C_1^2 - \frac{1}{32}\,\overline{A_2C_1} - \frac{1}{64}\,\overline{A_1C_2}$	C_1	-2	4	0	**(28)**
λ_1	A_0	-2	4	0	**(29)**
$-\frac{1}{8}\,C_1\overline{C_1}$	$\overline{A_2}$	-2	4	0	**(30)**
$A_0\alpha_2\overline{C_1} - \frac{9}{80}\,B_1\overline{C_1} - \frac{1}{8}\,C_1\overline{C_2}$	$\overline{A_1}$	-2	4	0	**(31)**

$$
\left(
\begin{array}{cccccc}
\lambda_2 & \overline{A_0} & -2 & 4 & 0 & \mathbf{(32)} \\
-\frac{1}{80}\,\overline{A_1 C_1} & B_1 & -2 & 4 & 0 & \mathbf{(33)} \\
-\frac{1}{48}\,C_1\overline{A_1} & \overline{C_2} & -2 & 4 & 0 & \mathbf{(34)} \\
\frac{1}{16}\,C_1\overline{A_0}\,\overline{\alpha_1} - \frac{1}{32}\,B_1\overline{A_1} - \frac{1}{16}\,C_1\overline{A_2} & \overline{C_1} & -2 & 4 & 0 & \mathbf{(35)} \\
-\frac{1}{4}\,C_1\gamma_1 & \overline{A_0} & -2 & 4 & 1 & \mathbf{(36)} \\
C_1\overline{A_0\zeta_0} - \frac{1}{2}\,C_1 E_1 & A_0 & -2 & 4 & 1 & \mathbf{(37)} \\
A_1 C_1 & \overline{A_0} & -1 & 0 & 0 & \mathbf{(38)} \\
A_1 C_1 & \overline{A_1} & -1 & 1 & 0 & \mathbf{(39)} \\
-2\,A_0 C_1 |A_1|^2 + A_1 B_1 & \overline{A_0} & -1 & 1 & 0 & \mathbf{(40)} \\
-C_1\overline{A_1} & A_1 & -1 & 1 & 0 & \mathbf{(41)} \\
2\,C_1|A_1|^2\overline{A_0} - C_2\overline{A_1} & A_0 & -1 & 1 & 0 & \mathbf{(42)} \\
-2\,A_0 B_1|A_1|^2 - A_1 C_1\overline{\alpha_1} - A_0 C_1\overline{\alpha_4} + A_1 B_2 - \frac{1}{8}\,C_1\overline{B_1} - \frac{3}{16}\,C_2\overline{C_1} & \overline{A_0} & -1 & 2 & 0 & \mathbf{(43)} \\
A_1 C_1 & \overline{A_2} & -1 & 2 & 0 & \mathbf{(44)} \\
-2\,A_0 C_1 |A_1|^2 + A_1 B_1 & \overline{A_1} & -1 & 2 & 0 & \mathbf{(45)} \\
B_1|A_1|^2\overline{A_0} + 2\,C_1|A_1|^2\overline{A_1} + C_2\overline{A_0}\,\overline{\alpha_1} + C_1\overline{A_0}\,\overline{\alpha_4} - \frac{1}{2}\,B_3\overline{A_1} - C_2\overline{A_2} & A_0 & -1 & 2 & 0 & \mathbf{(46)} \\
C_1\overline{A_0}\,\overline{\alpha_1} - \frac{1}{2}\,B_1\overline{A_1} - C_1\overline{A_2} & A_1 & -1 & 2 & 0 & \mathbf{(47)} \\
-2\,A_0 B_1|A_1|^2 - A_1 C_1\overline{\alpha_1} - A_0 C_1\overline{\alpha_4} + A_1 B_2 - \frac{1}{12}\,C_1\overline{B_1} - \frac{7}{48}\,C_2\overline{C_1} & \overline{A_1} & -1 & 3 & 0 & \mathbf{(48)} \\
\lambda_3 & \overline{A_0} & -1 & 3 & 0 & \mathbf{(49)} \\
A_1 C_1 & \overline{A_3} & -1 & 3 & 0 & \mathbf{(50)} \\
-\frac{1}{48}\,\overline{A_1 C_1} & C_2 & -1 & 3 & 0 & \mathbf{(51)} \\
-2\,A_0 C_1 |A_1|^2 + A_1 B_1 & \overline{A_2} & -1 & 3 & 0 & \mathbf{(52)} \\
\frac{1}{24}\,|A_1|^2\overline{A_0 C_1} - \frac{1}{48}\,\overline{A_1 B_1} & C_1 & -1 & 3 & 0 & \mathbf{(53)} \\
\lambda_4 & A_0 & -1 & 3 & 0 & \mathbf{(54)} \\
\frac{5}{3}\,\alpha_2\overline{A_0 C_1} + C_1\overline{A_0}\,\overline{\alpha_3} - \frac{1}{3}\,B_2\overline{A_1} - \frac{2}{3}\,B_1\overline{A_2} - C_1\overline{A_3} + \frac{1}{3}\left(2\,B_1\overline{A_0} + 3\,C_1\overline{A_1}\right)\overline{\alpha_1} & A_1 & -1 & 3 & 0 & \mathbf{(55)} \\
-\frac{1}{24}\,C_1\overline{A_1} & \overline{B_1} & -1 & 3 & 0 & \mathbf{(56)} \\
\frac{1}{12}\,C_1|A_1|^2\overline{A_0} - \frac{1}{24}\,C_2\overline{A_1} & \overline{C_1} & -1 & 3 & 0 & \mathbf{(57)} \\
\overline{A_1 C_1} & A_0 & 0 & -1 & 0 & \mathbf{(58)} \\
4\,\overline{A_0 C_1}\,\overline{\alpha_1} - \frac{1}{4}\,C_1^2 - 4\,\overline{A_2 C_1} - 2\,\overline{A_1 C_2} & A_0 & 0 & 0 & 1 & \mathbf{(59)} \\
4\,A_0 C_1\alpha_1 - 4\,A_2 C_1 - 2\,A_1 C_2 - \frac{1}{4}\,\overline{C_1}^2 & \overline{A_0} & 0 & 0 & 1 & \mathbf{(60)}
\end{array}
\right)
$$

$$
\begin{pmatrix}
-\frac{1}{8}\,\overline{C_1}^2 & \overline{A_1} & 0 & 1 & 0 & \textbf{(61)} \\
-\frac{1}{4}\,\overline{C_1 C_2} & \overline{A_0} & 0 & 1 & 0 & \textbf{(62)} \\
-\frac{1}{8}\,\overline{A_1 C_1} & \overline{C_1} & 0 & 1 & 0 & \textbf{(63)} \\
\lambda_5 & A_0 & 0 & 1 & 0 & \textbf{(64)} \\
-C_1\overline{A_1} & A_2 & 0 & 1 & 0 & \textbf{(65)} \\
2\,C_1|A_1|^2\overline{A_0} - C_2\overline{A_1} & A_1 & 0 & 1 & 0 & \textbf{(66)} \\
-\frac{1}{2}\,B_1C_1 - 2\,\gamma_1\overline{A_1} & A_0 & 0 & 1 & 1 & \textbf{(67)} \\
8\,A_1C_1|A_1|^2 + 4\,A_0C_2|A_1|^2 + 4\,A_0B_1\alpha_1 + 4\,A_0C_1\alpha_4 - 4\,A_2B_1 - 2\,A_1B_3 & \overline{A_0} & 0 & 1 & 1 & \textbf{(68)} \\
4\,A_0C_1\alpha_1 - 4\,A_2C_1 - 2\,A_1C_2 & \overline{A_1} & 0 & 1 & 1 & \textbf{(69)} \\
-\frac{1}{16}\,\overline{C_1}^2 & \overline{A_2} & 0 & 2 & 0 & \textbf{(70)} \\
-\frac{1}{8}\,\overline{C_1 C_2} & \overline{A_1} & 0 & 2 & 0 & \textbf{(71)} \\
-\frac{1}{8}\,C_2\overline{B_1} - \frac{1}{8}\,C_1\overline{B_2} - \frac{1}{4}\,C_3\overline{C_1} + \frac{1}{16}\,\gamma_1\overline{C_1} - \frac{1}{16}\,\overline{C_2}^2 & \overline{A_0} & 0 & 2 & 0 & \textbf{(72)} \\
-\frac{3}{256}\,C_1\overline{C_1} & C_1 & 0 & 2 & 0 & \textbf{(73)} \\
\frac{1}{8}\,\overline{A_0 C_1}\,\overline{\alpha_1} - \frac{1}{256}\,C_1^2 - \frac{1}{8}\,\overline{A_2 C_1} - \frac{1}{16}\,\overline{A_1 C_2} & \overline{C_1} & 0 & 2 & 0 & \textbf{(74)} \\
-\frac{1}{24}\,\overline{A_1 C_1} & \overline{C_2} & 0 & 2 & 0 & \textbf{(75)} \\
\lambda_6 & A_0 & 0 & 2 & 0 & \textbf{(76)} \\
B_1|A_1|^2\overline{A_0} + 2\,C_1|A_1|^2\overline{A_1} + C_2\overline{A_0}\,\overline{\alpha_1} + C_1\overline{A_0}\,\overline{\alpha_4} - \frac{1}{2}\,B_3\overline{A_1} - C_2\overline{A_2} & A_1 & 0 & 2 & 0 & \textbf{(77)} \\
C_1\overline{A_0}\,\overline{\alpha_1} - \frac{1}{2}\,B_1\overline{A_1} - C_1\overline{A_2} & A_2 & 0 & 2 & 0 & \textbf{(78)} \\
-\frac{1}{64}\,C_1^2 & \overline{C_1} & 0 & 2 & 1 & \textbf{(79)} \\
2\,\overline{A_0 C_1 \zeta_0} - \frac{1}{4}\,B_1^2 - \frac{1}{2}\,B_2C_1 + 2\left(\overline{A_0}\,\overline{\alpha_1} - \overline{A_2}\right)\gamma_1 - \frac{3}{2}\,E_1\overline{C_1} - \frac{1}{2}\,\overline{A_1}\,\overline{\gamma_2} & A_0 & 0 & 2 & 1 & \textbf{(80)} \\
\lambda_7 & \overline{A_0} & 0 & 2 & 1 & \textbf{(81)} \\
8\,A_1C_1|A_1|^2 + 4\,A_0C_2|A_1|^2 + 4\,A_0B_1\alpha_1 + 4\,A_0C_1\alpha_4 - 4\,A_2B_1 - 2\,A_1B_3 & \overline{A_1} & 0 & 2 & 1 & \textbf{(82)} \\
4\,A_0C_1\alpha_1 - 4\,A_2C_1 - 2\,A_1C_2 & \overline{A_2} & 0 & 2 & 1 & \textbf{(83)} \\
-A_1\overline{C_1} & \overline{A_0} & 1 & -2 & 0 & \textbf{(84)} \\
\overline{A_1 C_1} & A_1 & 1 & -1 & 0 & \textbf{(85)} \\
-2\,|A_1|^2\overline{A_0 C_1} + \overline{A_1 B_1} & A_0 & 1 & -1 & 0 & \textbf{(86)} \\
-A_1\overline{C_1} & \overline{A_1} & 1 & -1 & 0 & \textbf{(87)} \\
2\,A_0|A_1|^2\overline{C_1} - A_1\overline{C_2} & \overline{A_0} & 1 & -1 & 0 & \textbf{(88)} \\
-\frac{1}{8}\,C_1^2 & A_1 & 1 & 0 & 0 & \textbf{(89)} \\
-\frac{1}{4}\,C_1C_2 & A_0 & 1 & 0 & 0 & \textbf{(90)}
\end{pmatrix}
$$

$-\frac{1}{8}\,A_1C_1$	C_1	1	0	0	**(91)**
λ_8	$\overline{A_0}$	1	0	0	**(92)**
$-A_1\overline{C_1}$	$\overline{A_2}$	1	0	0	**(93)**
$2\,A_0\lvert A_1\rvert^2\overline{C_1} - A_1\overline{C_2}$	$\overline{A_1}$	1	0	0	**(94)**
$-2\,A_1\gamma_1 - \frac{1}{2}\,\overline{B_1C_1}$	$\overline{A_0}$	1	0	1	**(95)**
$8\,\lvert A_1\rvert^2\overline{A_1C_1} + 4\,\lvert A_1\rvert^2\overline{A_0C_2} + 4\,\overline{A_0B_1}\overline{\alpha_1} + 4\,\overline{A_0C_1}\overline{\alpha_4} - 4\,\overline{A_2B_1} - 2\,\overline{A_1B_3}$	A_0	1	0	1	**(96)**
$4\,\overline{A_0C_1}\overline{\alpha_1} - 4\,\overline{A_2C_1} - 2\,\overline{A_1C_2}$	A_1	1	0	1	**(97)**
λ_9	A_1	1	1	0	**(98)**
λ_{10}	$\overline{A_1}$	1	1	0	**(99)**
λ_{11}	A_0	1	1	0	**(100)**
λ_{12}	$\overline{A_0}$	1	1	0	**(101)**
$-\frac{1}{8}\,A_1C_1$	B_1	1	1	0	**(102)**
$-\frac{1}{8}\,\overline{A_1C_1}$	$\overline{B_1}$	1	1	0	**(103)**
$\frac{1}{4}\,A_0C_1\lvert A_1\rvert^2 - \frac{1}{8}\,A_1B_1$	C_1	1	1	0	**(104)**
$\frac{1}{4}\,\lvert A_1\rvert^2\overline{A_0C_1} - \frac{1}{8}\,\overline{A_1B_1}$	$\overline{C_1}$	1	1	0	**(105)**
$-A_1\overline{C_1}$	$\overline{A_3}$	1	1	0	**(106)**
$-C_1\overline{A_1}$	A_3	1	1	0	**(107)**
$2\,A_0\lvert A_1\rvert^2\overline{C_1} - A_1\overline{C_2}$	$\overline{A_2}$	1	1	0	**(108)**
$2\,C_1\lvert A_1\rvert^2\overline{A_0} - C_2\overline{A_1}$	A_2	1	1	0	**(109)**
$4\,A_0\gamma_1\lvert A_1\rvert^2 - A_1\overline{\gamma_2}$	$\overline{A_0}$	1	1	1	**(110)**
$4\,\gamma_1\lvert A_1\rvert^2\overline{A_0} - \gamma_2\overline{A_1}$	A_0	1	1	1	**(111)**
$-2\,A_1\gamma_1$	$\overline{A_1}$	1	1	1	**(112)**
$-2\,\gamma_1\overline{A_1}$	A_1	1	1	1	**(113)**
$-\frac{3}{16}\,C_1\overline{C_1}$	A_0	2	−2	0	**(114)**
$A_0\alpha_1\overline{C_1} - \frac{1}{2}\,A_1\overline{B_1} - A_2\overline{C_1}$	$\overline{A_0}$	2	−2	0	**(115)**
$-2\,\lvert A_1\rvert^2\overline{A_0B_1} - \alpha_4\overline{A_0C_1} - \alpha_1\overline{A_1C_1} + \overline{A_1B_2} - \frac{1}{8}\,B_1\overline{C_1} - \frac{3}{16}\,C_1\overline{C_2}$	A_0	2	−1	0	**(116)**
$\overline{A_1C_1}$	A_2	2	−1	0	**(117)**
$-2\,\lvert A_1\rvert^2\overline{A_0C_1} + \overline{A_1B_1}$	A_1	2	−1	0	**(118)**
$A_0\lvert A_1\rvert^2\overline{B_1} + 2\,A_1\lvert A_1\rvert^2\overline{C_1} + A_0\alpha_4\overline{C_1} + A_0\alpha_1\overline{C_2} - \frac{1}{2}\,A_1\overline{B_3} - A_2\overline{C_2}$	$\overline{A_0}$	2	−1	0	**(119)**
$A_0\alpha_1\overline{C_1} - \frac{1}{2}\,A_1\overline{B_1} - A_2\overline{C_1}$	$\overline{A_1}$	2	−1	0	**(120)**

$$
\begin{pmatrix}
-\frac{1}{16}\,C_1^2 & A_2 & 2 & 0 & 0 & \textbf{(121)} \\
-\frac{1}{8}\,C_1C_2 & A_1 & 2 & 0 & 0 & \textbf{(122)} \\
-\frac{1}{16}\,C_2^2-\frac{1}{4}\,C_1C_3+\frac{1}{16}\,C_1\gamma_1-\frac{1}{8}\,B_2\overline{C_1}-\frac{1}{8}\,B_1\overline{C_2} & A_0 & 2 & 0 & 0 & \textbf{(123)} \\
-\frac{3}{256}\,C_1\overline{C_1} & \overline{C_1} & 2 & 0 & 0 & \textbf{(124)} \\
\frac{1}{8}\,A_0C_1\alpha_1-\frac{1}{8}\,A_2C_1-\frac{1}{16}\,A_1C_2-\frac{1}{256}\,\overline{C_1}^2 & C_1 & 2 & 0 & 0 & \textbf{(125)} \\
-\frac{1}{24}\,A_1C_1 & C_2 & 2 & 0 & 0 & \textbf{(126)} \\
\lambda_{13} & \overline{A_0} & 2 & 0 & 0 & \textbf{(127)} \\
A_0|A_1|^2\overline{B_1}+2\,A_1|A_1|^2\overline{C_1}+A_0\alpha_4\overline{C_1}+A_0\alpha_1\overline{C_2}-\frac{1}{2}\,A_1\overline{B_3}-A_2\overline{C_2} & \overline{A_1} & 2 & 0 & 0 & \textbf{(128)} \\
A_0\alpha_1\overline{C_1}-\frac{1}{2}\,A_1\overline{B_1}-A_2\overline{C_1} & \overline{A_2} & 2 & 0 & 0 & \textbf{(129)} \\
-\frac{1}{64}\,\overline{C_1}^2 & C_1 & 2 & 0 & 1 & \textbf{(130)} \\
2\,A_0C_1\zeta_0+2\,(A_0\alpha_1-A_2)\gamma_1-\frac{1}{2}\,A_1\gamma_2-\frac{1}{4}\,\overline{B_1}^2-\frac{1}{2}\,\overline{B_2C_1}-\frac{3}{2}\,C_1\overline{E_1} & \overline{A_0} & 2 & 0 & 1 & \textbf{(131)} \\
\lambda_{14} & A_0 & 2 & 0 & 1 & \textbf{(132)} \\
8\,|A_1|^2\overline{A_1C_1}+4\,|A_1|^2\overline{A_0C_2}+4\,\overline{A_0B_1}\,\overline{\alpha_1}+4\,\overline{A_0C_1}\,\overline{\alpha_4}-4\,\overline{A_2B_1}-2\,\overline{A_1B_3} & A_1 & 2 & 0 & 1 & \textbf{(133)} \\
4\,\overline{A_0C_1}\,\overline{\alpha_1}-4\,\overline{A_2C_1}-2\,\overline{A_1C_2} & A_2 & 2 & 0 & 1 & \textbf{(134)} \\
-\frac{7}{48}\,C_1\overline{C_1} & A_1 & 3 & -2 & 0 & \textbf{(135)} \\
C_1\overline{A_0}\,\overline{\alpha_2}-\frac{2}{15}\,C_1\overline{B_1}-\frac{7}{48}\,C_2\overline{C_1} & A_0 & 3 & -2 & 0 & \textbf{(136)} \\
-\frac{1}{48}\,A_1C_1 & \overline{C_1} & 3 & -2 & 0 & \textbf{(137)} \\
A_0\alpha_3\overline{C_1}+\frac{5}{3}\,A_0C_1\overline{\alpha_2}+\frac{1}{3}\left(2\,A_0\overline{B_1}+3\,A_1\overline{C_1}\right)\alpha_1-\frac{2}{3}\,A_2\overline{B_1}-\frac{1}{3}\,A_1\overline{B_2}-A_3\overline{C_1} & \overline{A_0} & 3 & -2 & 0 & \textbf{(138)} \\
-\frac{1}{24}\,A_1\overline{C_1} & C_1 & 3 & -2 & 0 & \textbf{(139)} \\
-2\,|A_1|^2\overline{A_0B_1}-\alpha_4\overline{A_0C_1}-\alpha_1\overline{A_1C_1}+\overline{A_1B_2}-\frac{1}{12}\,B_1\overline{C_1}-\frac{7}{48}\,C_1\overline{C_2} & A_1 & 3 & -1 & 0 & \textbf{(140)} \\
\lambda_{15} & A_0 & 3 & -1 & 0 & \textbf{(141)} \\
\overline{A_1C_1} & A_3 & 3 & -1 & 0 & \textbf{(142)} \\
-\frac{1}{48}\,A_1C_1 & \overline{C_2} & 3 & -1 & 0 & \textbf{(143)} \\
-2\,|A_1|^2\overline{A_0C_1}+\overline{A_1B_1} & A_2 & 3 & -1 & 0 & \textbf{(144)} \\
\frac{1}{24}\,A_0C_1|A_1|^2-\frac{1}{48}\,A_1B_1 & \overline{C_1} & 3 & -1 & 0 & \textbf{(145)} \\
\lambda_{16} & \overline{A_0} & 3 & -1 & 0 & \textbf{(146)} \\
A_0\alpha_3\overline{C_1}+\frac{5}{3}\,A_0C_1\overline{\alpha_2}+\frac{1}{3}\left(2\,A_0\overline{B_1}+3\,A_1\overline{C_1}\right)\alpha_1-\frac{2}{3}\,A_2\overline{B_1}-\frac{1}{3}\,A_1\overline{B_2}-A_3\overline{C_1} & \overline{A_1} & 3 & -1 & 0 & \textbf{(147)} \\
-\frac{1}{24}\,A_1\overline{C_1} & B_1 & 3 & -1 & 0 & \textbf{(148)} \\
\frac{1}{12}\,A_0|A_1|^2\overline{C_1}-\frac{1}{24}\,A_1\overline{C_2} & C_1 & 3 & -1 & 0 & \textbf{(149)} \\
-\frac{1}{16}\,\overline{C_1}^2 & A_0 & 4 & -4 & 0 & \textbf{(150)}
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{1}{8}\,\overline{C_1C_2}+\frac{1}{3}\,\overline{A_1E_1} & A_0 & 4 & -3 & 0 & \mathbf{(151)}\\
\frac{1}{32}\,A_0C_1\alpha_1-\frac{1}{32}\,A_2C_1-\frac{1}{64}\,A_1C_2-\frac{1}{128}\,\overline{C_1}^2 & \overline{C_1} & 4 & -2 & 0 & \mathbf{(152)}\\
\lambda_{17} & \overline{A_0} & 4 & -2 & 0 & \mathbf{(153)}\\
-\frac{1}{8}\,C_1\overline{C_1} & A_2 & 4 & -2 & 0 & \mathbf{(154)}\\
C_1\overline{A_0}\,\overline{\alpha_2}-\frac{9}{80}\,C_1\overline{B_1}-\frac{1}{8}\,C_2\overline{C_1} & A_1 & 4 & -2 & 0 & \mathbf{(155)}\\
\lambda_{18} & A_0 & 4 & -2 & 0 & \mathbf{(156)}\\
-\frac{1}{80}\,A_1C_1 & \overline{B_1} & 4 & -2 & 0 & \mathbf{(157)}\\
-\frac{1}{48}\,A_1\overline{C_1} & C_2 & 4 & -2 & 0 & \mathbf{(158)}\\
\frac{1}{16}\,A_0\alpha_1\overline{C_1}-\frac{1}{32}\,A_1\overline{B_1}-\frac{1}{16}\,A_2\overline{C_1} & C_1 & 4 & -2 & 0 & \mathbf{(159)}\\
-\frac{1}{4}\,\gamma_1\overline{C_1} & A_0 & 4 & -2 & 1 & \mathbf{(160)}\\
A_0\zeta_0\overline{C_1}-\frac{1}{2}\,\overline{C_1E_1} & \overline{A_0} & 4 & -2 & 1 & \mathbf{(161)}\\
-\frac{9}{160}\,\overline{C_1}^2 & A_1 & 5 & -4 & 0 & \mathbf{(162)}\\
\frac{1}{2}\,\overline{A_0C_1\alpha_2}-\frac{17}{160}\,\overline{B_1C_1} & A_0 & 5 & -4 & 0 & \mathbf{(163)}\\
-\frac{1}{80}\,A_1\overline{C_1} & \overline{C_1} & 5 & -4 & 0 & \mathbf{(164)}\\
A_0\overline{C_1}\,\overline{\alpha_2}-\frac{1}{5}\,A_1\overline{E_1} & \overline{A_0} & 5 & -4 & 0 & \mathbf{(165)}\\
-\frac{9}{80}\,\overline{C_1C_2}+\frac{1}{3}\,\overline{A_1E_1} & A_1 & 5 & -3 & 0 & \mathbf{(166)}\\
\frac{1}{120}\,|A_1|^2\overline{C_1}^2-\frac{2}{3}\,|A_1|^2\overline{A_0E_1}+\frac{1}{3}\,\alpha_6\overline{A_0C_1}-\frac{13}{120}\,\overline{B_3C_1}-\frac{5}{48}\,\overline{B_1C_2}+\frac{1}{3}\,\overline{A_1E_2}+\frac{1}{3}\left(\overline{A_1C_1}+2\,\overline{A_0C_2}\right)\overline{\alpha_2} & A_0 & 5 & -3 & 0 & \mathbf{(167)}\\
-\frac{1}{80}\,A_1\overline{C_1} & \overline{C_2} & 5 & -3 & 0 & \mathbf{(168)}\\
\frac{1}{40}\,A_0|A_1|^2\overline{C_1}-\frac{1}{80}\,A_1\overline{C_2} & \overline{C_1} & 5 & -3 & 0 & \mathbf{(169)}\\
\frac{2}{5}\,A_0|A_1|^2\overline{E_1}+A_0\alpha_6\overline{C_1}+A_0\overline{C_2}\,\overline{\alpha_2}-\frac{1}{5}\,A_1\overline{E_3} & \overline{A_0} & 5 & -3 & 0 & \mathbf{(170)}\\
A_0\overline{C_1}\,\overline{\alpha_2}-\frac{1}{5}\,A_1\overline{E_1} & \overline{A_1} & 5 & -3 & 0 & \mathbf{(171)}\\
\frac{1}{2}\,\alpha_7\overline{A_0C_1}+\frac{1}{96}\,\alpha_1\overline{C_1}^2+\frac{1}{2}\,\overline{A_0B_1}\,\overline{\alpha_2}-\frac{11}{240}\,\overline{B_1}^2-\frac{3}{32}\,\overline{B_2C_1}-\frac{7}{96}\,C_1\overline{E_1} & A_0 & 6 & -4 & 0 & \mathbf{(172)}\\
-\frac{5}{96}\,\overline{C_1}^2 & A_2 & 6 & -4 & 0 & \mathbf{(173)}\\
\frac{1}{2}\,\overline{A_0C_1}\,\overline{\alpha_2}-\frac{47}{480}\,\overline{B_1C_1} & A_1 & 6 & -4 & 0 & \mathbf{(174)}\\
-\frac{1}{120}\,A_1\overline{C_1} & \overline{B_1} & 6 & -4 & 0 & \mathbf{(175)}\\
A_0\alpha_7\overline{C_1}+\frac{1}{3}\,A_0\alpha_1\overline{E_1}-\frac{1}{3}\,A_2\overline{E_1}-\frac{1}{6}\,A_1\overline{E_2}+\frac{1}{6}\left(5\,A_0\overline{B_1}+6\,A_1\overline{C_1}\right)\overline{\alpha_2} & \overline{A_0} & 6 & -4 & 0 & \mathbf{(176)}\\
\frac{1}{48}\,A_0\alpha_1\overline{C_1}-\frac{1}{96}\,A_1\overline{B_1}-\frac{1}{48}\,A_2\overline{C_1} & \overline{C_1} & 6 & -4 & 0 & \mathbf{(177)}\\
-\frac{7}{96}\,\overline{C_1E_1} & A_0 & 8 & -6 & 0 & \mathbf{(178)}
\end{pmatrix}
\tag{6.1.43}
$$

where

$$
\begin{aligned}
\lambda_1 &= -C_1\overline{A_0}\,\overline{\alpha_1}^2+\frac{3}{2}\,\overline{A_0C_1}\,\overline{\alpha_7}+\frac{1}{8}\,C_1E_1+\frac{1}{4}\left(6\,\overline{A_1C_1}+5\,\overline{A_0C_2}\right)\alpha_2-\frac{1}{4}\,B_4\overline{A_1}-\frac{1}{2}\,B_2\overline{A_2}-\frac{3}{4}\,B_1\overline{A_3}-C_1\overline{A_4}\\
&\quad+\frac{1}{4}\left(2\,B_2\overline{A_0}+3\,B_1\overline{A_1}+4\,C_1\overline{A_2}\right)\overline{\alpha_1}+\frac{1}{4}\left(3\,B_1\overline{A_0}+4\,C_1\overline{A_1}\right)\overline{\alpha_3}\\
\lambda_2 &= -A_0E_1\alpha_1+A_0\alpha_2\overline{C_2}+\frac{1}{48}\,C_1\overline{C_1}\,\overline{\alpha_1}+A_0\overline{C_1}\,\overline{\alpha_7}-\frac{1}{4}\,A_0C_1\overline{\zeta_0}-\frac{1}{16}\,C_2^2-\frac{1}{4}\,C_1C_3+A_2E_1+\frac{1}{2}\,A_1E_3+\frac{1}{64}\,C_1\gamma_1\\
&\quad-\frac{5}{48}\,B_2\overline{C_1}-\frac{9}{80}\,B_1\overline{C_2}\\
\lambda_3 &= -2\,A_0B_2|A_1|^2-A_1C_1\overline{\alpha_3}-A_0B_1\overline{\alpha_4}-A_0C_1\overline{\alpha_5}+A_0\overline{C_1}\,\overline{\alpha_6}+A_1B_4+\left(2\,A_0\overline{B_1}+A_1\overline{C_1}\right)\alpha_2-\frac{7}{120}\,B_1\overline{B_1}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{8}\,C_1\overline{B_3}-\frac{1}{120}\left(7\,C_1|A_1|^2+16\,B_3\right)\overline{C_1}-\frac{7}{48}\,C_2\overline{C_2}+\left(4\,A_0C_1|A_1|^2-A_1B_1\right)\overline{\alpha_1}\\
\lambda_4&=\frac{2}{3}\,B_2|A_1|^2\overline{A_0}+\frac{4}{3}\,B_1|A_1|^2\overline{A_1}+2\,C_1|A_1|^2\overline{A_2}+\frac{5}{3}\,\alpha_2\overline{A_0B_1}+C_2\overline{A_0}\overline{\alpha_3}+C_1\overline{A_0}\overline{\alpha_5}+\frac{5}{3}\,\overline{A_0C_1}\overline{\alpha_6}-\frac{1}{3}\,B_5\overline{A_1}\\
&-\frac{2}{3}\,B_3\overline{A_2}-C_2\overline{A_3}-\frac{1}{3}\left(12\,C_1|A_1|^2\overline{A_0}-2\,B_3\overline{A_0}-3\,C_2\overline{A_1}\right)\overline{\alpha_1}+\frac{1}{3}\left(2\,B_1\overline{A_0}+3\,C_1\overline{A_1}\right)\overline{\alpha_4}\\
\lambda_5&=2\,C_2|A_1|^2\overline{A_0}+C_1\alpha_4\overline{A_0}+C_1\alpha_1\overline{A_1}+3\,\overline{A_0C_1}\overline{\alpha_3}-\frac{1}{8}\,B_1C_1-2\,C_3\overline{A_1}+\gamma_1\overline{A_1}-3\,\overline{A_3C_1}-2\,\overline{A_2C_2}\\
&+\left(3\,\overline{A_1C_1}+2\,\overline{A_0C_2}\right)\overline{\alpha_1}\\
\lambda_6&=B_3|A_1|^2\overline{A_0}+2\,C_2|A_1|^2\overline{A_1}-2\,\overline{A_0C_1}\overline{\alpha_1}^2+C_1\beta\overline{A_0}+C_2\overline{A_0}\overline{\alpha_4}-\frac{1}{4}\,\overline{A_0C_1\zeta_0}-\frac{1}{16}\,B_1^2-\frac{1}{8}\left(32\,|A_1|^4\overline{A_0}+B_2\right)C_1\\
&+\frac{1}{2}\left(B_1\overline{A_1}+2\,C_1\overline{A_2}\right)\alpha_1+\frac{1}{2}\left(B_1\overline{A_0}+2\,C_1\overline{A_1}\right)\alpha_4-\frac{1}{2}\left(\overline{A_0}\overline{\alpha_1}-\overline{A_2}\right)\gamma_1-2\,C_3\overline{A_2}+\frac{3}{16}\,E_1\overline{C_1}-2\,\overline{A_4C_1}-\frac{3}{2}\,\overline{A_3C_2}\\
&-\frac{1}{2}\,\overline{A_1C_4}-\frac{1}{2}\left(4\,C_1\alpha_1\overline{A_0}-4\,C_3\overline{A_0}-4\,\overline{A_2C_1}-3\,\overline{A_1C_2}\right)\overline{\alpha_1}+\frac{1}{2}\left(4\,\overline{A_1C_1}+3\,\overline{A_0C_2}\right)\overline{\alpha_3}+\frac{1}{8}\,\overline{A_1}\overline{\gamma_2}\\
\lambda_7&=8\,A_1B_1|A_1|^2+4\,A_0B_3|A_1|^2+4\,A_0B_2\alpha_1+4\,A_0B_1\alpha_4+4\,A_0C_1\beta+A_0\overline{C_1\zeta_0}-4\,A_2B_2-2\,A_1B_5\\
&-\frac{1}{4}\left(64\,A_0|A_1|^4+\overline{B_2}\right)C_1-\frac{1}{8}\,C_2\overline{B_1}-\frac{3}{8}\,\gamma_1\overline{C_1}-2\left(4\,A_0C_1\alpha_1-2\,A_2C_1-A_1C_2\right)\overline{\alpha_1}+2\left(2\,A_1C_1+A_0C_2\right)\overline{\alpha_4}\\
\lambda_8&=2\,A_0|A_1|^2\overline{C_2}+3\,A_0C_1\alpha_3+A_1\overline{C_1}\overline{\alpha_1}+A_0\overline{C_1}\overline{\alpha_4}-3\,A_3C_1-2\,A_2C_2-2\,A_1C_3+\left(3\,A_1C_1+2\,A_0C_2\right)\alpha_1+A_1\gamma_1\\
&-\frac{1}{8}\,\overline{B_1C_1}\\
\lambda_9&=2\,C_2|A_1|^2\overline{A_0}+C_1\alpha_4\overline{A_0}+C_1\alpha_1\overline{A_1}+3\,\overline{A_0C_1}\overline{\alpha_3}-\frac{3}{8}\,B_1C_1-2\,C_3\overline{A_1}+\gamma_1\overline{A_1}-3\,\overline{A_3C_1}-2\,\overline{A_2C_2}\\
&+\left(3\,\overline{A_1C_1}+2\,\overline{A_0C_2}\right)\overline{\alpha_1}\\
\lambda_{10}&=2\,A_0|A_1|^2\overline{C_2}+3\,A_0C_1\alpha_3+A_1\overline{C_1}\overline{\alpha_1}+A_0\overline{C_1}\overline{\alpha_4}-3\,A_3C_1-2\,A_2C_2-2\,A_1C_3+\left(3\,A_1C_1+2\,A_0C_2\right)\alpha_1+A_1\gamma_1\\
&-\frac{3}{8}\,\overline{B_1C_1}\\
\lambda_{11}&=\frac{1}{12}\,C_1^2|A_1|^2+4\,C_3|A_1|^2\overline{A_0}-2\,\gamma_1|A_1|^2\overline{A_0}+6\,|A_1|^2\overline{A_2C_1}+4\,|A_1|^2\overline{A_1C_2}+C_2\alpha_4\overline{A_0}+C_1\alpha_5\overline{A_0}+C_1\alpha_3\overline{A_1}\\
&+3\,\overline{A_0B_1}\overline{\alpha_3}+3\,\overline{A_0C_1}\overline{\alpha_5}-\frac{1}{3}\,B_3C_1-\frac{3}{8}\,B_1C_2-\left(4\,C_1|A_1|^2\overline{A_0}-C_2\overline{A_1}\right)\alpha_1-C_4\overline{A_1}+\frac{1}{2}\,\gamma_2\overline{A_1}-3\,\overline{A_3B_1}-2\,\overline{A_2B_3}\\
&-\left(12\,|A_1|^2\overline{A_0C_1}-3\,\overline{A_1B_1}-2\,\overline{A_0B_3}\right)\overline{\alpha_1}+\left(3\,\overline{A_1C_1}+2\,\overline{A_0C_2}\right)\overline{\alpha_4}\\
\lambda_{12}&=6\,A_2C_1|A_1|^2+4\,A_1C_2|A_1|^2+4\,A_0C_3|A_1|^2-2\,A_0\gamma_1|A_1|^2+\frac{1}{12}\,|A_1|^2\overline{C_1}^2+3\,A_0B_1\alpha_3+3\,A_0C_1\alpha_5+A_1\overline{C_1}\overline{\alpha_3}\\
&+A_0\overline{C_2}\overline{\alpha_4}+A_0\overline{C_1}\overline{\alpha_5}-3\,A_3B_1-2\,A_2B_3-\left(12\,A_0C_1|A_1|^2-3\,A_1B_1-2\,A_0B_3\right)\alpha_1+\left(3\,A_1C_1+2\,A_0C_2\right)\alpha_4\\
&-\frac{1}{3}\,\overline{B_3C_1}-\frac{3}{8}\,\overline{B_1C_2}-A_1\overline{C_4}-\left(4\,A_0|A_1|^2\overline{C_1}-A_1\overline{C_2}\right)\overline{\alpha_1}+\frac{1}{2}\,A_1\overline{\gamma_2}\\
\lambda_{13}&=-2\,A_0C_1\alpha_1^2+A_0|A_1|^2\overline{B_3}+2\,A_1|A_1|^2\overline{C_2}-\frac{1}{4}\,A_0C_1\zeta_0+A_0\beta\overline{C_1}+A_0\alpha_4\overline{C_2}-2\,A_4C_1-\frac{3}{2}\,A_3C_2-2\,A_2C_3\\
&-\frac{1}{2}\,A_1C_4+\frac{1}{2}\left(4\,A_2C_1+3\,A_1C_2+4\,A_0C_3\right)\alpha_1+\frac{1}{2}\left(4\,A_1C_1+3\,A_0C_2\right)\alpha_3-\frac{1}{2}\left(A_0\alpha_1-A_2\right)\gamma_1+\frac{1}{8}\,A_1\gamma_2-\frac{1}{16}\,\overline{B_1}^2\\
&-\frac{1}{8}\left(32\,A_0|A_1|^4+\overline{B_2}\right)\overline{C_1}+\frac{3}{16}\,C_1\overline{E_1}-\frac{1}{2}\left(4\,A_0\alpha_1\overline{C_1}-A_1\overline{B_1}-2\,A_2\overline{C_1}\right)\overline{\alpha_1}+\frac{1}{2}\left(A_0\overline{B_1}+2\,A_1\overline{C_1}\right)\overline{\alpha_4}\\
\lambda_{14}&=8\,|A_1|^2\overline{A_1B_1}+4\,|A_1|^2\overline{A_0B_3}+C_1\zeta_0\overline{A_0}+4\,\beta\overline{A_0C_1}+4\,\overline{A_0B_1}\overline{\alpha_4}+2\left(2\,\overline{A_2C_1}+\overline{A_1C_2}\right)\alpha_1+2\left(2\,\overline{A_1C_1}+\overline{A_0C_2}\right)\alpha_4\\
&-\frac{3}{8}\,C_1\gamma_1-4\,\overline{A_2B_2}-2\,\overline{A_1B_5}-\frac{1}{4}\left(64\,|A_1|^4\overline{A_0}+B_2\right)\overline{C_1}-\frac{1}{8}\,B_1\overline{C_2}-4\left(2\,\alpha_1\overline{A_0C_1}-\overline{A_0B_2}\right)\overline{\alpha_1}\\
\lambda_{15}&=-2\,|A_1|^2\overline{A_0B_2}+C_1\alpha_6\overline{A_0}-\alpha_4\overline{A_0B_1}-\alpha_5\overline{A_0C_1}-\alpha_3\overline{A_1C_1}+\left(4\,|A_1|^2\overline{A_0C_1}-\overline{A_1B_1}\right)\alpha_1-\frac{7}{120}\,B_1\overline{B_1}-\frac{2}{15}\,C_1\overline{B_3}\\
&+\overline{A_1B_4}-\frac{1}{120}\left(7\,C_1|A_1|^2+15\,B_3\right)\overline{C_1}-\frac{7}{48}\,C_2\overline{C_2}+\left(2\,B_1\overline{A_0}+C_1\overline{A_1}\right)\overline{\alpha_2}\\
\lambda_{16}&=\frac{4}{3}\,A_1|A_1|^2\overline{B_1}+\frac{2}{3}\,A_0|A_1|^2\overline{B_2}+2\,A_2|A_1|^2\overline{C_1}+\frac{5}{3}\,A_0C_1\alpha_6+A_0\alpha_5\overline{C_1}+A_0\alpha_3\overline{C_2}+\frac{5}{3}\,A_0B_1\overline{\alpha_2}\\
&-\frac{1}{3}\left(12\,A_0|A_1|^2\overline{C_1}-2\,A_0\overline{B_3}-3\,A_1\overline{C_2}\right)\alpha_1+\frac{1}{3}\left(2\,A_0\overline{B_1}+3\,A_1\overline{C_1}\right)\alpha_4-\frac{2}{3}\,A_2\overline{B_3}-\frac{1}{3}\,A_1\overline{B_5}-A_3\overline{C_2}\\
\lambda_{17}&=-A_0\alpha_1^2\overline{C_1}+\frac{3}{2}\,A_0C_1\alpha_7+\frac{1}{4}\left(3\,A_1\overline{B_1}+2\,A_0\overline{B_2}+4\,A_2\overline{C_1}\right)\alpha_1+\frac{1}{4}\left(3\,A_0\overline{B_1}+4\,A_1\overline{C_1}\right)\alpha_3-\frac{3}{4}\,A_3\overline{B_1}\\
&-\frac{1}{2}\,A_2\overline{B_2}-\frac{1}{4}\,A_1\overline{B_4}-A_4\overline{C_1}+\frac{1}{8}\,\overline{C_1E_1}+\frac{1}{4}\left(6\,A_1C_1+5\,A_0C_2\right)\overline{\alpha_2}
\end{aligned}
$$

$$\lambda_{18}=C_1\alpha_7\overline{A_0}+\frac{1}{48}\,C_1\alpha_1\overline{C_1}-\frac{1}{4}\,\zeta_0\overline{A_0C_1}-\overline{A_0E_1}\overline{\alpha_1}+C_2\overline{A_0}\overline{\alpha_2}-\frac{9}{80}\,C_2\overline{B_1}-\frac{5}{48}\,C_1\overline{B_2}-\frac{1}{4}\,C_3\overline{C_1}+\frac{1}{64}\,\gamma_1\overline{C_1}-\frac{1}{16}\,\overline{C_2}^2$$
$$+\overline{A_2E_1}+\frac{1}{2}\,\overline{A_1E_3}$$

We see that the new powers are as $\langle\vec{C}_1,\vec{E}_1\rangle=0$ (recall that $\vec{E}_1\in\mathrm{Span}(\overline{\vec{A}_0})$)

$$\begin{pmatrix}-4 & 6 & 0\\ -3 & 5 & 0\\ -2 & 4 & 0\\ -1 & 3 & 0\end{pmatrix}\begin{pmatrix}2 & 0 & 0\\ 2 & 0 & 1\\ 1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}$$

and their conjugates, so there exists $\vec{C}_5,\vec{C}_6,\vec{B}_6,\vec{B}_7,\vec{E}_4,\vec{E}_5,\vec{\gamma}_2\in\mathbb{C}^n$ and $\vec{\gamma}_4\in\mathbb{R}^n$ such that

$$\begin{aligned}\vec{H}&=\mathrm{Re}\left(\frac{\vec{C}_1}{z^2}+\frac{\vec{C}_2}{z}+\vec{C}_4z+\vec{C}_5z^2+\left(\vec{B}_1\overline{z}+\vec{B}_2\overline{z}^2+\vec{B}_4\overline{z}^3+\vec{B}_6\overline{z}^4\right)\frac{1}{z^2}+\left(\vec{B}_3\overline{z}+\vec{B}_5\overline{z}^2+\vec{B}_7\overline{z}^3\right)\frac{1}{z}\right.\\&\left.\quad+\left(\vec{E}_1\overline{z}^4+\vec{E}_2\overline{z}^5+\vec{E}_4\overline{z}^6\right)\frac{1}{z^4}+\left(\vec{E}_3\overline{z}^4+\vec{E}_5\overline{z}^5\right)\frac{1}{z^3}\right)\\&\quad+\vec{C}_3+\vec{C}_6|z|^2+\vec{\gamma}_1\log|z|+\mathrm{Re}\left(\vec{\gamma}_2\,z+\vec{\gamma}_3z^2\right)\log|z|+\vec{\gamma}_4|z|^2\log|z|+O(|z|^{3-\varepsilon}).\end{aligned}\tag{6.1.44}$$

or

$$\vec{H}=\begin{pmatrix}\frac{1}{2} & C_1 & -2 & 0 & 0\\ \frac{1}{2} & C_2 & -1 & 0 & 0\\ 1 & C_3 & 0 & 0 & 0\\ \frac{1}{2} & C_4 & 1 & 0 & 0\\ \frac{1}{2} & C_5 & 2 & 0 & 0\\ 1 & C_6 & 1 & 1 & 0\\ \frac{1}{2} & B_1 & -2 & 1 & 0\\ \frac{1}{2} & B_2 & -2 & 2 & 0\\ \frac{1}{2} & B_4 & -2 & 3 & 0\\ \frac{1}{2} & B_6 & -2 & 4 & 0\\ \frac{1}{2} & B_3 & -1 & 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2} & B_5 & -1 & 2 & 0\\ \frac{1}{2} & B_7 & -1 & 3 & 0\\ \frac{1}{2} & E_1 & -4 & 4 & 0\\ \frac{1}{2} & E_2 & -4 & 5 & 0\\ \frac{1}{2} & E_4 & -4 & 6 & 0\\ \frac{1}{2} & E_3 & -3 & 4 & 0\\ \frac{1}{2} & E_5 & -3 & 5 & 0\\ 1 & \gamma_1 & 0 & 0 & 1\\ \frac{1}{2} & \gamma_2 & 1 & 0 & 1\\ \frac{1}{2} & \gamma_3 & 2 & 0 & 1\\ 1 & \gamma_4 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \overline{C_1} & 0 & -2 & 0\\ \frac{1}{2} & \overline{C_2} & 0 & -1 & 0\\ \frac{1}{2} & \overline{C_4} & 0 & 1 & 0\\ \frac{1}{2} & \overline{C_5} & 0 & 2 & 0\\ \frac{1}{2} & \overline{B_1} & 1 & -2 & 0\\ \frac{1}{2} & \overline{B_2} & 2 & -2 & 0\\ \frac{1}{2} & \overline{B_4} & 3 & -2 & 0\\ \frac{1}{2} & \overline{B_6} & 4 & -2 & 0\\ \frac{1}{2} & \overline{B_3} & 1 & -1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \overline{B_5} & 2 & -1 & 0\\ \frac{1}{2} & \overline{B_7} & 3 & -1 & 0\\ \frac{1}{2} & \overline{E_1} & 4 & -4 & 0\\ \frac{1}{2} & \overline{E_2} & 5 & -4 & 0\\ \frac{1}{2} & \overline{E_4} & 6 & -4 & 0\\ \frac{1}{2} & \overline{E_3} & 4 & -3 & 0\\ \frac{1}{2} & \overline{E_5} & 5 & -3 & 0\\ \frac{1}{2} & \overline{\gamma_2} & 0 & 1 & 1\\ \frac{1}{2} & \overline{\gamma_3} & 0 & 2 & 1\end{pmatrix}$$

We can easily check that both developments coincide. Then, we have

$$
\partial_z\vec{\Phi} = \begin{pmatrix}
1 & A_0 & 3 & 0 & 0 \\
1 & A_1 & 4 & 0 & 0 \\
1 & A_2 & 5 & 0 & 0 \\
1 & A_3 & 6 & 0 & 0 \\
1 & A_4 & 7 & 0 & 0 \\
1 & A_5 & 8 & 0 & 0 \\
1 & A_6 & 9 & 0 & 0 \\
\frac{1}{16} & C_1 & 1 & 4 & 0 \\
\frac{1}{16} & C_2 & 2 & 4 & 0 \\
\frac{1}{8} & C_3 & 3 & 4 & 0 \\
\frac{1}{16} & C_4 & 4 & 4 & 0 \\
\frac{1}{16} & C_5 & 5 & 4 & 0 \\
\frac{1}{10} & C_6 & 4 & 5 & 0 \\
\frac{1}{20} & B_1 & 1 & 5 & 0 \\
\frac{1}{24} & B_2 & 1 & 6 & 0 \\
\frac{1}{28} & B_4 & 1 & 7 & 0 \\
\frac{1}{32} & B_6 & 1 & 8 & 0 \\
\frac{1}{20} & B_3 & 2 & 5 & 0 \\
\frac{1}{24} & B_5 & 2 & 6 & 0 \\
\frac{1}{28} & B_7 & 2 & 7 & 0 \\
\frac{1}{32} & E_1 & -1 & 8 & 0 \\
\frac{1}{36} & E_2 & -1 & 9 & 0 \\
\frac{1}{40} & E_4 & -1 & 10 & 0 \\
\frac{1}{32} & E_3 & 0 & 8 & 0 \\
\frac{1}{36} & E_5 & 0 & 9 & 0 \\
\frac{1}{8} & \gamma_1 & 3 & 4 & 1 \\
-\frac{1}{64} & \gamma_1 & 3 & 4 & 0 \\
\frac{1}{16} & \gamma_2 & 4 & 4 & 1
\end{pmatrix}
\begin{pmatrix}
-\frac{1}{128} & \gamma_2 & 4 & 4 & 0 \\
\frac{1}{16} & \gamma_3 & 5 & 4 & 1 \\
-\frac{1}{128} & \gamma_3 & 5 & 4 & 0 \\
\frac{1}{10} & \gamma_4 & 4 & 5 & 1 \\
-\frac{1}{100} & \gamma_4 & 4 & 5 & 0 \\
\frac{1}{8} & \overline{C_1} & 3 & 2 & 0 \\
\frac{1}{12} & \overline{C_2} & 3 & 3 & 0 \\
\frac{1}{20} & \overline{C_4} & 3 & 5 & 0 \\
\frac{1}{24} & \overline{C_5} & 3 & 6 & 0 \\
\frac{1}{8} & \overline{B_1} & 4 & 2 & 0 \\
\frac{1}{8} & \overline{B_2} & 5 & 2 & 0 \\
\frac{1}{8} & \overline{B_4} & 6 & 2 & 0 \\
\frac{1}{8} & \overline{B_6} & 7 & 2 & 0 \\
\frac{1}{12} & \overline{B_3} & 4 & 3 & 0 \\
\frac{1}{12} & \overline{B_5} & 5 & 3 & 0 \\
\frac{1}{12} & \overline{B_7} & 6 & 3 & 0 \\
\frac{1}{2} & \overline{E_1} & 7 & 0 & 1 \\
\frac{1}{2} & \overline{E_2} & 8 & 0 & 1 \\
\frac{1}{2} & \overline{E_4} & 9 & 0 & 1 \\
\frac{1}{4} & \overline{E_3} & 7 & 1 & 0 \\
\frac{1}{4} & \overline{E_5} & 8 & 1 & 0 \\
\frac{1}{20} & \overline{\gamma_2} & 3 & 5 & 1 \\
-\frac{1}{200} & \overline{\gamma_2} & 3 & 5 & 0 \\
\frac{1}{24} & \overline{\gamma_3} & 3 & 6 & 1 \\
-\frac{1}{288} & \overline{\gamma_3} & 3 & 6 & 0 \\
\frac{1}{10}\,|A_1|^2 & C_1 & 2 & 5 & 0 \\
\frac{1}{10}\,|A_1|^2 & C_2 & 3 & 5 & 0 \\
\frac{1}{5}\,|A_1|^2 & C_3 & 4 & 5 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{12}\,|A_1|^2 & B_1 & 2 & 6 & 0 \\
\frac{1}{14}\,|A_1|^2 & B_2 & 2 & 7 & 0 \\
\frac{1}{12}\,|A_1|^2 & B_3 & 3 & 6 & 0 \\
\frac{1}{18}\,|A_1|^2 & E_1 & 0 & 9 & 0 \\
\frac{1}{5}\,|A_1|^2 & \gamma_1 & 4 & 5 & 1 \\
-\frac{1}{50}\,|A_1|^2 & \gamma_1 & 4 & 5 & 0 \\
\frac{1}{6}\,|A_1|^2 & \overline{C_1} & 4 & 3 & 0 \\
\frac{1}{8}\,|A_1|^2 & \overline{C_2} & 4 & 4 & 0 \\
\frac{1}{6}\,|A_1|^2 & \overline{B_1} & 5 & 3 & 0 \\
\frac{1}{6}\,|A_1|^2 & \overline{B_2} & 6 & 3 & 0 \\
\frac{1}{8}\,|A_1|^2 & \overline{B_3} & 5 & 4 & 0 \\
\frac{1}{2}\,|A_1|^2 & \overline{E_1} & 8 & 1 & 0 \\
\frac{1}{24}\,\beta & C_1 & 3 & 6 & 0 \\
\frac{1}{16}\,\beta & \overline{C_1} & 5 & 4 & 0 \\
\frac{1}{16}\,\alpha_1 & C_1 & 3 & 4 & 0 \\
\frac{1}{16}\,\alpha_1 & C_2 & 4 & 4 & 0 \\
\frac{1}{8}\,\alpha_1 & C_3 & 5 & 4 & 0 \\
\frac{1}{20}\,\alpha_1 & B_1 & 3 & 5 & 0 \\
\frac{1}{24}\,\alpha_1 & B_2 & 3 & 6 & 0 \\
\frac{1}{20}\,\alpha_1 & B_3 & 4 & 5 & 0 \\
\frac{1}{32}\,\alpha_1 & E_1 & 1 & 8 & 0 \\
\frac{1}{8}\,\alpha_1 & \gamma_1 & 5 & 4 & 1 \\
-\frac{1}{64}\,\alpha_1 & \gamma_1 & 5 & 4 & 0 \\
\frac{1}{8}\,\alpha_1 & \overline{C_1} & 5 & 2 & 0 \\
\frac{1}{12}\,\alpha_1 & \overline{C_2} & 5 & 3 & 0 \\
\frac{1}{8}\,\alpha_1 & \overline{B_1} & 6 & 2 & 0 \\
\frac{1}{8}\,\alpha_1 & \overline{B_2} & 7 & 2 & 0 \\
\frac{1}{12}\,\alpha_1 & \overline{B_3} & 6 & 3 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{1}{2}\,\alpha_1 & \overline{E_1} & 9 & 0 & 1\\
\frac{1}{36}\,\alpha_2 & C_1 & -1 & 9 & 0\\
\frac{1}{36}\,\alpha_2 & C_2 & 0 & 9 & 0\\
\frac{1}{40}\,\alpha_2 & B_1 & -1 & 10 & 0\\
\frac{1}{28}\,\alpha_2 & \overline{C_1} & 1 & 7 & 0\\
\frac{1}{32}\,\alpha_2 & \overline{C_2} & 1 & 8 & 0\\
\frac{1}{28}\,\alpha_2 & \overline{B_1} & 2 & 7 & 0\\
\frac{1}{16}\,\alpha_3 & C_1 & 4 & 4 & 0\\
\frac{1}{16}\,\alpha_3 & C_2 & 5 & 4 & 0\\
\frac{1}{20}\,\alpha_3 & B_1 & 4 & 5 & 0\\
\frac{1}{8}\,\alpha_3 & \overline{C_1} & 6 & 2 & 0\\
\frac{1}{12}\,\alpha_3 & \overline{C_2} & 6 & 3 & 0\\
\frac{1}{8}\,\alpha_3 & \overline{B_1} & 7 & 2 & 0\\
\frac{1}{20}\,\alpha_4 & C_1 & 3 & 5 & 0\\
\frac{1}{20}\,\alpha_4 & C_2 & 4 & 5 & 0\\
\frac{1}{24}\,\alpha_4 & B_1 & 3 & 6 & 0\\
\frac{1}{12}\,\alpha_4 & \overline{C_1} & 5 & 3 & 0\\
\frac{1}{16}\,\alpha_4 & \overline{C_2} & 5 & 4 & 0\\
\frac{1}{12}\,\alpha_4 & \overline{B_1} & 6 & 3 & 0\\
\frac{1}{20}\,\alpha_5 & C_1 & 4 & 5 & 0\\
\frac{1}{12}\,\alpha_5 & \overline{C_1} & 6 & 3 & 0\\
\frac{1}{12}\,\alpha_6 & C_1 & 6 & 3 & 0\\
\frac{1}{4}\,\alpha_6 & \overline{C_1} & 8 & 1 & 0\\
\frac{1}{8}\,\alpha_7 & C_1 & 7 & 2 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2}\,\alpha_7 & \overline{C_1} & 9 & 0 & 1\\
\frac{1}{16}\,\zeta_0 & C_1 & 5 & 4 & 1\\
-\frac{1}{128}\,\zeta_0 & C_1 & 5 & 4 & 0\\
\frac{1}{8}\,\zeta_0 & \overline{C_1} & 7 & 2 & 1\\
-\frac{1}{32}\,\zeta_0 & \overline{C_1} & 7 & 2 & 0\\
\frac{1}{24}\,\overline{\alpha_1} & C_1 & 1 & 6 & 0\\
\frac{1}{24}\,\overline{\alpha_1} & C_2 & 2 & 6 & 0\\
\frac{1}{12}\,\overline{\alpha_1} & C_3 & 3 & 6 & 0\\
\frac{1}{28}\,\overline{\alpha_1} & B_1 & 1 & 7 & 0\\
\frac{1}{32}\,\overline{\alpha_1} & B_2 & 1 & 8 & 0\\
\frac{1}{28}\,\overline{\alpha_1} & B_3 & 2 & 7 & 0\\
\frac{1}{40}\,\overline{\alpha_1} & E_1 & -1 & 10 & 0\\
\frac{1}{12}\,\overline{\alpha_1} & \gamma_1 & 3 & 6 & 1\\
-\frac{1}{144}\,\overline{\alpha_1} & \gamma_1 & 3 & 6 & 0\\
\frac{1}{16}\,\overline{\alpha_1} & \overline{C_1} & 3 & 4 & 0\\
\frac{1}{20}\,\overline{\alpha_1} & \overline{C_2} & 3 & 5 & 0\\
\frac{1}{16}\,\overline{\alpha_1} & \overline{B_1} & 4 & 4 & 0\\
\frac{1}{16}\,\overline{\alpha_1} & \overline{B_2} & 5 & 4 & 0\\
\frac{1}{20}\,\overline{\alpha_1} & \overline{B_3} & 4 & 5 & 0\\
\frac{1}{8}\,\overline{\alpha_1} & \overline{E_1} & 7 & 2 & 0\\
\frac{1}{8}\,\overline{\alpha_2} & C_1 & 6 & 2 & 0\\
\frac{1}{8}\,\overline{\alpha_2} & C_2 & 7 & 2 & 0\\
\frac{1}{12}\,\overline{\alpha_2} & B_1 & 6 & 3 & 0\\
\frac{1}{2}\,\overline{\alpha_2} & \overline{C_1} & 8 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{4}\,\overline{\alpha_2} & \overline{C_2} & 8 & 1 & 0\\
\frac{1}{2}\,\overline{\alpha_2} & \overline{B_1} & 9 & 0 & 1\\
\frac{1}{28}\,\overline{\alpha_3} & C_1 & 1 & 7 & 0\\
\frac{1}{28}\,\overline{\alpha_3} & C_2 & 2 & 7 & 0\\
\frac{1}{32}\,\overline{\alpha_3} & B_1 & 1 & 8 & 0\\
\frac{1}{20}\,\overline{\alpha_3} & \overline{C_1} & 3 & 5 & 0\\
\frac{1}{24}\,\overline{\alpha_3} & \overline{C_2} & 3 & 6 & 0\\
\frac{1}{20}\,\overline{\alpha_3} & \overline{B_1} & 4 & 5 & 0\\
\frac{1}{24}\,\overline{\alpha_4} & C_1 & 2 & 6 & 0\\
\frac{1}{24}\,\overline{\alpha_4} & C_2 & 3 & 6 & 0\\
\frac{1}{28}\,\overline{\alpha_4} & B_1 & 2 & 7 & 0\\
\frac{1}{16}\,\overline{\alpha_4} & \overline{C_1} & 4 & 4 & 0\\
\frac{1}{20}\,\overline{\alpha_4} & \overline{C_2} & 4 & 5 & 0\\
\frac{1}{16}\,\overline{\alpha_4} & \overline{B_1} & 5 & 4 & 0\\
\frac{1}{28}\,\overline{\alpha_5} & C_1 & 2 & 7 & 0\\
\frac{1}{20}\,\overline{\alpha_5} & \overline{C_1} & 4 & 5 & 0\\
\frac{1}{36}\,\overline{\alpha_6} & C_1 & 0 & 9 & 0\\
\frac{1}{28}\,\overline{\alpha_6} & \overline{C_1} & 2 & 7 & 0\\
\frac{1}{40}\,\overline{\alpha_7} & C_1 & -1 & 10 & 0\\
\frac{1}{32}\,\overline{\alpha_7} & \overline{C_1} & 1 & 8 & 0\\
\frac{1}{32}\,\overline{\zeta_0} & C_1 & 1 & 8 & 1\\
-\frac{1}{512}\,\overline{\zeta_0} & C_1 & 1 & 8 & 0\\
\frac{1}{24}\,\overline{\zeta_0} & \overline{C_1} & 3 & 6 & 1\\
-\frac{1}{288}\,\overline{\zeta_0} & \overline{C_1} & 3 & 6 & 0
\end{pmatrix}
\tag{6.1.45}
$$

By conformality of $\vec{\Phi}$, we have

$$0 = \langle \partial_z\vec{\Phi}, \partial_z\vec{\Phi}\rangle =$$

$$
\begin{pmatrix}
A_0^2 & 6 & 0 & 0 \\
2\,A_0A_1 & 7 & 0 & 0 \\
A_1^2 + 2\,A_0A_2 & 8 & 0 & 0 \\
2\,A_1A_2 + 2\,A_0A_3 & 9 & 0 & 0 \\
A_2^2 + 2\,A_1A_3 + 2\,A_0A_4 & 10 & 0 & 0 \\
2\,A_2A_3 + 2\,A_1A_4 + 2\,A_0A_5 & 11 & 0 & 0 \\
A_3^2 + 2\,A_2A_4 + 2\,A_1A_5 + 2\,A_0A_6 & 12 & 0 & 0 \\
\frac{1}{8}\,A_0C_1 & 4 & 4 & 0 \\
\frac{1}{8}\,A_1C_1 + \frac{1}{8}\,A_0C_2 & 5 & 4 & 0 \\
\frac{1}{8}\,A_0C_1\alpha_1 + \frac{1}{8}\,A_0\overline{C_1}\,\overline{\alpha_1} + \frac{1}{8}\,A_2C_1 + \frac{1}{8}\,A_1C_2 + \frac{1}{4}\,A_0C_3 - \frac{1}{32}\,A_0\gamma_1 + \frac{1}{64}\,\overline{C_1}^2 & 6 & 4 & 0 \\
\mu_1 & 7 & 4 & 0 \\
\mu_2 & 8 & 4 & 0 \\
\mu_3 & 7 & 5 & 0 \\
\frac{1}{10}\,A_0B_1 & 4 & 5 & 0 \\
\frac{1}{12}\,A_0C_1\overline{\alpha_1} + \frac{1}{12}\,A_0B_2 + \frac{1}{64}\,C_1\overline{C_1} & 4 & 6 & 0 \\
\frac{1}{14}\,A_0\alpha_2\overline{C_1} + \frac{1}{14}\,A_0B_1\overline{\alpha_1} + \frac{1}{14}\,A_0C_1\overline{\alpha_3} + \frac{1}{14}\,A_0B_4 + \frac{1}{80}\,B_1\overline{C_1} + \frac{1}{96}\,C_1\overline{C_2} & 4 & 7 & 0 \\
\mu_4 & 4 & 8 & 0 \\
\frac{1}{5}\,A_0C_1|A_1|^2 + \frac{1}{10}\,A_1B_1 + \frac{1}{10}\,A_0B_3 & 5 & 5 & 0 \\
\frac{1}{6}\,A_0B_1|A_1|^2 + \frac{1}{12}\,A_0C_1\overline{\alpha_4} + \frac{1}{12}\,A_1B_2 + \frac{1}{12}\,A_0B_5 + \frac{1}{64}\,C_1\overline{B_1} + \frac{1}{64}\,C_2\overline{C_1} + \frac{1}{12}\,(A_1C_1 + A_0C_2)\overline{\alpha_1} & 5 & 6 & 0 \\
\mu_5 & 5 & 7 & 0 \\
\frac{1}{256}\,C_1^2 + \frac{1}{16}\,A_0E_1 & 2 & 8 & 0 \\
\frac{1}{18}\,A_0C_1\alpha_2 + \frac{1}{160}\,B_1C_1 + \frac{1}{18}\,A_0E_2 & 2 & 9 & 0 \\
\frac{1}{20}\,A_0B_1\alpha_2 + \frac{1}{20}\,A_0C_1\overline{\alpha_7} + \frac{1}{400}\,B_1^2 + \frac{1}{192}\,B_2C_1 + \frac{1}{20}\,A_0E_4 + \frac{1}{128}\,E_1\overline{C_1} + \frac{1}{960}\left(5\,C_1^2 + 48\,A_0E_1\right)\overline{\alpha_1} & 2 & 10 & 0 \\
\frac{1}{128}\,C_1C_2 + \frac{1}{16}\,A_1E_1 + \frac{1}{16}\,A_0E_3 & 3 & 8 & 0 \\
\mu_6 & 3 & 9 & 0 \\
\frac{1}{4}\,A_0\gamma_1 & 6 & 4 & 1 \\
\frac{1}{4}\,A_1\gamma_1 + \frac{1}{8}\,A_0\gamma_2 & 7 & 4 & 1 \\
\frac{1}{8}\,A_0C_1\zeta_0 + \frac{1}{4}\,(A_0\alpha_1 + A_2)\gamma_1 + \frac{1}{8}\,A_1\gamma_2 + \frac{1}{8}\,A_0\gamma_3 + \frac{1}{16}\,C_1\overline{E_1} & 8 & 4 & 1
\end{pmatrix}
\begin{pmatrix}
\mathbf{(1)} \\ \mathbf{(2)} \\ \mathbf{(3)} \\ \mathbf{(4)} \\ \mathbf{(5)} \\ \mathbf{(6)} \\ \mathbf{(7)} \\ \mathbf{(8)} \\ \mathbf{(9)} \\ \mathbf{(10)} \\ \mathbf{(11)} \\ \mathbf{(12)} \\ \mathbf{(13)} \\ \mathbf{(14)} \\ \mathbf{(15)} \\ \mathbf{(16)} \\ \mathbf{(17)} \\ \mathbf{(18)} \\ \mathbf{(19)} \\ \mathbf{(20)} \\ \mathbf{(21)} \\ \mathbf{(22)} \\ \mathbf{(23)} \\ \mathbf{(24)} \\ \mathbf{(25)} \\ \mathbf{(26)} \\ \mathbf{(27)} \\ \mathbf{(28)}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{2}{5}\,A_0\gamma_1|A_1|^2+\frac{1}{5}\,A_0\gamma_4+\frac{1}{10}\,A_1\overline{\gamma_2} & 7 & 5 & 1\\
\frac{1}{4}\,A_0\overline{C_1} & 6 & 2 & 0\\
\frac{1}{6}\,A_0\overline{C_2} & 6 & 3 & 0\\
\mu_7 & 6 & 5 & 0\\
\mu_8 & 6 & 6 & 0\\
\frac{1}{4}\,A_0\overline{B_1}+\frac{1}{4}\,A_1\overline{C_1} & 7 & 2 & 0\\
\frac{1}{4}\,A_0\alpha_1\overline{C_1}+\frac{1}{4}\,A_1\overline{B_1}+\frac{1}{4}\,A_0\overline{B_2}+\frac{1}{4}\,A_2\overline{C_1} & 8 & 2 & 0\\
\frac{1}{4}\,A_0\alpha_3\overline{C_1}+\frac{1}{4}\,A_0C_1\overline{\alpha_2}+\frac{1}{4}\left(A_0\overline{B_1}+A_1\overline{C_1}\right)\alpha_1+\frac{1}{4}\,A_2\overline{B_1}+\frac{1}{4}\,A_1\overline{B_2}+\frac{1}{4}\,A_0\overline{B_4}+\frac{1}{4}\,A_3\overline{C_1} & 9 & 2 & 0\\
\mu_9 & 10 & 2 & 0\\
\frac{1}{3}\,A_0|A_1|^2\overline{C_1}+\frac{1}{6}\,A_0\overline{B_3}+\frac{1}{6}\,A_1\overline{C_2} & 7 & 3 & 0\\
\frac{1}{3}\,A_0|A_1|^2\overline{B_1}+\frac{1}{3}\,A_1|A_1|^2\overline{C_1}+\frac{1}{6}\,A_0\alpha_4\overline{C_1}+\frac{1}{6}\,A_0\alpha_1\overline{C_2}+\frac{1}{6}\,A_1\overline{B_3}+\frac{1}{6}\,A_0\overline{B_5}+\frac{1}{6}\,A_2\overline{C_2} & 8 & 3 & 0\\
\mu_{10} & 9 & 3 & 0\\
A_0\overline{E_1} & 10 & 0 & 1\\
A_0\overline{C_1}\overline{\alpha_2}+A_1\overline{E_1}+A_0\overline{E_2} & 11 & 0 & 1\\
A_0\alpha_7\overline{C_1}+A_0\alpha_1\overline{E_1}+A_2\overline{E_1}+A_1\overline{E_2}+A_0\overline{E_4}+\left(A_0\overline{B_1}+A_1\overline{C_1}\right)\overline{\alpha_2} & 12 & 0 & 1\\
\frac{1}{2}\,A_0\overline{E_3} & 10 & 1 & 0\\
A_0|A_1|^2\overline{E_1}+\frac{1}{2}\,A_0\alpha_6\overline{C_1}+\frac{1}{2}\,A_0\overline{C_2}\overline{\alpha_2}+\frac{1}{2}\,A_1\overline{E_3}+\frac{1}{2}\,A_0\overline{E_5} & 11 & 1 & 0\\
\frac{1}{10}\,A_0\overline{\gamma_2} & 6 & 5 & 1\\
\frac{1}{12}\,A_0\overline{C_1\zeta_0}+\frac{1}{96}\left(16\,A_0\overline{\alpha_1}+3\,\overline{C_1}\right)\gamma_1+\frac{1}{12}\,A_0\overline{\gamma_3} & 6 & 6 & 1\\
\frac{1}{4}\,A_0\zeta_0\overline{C_1}+\frac{1}{8}\,\overline{C_1E_1} & 10 & 2 & 1\\
\frac{1}{16}\,A_0C_1\overline{\zeta_0}+\frac{1}{64}\,C_1\gamma_1 & 4 & 8 & 1\\
\frac{1}{256}\,C_1E_1 & 0 & 12 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{(29)}\\ \mathbf{(30)}\\ \mathbf{(31)}\\ \mathbf{(32)}\\ \mathbf{(33)}\\ \mathbf{(34)}\\ \mathbf{(35)}\\ \mathbf{(36)}\\ \mathbf{(37)}\\ \mathbf{(38)}\\ \mathbf{(39)}\\ \mathbf{(40)}\\ \mathbf{(41)}\\ \mathbf{(42)}\\ \mathbf{(43)}\\ \mathbf{(44)}\\ \mathbf{(45)}\\ \mathbf{(46)}\\ \mathbf{(47)}\\ \mathbf{(48)}\\ \mathbf{(49)}\\ \mathbf{(50)}
\end{pmatrix}
\tag{6.1.46}
$$

where

$$
\begin{aligned}
\mu_1 &= \frac{1}{4}\,A_0|A_1|^2\overline{C_2}+\frac{1}{8}\,A_0C_1\alpha_3+\frac{1}{8}\,A_0\overline{C_1}\overline{\alpha_4}+\frac{1}{8}\,A_3C_1+\frac{1}{8}\,A_2C_2+\frac{1}{4}\,A_1C_3+\frac{1}{8}\,A_0C_4+\frac{1}{8}\,(A_1C_1+A_0C_2)\alpha_1-\frac{1}{32}\,A_1\gamma_1\\
&\quad-\frac{1}{64}\,A_0\gamma_2+\frac{1}{32}\,\overline{B_1C_1}+\frac{1}{8}\left(A_0\overline{B_1}+A_1\overline{C_1}\right)\overline{\alpha_1}\\
\mu_2 &= \frac{1}{4}\,A_0|A_1|^2\overline{B_3}+\frac{1}{4}\,A_1|A_1|^2\overline{C_2}-\frac{1}{64}\,A_0C_1\zeta_0+\frac{1}{8}\,A_0\beta\overline{C_1}+\frac{1}{8}\,A_0\alpha_4\overline{C_2}+\frac{1}{8}\,A_4C_1+\frac{1}{8}\,A_3C_2+\frac{1}{4}\,A_2C_3+\frac{1}{8}\,A_1C_4\\
&\quad+\frac{1}{8}\,A_0C_5+\frac{1}{32}\left(4\,A_2C_1+4\,A_1C_2+8\,A_0C_3+\overline{C_1}^2\right)\alpha_1+\frac{1}{8}\,(A_1C_1+A_0C_2)\alpha_3-\frac{1}{32}\,(A_0\alpha_1+A_2)\gamma_1\\
&\quad-\frac{1}{64}\,A_1\gamma_2-\frac{1}{64}\,A_0\gamma_3+\frac{1}{64}\,\overline{B_1}^2+\frac{1}{32}\,\overline{B_2C_1}+\frac{1}{8}\left(A_1\overline{B_1}+A_0\overline{B_2}+A_2\overline{C_1}\right)\overline{\alpha_1}+\frac{1}{8}\left(A_0\overline{B_1}+A_1\overline{C_1}\right)\overline{\alpha_4}\\
\mu_3 &= \frac{1}{5}\,A_2C_1|A_1|^2+\frac{1}{5}\,A_1C_2|A_1|^2+\frac{2}{5}\,A_0C_3|A_1|^2-\frac{1}{25}\,A_0\gamma_1|A_1|^2+\frac{1}{24}\,|A_1|^2\overline{C_1}^2+\frac{1}{10}\,A_0B_1\alpha_3+\frac{1}{10}\,A_0C_1\alpha_5\\
&\quad+\frac{1}{10}\,A_0\overline{C_2}\overline{\alpha_4}+\frac{1}{10}\,A_0\overline{C_1}\overline{\alpha_5}+\frac{1}{10}\,A_3B_1+\frac{1}{10}\,A_2B_3+\frac{1}{5}\,A_0C_6+\frac{1}{10}\,(A_1B_1+A_0B_3)\alpha_1+\frac{1}{10}\,(A_1C_1+A_0C_2)\alpha_4
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{50}\, A_0\gamma_4 + \frac{1}{48}\, \overline{B_3 C_1} + \frac{1}{48}\, \overline{B_1 C_2} + \frac{1}{10}\, A_1\overline{C_4} + \frac{1}{10}\left(A_0\overline{B_3} + A_1\overline{C_2}\right)\overline{\alpha_1} + \frac{1}{10}\left(A_0\overline{B_1} + A_1\overline{C_1}\right)\overline{\alpha_3} - \frac{1}{100}\, A_1\overline{\gamma_2} \\
\mu_4 &= \frac{1}{16}\, A_0\alpha_2\overline{C_2} + \frac{1}{16}\, A_0 B_1\overline{\alpha_3} + \frac{1}{16}\, A_0\overline{C_1}\overline{\alpha_7} - \frac{1}{256}\, A_0 C_1\overline{\zeta_0} + \frac{1}{16}\, A_0 B_6 + \frac{1}{256}\, C_2^2 + \frac{1}{64}\, C_1 C_3 + \frac{1}{16}\, A_2 E_1 + \frac{1}{16}\, A_1 E_3 \\
&+ \frac{1}{128}\left(C_1^2 + 8\, A_0 E_1\right)\alpha_1 - \frac{1}{512}\, C_1\gamma_1 + \frac{1}{96}\, B_2\overline{C_1} + \frac{1}{120}\, B_1\overline{C_2} + \frac{1}{384}\left(24\, A_0 B_2 + 7\, C_1\overline{C_1}\right)\overline{\alpha_1} \\
\mu_5 &= \frac{1}{7}\, A_0 B_2 |A_1|^2 + \frac{1}{14}\, A_0 B_1\overline{\alpha_4} + \frac{1}{14}\, A_0 C_1\overline{\alpha_5} + \frac{1}{14}\, A_0\overline{C_1}\overline{\alpha_6} + \frac{1}{14}\, A_1 B_4 + \frac{1}{14}\, A_0 B_7 + \frac{1}{14}\left(A_0\overline{B_1} + A_1\overline{C_1}\right)\alpha_2 + \frac{1}{80}\, B_1\overline{B_1} \\
&+ \frac{1}{96}\, C_1\overline{B_3} + \frac{1}{240}\left(11\, C_1|A_1|^2 + 3\, B_3\right)\overline{C_1} + \frac{1}{96}\, C_2\overline{C_2} + \frac{1}{14}\left(A_1 B_1 + A_0 B_3\right)\overline{\alpha_1} + \frac{1}{14}\left(A_1 C_1 + A_0 C_2\right)\overline{\alpha_3} \\
\mu_6 &= \frac{1}{80}\, C_1^2|A_1|^2 + \frac{1}{9}\, A_0 E_1|A_1|^2 + \frac{1}{18}\, A_0 C_1\overline{\alpha_6} + \frac{1}{160}\, B_3 C_1 + \frac{1}{160}\, B_1 C_2 + \frac{1}{18}\, A_1 E_2 + \frac{1}{18}\, A_0 E_5 + \frac{1}{18}\left(A_1 C_1 + A_0 C_2\right)\alpha_2 \\
\mu_7 &= \frac{1}{5}\, A_1 C_1|A_1|^2 + \frac{1}{5}\, A_0 C_2|A_1|^2 + \frac{1}{10}\, A_0 B_1\alpha_1 + \frac{1}{10}\, A_0 C_1\alpha_4 + \frac{1}{10}\, A_0\overline{C_2}\overline{\alpha_1} + \frac{1}{10}\, A_0\overline{C_1}\overline{\alpha_3} + \frac{1}{10}\, A_2 B_1 + \frac{1}{10}\, A_1 B_3 \\
&+ \frac{1}{48}\, \overline{C_1 C_2} + \frac{1}{10}\, A_0\overline{C_4} - \frac{1}{100}\, A_0\overline{\gamma_2} \\
\mu_8 &= \frac{1}{6}\, A_1 B_1|A_1|^2 + \frac{1}{6}\, A_0 B_3|A_1|^2 + \frac{1}{12}\, A_0 B_1\alpha_4 + \frac{1}{12}\, A_0 C_1\beta + \frac{1}{12}\, A_0\overline{C_2}\overline{\alpha_3} - \frac{1}{144}\, A_0\overline{C_1\zeta_0} + \frac{1}{12}\, A_2 B_2 + \frac{1}{12}\, A_1 B_5 \\
&+ \frac{1}{96}\left(8\, A_0 B_2 + 3\, C_1\overline{C_1}\right)\alpha_1 - \frac{1}{2304}\left(32\, A_0\overline{\alpha_1} + 9\,\overline{C_1}\right)\gamma_1 + \frac{1}{64}\, C_2\overline{B_1} + \frac{1}{64}\, C_1\overline{B_2} + \frac{1}{32}\, C_3\overline{C_1} + \frac{1}{144}\, \overline{C_2}^2 + \frac{1}{12}\, A_0\overline{C_5} \\
&+ \frac{1}{192}\left(16\, A_2 C_1 + 16\, A_1 C_2 + 32\, A_0 C_3 + 3\,\overline{C_1}^2\right)\overline{\alpha_1} + \frac{1}{12}\left(A_1 C_1 + A_0 C_2\right)\overline{\alpha_4} - \frac{1}{144}\, A_0\overline{\gamma_3} \\
\mu_9 &= \frac{1}{4}\, A_0 C_1\alpha_7 - \frac{1}{16}\, A_0\zeta_0\overline{C_1} + \frac{1}{4}\, A_0\overline{E_1}\overline{\alpha_1} + \frac{1}{4}\left(A_1\overline{B_1} + A_0\overline{B_2} + A_2\overline{C_1}\right)\alpha_1 + \frac{1}{4}\left(A_0\overline{B_1} + A_1\overline{C_1}\right)\alpha_3 + \frac{1}{4}\, A_3\overline{B_1} + \frac{1}{4}\, A_2\overline{B_2} \\
&+ \frac{1}{4}\, A_1\overline{B_4} + \frac{1}{4}\, A_0\overline{B_6} + \frac{1}{4}\, A_4\overline{C_1} + \frac{1}{4}\left(A_1 C_1 + A_0 C_2\right)\overline{\alpha_2} \\
\mu_{10} &= \frac{1}{3}\, A_1|A_1|^2\overline{B_1} + \frac{1}{3}\, A_0|A_1|^2\overline{B_2} + \frac{1}{3}\, A_2|A_1|^2\overline{C_1} + \frac{1}{6}\, A_0 C_1\alpha_6 + \frac{1}{6}\, A_0\alpha_5\overline{C_1} + \frac{1}{6}\, A_0\alpha_3\overline{C_2} + \frac{1}{6}\, A_0 B_1\overline{\alpha_2} \\
&+ \frac{1}{6}\left(A_0\overline{B_3} + A_1\overline{C_2}\right)\alpha_1 + \frac{1}{6}\left(A_0\overline{B_1} + A_1\overline{C_1}\right)\alpha_4 + \frac{1}{6}\, A_2\overline{B_3} + \frac{1}{6}\, A_1\overline{B_5} + \frac{1}{6}\, A_0\overline{B_7} + \frac{1}{6}\, A_3\overline{C_2}
\end{aligned}
$$

Then we compute the metric

$e^{2\lambda} =$

$$
\begin{pmatrix}
2\,A_0\overline{A_0} & 3 & 3 & 0 \\
2\,A_0\overline{A_1} & 3 & 4 & 0 \\
2\,A_0\overline{A_2} + \frac{1}{4}\,\overline{A_0C_1} & 3 & 5 & 0 \\
2\,A_0\overline{A_3} + \frac{1}{4}\,\overline{A_1C_1} + \frac{1}{6}\,\overline{A_0C_2} & 3 & 6 & 0 \\
\frac{1}{8}\,C_1\alpha_1\overline{A_0} + \frac{1}{8}\,\overline{A_0C_1}\overline{\alpha_1} + \frac{1}{64}\,C_1^2 + \frac{1}{4}\,C_3\overline{A_0} - \frac{1}{32}\,\gamma_1\overline{A_0} + 2\,A_0\overline{A_4} + \frac{1}{4}\,\overline{A_2C_1} + \frac{1}{6}\,\overline{A_1C_2} & 3 & 7 & 0 \\
\nu_1 & 3 & 8 & 0 \\
\nu_2 & 3 & 9 & 0 \\
\frac{1}{8}\,A_0\overline{C_1} & 7 & 1 & 0 \\
\frac{1}{8}\,A_0\overline{C_2} & 7 & 2 & 0 \\
\frac{1}{8}\,A_0C_1\alpha_1 + \frac{1}{8}\,A_0\overline{C_1}\overline{\alpha_1} + \frac{1}{4}\,A_2C_1 + \frac{1}{6}\,A_1C_2 + \frac{1}{4}\,A_0C_3 - \frac{1}{32}\,A_0\gamma_1 + 2\,A_4\overline{A_0} + \frac{1}{64}\,\overline{C_1}^2 & 7 & 3 & 0 \\
\nu_3 & 7 & 4 & 0 \\
\nu_4 & 7 & 5 & 0 \\
\nu_5 & 8 & 4 & 0 \\
\frac{1}{10}\,A_0\overline{B_1} + \frac{1}{8}\,A_1\overline{C_1} & 8 & 1 & 0 \\
\frac{1}{12}\,A_0\alpha_1\overline{C_1} + \frac{1}{10}\,A_1\overline{B_1} + \frac{1}{12}\,A_0\overline{B_2} + \frac{1}{8}\,A_2\overline{C_1} & 9 & 1 & 0 \\
\frac{1}{14}\,A_0\alpha_3\overline{C_1} + \frac{1}{14}\,A_0C_1\overline{\alpha_2} + \frac{1}{84}\left(6\,A_0\overline{B_1} + 7\,A_1\overline{C_1}\right)\alpha_1 + \frac{1}{10}\,A_2\overline{B_1} + \frac{1}{12}\,A_1\overline{B_2} + \frac{1}{14}\,A_0\overline{B_4} + \frac{1}{8}\,A_3\overline{C_1} & 10 & 1 & 0 \\
\nu_6 & 11 & 1 & 0 \\
\frac{1}{5}\,A_0|A_1|^2\overline{C_1} + \frac{1}{10}\,A_0\overline{B_3} + \frac{1}{8}\,A_1\overline{C_2} & 8 & 2 & 0 \\
\frac{1}{6}\,A_0|A_1|^2\overline{B_1} + \frac{1}{5}\,A_1|A_1|^2\overline{C_1} + \frac{1}{12}\,A_0\alpha_4\overline{C_1} + \frac{1}{12}\,A_0\alpha_1\overline{C_2} + \frac{1}{10}\,A_1\overline{B_3} + \frac{1}{12}\,A_0\overline{B_5} + \frac{1}{8}\,A_2\overline{C_2} & 9 & 2 & 0 \\
\nu_7 & 10 & 2 & 0 \\
\frac{1}{16}\,A_0\overline{E_1} & 11 & -1 & 0 \\
\frac{1}{18}\,A_0\overline{C_1}\overline{\alpha_2} + \frac{1}{16}\,A_1\overline{E_1} + \frac{1}{18}\,A_0\overline{E_2} & 12 & -1 & 0 \\
\frac{1}{20}\,A_0\alpha_7\overline{C_1} + \frac{1}{20}\,A_0\alpha_1\overline{E_1} + \frac{1}{16}\,A_2\overline{E_1} + \frac{1}{18}\,A_1\overline{E_2} + \frac{1}{20}\,A_0\overline{E_4} + \frac{1}{180}\left(9\,A_0\overline{B_1} + 10\,A_1\overline{C_1}\right)\overline{\alpha_2} & 13 & -1 & 0 \\
\frac{1}{16}\,A_0\overline{E_3} & 11 & 0 & 0 \\
\frac{1}{9}\,A_0|A_1|^2\overline{E_1} + \frac{1}{18}\,A_0\alpha_6\overline{C_1} + \frac{1}{18}\,A_0\overline{C_2}\overline{\alpha_2} + \frac{1}{16}\,A_1\overline{E_3} + \frac{1}{18}\,A_0\overline{E_5} & 12 & 0 & 0 \\
\frac{1}{4}\,A_0\gamma_1 + \overline{A_0E_1} & 7 & 3 & 1 \\
\overline{A_1E_1} + \frac{1}{8}\,A_0\overline{\gamma_2} & 7 & 4 & 1 \\
\frac{1}{4}\,\zeta_0\overline{A_0C_1} + \frac{1}{8}\,A_0\overline{C_1}\overline{\zeta_0} + \frac{1}{64}\left(16\,A_0\overline{\alpha_1} + 3\,\overline{C_1}\right)\gamma_1 + \overline{A_2E_1} + \frac{1}{8}\,A_0\overline{\gamma_3} & 7 & 5 & 1
\end{pmatrix}
\begin{pmatrix}
\textbf{(1)} \\ \textbf{(2)} \\ \textbf{(3)} \\ \textbf{(4)} \\ \textbf{(5)} \\ \textbf{(6)} \\ \textbf{(7)} \\ \textbf{(8)} \\ \textbf{(9} \\ \textbf{(10)} \\ \textbf{(11)} \\ \textbf{(12)} \\ \textbf{(13)} \\ \textbf{(14)} \\ \textbf{(15)} \\ \textbf{(16)} \\ \textbf{(17)} \\ \textbf{(18)} \\ \textbf{(19)} \\ \textbf{(20)} \\ \textbf{(21)} \\ \textbf{(22)} \\ \textbf{(23)} \\ \textbf{(24)} \\ \textbf{(25)} \\ \textbf{(26)} \\ \textbf{(27)} \\ \textbf{(28)}
\end{pmatrix}
$$

$\frac{2}{5}\,A_0\gamma_1\lvert A_1\rvert^2 + \overline{A_1C_1}\overline{\alpha_2} + \frac{1}{5}\,A_0\gamma_4 + \overline{A_1E_2} + \frac{1}{8}\,A_1\overline{\gamma_2}$	8	4	1	**(29**
$\frac{1}{4}\,A_0C_1 + 2\,A_2\overline{A_0}$	5	3	0	**(30)**
$\frac{1}{4}\,A_1C_1 + \frac{1}{6}\,A_0C_2 + 2\,A_3\overline{A_0}$	6	3	0	**(31)**
ν_8	8	3	0	**(32)**
ν_9	9	3	0	**(33)**
$\frac{1}{4}\,A_0B_1 + 2\,A_2\overline{A_1}$	5	4	0	**(34)**
$\frac{1}{4}\,\alpha_1\overline{A_0C_1} + \frac{1}{4}\,A_0C_1\overline{\alpha_1} + \frac{1}{4}\,A_0B_2 + 2\,A_2\overline{A_2} + \frac{1}{4}\,\overline{A_0B_2} + \frac{5}{128}\,C_1\overline{C_1}$	5	5	0	**(35)**
ν_{10}	5	6	0	**(36)**
ν_{11}	5	7	0	**(37)**
$\frac{1}{3}\,A_0C_1\lvert A_1\rvert^2 + \frac{1}{4}\,A_1B_1 + \frac{1}{6}\,A_0B_3 + 2\,A_3\overline{A_1}$	6	4	0	**(38)**
ν_{12}	6	5	0	**(39)**
ν_{13}	6	6	0	**(40)**
$A_0E_1 + \frac{1}{4}\,\gamma_1\overline{A_0}$	3	7	1	**(41)**
$A_0C_1\alpha_2 + A_0E_2 + \frac{1}{4}\,\gamma_1\overline{A_1} + \frac{1}{10}\,\overline{A_0}\overline{\gamma_2}$	3	8	1	**(42)**
$A_0B_1\alpha_2 + A_0E_1\overline{\alpha_1} + A_0C_1\overline{\alpha_7} + \frac{1}{12}\,\overline{A_0C_1\zeta_0} + A_0E_4 + \frac{1}{12}\left(2\,\overline{A_0}\overline{\alpha_1} + 3\,\overline{A_2}\right)\gamma_1 + \frac{1}{8}\,E_1\overline{C_1} + \frac{1}{10}\,\overline{A_1}\overline{\gamma_2} + \frac{1}{12}\,\overline{A_0}\overline{\gamma_3}$	3	9	1	**(43)**
ν_{14}	4	7	0	**(44)**
ν_{15}	4	8	0	**(45)**
$\overline{A_0C_1}\overline{\alpha_2} + \frac{1}{4}\,A_1\gamma_1 + \frac{1}{10}\,A_0\gamma_2 + \overline{A_0E_2}$	8	3	1	**(46)**
$\frac{1}{12}\,A_0C_1\zeta_0 + \alpha_7\overline{A_0C_1} + \alpha_1\overline{A_0E_1} + \overline{A_0B_1}\overline{\alpha_2} + \frac{1}{12}\,(2\,A_0\alpha_1 + 3\,A_2)\gamma_1 + \frac{1}{10}\,A_1\gamma_2 + \frac{1}{12}\,A_0\gamma_3 + \frac{1}{8}\,C_1\overline{E_1} + \overline{A_0E_4}$	9	3	1	**(47)**
$\frac{1}{8}\,C_1\zeta_0\overline{A_0} + \frac{1}{4}\,A_0C_1\overline{\zeta_0} + A_2E_1 + \frac{1}{64}\left(16\,\alpha_1\overline{A_0} + 3\,C_1\right)\gamma_1 + \frac{1}{8}\,\gamma_3\overline{A_0}$	5	7	1	**(48)**
$\frac{1}{16}\,A_0\zeta_0\overline{C_1} + \frac{1}{16}\,\overline{C_1E_1}$	11	1	1	**(49)**
$2\,A_1\overline{A_0}$	4	3	0	**(50)**
$2\,A_1\overline{A_1}$	4	4	0	**(51)**
$2\,A_1\overline{A_2} + \frac{1}{4}\,\overline{A_0B_1}$	4	5	0	**(52)**
$\frac{1}{3}\,\lvert A_1\rvert^2\overline{A_0C_1} + 2\,A_1\overline{A_3} + \frac{1}{4}\,\overline{A_1B_1} + \frac{1}{6}\,\overline{A_0B_3}$	4	6	0	**(53)**
$A_1E_1 + \frac{1}{8}\,\gamma_2\overline{A_0}$	4	7	1	**(54)**
$\frac{2}{5}\,\gamma_1\lvert A_1\rvert^2\overline{A_0} + A_1C_1\alpha_2 + A_1E_2 + \frac{1}{5}\,\gamma_4\overline{A_0} + \frac{1}{8}\,\gamma_2\overline{A_1}$	4	8	1	**(55)**
$\frac{1}{8}\,C_1\overline{A_0}$	1	7	0	**(56)**
$\frac{1}{10}\,B_1\overline{A_0} + \frac{1}{8}\,C_1\overline{A_1}$	1	8	0	**(57)**
$\frac{1}{12}\,C_1\overline{A_0}\overline{\alpha_1} + \frac{1}{12}\,B_2\overline{A_0} + \frac{1}{10}\,B_1\overline{A_1} + \frac{1}{8}\,C_1\overline{A_2}$	1	9	0	**(58)**

$$
\begin{pmatrix}
\frac{1}{14}\,\alpha_2\overline{A_0C_1} + \frac{1}{14}\,C_1\overline{A_0}\,\overline{\alpha_3} + \frac{1}{14}\,B_4\overline{A_0} + \frac{1}{12}\,B_2\overline{A_1} + \frac{1}{10}\,B_1\overline{A_2} + \frac{1}{8}\,C_1\overline{A_3} + \frac{1}{84}\left(6\,B_1\overline{A_0} + 7\,C_1\overline{A_1}\right)\overline{\alpha_1} & 1 & 10 & 0 \\
\nu_{16} & 1 & 11 & 0 \\
\frac{1}{16}\,C_1\overline{A_0\zeta_0} + \frac{1}{16}\,C_1E_1 & 1 & 11 & 1 \\
\frac{1}{8}\,C_2\overline{A_0} & 2 & 7 & 0 \\
\frac{1}{5}\,C_1|A_1|^2\overline{A_0} + \frac{1}{10}\,B_3\overline{A_0} + \frac{1}{8}\,C_2\overline{A_1} & 2 & 8 & 0 \\
\frac{1}{6}\,B_1|A_1|^2\overline{A_0} + \frac{1}{5}\,C_1|A_1|^2\overline{A_1} + \frac{1}{12}\,C_2\overline{A_0}\,\overline{\alpha_1} + \frac{1}{12}\,C_1\overline{A_0}\,\overline{\alpha_4} + \frac{1}{12}\,B_5\overline{A_0} + \frac{1}{10}\,B_3\overline{A_1} + \frac{1}{8}\,C_2\overline{A_2} & 2 & 9 & 0 \\
\nu_{17} & 2 & 10 & 0 \\
\frac{1}{16}\,E_1\overline{A_0} & -1 & 11 & 0 \\
\frac{1}{18}\,C_1\alpha_2\overline{A_0} + \frac{1}{18}\,E_2\overline{A_0} + \frac{1}{16}\,E_1\overline{A_1} & -1 & 12 & 0 \\
\frac{1}{20}\,E_1\overline{A_0}\,\overline{\alpha_1} + \frac{1}{20}\,C_1\overline{A_0}\,\overline{\alpha_7} + \frac{1}{180}\left(9\,B_1\overline{A_0} + 10\,C_1\overline{A_1}\right)\alpha_2 + \frac{1}{20}\,E_4\overline{A_0} + \frac{1}{18}\,E_2\overline{A_1} + \frac{1}{16}\,E_1\overline{A_2} & -1 & 13 & 0 \\
\frac{1}{16}\,E_3\overline{A_0} & 0 & 11 & 0 \\
\frac{1}{9}\,E_1|A_1|^2\overline{A_0} + \frac{1}{18}\,C_2\alpha_2\overline{A_0} + \frac{1}{18}\,C_1\overline{A_0}\,\overline{\alpha_6} + \frac{1}{18}\,E_5\overline{A_0} + \frac{1}{16}\,E_3\overline{A_1} & 0 & 12 & 0
\end{pmatrix}
\begin{pmatrix}
\textbf{(59)} \\ \textbf{(60)} \\ \textbf{(61)} \\ \textbf{(62)} \\ \textbf{(63)} \\ \textbf{(64)} \\ \textbf{(65)} \\ \textbf{(66)} \\ \textbf{(67)} \\ \textbf{(68)} \\ \textbf{(69)} \\ \textbf{(70)}
\end{pmatrix}
$$

where

$$
\begin{aligned}
\nu_1 = {} & \frac{1}{5}\,C_2|A_1|^2\overline{A_0} + \frac{1}{10}\,C_1\alpha_4\overline{A_0} + \frac{1}{10}\,\overline{A_0C_1}\,\overline{\alpha_3} + \frac{9}{320}\,B_1C_1 + \frac{1}{40}\left(4\,B_1\overline{A_0} + 5\,C_1\overline{A_1}\right)\alpha_1 + \frac{1}{4}\,C_3\overline{A_1} - \frac{1}{32}\,\gamma_1\overline{A_1} + 2\,A_0\overline{A_5} \\
& + \frac{1}{4}\,\overline{A_3C_1} + \frac{1}{6}\,\overline{A_2C_2} + \frac{1}{10}\,\overline{A_0C_4} + \frac{1}{40}\left(5\,\overline{A_1C_1} + 4\,\overline{A_0C_2}\right)\overline{\alpha_1} - \frac{1}{100}\,\overline{A_0}\,\overline{\gamma_2} \\
\nu_2 = {} & \frac{1}{6}\,B_3|A_1|^2\overline{A_0} + \frac{1}{5}\,C_2|A_1|^2\overline{A_1} + \frac{1}{12}\,C_1\beta\overline{A_0} + \frac{1}{12}\,C_2\overline{A_0}\,\overline{\alpha_4} - \frac{1}{144}\,\overline{A_0C_1\zeta_0} + \frac{1}{80}\,B_1^2 + \frac{5}{192}\,B_2C_1 \\
& + \frac{1}{120}\left(10\,B_2\overline{A_0} + 12\,B_1\overline{A_1} + 15\,C_1\overline{A_2}\right)\alpha_1 + \frac{1}{60}\left(5\,B_1\overline{A_0} + 6\,C_1\overline{A_1}\right)\alpha_4 - \frac{1}{288}\left(4\,\overline{A_0}\,\overline{\alpha_1} + 9\,\overline{A_2}\right)\gamma_1 + \frac{1}{4}\,C_3\overline{A_2} \\
& + 2\,A_0\overline{A_6} + \frac{1}{256}\,E_1\overline{C_1} + \frac{1}{4}\,\overline{A_4C_1} + \frac{1}{6}\,\overline{A_3C_2} + \frac{1}{10}\,\overline{A_1C_4} + \frac{1}{12}\,\overline{A_0C_5} + \frac{1}{960}\left(25\,C_1^2 + 160\,C_3\overline{A_0} + 120\,\overline{A_2C_1} + 96\,\overline{A_1C_2}\right)\overline{\alpha_1} \\
& + \frac{1}{60}\left(6\,\overline{A_1C_1} + 5\,\overline{A_0C_2}\right)\overline{\alpha_3} - \frac{1}{100}\,\overline{A_1}\,\overline{\gamma_2} - \frac{1}{144}\,\overline{A_0}\,\overline{\gamma_3} \\
\nu_3 = {} & \frac{1}{3}\,A_1C_1|A_1|^2 + \frac{1}{4}\,A_0C_2|A_1|^2 + \frac{1}{8}\,A_0B_1\alpha_1 + \frac{1}{8}\,A_0C_1\alpha_4 + \frac{1}{8}\,A_0\overline{C_2}\,\overline{\alpha_1} + \frac{1}{8}\,A_0\overline{C_1}\,\overline{\alpha_3} + \frac{1}{4}\,A_2B_1 + \frac{1}{6}\,A_1B_3 + 2\,A_4\overline{A_1} \\
& + \frac{5}{192}\,\overline{C_1C_2} + \frac{1}{8}\,A_0\overline{C_4} + \frac{1}{2}\,\overline{A_0E_3} - \frac{1}{64}\,A_0\overline{\gamma_2} \\
\nu_4 = {} & \frac{1}{3}\,A_1B_1|A_1|^2 + \frac{1}{4}\,A_0B_3|A_1|^2 + \frac{1}{8}\,A_0B_1\alpha_4 + \frac{1}{8}\,A_0C_1\beta + \frac{1}{4}\,C_1\alpha_7\overline{A_0} + \frac{1}{4}\,\alpha_3\overline{A_0B_1} - \frac{1}{16}\,\zeta_0\overline{A_0C_1} + \frac{1}{4}\,C_2\overline{A_0}\,\overline{\alpha_2} \\
& + \frac{1}{8}\,A_0\overline{C_2}\,\overline{\alpha_3} - \frac{1}{64}\,A_0\overline{C_1\zeta_0} + \frac{1}{4}\,A_2B_2 + \frac{1}{6}\,A_1B_5 + \frac{1}{384}\left(48\,A_0B_2 + 96\,\overline{A_0B_2} + 23\,C_1\overline{C_1}\right)\alpha_1 - \frac{1}{512}\left(16\,A_0\overline{\alpha_1} + 3\,\overline{C_1}\right)\gamma_1 \\
& + 2\,A_4\overline{A_2} + \frac{13}{480}\,C_2\overline{B_1} + \frac{7}{192}\,C_1\overline{B_2} + \frac{1}{4}\,\overline{A_0B_6} + \frac{3}{64}\,C_3\overline{C_1} + \frac{1}{96}\,\overline{C_2}^2 + \frac{1}{8}\,A_0\overline{C_5} + \frac{1}{2}\,\overline{A_1E_3} \\
& + \frac{1}{384}\left(96\,A_2C_1 + 64\,A_1C_2 + 96\,A_0C_3 + 9\,\overline{C_1}^2 + 96\,\overline{A_0E_1}\right)\overline{\alpha_1} + \frac{1}{24}\,(4\,A_1C_1 + 3\,A_0C_2)\overline{\alpha_4} - \frac{1}{64}\,A_0\overline{\gamma_3} \\
\nu_5 = {} & \frac{1}{3}\,A_2C_1|A_1|^2 + \frac{1}{4}\,A_1C_2|A_1|^2 + \frac{2}{5}\,A_0C_3|A_1|^2 - \frac{1}{25}\,A_0\gamma_1|A_1|^2 + \frac{11}{240}\,|A_1|^2\overline{C_1}^2 + |A_1|^2\overline{A_0E_1} + \frac{1}{10}\,A_0B_1\alpha_3 \\
& + \frac{1}{10}\,A_0C_1\alpha_5 + \frac{1}{2}\,\alpha_6\overline{A_0C_1} + \frac{1}{2}\,\overline{A_0C_2}\,\overline{\alpha_2} + \frac{1}{10}\,A_0\overline{C_2}\,\overline{\alpha_4} + \frac{1}{10}\,A_0\overline{C_1}\,\overline{\alpha_5} + \frac{1}{4}\,A_3B_1 + \frac{1}{6}\,A_2B_3 + \frac{1}{5}\,A_0C_6 \\
& + \frac{1}{40}\,(5\,A_1B_1 + 4\,A_0B_3)\alpha_1 + \frac{1}{40}\,(5\,A_1C_1 + 4\,A_0C_2)\alpha_4 - \frac{1}{50}\,A_0\gamma_4 + 2\,A_5\overline{A_1} + \frac{11}{480}\,\overline{B_3C_1} + \frac{23}{960}\,\overline{B_1C_2} + \frac{1}{8}\,A_1\overline{C_4} \\
& + \frac{1}{2}\,\overline{A_0E_5} + \frac{1}{40}\left(4\,A_0\overline{B_3} + 5\,A_1\overline{C_2}\right)\overline{\alpha_1} + \frac{1}{40}\left(4\,A_0\overline{B_1} + 5\,A_1\overline{C_1}\right)\overline{\alpha_3} - \frac{1}{64}\,A_1\overline{\gamma_2} \\
\nu_6 = {} & \frac{1}{16}\,A_0C_1\alpha_7 - \frac{1}{256}\,A_0\zeta_0\overline{C_1} + \frac{1}{16}\,A_0\overline{E_1}\,\overline{\alpha_1} + \frac{1}{336}\left(24\,A_1\overline{B_1} + 21\,A_0\overline{B_2} + 28\,A_2\overline{C_1}\right)\alpha_1 + \frac{1}{112}\left(7\,A_0\overline{B_1} + 8\,A_1\overline{C_1}\right)\alpha_3 \\
& + \frac{1}{10}\,A_3\overline{B_1} + \frac{1}{12}\,A_2\overline{B_2} + \frac{1}{14}\,A_1\overline{B_4} + \frac{1}{16}\,A_0\overline{B_6} + \frac{1}{8}\,A_4\overline{C_1} + \frac{1}{128}\,\overline{C_1E_1} + \frac{1}{112}\,(8\,A_1C_1 + 7\,A_0C_2)\overline{\alpha_2} \\
\nu_7 = {} & \frac{1}{6}\,A_1|A_1|^2\overline{B_1} + \frac{1}{7}\,A_0|A_1|^2\overline{B_2} + \frac{1}{5}\,A_2|A_1|^2\overline{C_1} + \frac{1}{14}\,A_0C_1\alpha_6 + \frac{1}{14}\,A_0\alpha_5\overline{C_1} + \frac{1}{14}\,A_0\alpha_3\overline{C_2} + \frac{1}{14}\,A_0B_1\overline{\alpha_2} \\
& + \frac{1}{84}\left(6\,A_0\overline{B_3} + 7\,A_1\overline{C_2}\right)\alpha_1 + \frac{1}{84}\left(6\,A_0\overline{B_1} + 7\,A_1\overline{C_1}\right)\alpha_4 + \frac{1}{10}\,A_2\overline{B_3} + \frac{1}{12}\,A_1\overline{B_5} + \frac{1}{14}\,A_0\overline{B_7} + \frac{1}{8}\,A_3\overline{C_2}
\end{aligned}
$$

$$
\begin{aligned}
\nu_8 &= \frac{1}{5}A_0|A_1|^2\overline{C_2} + \frac{1}{10}A_0C_1\alpha_3 + \frac{1}{10}A_0\overline{C_1}\,\overline{\alpha_4} + \frac{1}{4}A_3C_1 + \frac{1}{6}A_2C_2 + \frac{1}{4}A_1C_3 + \frac{1}{10}A_0C_4 + \frac{1}{40}(5A_1C_1 + 4A_0C_2)\alpha_1 \\
&\quad - \frac{1}{32}A_1\gamma_1 - \frac{1}{100}A_0\gamma_2 + 2A_5\overline{A_0} + \frac{9}{320}\overline{B_1C_1} + \frac{1}{40}\left(4A_0\overline{B_1} + 5A_1\overline{C_1}\right)\overline{\alpha_1}
\end{aligned}
$$

$$
\begin{aligned}
\nu_9 &= \frac{1}{6}A_0|A_1|^2\overline{B_3} + \frac{1}{5}A_1|A_1|^2\overline{C_2} - \frac{1}{144}A_0C_1\zeta_0 + \frac{1}{12}A_0\beta\overline{C_1} + \frac{1}{12}A_0\alpha_4\overline{C_2} + \frac{1}{4}A_4C_1 + \frac{1}{6}A_3C_2 + \frac{1}{4}A_2C_3 + \frac{1}{10}A_1C_4 \\
&\quad + \frac{1}{12}A_0C_5 + \frac{1}{960}\left(120A_2C_1 + 96A_1C_2 + 160A_0C_3 + 25\overline{C_1}^2\right)\alpha_1 + \frac{1}{60}(6A_1C_1 + 5A_0C_2)\alpha_3 - \frac{1}{288}(4A_0\alpha_1 + 9A_2)\gamma_1 \\
&\quad - \frac{1}{100}A_1\gamma_2 - \frac{1}{144}A_0\gamma_3 + 2A_6\overline{A_0} + \frac{1}{80}\overline{B_1}^2 + \frac{5}{192}\overline{B_2C_1} + \frac{1}{256}C_1\overline{E_1} + \frac{1}{120}\left(12A_1\overline{B_1} + 10A_0\overline{B_2} + 15A_2\overline{C_1}\right)\overline{\alpha_1} \\
&\quad + \frac{1}{60}\left(5A_0\overline{B_1} + 6A_1\overline{C_1}\right)\overline{\alpha_4}
\end{aligned}
$$

$$
\begin{aligned}
\nu_{10} &= \frac{1}{3}|A_1|^2\overline{A_0B_1} + \frac{1}{4}A_0\alpha_2\overline{C_1} + \frac{1}{6}\alpha_4\overline{A_0C_1} + \frac{1}{4}A_0B_1\overline{\alpha_1} + \frac{1}{4}A_0C_1\overline{\alpha_3} + \frac{1}{4}A_0B_4 + \frac{1}{12}\left(3\overline{A_1C_1} + 2\overline{A_0C_2}\right)\alpha_1 + 2A_2\overline{A_3} \\
&\quad + \frac{1}{4}\overline{A_1B_2} + \frac{1}{6}\overline{A_0B_5} + \frac{3}{80}B_1\overline{C_1} + \frac{11}{384}C_1\overline{C_2}
\end{aligned}
$$

$$
\begin{aligned}
\nu_{11} &= \frac{1}{3}|A_1|^2\overline{A_1B_1} + \frac{1}{4}|A_1|^2\overline{A_0B_3} + \frac{1}{8}C_2\alpha_3\overline{A_0} - \frac{1}{64}C_1\zeta_0\overline{A_0} + \frac{1}{8}\beta\overline{A_0C_1} + \frac{1}{4}A_0\alpha_2\overline{C_2} + \frac{1}{4}A_0B_1\overline{\alpha_3} + \frac{1}{8}\overline{A_0B_1}\,\overline{\alpha_4} \\
&\quad + \frac{1}{4}A_0\overline{C_1}\,\overline{\alpha_7} - \frac{1}{16}A_0C_1\overline{\zeta_0} + \frac{1}{4}A_0B_6 + \frac{1}{96}C_2^2 + \frac{3}{64}C_1C_3 + \frac{1}{2}A_1E_3 \\
&\quad + \frac{1}{384}\left(9C_1^2 + 96A_0E_1 + 96C_3\overline{A_0} + 96\overline{A_2C_1} + 64\overline{A_1C_2}\right)\alpha_1 + \frac{1}{24}\left(4\overline{A_1C_1} + 3\overline{A_0C_2}\right)\alpha_4 - \frac{1}{512}\left(16\alpha_1\overline{A_0} + 3C_1\right)\gamma_1 \\
&\quad + \frac{1}{8}C_5\overline{A_0} - \frac{1}{64}\gamma_3\overline{A_0} + 2A_2\overline{A_4} + \frac{1}{4}\overline{A_2B_2} + \frac{1}{6}\overline{A_1B_5} + \frac{7}{192}B_2\overline{C_1} + \frac{13}{480}B_1\overline{C_2} + \frac{1}{384}\left(96A_0B_2 + 48\overline{A_0B_2} + 23C_1\overline{C_1}\right)\overline{\alpha_1}
\end{aligned}
$$

$$
\begin{aligned}
\nu_{12} &= \frac{1}{3}A_0B_1|A_1|^2 + \frac{1}{4}\alpha_1\overline{A_0B_1} + \frac{1}{4}\alpha_3\overline{A_0C_1} + \frac{1}{4}C_1\overline{A_0}\,\overline{\alpha_2} + \frac{1}{6}A_0C_1\overline{\alpha_4} + \frac{1}{4}A_1B_2 + \frac{1}{6}A_0B_5 + 2A_3\overline{A_2} + \frac{3}{80}C_1\overline{B_1} \\
&\quad + \frac{1}{4}\overline{A_0B_4} + \frac{11}{384}C_2\overline{C_1} + \frac{1}{12}(3A_1C_1 + 2A_0C_2)\overline{\alpha_1}
\end{aligned}
$$

$$
\begin{aligned}
\nu_{13} &= \frac{1}{3}A_0B_2|A_1|^2 + \frac{1}{3}|A_1|^2\overline{A_0B_2} + \frac{1}{6}C_1\alpha_6\overline{A_0} + \frac{1}{6}\alpha_4\overline{A_0B_1} + \frac{1}{6}\alpha_5\overline{A_0C_1} + \frac{1}{6}A_0B_1\overline{\alpha_4} + \frac{1}{6}A_0C_1\overline{\alpha_5} + \frac{1}{6}A_0\overline{C_1}\,\overline{\alpha_6} \\
&\quad + \frac{1}{4}A_1B_4 + \frac{1}{6}A_0B_7 + \frac{1}{12}\left(3\overline{A_1B_1} + 2\overline{A_0B_3}\right)\alpha_1 + \frac{1}{12}\left(2A_0\overline{B_1} + 3A_1\overline{C_1}\right)\alpha_2 + \frac{1}{12}\left(3\overline{A_1C_1} + 2\overline{A_0C_2}\right)\alpha_3 + 2A_3\overline{A_3} \\
&\quad + \frac{29}{800}B_1\overline{B_1} + \frac{13}{480}C_1\overline{B_3} + \frac{1}{4}\overline{A_1B_4} + \frac{1}{6}\overline{A_0B_7} + \frac{13}{480}\left(4C_1|A_1|^2 + B_3\right)\overline{C_1} + \frac{25}{1152}C_2\overline{C_2} + \frac{1}{12}(3A_1B_1 + 2A_0B_3)\overline{\alpha_1} \\
&\quad + \frac{1}{12}\left(2B_1\overline{A_0} + 3C_1\overline{A_1}\right)\overline{\alpha_2} + \frac{1}{12}(3A_1C_1 + 2A_0C_2)\overline{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
\nu_{14} &= \frac{1}{3}|A_1|^2\overline{A_1C_1} + \frac{1}{4}|A_1|^2\overline{A_0C_2} + \frac{1}{8}C_2\alpha_1\overline{A_0} + \frac{1}{8}C_1\alpha_3\overline{A_0} + \frac{1}{8}\overline{A_0B_1}\,\overline{\alpha_1} + \frac{1}{8}\overline{A_0C_1}\,\overline{\alpha_4} + \frac{5}{192}C_1C_2 + \frac{1}{2}A_0E_3 + \frac{1}{8}C_4\overline{A_0} \\
&\quad - \frac{1}{64}\gamma_2\overline{A_0} + 2A_1\overline{A_4} + \frac{1}{4}\overline{A_2B_1} + \frac{1}{6}\overline{A_1B_3}
\end{aligned}
$$

$$
\begin{aligned}
\nu_{15} &= \frac{11}{240}C_1^2|A_1|^2 + A_0E_1|A_1|^2 + \frac{2}{5}C_3|A_1|^2\overline{A_0} - \frac{1}{25}\gamma_1|A_1|^2\overline{A_0} + \frac{1}{3}|A_1|^2\overline{A_2C_1} + \frac{1}{4}|A_1|^2\overline{A_1C_2} + \frac{1}{2}A_0C_2\alpha_2 + \frac{1}{10}C_2\alpha_4\overline{A_0} \\
&\quad + \frac{1}{10}C_1\alpha_5\overline{A_0} + \frac{1}{10}\overline{A_0B_1}\,\overline{\alpha_3} + \frac{1}{10}\overline{A_0C_1}\,\overline{\alpha_5} + \frac{1}{2}A_0C_1\overline{\alpha_6} + \frac{11}{480}B_3C_1 + \frac{23}{960}B_1C_2 + \frac{1}{2}A_0E_5 + \frac{1}{40}\left(4B_3\overline{A_0} + 5C_2\overline{A_1}\right)\alpha_1 \\
&\quad + \frac{1}{40}\left(4B_1\overline{A_0} + 5C_1\overline{A_1}\right)\alpha_3 + \frac{1}{5}C_6\overline{A_0} - \frac{1}{50}\gamma_4\overline{A_0} + \frac{1}{8}C_4\overline{A_1} - \frac{1}{64}\gamma_2\overline{A_1} + 2A_1\overline{A_5} + \frac{1}{4}\overline{A_3B_1} + \frac{1}{6}\overline{A_2B_3} \\
&\quad + \frac{1}{40}\left(5\overline{A_1B_1} + 4\overline{A_0B_3}\right)\overline{\alpha_1} + \frac{1}{40}\left(5\overline{A_1C_1} + 4\overline{A_0C_2}\right)\overline{\alpha_4}
\end{aligned}
$$

$$
\begin{aligned}
\nu_{16} &= \frac{1}{16}E_1\alpha_1\overline{A_0} + \frac{1}{16}\overline{A_0C_1}\,\overline{\alpha_7} - \frac{1}{256}C_1\overline{A_0}\,\overline{\zeta_0} + \frac{1}{128}C_1E_1 + \frac{1}{112}\left(8\overline{A_1C_1} + 7\overline{A_0C_2}\right)\alpha_2 + \frac{1}{16}B_6\overline{A_0} + \frac{1}{14}B_4\overline{A_1} \\
&\quad + \frac{1}{12}B_2\overline{A_2} + \frac{1}{10}B_1\overline{A_3} + \frac{1}{8}C_1\overline{A_4} + \frac{1}{336}\left(21B_2\overline{A_0} + 24B_1\overline{A_1} + 28C_1\overline{A_2}\right)\overline{\alpha_1} + \frac{1}{112}\left(7B_1\overline{A_0} + 8C_1\overline{A_1}\right)\overline{\alpha_3}
\end{aligned}
$$

$$
\begin{aligned}
\nu_{17} &= \frac{1}{7}B_2|A_1|^2\overline{A_0} + \frac{1}{6}B_1|A_1|^2\overline{A_1} + \frac{1}{5}C_1|A_1|^2\overline{A_2} + \frac{1}{14}\alpha_2\overline{A_0B_1} + \frac{1}{14}C_2\overline{A_0}\,\overline{\alpha_3} + \frac{1}{14}C_1\overline{A_0}\,\overline{\alpha_5} + \frac{1}{14}\overline{A_0C_1}\,\overline{\alpha_6} + \frac{1}{14}B_7\overline{A_0} \\
&\quad + \frac{1}{12}B_5\overline{A_1} + \frac{1}{10}B_3\overline{A_2} + \frac{1}{8}C_2\overline{A_3} + \frac{1}{84}\left(6B_3\overline{A_0} + 7C_2\overline{A_1}\right)\overline{\alpha_1} + \frac{1}{84}\left(6B_1\overline{A_0} + 7C_1\overline{A_1}\right)\overline{\alpha_4}
\end{aligned}
$$

There are some non-trivial cancellations which we used to rule out the dual **(22)** − **(67)**, which comes from the fact that thanks of (6.1.43), we have

$$\frac{1}{2}\vec{E}_2 = \begin{pmatrix} -\dfrac{9}{160}\,C_1^2 & \overline{A_1} & -4 & 5 & 0 & \mathbf{(3)} \\ \dfrac{1}{2}\cancel{A_0C_1\alpha_2} - \dfrac{17}{160}\cancel{B_1C_1} & \overline{A_0} & -4 & 5 & 0 & \mathbf{(4)} \\ -\dfrac{1}{80}\,C_1\overline{A_1} & C_1 & -4 & 5 & 0 & \mathbf{(5)} \\ \cancel{C_1\alpha_2\overline{A_0}} - \dfrac{1}{5}\cancel{E_1\overline{A_1}} & A_0 & -4 & 5 & 0 & \mathbf{(6)} \end{pmatrix} = -\frac{9}{160}\langle\vec{C}_1,\overline{\vec{C}_1}\rangle\overline{\vec{A}_1} - \frac{1}{80}\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{C}_1$$

so that

$$\langle\vec{A}_0,\vec{E}_2\rangle = \langle\overline{\vec{A}_0},\vec{E}_2\rangle = 0.$$

Furthermore, as by (6.1.43), we have

$$\frac{1}{2}\vec{E}_3 = \begin{pmatrix} -\dfrac{1}{8}\,C_1C_2 + \dfrac{1}{3}\cancel{A_1E_1} & \overline{A_0} & -3 & 4 & 0 & \mathbf{(13)} \end{pmatrix} = -\frac{1}{8}\langle\vec{C}_1,\vec{C}_2\rangle\overline{\vec{A}_0}$$

so that (see **(24)** – **(69)**)

$$\langle\overline{\vec{A}_0},\vec{E}_3\rangle = \langle\vec{A}_1,\vec{E}_3\rangle = \langle\overline{\vec{A}_1},\vec{E}_3\rangle = 0$$

The relation for **(27)** – **(54)** we used is

$$\langle\vec{A}_0,\overline{\vec{\gamma}_2}\rangle = 0$$

and comes from the line **(46)** of (6.1.46). The new powers of order 11 are

$$\begin{pmatrix} 10 & 1 & 0 \\ 9 & 2 & 0 \\ 8 & 3 & 0 \end{pmatrix} \begin{pmatrix} 7 & 4 & 0 \\ 6 & 5 & 0 \\ 8 & 3 & 1 \end{pmatrix}$$

while the ones of order 12 are

$$\begin{pmatrix} 13 & -1 & 0 \\ 12 & 0 & 0 \\ 11 & 1 & 0 \\ 10 & 2 & 0 \\ 9 & 3 & 0 \\ 8 & 4 & 0 \end{pmatrix} \begin{pmatrix} 7 & 5 & 0 \\ 6 & 6 & 0 \\ 9 & 3 & 1 \\ 8 & 4 & 1 \\ 7 & 5 & 1 \end{pmatrix}$$

Remembering that

$$e^{2\lambda} = \begin{pmatrix} 1 & 3 & 3 & 0 \\ 2|A_1|^2 & 4 & 4 & 0 \\ \beta & 5 & 5 & 0 \\ \alpha_1 & 5 & 3 & 0 \\ \alpha_2 & 1 & 8 & 0 \\ \alpha_3 & 6 & 3 & 0 \end{pmatrix} \begin{pmatrix} \alpha_4 & 5 & 4 & 0 \\ \alpha_5 & 6 & 4 & 0 \\ \alpha_6 & 8 & 2 & 0 \\ \alpha_7 & 9 & 1 & 0 \\ \zeta_0 & 7 & 3 & 1 \\ \overline{\alpha_1} & 3 & 5 & 0 \end{pmatrix} \begin{pmatrix} \overline{\alpha_2} & 8 & 1 & 0 \\ \overline{\alpha_3} & 3 & 6 & 0 \\ \overline{\alpha_4} & 4 & 5 & 0 \\ \overline{\alpha_5} & 4 & 6 & 0 \\ \overline{\alpha_6} & 2 & 8 & 0 \\ \overline{\alpha_7} & 1 & 9 & 0 \\ \overline{\zeta_0} & 3 & 7 & 1 \end{pmatrix} \tag{6.1.47}$$

we see that there exists

$$\alpha_j \in \mathbb{C},\ \ 8 \leq j \leq 19, \quad \zeta_j \in \mathbb{C},\ \ 1 \leq j \leq 4, \quad \delta \in \mathbb{R},$$

such that

$$e^{2\lambda} = \begin{pmatrix} 1 & 3 & 3 & 0 \\ 2|A_1|^2 & 4 & 4 & 0 \\ \beta & 5 & 5 & 0 \\ \delta & 6 & 6 & 0 \end{pmatrix} + 2\,\mathrm{Re}\left\{ \begin{pmatrix} \alpha_1 & 5 & 3 & 0 \\ \alpha_2 & 1 & 8 & 0 \\ \alpha_3 & 6 & 3 & 0 \\ \alpha_4 & 5 & 4 & 0 \\ \alpha_5 & 6 & 4 & 0 \\ \alpha_6 & 8 & 2 & 0 \\ \alpha_7 & 9 & 1 & 0 \\ \alpha_8 & 10 & 1 & 0 \\ \alpha_9 & 9 & 2 & 0 \\ \alpha_{10} & 8 & 3 & 0 \\ \alpha_{11} & 7 & 4 & 0 \\ \alpha_{12} & 6 & 5 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{13} & 13 & -1 & 0 \\ \alpha_{14} & 12 & 0 & 0 \\ \alpha_{15} & 11 & 1 & 0 \\ \alpha_{16} & 10 & 2 & 0 \\ \alpha_{17} & 9 & 3 & 0 \\ \alpha_{18} & 8 & 4 & 0 \\ \alpha_{19} & 7 & 5 & 0 \\ \zeta_0 & 7 & 3 & 1 \\ \zeta_1 & 8 & 3 & 1 \\ \zeta_2 & 9 & 3 & 1 \\ \zeta_3 & 8 & 4 & 1 \\ \zeta_4 & 7 & 5 & 1 \end{pmatrix} \right\} \tag{6.1.48}$$

Comparison TEX (left, we do not write the complex conjugate part of the real part arising in (6.1.48)) and Sage (right)

$$\begin{pmatrix} 1 & 3 & 3 & 0 \\ 2|A_1|^2 & 4 & 4 & 0 \\ \beta & 5 & 5 & 0 \\ \delta & 6 & 6 & 0 \\ \alpha_1 & 5 & 3 & 0 \\ \alpha_2 & 1 & 8 & 0 \\ \alpha_3 & 6 & 3 & 0 \\ \alpha_4 & 5 & 4 & 0 \\ \alpha_5 & 6 & 4 & 0 \\ \alpha_6 & 8 & 2 & 0 \\ \alpha_7 & 9 & 1 & 0 \\ \alpha_8 & 10 & 1 & 0 \\ \alpha_9 & 9 & 2 & 0 \\ \alpha_{10} & 8 & 3 & 0 \\ \alpha_{11} & 7 & 4 & 0 \\ \alpha_{12} & 6 & 5 & 0 \\ \alpha_{13} & 13 & -1 & 0 \\ \alpha_{14} & 12 & 0 & 0 \\ \alpha_{15} & 11 & 1 & 0 \\ \alpha_{16} & 10 & 2 & 0 \\ \alpha_{17} & 9 & 3 & 0 \\ \alpha_{18} & 8 & 4 & 0 \\ \alpha_{19} & 7 & 5 & 0 \\ \zeta_0 & 7 & 3 & 1 \\ \zeta_1 & 8 & 3 & 1 \\ \zeta_2 & 9 & 3 & 1 \\ \zeta_3 & 8 & 4 & 1 \\ \zeta_4 & 7 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 & 0 \\ 2\,|A_1|^2 & 4 & 4 & 0 \\ \beta & 5 & 5 & 0 \\ \delta & 6 & 6 & 0 \\ \alpha_1 & 5 & 3 & 0 \\ \alpha_2 & 1 & 8 & 0 \\ \alpha_3 & 6 & 3 & 0 \\ \alpha_4 & 5 & 4 & 0 \\ \alpha_5 & 6 & 4 & 0 \\ \alpha_6 & 8 & 2 & 0 \\ \alpha_7 & 9 & 1 & 0 \\ \alpha_8 & 10 & 1 & 0 \\ \alpha_9 & 9 & 2 & 0 \\ \alpha_{10} & 8 & 3 & 0 \\ \alpha_{11} & 7 & 4 & 0 \\ \alpha_{12} & 6 & 5 & 0 \\ \alpha_{13} & 13 & -1 & 0 \\ \alpha_{14} & 12 & 0 & 0 \\ \alpha_{15} & 11 & 1 & 0 \\ \alpha_{16} & 10 & 2 & 0 \\ \alpha_{17} & 9 & 3 & 0 \\ \alpha_{18} & 8 & 4 & 0 \\ \alpha_{19} & 7 & 5 & 0 \\ \zeta_0 & 7 & 3 & 1 \\ \zeta_1 & 8 & 3 & 1 \\ \zeta_2 & 9 & 3 & 1 \\ \zeta_3 & 8 & 4 & 1 \\ \zeta_4 & 7 & 5 & 1 \end{pmatrix} \begin{pmatrix} \overline{\alpha_1} & 3 & 5 & 0 \\ \overline{\alpha_2} & 8 & 1 & 0 \\ \overline{\alpha_3} & 3 & 6 & 0 \\ \overline{\alpha_4} & 4 & 5 & 0 \\ \overline{\alpha_5} & 4 & 6 & 0 \\ \overline{\alpha_6} & 2 & 8 & 0 \\ \overline{\alpha_7} & 1 & 9 & 0 \\ \overline{\alpha_8} & 1 & 10 & 0 \\ \overline{\alpha_9} & 2 & 9 & 0 \\ \overline{\alpha_{10}} & 3 & 8 & 0 \\ \overline{\alpha_{11}} & 4 & 7 & 0 \\ \overline{\alpha_{12}} & 5 & 6 & 0 \\ \overline{\alpha_{13}} & -1 & 13 & 0 \\ \overline{\alpha_{14}} & 0 & 12 & 0 \\ \overline{\alpha_{15}} & 1 & 11 & 0 \\ \overline{\alpha_{16}} & 2 & 10 & 0 \\ \overline{\alpha_{17}} & 3 & 9 & 0 \\ \overline{\alpha_{18}} & 4 & 8 & 0 \\ \overline{\alpha_{19}} & 5 & 7 & 0 \\ \overline{\zeta_0} & 3 & 7 & 1 \\ \overline{\zeta_1} & 3 & 8 & 1 \\ \overline{\zeta_2} & 3 & 9 & 1 \\ \overline{\zeta_3} & 4 & 8 & 1 \\ \overline{\zeta_4} & 5 & 7 & 1 \end{pmatrix}$$

Then we have

$$
\vec{h}_0 = \begin{pmatrix}
-\frac{1}{4} & E_1 & -2 & 8 & 0 \\
-\frac{2}{9} & E_2 & -2 & 9 & 0 \\
\frac{1}{36}\,\alpha_2 & C_1 & -2 & 9 & 0 \\
-\frac{1}{5} & E_4 & -2 & 10 & 0 \\
-\frac{1}{5}\,\overline{\alpha_1} & E_1 & -2 & 10 & 0 \\
\frac{1}{20}\,\overline{\alpha_7} & C_1 & -2 & 10 & 0 \\
-4\,\alpha_2^2+8\,\overline{\alpha_{13}} & A_0 & -2 & 10 & 0 \\
-\frac{3}{16} & E_3 & -1 & 8 & 0 \\
-\frac{1}{6} & E_5 & -1 & 9 & 0 \\
-\frac{11}{24}\,|A_1|^2 & E_1 & -1 & 9 & 0 \\
\frac{1}{12}\,\alpha_2 & C_2 & -1 & 9 & 0 \\
-\frac{1}{24}\,\overline{\alpha_6} & C_1 & -1 & 9 & 0 \\
6\,\overline{\alpha_{14}} & A_0 & -1 & 9 & 0 \\
-\frac{1}{4} & C_1 & 0 & 4 & 0 \\
-\frac{1}{5} & B_1 & 0 & 5 & 0 \\
4\,\alpha_2 & A_0 & 0 & 5 & 0 \\
-\frac{1}{6} & B_2 & 0 & 6 & 0 \\
-\frac{1}{6}\,\overline{\alpha_1} & C_1 & 0 & 6 & 0 \\
4\,\overline{\alpha_7} & A_0 & 0 & 6 & 0 \\
-\frac{1}{7} & B_4 & 0 & 7 & 0 \\
\frac{5}{14}\,\alpha_2 & \overline{C_1} & 0 & 7 & 0 \\
-\frac{1}{7}\,\overline{\alpha_1} & B_1 & 0 & 7 & 0 \\
-\frac{1}{7}\,\overline{\alpha_3} & C_1 & 0 & 7 & 0 \\
-4\,\alpha_2\overline{\alpha_1}+4\,\overline{\alpha_8} & A_0 & 0 & 7 & 0 \\
-\frac{1}{8} & B_6 & 0 & 8 & 0 \\
-\frac{1}{4}\,\alpha_1 & E_1 & 0 & 8 & 0 \\
\frac{5}{24}\,\alpha_2 & \overline{C_2} & 0 & 8 & 0 \\
-\frac{1}{8}\,\overline{\alpha_1} & B_2 & 0 & 8 & 0 \\
-\frac{1}{8}\,\overline{\alpha_3} & B_1 & 0 & 8 & 0 \\
\frac{3}{8}\,\overline{\alpha_7} & \overline{C_1} & 0 & 8 & 0
\end{pmatrix}
\begin{pmatrix}
-\frac{3}{128}\,\overline{\zeta_0} & C_1 & 0 & 8 & 0 \\
-4\,\alpha_2\overline{\alpha_3}-4\,\overline{\alpha_1\alpha_7}+4\,\overline{\alpha_{15}} & A_0 & 0 & 8 & 0 \\
-\frac{1}{8}\,\overline{\zeta_0} & C_1 & 0 & 8 & 1 \\
-\frac{1}{8} & C_2 & 1 & 4 & 0 \\
-\frac{1}{10} & B_3 & 1 & 5 & 0 \\
-\frac{9}{20}\,|A_1|^2 & C_1 & 1 & 5 & 0 \\
4\,\alpha_2 & A_1 & 1 & 5 & 0 \\
2\,\overline{\alpha_6} & A_0 & 1 & 5 & 0 \\
-\frac{1}{12} & B_5 & 1 & 6 & 0 \\
-\frac{11}{30}\,|A_1|^2 & B_1 & 1 & 6 & 0 \\
-\frac{1}{12}\,\overline{\alpha_1} & C_2 & 1 & 6 & 0 \\
-\frac{5}{24}\,\overline{\alpha_4} & C_1 & 1 & 6 & 0 \\
4\,\overline{\alpha_7} & A_1 & 1 & 6 & 0 \\
-4\,\alpha_2|A_1|^2+2\,\overline{\alpha_9} & A_0 & 1 & 6 & 0 \\
-\frac{1}{14} & B_7 & 1 & 7 & 0 \\
-\frac{13}{42}\,|A_1|^2 & B_2 & 1 & 7 & 0 \\
\frac{3}{7}\,\alpha_2 & \overline{B_1} & 1 & 7 & 0 \\
-\frac{1}{14}\,\overline{\alpha_1} & B_3 & 1 & 7 & 0 \\
-\frac{1}{14}\,\overline{\alpha_3} & C_2 & 1 & 7 & 0 \\
-\frac{6}{35}\,\overline{\alpha_4} & B_1 & 1 & 7 & 0 \\
\frac{1}{12}\,|A_1|^2\overline{\alpha_1}-\frac{11}{56}\,\overline{\alpha_5} & C_1 & 1 & 7 & 0 \\
\frac{5}{28}\,\overline{\alpha_6} & \overline{C_1} & 1 & 7 & 0 \\
-4\,\alpha_2\overline{\alpha_1}+4\,\overline{\alpha_8} & A_1 & 1 & 7 & 0 \\
-4\,|A_1|^2\overline{\alpha_7}-2\,\alpha_2\overline{\alpha_4}-2\,\overline{\alpha_1\alpha_6}+2\,\overline{\alpha_{16}} & A_0 & 1 & 7 & 0 \\
\frac{1}{8} & \gamma_1 & 2 & 4 & 0 \\
-\frac{1}{4}\,\alpha_1 & C_1 & 2 & 4 & 0 \\
-\overline{\zeta_0} & A_0 & 2 & 4 & 0 \\
-\frac{1}{4}\,|A_1|^2 & C_2 & 2 & 5 & 0 \\
\frac{1}{20} & \overline{\gamma_2} & 2 & 5 & 0 \\
-\frac{1}{5}\,\alpha_1 & B_1 & 2 & 5 & 0
\end{pmatrix}
$$

$$\begin{pmatrix}
-\frac{1}{4}\,\alpha_4 & C_1 & 2 & 5 & 0 \\
4\,\alpha_2 & A_2 & 2 & 5 & 0 \\
2\,\overline{\alpha_6} & A_1 & 2 & 5 & 0 \\
-\overline{\zeta_1} & A_0 & 2 & 5 & 0 \\
-\frac{1}{5}\,|A_1|^2 & B_3 & 2 & 6 & 0 \\
\frac{1}{24} & \overline{\gamma_3} & 2 & 6 & 0 \\
\frac{1}{10}\,|A_1|^4 + \frac{1}{12}\,\alpha_1\overline{\alpha_1} - \frac{1}{4}\,\beta & C_1 & 2 & 6 & 0 \\
-\frac{1}{6}\,\alpha_1 & B_2 & 2 & 6 & 0 \\
-\frac{1}{5}\,\alpha_4 & B_1 & 2 & 6 & 0 \\
\frac{1}{12}\,\overline{\alpha_1} & \gamma_1 & 2 & 6 & 0 \\
-\frac{1}{8}\,\overline{\alpha_4} & C_2 & 2 & 6 & 0 \\
-\frac{1}{12}\,\overline{\zeta_0} & \overline{C_1} & 2 & 6 & 0 \\
4\,\overline{\alpha_7} & A_2 & 2 & 6 & 0 \\
-4\,\alpha_2|A_1|^2 + 2\,\overline{\alpha_9} & A_1 & 2 & 6 & 0 \\
-12\,\overline{\alpha_1}^3 + \overline{\alpha_1}\overline{\zeta_0} - \overline{\zeta_2} & A_0 & 2 & 6 & 0 \\
2 & A_1 & 3 & 0 & 0 \\
-4\,|A_1|^2 & A_0 & 3 & 1 & 0 \\
\frac{1}{4} & \overline{B_1} & 3 & 2 & 0 \\
-2\,\overline{\alpha_4} & A_0 & 3 & 2 & 0 \\
-\frac{1}{6}\,|A_1|^2 & \overline{C_1} & 3 & 3 & 0 \\
\frac{1}{6} & \overline{B_3} & 3 & 3 & 0 \\
4\,|A_1|^2\overline{\alpha_1} - 2\,\overline{\alpha_5} & A_0 & 3 & 3 & 0 \\
-\frac{1}{12}\,|A_1|^2 & \overline{C_2} & 3 & 4 & 0 \\
\frac{3}{64} & \gamma_2 & 3 & 4 & 0 \\
\frac{1}{8} & C_4 & 3 & 4 & 0 \\
-\frac{1}{8}\,\alpha_1 & C_2 & 3 & 4 & 0 \\
-\frac{1}{4}\,\alpha_3 & C_1 & 3 & 4 & 0 \\
\frac{1}{8}\,\overline{\alpha_1} & \overline{B_1} & 3 & 4 & 0 \\
-\frac{1}{8}\,\overline{\alpha_4} & \overline{C_1} & 3 & 4 & 0 \\
4\,|A_1|^2\overline{\alpha_3} + 2\,\overline{\alpha_1\alpha_4} - 2\,\overline{\alpha_{11}} & A_0 & 3 & 4 & 0 \\
-\overline{\zeta_0} & A_1 & 3 & 4 & 0
\end{pmatrix}$$

$$\begin{pmatrix}
\frac{1}{8} & \gamma_2 & 3 & 4 & 1 \\
-\frac{1}{10}\,|A_1|^2 & C_3 & 3 & 5 & 0 \\
\frac{2}{25} & \gamma_4 & 3 & 5 & 0 \\
\frac{1}{5} & C_6 & 3 & 5 & 0 \\
\frac{89}{400}\,|A_1|^2 & \gamma_1 & 3 & 5 & 0 \\
-\frac{1}{10}\,\alpha_1 & B_3 & 3 & 5 & 0 \\
-\frac{1}{5}\,\alpha_3 & B_1 & 3 & 5 & 0 \\
-\frac{3}{20}\,\alpha_4 & C_2 & 3 & 5 & 0 \\
\frac{1}{10}\,\alpha_1|A_1|^2 - \frac{11}{40}\,\alpha_5 & C_1 & 3 & 5 & 0 \\
\frac{1}{10}\,\overline{\alpha_1} & \overline{B_3} & 3 & 5 & 0 \\
\frac{1}{10}\,\overline{\alpha_3} & \overline{B_1} & 3 & 5 & 0 \\
-\frac{1}{15}\,\overline{\alpha_4} & \overline{C_2} & 3 & 5 & 0 \\
\frac{1}{4}\,|A_1|^2\overline{\alpha_1} - \frac{3}{20}\,\overline{\alpha_5} & \overline{C_1} & 3 & 5 & 0 \\
4\,\alpha_2 & A_3 & 3 & 5 & 0 \\
2\,\overline{\alpha_6} & A_2 & 3 & 5 & 0 \\
-\overline{\zeta_1} & A_1 & 3 & 5 & 0 \\
-76\,|A_1|^2\overline{\alpha_1}^2 + 2\,|A_1|^2\overline{\zeta_0} + 2\,\alpha_2\alpha_3 + 2\,\overline{\alpha_3\alpha_4} + 2\,\overline{\alpha_1\alpha_5} + 2\,\alpha_1\overline{\alpha_6} - 2\,\overline{\alpha_{18}} - \overline{\zeta_3} & A_0 & 3 & 5 & 0 \\
-\frac{1}{10}\,|A_1|^2 & \gamma_1 & 3 & 5 & 1 \\
\frac{1}{5} & \gamma_4 & 3 & 5 & 1 \\
4\,|A_1|^2\overline{\zeta_0} - 2\,\overline{\zeta_3} & A_0 & 3 & 5 & 1 \\
4 & A_2 & 4 & 0 & 0 \\
-4\,\alpha_1 & A_0 & 4 & 0 & 0 \\
-4\,|A_1|^2 & A_1 & 4 & 1 & 0 \\
-4\,\alpha_4 & A_0 & 4 & 1 & 0 \\
\frac{1}{2} & \overline{B_2} & 4 & 2 & 0 \\
8\,|A_1|^4 + 4\,\alpha_1\overline{\alpha_1} - 4\,\beta & A_0 & 4 & 2 & 0 \\
-2\,\overline{\alpha_4} & A_1 & 4 & 2 & 0 \\
\frac{1}{3} & \overline{B_5} & 4 & 3 & 0 \\
\frac{1}{6}\,|A_1|^2 & \overline{B_1} & 4 & 3 & 0 \\
-\frac{1}{6}\,\alpha_4 & \overline{C_1} & 4 & 3 & 0 \\
4\,|A_1|^2\overline{\alpha_1} - 2\,\overline{\alpha_5} & A_1 & 4 & 3 & 0
\end{pmatrix}$$

$$
\begin{pmatrix}
8\,|A_1|^2\overline{\alpha_4}+4\,\alpha_4\overline{\alpha_1}+4\,\alpha_1\overline{\alpha_3}-4\,\overline{\alpha_{12}} & A_0 & 4 & 3 & 0 \\
\frac{1}{32} & \gamma_3 & 4 & 4 & 0 \\
\frac{1}{4} & C_5 & 4 & 4 & 0 \\
\frac{1}{6}\,|A_1|^2 & \overline{B_3} & 4 & 4 & 0 \\
\frac{1}{3}\,|A_1|^4+\frac{1}{4}\,\alpha_1\overline{\alpha_1}-\frac{1}{4}\,\beta & \overline{C_1} & 4 & 4 & 0 \\
\frac{1}{8}\,\alpha_1 & \gamma_1 & 4 & 4 & 0 \\
-\frac{1}{8}\,\alpha_3 & C_2 & 4 & 4 & 0 \\
-\frac{1}{12}\,\alpha_4 & \overline{C_2} & 4 & 4 & 0 \\
-\frac{1}{32}\,\zeta_0 & C_1 & 4 & 4 & 0 \\
\frac{1}{4}\,\overline{\alpha_1} & \overline{B_2} & 4 & 4 & 0 \\
4\,|A_1|^2\overline{\alpha_3}+2\,\overline{\alpha_1\alpha_4}-2\,\overline{\alpha_{11}} & A_1 & 4 & 4 & 0 \\
-160\,|A_1|^4\overline{\alpha_1}-40\,\alpha_1\overline{\alpha_1}^2+8\,|A_1|^2\overline{\alpha_5}+4\,\beta\overline{\alpha_1}+4\,\alpha_4\overline{\alpha_3}+2\,\overline{\alpha_4}^2+\alpha_1\overline{\zeta_0}-4\,\overline{\alpha_{19}}-\overline{\zeta_4} & A_0 & 4 & 4 & 0 \\
-\overline{\zeta_0} & A_2 & 4 & 4 & 0 \\
\frac{1}{4} & \gamma_3 & 4 & 4 & 1 \\
-\frac{1}{4}\,\zeta_0 & C_1 & 4 & 4 & 1 \\
4\,\alpha_1\overline{\zeta_0}-4\,\overline{\zeta_4} & A_0 & 4 & 4 & 1 \\
6 & A_3 & 5 & 0 & 0 \\
-4\,\alpha_1 & A_1 & 5 & 0 & 0 \\
-6\,\alpha_3 & A_0 & 5 & 0 & 0 \\
-4\,|A_1|^2 & A_2 & 5 & 1 & 0 \\
-4\,\alpha_4 & A_1 & 5 & 1 & 0 \\
12\,\alpha_1|A_1|^2-6\,\alpha_5 & A_0 & 5 & 1 & 0 \\
\frac{3}{4} & \overline{B_4} & 5 & 2 & 0 \\
\frac{1}{4}\,\alpha_1 & \overline{B_1} & 5 & 2 & 0 \\
\frac{1}{8}\,\overline{\alpha_2} & C_1 & 5 & 2 & 0 \\
8\,|A_1|^4+4\,\alpha_1\overline{\alpha_1}-4\,\beta & A_1 & 5 & 2 & 0 \\
12\,\alpha_4|A_1|^2+6\,\alpha_3\overline{\alpha_1}+6\,\alpha_1\overline{\alpha_4}-6\,\alpha_{12} & A_0 & 5 & 2 & 0 \\
-2\,\overline{\alpha_4} & A_2 & 5 & 2 & 0 \\
\frac{1}{2} & \overline{B_7} & 5 & 3 & 0 \\
\frac{1}{2}\,|A_1|^2 & \overline{B_2} & 5 & 3 & 0 \\
\frac{1}{6}\,\alpha_1 & \overline{B_3} & 5 & 3 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{1}{3}\,\alpha_1|A_1|^2 - \frac{1}{4}\,\alpha_5 & \overline{C_1} & 5 & 3 & 0 \\
-\frac{1}{8}\,\alpha_6 & C_1 & 5 & 3 & 0 \\
-112\,|A_1|^6 - 168\,\alpha_1|A_1|^2\overline{\alpha_1} + 12\,\beta|A_1|^2 + 6\,\alpha_5\overline{\alpha_1} + 6\,\alpha_2\overline{\alpha_2} + 6\,\alpha_3\overline{\alpha_3} + 6\,\alpha_4\overline{\alpha_4} + 6\,\alpha_1\overline{\alpha_5} - 6\,\delta & A_0 & 5 & 3 & 0 \\
4\,|A_1|^2\overline{\alpha_1} - 2\,\overline{\alpha_5} & A_2 & 5 & 3 & 0 \\
8\,|A_1|^2\overline{\alpha_4} + 4\,\alpha_4\overline{\alpha_1} + 4\,\alpha_1\overline{\alpha_3} - 4\,\overline{\alpha_{12}} & A_1 & 5 & 3 & 0 \\
\frac{1}{2} & \overline{E_1} & 6 & 0 & 0 \\
8 & A_4 & 6 & 0 & 0 \\
-4\,\alpha_1 & A_2 & 6 & 0 & 0 \\
-6\,\alpha_3 & A_1 & 6 & 0 & 0 \\
4\,\alpha_1^2 - \zeta_0 & A_0 & 6 & 0 & 0 \\
4 & \overline{E_1} & 6 & 0 & 1 \\
-8\,\zeta_0 & A_0 & 6 & 0 & 1 \\
-4\,|A_1|^2 & A_3 & 6 & 1 & 0 \\
2 & \overline{E_3} & 6 & 1 & 0 \\
-4\,\alpha_4 & A_2 & 6 & 1 & 0 \\
12\,\alpha_1|A_1|^2 - 6\,\alpha_5 & A_1 & 6 & 1 & 0 \\
16\,\alpha_3|A_1|^2 + 8\,\alpha_1\alpha_4 - 8\,\alpha_{11} & A_0 & 6 & 1 & 0 \\
1 & \overline{B_6} & 6 & 2 & 0 \\
\frac{1}{2}\,\alpha_1 & \overline{B_2} & 6 & 2 & 0 \\
\frac{1}{4}\,\alpha_3 & \overline{B_1} & 6 & 2 & 0 \\
\frac{1}{4}\,\alpha_7 & C_1 & 6 & 2 & 0 \\
-\frac{1}{4}\,\zeta_0 & \overline{C_1} & 6 & 2 & 0 \\
\overline{\alpha_1} & \overline{E_1} & 6 & 2 & 0 \\
\frac{3}{8}\,\overline{\alpha_2} & C_2 & 6 & 2 & 0 \\
8\,|A_1|^4 + 4\,\alpha_1\overline{\alpha_1} - 4\,\beta & A_2 & 6 & 2 & 0 \\
12\,\alpha_4|A_1|^2 + 6\,\alpha_3\overline{\alpha_1} + 6\,\alpha_1\overline{\alpha_4} - 6\,\alpha_{12} & A_1 & 6 & 2 & 0 \\
-176\,\alpha_1|A_1|^4 + 16\,\alpha_5|A_1|^2 - 44\,\alpha_1^2\overline{\alpha_1} + 4\,\alpha_4^2 + 8\,\alpha_1\beta + \zeta_0\overline{\alpha_1} + 8\,\alpha_3\overline{\alpha_4} - 8\,\alpha_{19} - \zeta_4 & A_0 & 6 & 2 & 0 \\
-2\,\overline{\alpha_4} & A_3 & 6 & 2 & 0 \\
8\,\zeta_0\overline{\alpha_1} - 8\,\zeta_4 & A_0 & 6 & 2 & 1 \\
-10\,\overline{\alpha_2} & A_0 & 7 & -2 & 0 \\
-10\,\alpha_6 & A_0 & 7 & -1 & 0
\end{pmatrix}
$$

$$\begin{pmatrix}
\frac{1}{2} & \overline{E_2} & 7 & 0 & 0 \\
10 & A_5 & 7 & 0 & 0 \\
-\frac{3}{4}\,\overline{\alpha_2} & \overline{C_1} & 7 & 0 & 0 \\
-4\,\alpha_1 & A_3 & 7 & 0 & 0 \\
-6\,\alpha_3 & A_2 & 7 & 0 & 0 \\
10\,\alpha_1\alpha_3 + 10\,\overline{\alpha_1\alpha_2} - 10\,\alpha_{10} - \zeta_1 & A_0 & 7 & 0 & 0 \\
4\,\alpha_1^2 - \zeta_0 & A_1 & 7 & 0 & 0 \\
5 & \overline{E_2} & 7 & 0 & 1 \\
5\,\overline{\alpha_2} & \overline{C_1} & 7 & 0 & 1 \\
-8\,\zeta_0 & A_1 & 7 & 0 & 1 \\
-10\,\zeta_1 & A_0 & 7 & 0 & 1 \\
\frac{5}{2} & \overline{E_5} & 7 & 1 & 0 \\
5\,|A_1|^2 & \overline{E_1} & 7 & 1 & 0 \\
\frac{5}{4}\,\alpha_6 & \overline{C_1} & 7 & 1 & 0 \\
\frac{5}{3}\,\overline{\alpha_2} & \overline{C_2} & 7 & 1 & 0 \\
-4\,|A_1|^2 & A_4 & 7 & 1 & 0 \\
-4\,\alpha_4 & A_3 & 7 & 1 & 0 \\
12\,\alpha_1|A_1|^2 - 6\,\alpha_5 & A_2 & 7 & 1 & 0 \\
16\,\alpha_3|A_1|^2 + 8\,\alpha_1\alpha_4 - 8\,\alpha_{11} & A_1 & 7 & 1 & 0 \\
-92\,\alpha_1^2|A_1|^2 + 2\,\zeta_0|A_1|^2 + 10\,\alpha_3\alpha_4 + 10\,\alpha_1\alpha_5 + 10\,\alpha_6\overline{\alpha_1} + 10\,\overline{\alpha_2\alpha_3} - 10\,\alpha_{18} - \zeta_3 & A_0 & 7 & 1 & 0 \\
-2\,|A_1|^2 & \overline{E_1} & 7 & 1 & 1 \\
20\,\zeta_0|A_1|^2 - 10\,\zeta_3 & A_0 & 7 & 1 & 1 \\
-12\,\alpha_7 & A_0 & 8 & -2 & 0 \\
-10\,\overline{\alpha_2} & A_1 & 8 & -2 & 0 \\
-10\,\alpha_6 & A_1 & 8 & -1 & 0 \\
24\,|A_1|^2\overline{\alpha_2} - 12\,\alpha_9 & A_0 & 8 & -1 & 0 \\
\frac{1}{2} & \overline{E_4} & 8 & 0 & 0 \\
12 & A_6 & 8 & 0 & 0 \\
\frac{1}{2}\,\alpha_1 & \overline{E_1} & 8 & 0 & 0 \\
-\alpha_7 & \overline{C_1} & 8 & 0 & 0 \\
-\frac{3}{4}\,\overline{\alpha_2} & \overline{B_1} & 8 & 0 & 0
\end{pmatrix}$$

$$\begin{pmatrix}
-4\,\alpha_1 & A_4 & 8 & 0 & 0\\
-6\,\alpha_3 & A_3 & 8 & 0 & 0\\
10\,\alpha_1\alpha_3 + 10\,\overline{\alpha_1\alpha_2} - 10\,\alpha_{10} - \zeta_1 & A_1 & 8 & 0 & 0\\
-16\,\alpha_1^3 + 24\,\alpha_6|A_1|^2 + 6\,\alpha_3^2 + \alpha_1\zeta_0 + 12\,\alpha_7\overline{\alpha_1} + 12\,\overline{\alpha_2\alpha_4} - 12\,\alpha_{17} - \zeta_2 & A_0 & 8 & 0 & 0\\
4\,\alpha_1^2 - \zeta_0 & A_2 & 8 & 0 & 0\\
6 & \overline{E_4} & 8 & 0 & 1\\
4\,\alpha_1 & \overline{E_1} & 8 & 0 & 1\\
6\,\alpha_7 & \overline{C_1} & 8 & 0 & 1\\
6\,\overline{\alpha_2} & \overline{B_1} & 8 & 0 & 1\\
-8\,\zeta_0 & A_2 & 8 & 0 & 1\\
-10\,\zeta_1 & A_1 & 8 & 0 & 1\\
12\,\alpha_1\zeta_0 - 12\,\zeta_2 & A_0 & 8 & 0 & 1\\
-12\,\alpha_7 & A_1 & 9 & -2 & 0\\
14\,\alpha_1\overline{\alpha_2} - 14\,\alpha_8 & A_0 & 9 & -2 & 0\\
-10\,\overline{\alpha_2} & A_2 & 9 & -2 & 0\\
-10\,\alpha_6 & A_2 & 9 & -1 & 0\\
24\,|A_1|^2\overline{\alpha_2} - 12\,\alpha_9 & A_1 & 9 & -1 & 0\\
28\,\alpha_7|A_1|^2 + 14\,\alpha_1\alpha_6 + 14\,\alpha_4\overline{\alpha_2} - 14\,\alpha_{16} & A_0 & 9 & -1 & 0\\
-12\,\alpha_7 & A_2 & 10 & -2 & 0\\
14\,\alpha_1\overline{\alpha_2} - 14\,\alpha_8 & A_1 & 10 & -2 & 0\\
16\,\alpha_1\alpha_7 + 16\,\alpha_3\overline{\alpha_2} - 16\,\alpha_{15} & A_0 & 10 & -2 & 0\\
-10\,\overline{\alpha_2} & A_3 & 10 & -2 & 0\\
-18\,\alpha_{14} & A_0 & 11 & -3 & 0\\
10\,\overline{\alpha_2}^2 - 20\,\alpha_{13} & A_0 & 12 & -4 & 0
\end{pmatrix}$$

Now, we have

$g^{-1} \otimes Q(\vec{h}_0) =$

$$
\begin{pmatrix}
-16\,A_0C_1\alpha_1+16\,A_2C_1+2\,A_1C_2 & 0 & 0 & 0\\
-30\,A_0C_1\alpha_3+8\,A_0A_1\overline{\zeta_0}+30\,A_3C_1+6\,A_2C_2-6\,(4\,A_1C_1+A_0C_2)\alpha_1-A_1\gamma_1 & 1 & 0 & 0\\
\omega_1 & 2 & 0 & 0\\
\omega_2 & 3 & 0 & 0\\
20\,A_1E_1 & -3 & 4 & 0\\
-48\,A_0E_1\alpha_1+48\,A_2E_1+12\,A_1E_3 & -2 & 4 & 0\\
\frac{9}{80}\,C_1^2|A_1|^2-35\,A_0E_1|A_1|^2-\frac{1}{2}\,A_0C_1\overline{\alpha_6}+\frac{1}{40}\,B_3C_1-\frac{1}{40}\,B_1C_2+20\,A_1E_2-\frac{1}{2}\,(19\,A_1C_1-A_0C_2)\alpha_2 & -3 & 5 & 0\\
\omega_3 & -3 & 6 & 0\\
\omega_4 & -2 & 5 & 0\\
-\frac{3}{8}\,C_2E_1-\frac{3}{16}\,C_1E_3 & -5 & 8 & 0\\
\omega_5 & -1 & 4 & 0\\
\omega_6 & 0 & 2 & 0\\
\omega_7 & 1 & 2 & 0\\
\omega_8 & 0 & 3 & 0\\
\omega_9 & 3 & 0 & 1\\
\omega_{10} & 0 & 1 & 0\\
\omega_{11} & 1 & 1 & 0\\
\omega_{12} & 2 & 1 & 0\\
\omega_{13} & 4 & -1 & 0\\
\omega_{14} & 5 & -2 & 0\\
\omega_{15} & 4 & -2 & 0\\
-\frac{1}{2}\,C_1E_1 & -6 & 8 & 0\\
-\frac{3}{10}\,B_1E_1-\frac{5}{9}\,C_1E_2+\frac{1}{72}\left(5\,C_1^2+432\,A_0E_1\right)\alpha_2 & -6 & 9 & 0\\
-48\,A_0C_1\zeta_0+24\,C_1\overline{E_1} & 2 & 0 & 1\\
-480\,A_0\alpha_2\overline{E_1}+48\left(20\,A_0^2\alpha_2-A_0B_1\right)\zeta_0+24\,B_1\overline{E_1} & 2 & 1 & 1\\
\omega_{16} & 3 & -2 & 0\\
192\,A_0^2\alpha_2|A_1|^2-\frac{48}{5}\,A_0B_1|A_1|^2-3\,A_0C_1\overline{\alpha_4}-144\,A_0A_1\overline{\alpha_7}+6\,A_1B_2+\frac{3}{8}\,C_1\overline{B_1} & -1 & 2 & 0\\
\omega_{17} & -1 & 3 & 0\\
-96\,A_0A_1\zeta_4+48\,A_0\overline{E_1}\,\overline{\alpha_4}+12\left(8\,A_0A_1\overline{\alpha_1}-8\,A_0^2\overline{\alpha_4}+A_0\overline{B_1}\right)\zeta_0-6\,\overline{B_1E_1} & 5 & -2 & 1\\
\omega_{18} & 3 & -1 & 0\\
\omega_{19} & 6 & -3 & 0\\
\omega_{20} & 7 & -4 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
\omega_{21} & 6 & -4 & 0 \\
-24\,A_0A_1\zeta_0 + 12\,A_1\overline{E_1} & 5 & -4 & 0 \\
\omega_{22} & 5 & -3 & 0 \\
\omega_{23} & 9 & -6 & 0 \\
-16\,A_0^2\alpha_1|A_1|^2 - 8\,A_1^2|A_1|^2 + 16\,A_0A_2|A_1|^2 - 8\,A_0A_1\alpha_4 & 3 & -3 & 0 \\
16\,A_0A_1\alpha_1|A_1|^2 - 48\,A_0^2\alpha_3|A_1|^2 - 16\,A_1A_2|A_1|^2 + 48\,A_0A_3|A_1|^2 - 16\,A_1^2\alpha_4 - 24\,A_0A_1\alpha_5 & 4 & -3 & 0 \\
-9\,A_0C_1|A_1|^2 - 120\,A_0A_1\alpha_2 + 6\,A_1B_1 & -1 & 1 & 0 \\
-96\,A_0^2\zeta_0|A_1|^2 + 48\,A_0|A_1|^2\overline{E_1} & 5 & -3 & 1 \\
\omega_{24} & 6 & -3 & 1 \\
-720\,A_0^2\alpha_7|A_1|^2 - 1520\,A_0A_1|A_1|^2\overline{\alpha_2} - 360\,A_0^2\alpha_4\overline{\alpha_2} + 120\,A_0A_1\alpha_9 - 20\left(6\,A_0^2\alpha_1 - 5\,A_1^2 - 6\,A_0A_2\right)\alpha_6 & 7 & -5 & 0 \\
\omega_{25} & 8 & -5 & 0 \\
-480\,A_0^2|A_1|^2\overline{\alpha_2} + 80\,A_0A_1\alpha_6 & 6 & -5 & 0 \\
864\,A_0A_1\alpha_{14} & 10 & -7 & 0 \\
240\,A_0A_1\alpha_7 - 40\left(6\,A_0^2\alpha_1 - 5\,A_1^2 - 6\,A_0A_2\right)\overline{\alpha_2} & 7 & -6 & 0 \\
-240\,A_0^2\alpha_3\overline{\alpha_2} + 336\,A_0A_1\alpha_8 - 96\left(4\,A_0^2\alpha_1 - 3\,A_1^2 - 4\,A_0A_2\right)\alpha_7 - 16\,(61\,A_0A_1\alpha_1 - 35\,A_1A_2 - 15\,A_0A_3)\overline{\alpha_2} & 8 & -6 & 0 \\
-880\,A_0A_1\overline{\alpha_2}^2 + 1440\,A_0A_1\alpha_{13} & 11 & -8 & 0 \\
160\,A_0A_1\overline{\alpha_2} & 6 & -6 & 0 \\
-160\,A_0^2\zeta_0\overline{\alpha_2} + 80\,A_0\overline{E_1}\,\overline{\alpha_2} & 9 & -6 & 1 \\
6\,A_1C_1 & -1 & 0 & 0 \\
-6\,A_1C_1\overline{\zeta_0} & -1 & 4 & 1
\end{pmatrix}
$$

where

$$
\begin{aligned}
\omega_1 &= 32\,A_0C_1\alpha_1^2 - 4\,A_0C_1\zeta_0 + 48\,A_4C_1 + 12\,A_3C_2 - 2\,(16\,A_2C_1 + 5\,A_1C_2)\alpha_1 - 6\,(7\,A_1C_1 + 2\,A_0C_2)\alpha_3 + 4\,(A_0\alpha_1 - A_2)\gamma_1 \\
&\quad + \frac{1}{2}\,A_1\gamma_2 + 2\,C_1\overline{E_1} - 32\left(A_0^2\alpha_1 - A_0A_2\right)\overline{\zeta_0} \\
\omega_2 &= -\frac{4}{3}\,A_0|A_1|^4\overline{B_1} + \frac{4}{3}\,A_1|A_1|^4\overline{C_1} - 336\,A_0A_1\alpha_1\overline{\alpha_1}^2 + 32\,A_0^2|A_1|^2\overline{\alpha_{12}} + \frac{8}{3}\,A_1|A_1|^2\overline{B_3} - \frac{8}{3}\,A_0|A_1|^2\overline{B_5} + \frac{4}{3}\,A_2|A_1|^2\overline{C_2} \\
&\quad + 16\,A_0A_1\overline{\alpha_4}^2 + 2240\,A_0^2\overline{\alpha_2\alpha_7} + 2\,(20\,A_1C_1 + 7\,A_0C_2)\alpha_1^2 - 70\,A_0C_1\alpha_{10} - \frac{13}{4}\,A_0C_1\zeta_1 - 32\,A_0A_1\overline{\alpha_{19}} - 28\,A_0A_1\overline{\zeta_4} \\
&\quad + 70\,A_5C_1 + 20\,A_4C_2 - 2\,A_2C_4 + 2\,A_1C_5 - \frac{2}{3}\left(2\,A_0|A_1|^2\overline{C_2} + 60\,A_3C_1 + 21\,A_2C_2 - 3\,A_0C_4\right)\alpha_1 \\
&\quad + 2\,(47\,A_0C_1\alpha_1 - 27\,A_2C_1 - 9\,A_1C_2)\alpha_3 + \frac{2}{3}\left(60\,A_0A_1\overline{\alpha_3} + 2\,A_0\overline{B_3} - A_1\overline{C_2}\right)\alpha_4 + 84\left(20\,A_0^2\alpha_2 - A_0B_1\right)\alpha_6 \\
&\quad + 2\left(24\,A_0A_1\overline{\alpha_1} - A_1\overline{C_1}\right)\beta + (8\,A_1\alpha_1 + 9\,A_0\alpha_3 - 9\,A_3)\gamma_1 + \frac{3}{2}\,A_1\gamma_3 - \frac{1}{2}\,(11\,A_1C_1 + 4\,A_0C_2)\zeta_0 + C_2\overline{E_1} + \frac{13}{8}\,C_1\overline{E_2} \\
&\quad - 2\left(656\,A_0A_1|A_1|^4 - A_1\alpha_1\overline{C_1}\right)\overline{\alpha_1} - 16\left(2\,A_0^2\alpha_1 + A_1^2 - 2\,A_0A_2\right)\overline{\alpha_{11}} + \frac{1}{24}\left(1960\,A_0C_1\overline{\alpha_1} - 2240\,A_0B_2 - 171\,C_1\overline{C_1}\right)\overline{\alpha_2} \\
&\quad + 8\left(6\,A_0^2\alpha_1|A_1|^2 + 5\,A_1^2|A_1|^2 - 10\,A_0A_2|A_1|^2\right)\overline{\alpha_3} - 2\left(32\,A_0^2|A_1|^4 + A_0\alpha_1\overline{C_1} - A_2\overline{C_1} - 12\left(2\,A_0^2\alpha_1 + A_1^2 - 2\,A_0A_2\right)\overline{\alpha_1}\right)\overline{\alpha_4} \\
&\quad + 16\left(4\,A_0A_1|A_1|^2 - A_0^2\alpha_4\right)\overline{\alpha_5} - 4\left(11\,A_0A_1\alpha_1 + 18\,A_0^2\alpha_3 - 2\,A_1A_2 - 18\,A_0A_3\right)\overline{\zeta_0} \\
\omega_3 &= 520\,A_0A_1\alpha_2^2 - \frac{320}{9}\,A_0E_2|A_1|^2 - 800\,A_0A_1\overline{\alpha_{13}} - A_0C_1\overline{\alpha_9} + \frac{1}{120}\left(22\,B_1|A_1|^2 + 5\,B_5\right)C_1 - \frac{1}{24}\,B_2C_2 + 20\,A_1E_4 \\
&\quad + \frac{1}{9}\left(139\,A_0C_1|A_1|^2 - 54\,A_1B_1\right)\alpha_2 + \frac{15}{8}\,E_1\overline{B_1} + \frac{5}{48}\left(C_1^2 - 144\,A_0E_1\right)\overline{\alpha_4} - (13\,A_1C_1 - A_0C_2)\overline{\alpha_7} \\
\omega_4 &= \frac{1}{8}\,C_1C_2|A_1|^2 - 49\,A_1E_1|A_1|^2 - 21\,A_0E_3|A_1|^2 - 48\,A_0E_2\alpha_1 - 432\,A_0A_1\overline{\alpha_{14}} - 4\,A_1C_1\overline{\alpha_6} + \frac{1}{2}\,A_0C_1\overline{\zeta_1} + 48\,A_2E_2
\end{aligned}
$$

$$+\,12\,A_1E_5+8\,(3\,A_0C_1\alpha_1-3\,A_2C_1-A_1C_2)\alpha_2+\frac{1}{8}\left(C_1^2-336\,A_0E_1\right)\alpha_4-\frac{1}{20}\,(20\,A_0\alpha_2-B_1)\gamma_1-\frac{1}{40}\,C_1\overline{\gamma_2}$$
$$+\,\frac{2}{5}\left(20\,A_0^2\alpha_2-A_0B_1\right)\overline{\zeta_0}$$
$$\omega_5=288\,A_0^2|A_1|^2\overline{\alpha_8}-\frac{72}{7}\,A_0B_4|A_1|^2-\frac{1}{5}\,B_1|A_1|^2\overline{C_1}-84\,A_0E_1\alpha_3-192\,A_0A_1\overline{\alpha_{15}}+\frac{1}{2}\,A_1C_1\overline{\zeta_0}+6\,A_1B_6+84\,A_3E_1$$
$$+\,30\,A_2E_3-2\,(32\,A_1E_1+15\,A_0E_3)\alpha_1-\frac{2}{7}\left(2016\,A_0^2|A_1|^2\overline{\alpha_1}-104\,A_0|A_1|^2\overline{C_1}+14\,A_0\overline{B_3}+35\,A_1\overline{C_2}\right)\alpha_2-\frac{3}{64}\,C_1\gamma_2$$
$$+\,\frac{1}{2}\,B_2\overline{B_1}+\frac{1}{5}\,B_1\overline{B_3}+\frac{1}{280}\left(1152\,A_0B_1|A_1|^2+35\,C_1\overline{B_1}\right)\overline{\alpha_1}-\frac{3}{7}\left(3\,A_0C_1|A_1|^2-728\,A_0A_1\alpha_2\right)\overline{\alpha_3}$$
$$-\,(A_0C_1\overline{\alpha_1}+4\,A_0B_2)\overline{\alpha_4}+\frac{12}{5}\left(20\,A_0^2\alpha_2-A_0B_1\right)\overline{\alpha_5}+6\left(56\,A_0A_1\overline{\alpha_1}+16\,A_0^2\overline{\alpha_4}-2\,A_0\overline{B_1}-3\,A_1\overline{C_1}\right)\overline{\alpha_7}$$
$$\omega_6=720\,A_0A_1\alpha_2|A_1|^2+64\,A_0^2|A_1|^2\overline{\alpha_6}-16\,A_1B_1|A_1|^2-\frac{16}{5}\,A_0B_3|A_1|^2+8\,A_0C_1\alpha_1\overline{\alpha_1}-16\,A_0B_2\alpha_1-8\,A_0C_1\beta$$
$$-\,48\,A_0A_1\overline{\alpha_9}+16\,A_2B_2+2\,A_1B_5+\frac{1}{5}\left(98\,A_0|A_1|^4+5\,\overline{B_2}\right)C_1+\frac{64}{5}\left(20\,A_0^2\alpha_2-A_0B_1\right)\alpha_4+\frac{1}{8}\,C_2\overline{B_1}$$
$$-\,(5\,A_1C_1+A_0C_2)\overline{\alpha_4}+96\left(4\,A_0^2\alpha_1-A_1^2-4\,A_0A_2\right)\overline{\alpha_7}$$
$$\omega_7=144\,A_0A_1\overline{\alpha_1}^3-16\,A_0^2|A_1|^2\overline{\zeta_1}-\frac{108}{5}\,A_2B_1|A_1|^2-\frac{32}{5}\,A_1B_3|A_1|^2+\frac{4}{5}\,A_0|A_1|^2\overline{\gamma_2}-15\,A_0C_1\alpha_{12}$$
$$+\,(13\,A_1C_1+3\,A_0C_2)\alpha_1\overline{\alpha_1}+12\,A_0A_1\overline{\zeta_2}+30\,A_3B_2+6\,A_2B_5+\frac{1}{40}\left(1088\,A_1|A_1|^4+75\,\overline{B_4}\right)C_1$$
$$+\,\frac{1}{8}\left(64\,A_0|A_1|^4+3\,\overline{B_2}\right)C_2+\frac{3}{8}\left(160\,A_0B_1|A_1|^2-64\,A_1B_2-16\,A_0B_5+C_1\overline{B_1}\right)\alpha_1$$
$$-\,24\left(86\,A_0^2\alpha_1|A_1|^2-17\,A_1^2|A_1|^2-54\,A_0A_2|A_1|^2\right)\alpha_2+15\,(A_0C_1\overline{\alpha_1}-2\,A_0B_2)\alpha_3$$
$$+\,\frac{1}{5}\left(187\,A_0C_1|A_1|^2+3200\,A_0A_1\alpha_2-98\,A_1B_1-24\,A_0B_3\right)\alpha_4+24\left(20\,A_0^2\alpha_2-A_0B_1\right)\alpha_5-(13\,A_1C_1+3\,A_0C_2)\beta$$
$$+\,\frac{1}{16}\left(8\,A_0\overline{\alpha_4}-\overline{B_1}\right)\gamma_1+\frac{5}{16}\left(C_1^2-720\,A_0E_1\right)\overline{\alpha_2}+2\,(9\,A_0C_1\alpha_1-3\,A_2C_1-A_1C_2)\overline{\alpha_4}+16\left(13\,A_0A_1|A_1|^2+6\,A_0^2\alpha_4\right)\overline{\alpha_6}$$
$$+\,48\left(19\,A_0A_1\alpha_1+15\,A_0^2\alpha_3-7\,A_1A_2-15\,A_0A_3\right)\overline{\alpha_7}+24\left(6\,A_0^2\alpha_1-A_1^2-6\,A_0A_2\right)\overline{\alpha_9}-\frac{1}{2}\,A_1\overline{\gamma_3}$$
$$-\,\frac{1}{2}\left(40\,A_0A_1\overline{\alpha_1}+8\,A_0^2\overline{\alpha_4}-A_0\overline{B_1}-2\,A_1\overline{C_1}\right)\overline{\zeta_0}$$
$$\omega_8=\frac{356}{15}\,A_0B_1|A_1|^4+80\,A_0^2|A_1|^2\overline{\alpha_9}-\frac{50}{3}\,A_1B_2|A_1|^2-\frac{10}{3}\,A_0B_5|A_1|^2+4\,A_0C_1\alpha_1\overline{\alpha_3}-16\,A_0B_4\alpha_1-4\,A_0C_1\overline{\alpha_{12}}$$
$$-\,56\,A_0A_1\overline{\alpha_{16}}+16\,A_2B_4+2\,A_1B_7+\frac{1}{120}\left(11\,|A_1|^2\overline{B_1}+40\,\overline{B_5}\right)C_1-\frac{1}{24}\left(|A_1|^2\overline{C_1}-\overline{B_3}\right)C_2$$
$$-\,2\left(464\,A_0^2|A_1|^4-20\,A_0\alpha_1\overline{C_1}+9\,A_1\overline{B_1}+12\,A_0\overline{B_2}+20\,A_2\overline{C_1}+96\left(5\,A_0^2\alpha_1-A_1^2-4\,A_0A_2\right)\overline{\alpha_1}\right)\alpha_2$$
$$+\,\frac{1}{6}\left(16\,A_0C_1\overline{\alpha_1}-80\,A_0B_2-C_1\overline{C_1}\right)\alpha_4+\frac{48}{5}\left(20\,A_0^2\alpha_2-A_0B_1\right)\beta+\frac{3}{20}\,B_3\overline{B_1}+\frac{6}{5}\,B_1\overline{B_2}$$
$$+\,\frac{2}{15}\left(25\,A_1C_1|A_1|^2+5\,A_0C_2|A_1|^2+72\,A_0B_1\alpha_1\right)\overline{\alpha_1}+\frac{1}{15}\left(139\,A_0C_1|A_1|^2+4800\,A_0A_1\alpha_2-90\,A_1B_1-18\,A_0B_3\right)\overline{\alpha_4}$$
$$-\,\frac{1}{2}\,(5\,A_1C_1+A_0C_2)\overline{\alpha_5}+\left(96\,A_0A_1\overline{\alpha_1}+24\,A_0^2\overline{\alpha_4}-3\,A_0\overline{B_1}-5\,A_1\overline{C_1}\right)\overline{\alpha_6}+80\left(11\,A_0A_1|A_1|^2+4\,A_0^2\alpha_4\right)\overline{\alpha_7}$$
$$+\,112\left(4\,A_0^2\alpha_1-A_1^2-4\,A_0A_2\right)\overline{\alpha_8}$$
$$\omega_9=32\,A_0A_1\alpha_1\overline{\zeta_0}-70\,A_0C_1\zeta_1+35\,C_1\overline{C_1}\overline{\alpha_2}-32\,A_0A_1\overline{\zeta_4}+2\,(A_0\alpha_1-A_2)\gamma_2+2\,A_1\gamma_3-4\,(16\,A_1C_1+5\,A_0C_2)\zeta_0$$
$$+\,10\,C_2\overline{E_1}+35\,C_1\overline{E_2}$$
$$\omega_{10}=-15\,A_1C_1|A_1|^2-3\,A_0C_2|A_1|^2-16\,A_0B_1\alpha_1-12\,A_0C_1\alpha_4-40\,A_0A_1\overline{\alpha_6}+16\,A_2B_1+2\,A_1B_3$$
$$+\,80\left(4\,A_0^2\alpha_1-A_1^2-4\,A_0A_2\right)\alpha_2$$
$$\omega_{11}=-12\,A_0^2|A_1|^2\overline{\zeta_0}-20\,A_2C_1|A_1|^2-6\,A_1C_2|A_1|^2+\frac{3}{2}\,A_0\gamma_1|A_1|^2-\frac{45}{2}\,A_0C_1\alpha_5+10\,A_0A_1\overline{\zeta_1}+30\,A_3B_1+6\,A_2B_3$$
$$+\,2\left(28\,A_0C_1|A_1|^2-12\,A_1B_1-3\,A_0B_3\right)\alpha_1+40\,(19\,A_0A_1\alpha_1-7\,A_1A_2-15\,A_0A_3)\alpha_2+30\left(20\,A_0^2\alpha_2-A_0B_1\right)\alpha_3$$
$$-\,\frac{1}{2}\,(37\,A_1C_1+9\,A_0C_2)\alpha_4+20\left(6\,A_0^2\alpha_1-A_1^2-6\,A_0A_2\right)\overline{\alpha_6}-\frac{1}{2}\,A_1\overline{\gamma_2}$$
$$\omega_{12}=32\,A_0B_1\alpha_1^2-24\,A_3C_1|A_1|^2-8\,A_2C_2|A_1|^2-\frac{3}{4}\,A_0\gamma_2|A_1|^2-36\,A_0C_1\alpha_{11}-10\,A_0A_1\overline{\zeta_3}+48\,A_4B_1+12\,A_3B_3$$
$$+\left(75\,A_1C_1|A_1|^2+23\,A_0C_2|A_1|^2-32\,A_2B_1-10\,A_1B_3\right)\alpha_1-4\left\{200\,A_0^2\alpha_1^2+40\,A_2^2+120\,A_1A_3+240\,A_0A_4\right.$$

$$-20\left(5A_1^2+12A_0A_2\right)\alpha_1+13A_0\overline{E_1}\Big\}\alpha_2+3\left(29A_0C_1|A_1|^2+440A_0A_1\alpha_2-14A_1B_1-4A_0B_3\right)\alpha_3$$
$$+8\left(6A_0C_1\alpha_1-3A_2C_1-A_1C_2\right)\alpha_4-3\left(11A_1C_1+3A_0C_2\right)\alpha_5+\frac{3}{2}\left(3A_1|A_1|^2+2A_0\alpha_4\right)\gamma_1+A_1\gamma_4$$
$$+\frac{26}{5}\left(20A_0^2\alpha_2-A_0B_1\right)\zeta_0+\frac{13}{5}B_1\overline{E_1}+9C_1\overline{E_3}+40\left(7A_0A_1\alpha_1+6A_0^2\alpha_3-2A_1A_2-6A_0A_3\right)\overline{\alpha_6}+2\left(A_0\alpha_1-A_2\right)\overline{\gamma_2}$$
$$-4\left(5A_0A_1|A_1|^2+6A_0^2\alpha_4\right)\overline{\zeta_0}-40\left(A_0^2\alpha_1-A_0A_2\right)\overline{\zeta_1}$$
$$\omega_{13}=-1472A_0A_1|A_1|^6+48A_0^2\alpha_{12}|A_1|^2+208A_0A_1\beta|A_1|^2+6A_2|A_1|^2\overline{B_1}+2A_1|A_1|^2\overline{B_2}-6A_0|A_1|^2\overline{B_4}+6A_3|A_1|^2\overline{C_1}$$
$$+840A_0^2\overline{\alpha_2\alpha_6}-120A_0C_1\alpha_9-72A_0A_1\delta-48A_1^2\overline{\alpha_{12}}-6\left(2A_0|A_1|^2\overline{B_1}-A_1\overline{B_3}\right)\alpha_1$$
$$+6\left(24A_0^2|A_1|^2\overline{\alpha_1}-A_0|A_1|^2\overline{C_1}+A_0\overline{B_3}\right)\alpha_3-2\left(48A_0^2|A_1|^4-32A_1^2\overline{\alpha_1}-A_1\overline{B_1}\right)\alpha_4$$
$$+3\left(32A_0A_1\overline{\alpha_1}-8A_0^2\overline{\alpha_4}+A_0\overline{B_1}-A_1\overline{C_1}\right)\alpha_5-\frac{5}{2}\left(43A_1C_1+15A_0C_2\right)\alpha_6+\frac{672}{5}\left(20A_0^2\alpha_2-A_0B_1\right)\alpha_7$$
$$-6A_3\overline{B_3}+6A_1\overline{B_7}-16\left(125A_0A_1\alpha_1|A_1|^2-4A_1A_2|A_1|^2+12A_0A_3|A_1|^2\right)\overline{\alpha_1}$$
$$+\left(269A_0C_1|A_1|^2+3952A_0A_1\alpha_2-118A_1B_1-42A_0B_3\right)\overline{\alpha_2}+24\left(2A_1^2\alpha_1+3A_0A_1\alpha_3\right)\overline{\alpha_3}$$
$$+8\left(6A_0^2\alpha_1|A_1|^2+15A_1^2|A_1|^2-6A_0A_2|A_1|^2+8A_0A_1\alpha_4\right)\overline{\alpha_4}+24\left(A_0A_1\alpha_1-3A_0^2\alpha_3-A_1A_2+3A_0A_3\right)\overline{\alpha_5}$$
$$\omega_{14}=192A_1A_2|A_1|^4-192A_0A_3|A_1|^4+240A_0A_1\alpha_5|A_1|^2+56A_0A_1\alpha_4^2-10A_0\alpha_1^2\overline{B_1}-300A_0^2\overline{\alpha_2\zeta_0}-96A_0A_1\alpha_{19}$$
$$-189A_0C_1\alpha_8-52A_0A_1\zeta_4+\frac{75}{2}A_0\gamma_1\overline{\alpha_2}-2\left(1088A_0A_1|A_1|^4-5A_2\overline{B_1}-4A_1\overline{B_2}+3A_0\overline{B_4}\right)\alpha_1$$
$$+24\left(2A_0^2\alpha_1-3A_1^2-2A_0A_2\right)\alpha_{12}+6\left(32A_0^2|A_1|^4+2A_1\overline{B_1}+A_0\overline{B_2}+4\left(3A_1^2+2A_0A_2\right)\overline{\alpha_1}\right)\alpha_3$$
$$-8\left(10A_0^2\alpha_1|A_1|^2-23A_1^2|A_1|^2-10A_0A_2|A_1|^2\right)\alpha_4-3\left(55A_1C_1+21A_0C_2\right)\alpha_7$$
$$+16\left(7A_0A_1\alpha_1-3A_0^2\alpha_3-5A_1A_2+3A_0A_3\right)\beta+\frac{1}{2}\left(152A_0A_1\overline{\alpha_1}-8A_0^2\overline{\alpha_4}+A_0\overline{B_1}-6A_1\overline{C_1}\right)\zeta_0-12A_4\overline{B_1}$$
$$-6A_3\overline{B_2}+6A_2\overline{B_4}+12A_1\overline{B_6}-\frac{1}{4}\overline{B_1E_1}-16\left(34A_0A_1\alpha_1^2-\left(5A_1A_2-3A_0A_3\right)\alpha_1\right)\overline{\alpha_1}$$
$$+2\left(117A_0C_1\alpha_1-75A_2C_1-25A_1C_2\right)\overline{\alpha_2}+2\left(8A_0^2\alpha_1^2+32A_1^2\alpha_1-8A_2^2+48A_0A_4+A_0\overline{E_1}\right)\overline{\alpha_4}$$
$$\omega_{15}=80A_1^2|A_1|^4-32A_0A_2|A_1|^4+112A_0A_1\alpha_4|A_1|^2+32A_1^2\alpha_1\overline{\alpha_1}-48A_0A_1\alpha_{12}-144A_0C_1\alpha_7-32A_1^2\beta$$
$$+2\left(16A_0^2|A_1|^4+3A_1\overline{B_1}\right)\alpha_1+6\left(8A_0A_1\overline{\alpha_1}+A_0\overline{B_1}\right)\alpha_3-6A_3\overline{B_1}+6A_1\overline{B_4}-5\left(25A_1C_1+9A_0C_2\right)\overline{\alpha_2}$$
$$+16\left(A_0A_1\alpha_1-3A_0^2\alpha_3-A_1A_2+3A_0A_3\right)\overline{\alpha_4}$$
$$\omega_{16}=32A_0A_1|A_1|^4+16A_0A_1\alpha_1\overline{\alpha_1}-16A_0A_1\beta+2A_0\alpha_1\overline{B_1}-105A_0C_1\overline{\alpha_2}-2A_2\overline{B_1}+2A_1\overline{B_2}$$
$$-8\left(2A_0^2\alpha_1+A_1^2-2A_0A_2\right)\overline{\alpha_4}$$
$$\omega_{17}=2A_0C_1|A_1|^2\overline{\alpha_1}+240A_0^2|A_1|^2\overline{\alpha_7}-10A_0B_2|A_1|^2-\frac{1}{8}C_1|A_1|^2\overline{C_1}-\frac{3}{2}A_0C_1\overline{\alpha_5}-168A_0A_1\overline{\alpha_8}+6A_1B_4$$
$$+3\left(96A_0A_1\overline{\alpha_1}-3A_0\overline{B_1}-5A_1\overline{C_1}\right)\alpha_2+\frac{9}{20}B_1\overline{B_1}+\frac{1}{8}C_1\overline{B_3}+\frac{18}{5}\left(20A_0^2\alpha_2-A_0B_1\right)\overline{\alpha_4}$$
$$\omega_{18}=-32A_0^2|A_1|^6+16A_0^2\beta|A_1|^2+2A_1|A_1|^2\overline{B_1}-2A_0|A_1|^2\overline{B_2}+2A_2|A_1|^2\overline{C_1}+24A_0A_1\alpha_1\overline{\alpha_3}-\frac{175}{2}A_0C_1\alpha_6$$
$$-24A_0A_1\overline{\alpha_{12}}-2\left(A_0|A_1|^2\overline{C_1}-A_0\overline{B_3}\right)\alpha_1+\left(32A_0A_1\overline{\alpha_1}+A_0\overline{B_1}-A_1\overline{C_1}\right)\alpha_4-2A_2\overline{B_3}+2A_1\overline{B_5}$$
$$+16\left(3A_0^2\alpha_1|A_1|^2+2A_1^2|A_1|^2-4A_0A_2|A_1|^2\right)\overline{\alpha_1}+98\left(20A_0^2\alpha_2-A_0B_1\right)\overline{\alpha_2}+8\left(6A_0A_1|A_1|^2-A_0^2\alpha_4\right)\overline{\alpha_4}$$
$$-12\left(2A_0^2\alpha_1+A_1^2-2A_0A_2\right)\overline{\alpha_5}$$
$$\omega_{19}=-752A_0A_1\alpha_1^2|A_1|^2-160A_0^2\alpha_{10}|A_1|^2+44A_0^2\zeta_1|A_1|^2-32A_2A_3|A_1|^2+32A_1A_4|A_1|^2+160A_0A_5|A_1|^2$$
$$+2A_1|A_1|^2\overline{E_1}-22A_0|A_1|^2\overline{E_2}-80A_0A_1\overline{\alpha_2\alpha_3}-400A_0^2\overline{\alpha_2\alpha_5}-80A_0A_1\alpha_{18}-58A_0A_1\zeta_3$$
$$+16\left(11A_1A_2|A_1|^2-5A_0A_3|A_1|^2-A_0\overline{E_3}\right)\alpha_1+64\left(A_0^2\alpha_1-A_1^2-A_0A_2\right)\alpha_{11}$$
$$+8\left(12A_0^2\alpha_1|A_1|^2+11A_1^2|A_1|^2+2A_0A_2|A_1|^2\right)\alpha_3-4\Big\{8A_0^2\alpha_1^2-10A_0A_1\alpha_3+8A_2^2+8A_1A_3-16A_0A_4$$
$$-4\left(5A_1^2+4A_0A_2\right)\alpha_1+A_0\overline{E_1}\Big\}\alpha_4+8\left(19A_0A_1\alpha_1-12A_1A_2\right)\alpha_5-10\left(24A_0^2\overline{\alpha_4}-3A_0\overline{B_1}-A_1\overline{C_1}\right)\alpha_6$$
$$+4\left(51A_0A_1|A_1|^2+2A_0^2\alpha_4\right)\zeta_0+16A_2\overline{E_3}+20A_1\overline{E_5}+\frac{2}{3}\left(2160A_0^2|A_1|^2\overline{\alpha_1}-113A_0|A_1|^2\overline{C_1}+50A_0\overline{B_3}+20A_1\overline{C_2}\right)\overline{\alpha_2}$$
$$\omega_{20}=-480A_0^2\alpha_9|A_1|^2-480A_0^2\beta\overline{\alpha_2}-800A_0A_1\overline{\alpha_2\alpha_4}-60A_0A_1\zeta_2-12A_0\alpha_3\overline{E_1}-30A_0\alpha_1\overline{E_2}$$

$$- 80\left(10\,A_0A_1|A_1|^2 + 3\,A_0^2\alpha_4\right)\alpha_6 - 30\left(8\,A_0A_1\overline{\alpha_1} + 16\,A_0^2\overline{\alpha_4} - 2\,A_0\overline{B_1} - A_1\overline{C_1}\right)\alpha_7$$
$$+ 4\left(37\,A_0A_1\alpha_1 + 6\,A_0^2\alpha_3 - 22\,A_1A_2 - 6\,A_0A_3\right)\zeta_0 + 10\left(6\,A_0^2\alpha_1 - 5\,A_1^2 - 6\,A_0A_2\right)\zeta_1 + 12\,A_3\overline{E_1} + 30\,A_2\overline{E_2}$$
$$+ 30\,A_1\overline{E_4} + 10\left(288\,A_0^2|A_1|^4 - 3\,A_0\alpha_1\overline{C_1} + 8\,A_1\overline{B_1} + 6\,A_0\overline{B_2} + 3\,A_2\overline{C_1} + 4\left(18\,A_0^2\alpha_1 - 5\,A_1^2 - 6\,A_0A_2\right)\overline{\alpha_1}\right)\overline{\alpha_2}$$
$$\omega_{21} = -320\,A_0^2\alpha_6|A_1|^2 - 320\,A_0^2\overline{\alpha_2\alpha_4} - 40\,A_0A_1\zeta_1 - 16\,A_0\alpha_1\overline{E_1} + 32\left(A_0^2\alpha_1 - A_1^2 - A_0A_2\right)\zeta_0 + 16\,A_2\overline{E_1} + 20\,A_1\overline{E_2}$$
$$- 20\left(8\,A_0A_1\overline{\alpha_1} - 2\,A_0\overline{B_1} - A_1\overline{C_1}\right)\overline{\alpha_2}$$
$$\omega_{22} = 16\,A_0^2\alpha_1^2|A_1|^2 + 64\,A_1^2\alpha_1|A_1|^2 + 20\,A_0^2\zeta_0|A_1|^2 - 16\,A_2^2|A_1|^2 + 96\,A_0A_4|A_1|^2 - 10\,A_0|A_1|^2\overline{E_1} - 48\,A_0A_1\alpha_{11}$$
$$+ 8\left(7\,A_0A_1\alpha_1 - 3\,A_0^2\alpha_3 - 5\,A_1A_2 + 3\,A_0A_3\right)\alpha_4 + 12\left(2\,A_0^2\alpha_1 - 3\,A_1^2 - 2\,A_0A_2\right)\alpha_5 + 12\,A_1\overline{E_3}$$
$$\omega_{23} = -1088\,A_0A_1\alpha_3\overline{\alpha_2} + 20\,A_0^2\zeta_0\overline{\alpha_2} + 448\,A_0A_1\alpha_{15} - 16\left(91\,A_0A_1\alpha_1 + 27\,A_0^2\alpha_3 - 51\,A_1A_2 - 27\,A_0A_3\right)\alpha_7$$
$$- 56\left(10\,A_0^2\alpha_1 - 7\,A_1^2 - 10\,A_0A_2\right)\alpha_8 + 2\left\{440\,A_0^2\alpha_1^2 + 200\,A_2^2 + 320\,A_1A_3 + 80\,A_0A_4\right.$$
$$\left.- 32\left(13\,A_1^2 + 20\,A_0A_2\right)\alpha_1 - 5\,A_0\overline{E_1}\right\}\overline{\alpha_2}$$
$$\omega_{24} = -160\,A_0^2\zeta_1|A_1|^2 + 80\,A_0|A_1|^2\overline{C_1}\overline{\alpha_2} + 16\,A_1|A_1|^2\overline{E_1} + 80\,A_0|A_1|^2\overline{E_2} - 80\,A_0A_1\zeta_3 + 32\,A_0\alpha_4\overline{E_1}$$
$$- 32\left(A_0A_1|A_1|^2 + 2\,A_0^2\alpha_4\right)\zeta_0$$
$$\omega_{25} = -1008\,A_0^2\alpha_8|A_1|^2 - 1040\,A_0A_1\alpha_4\overline{\alpha_2} - 360\,A_0^2\alpha_5\overline{\alpha_2} + 168\,A_0A_1\alpha_{16} - 8\left\{61\,A_0A_1\alpha_1 + 15\,A_0^2\alpha_3 - 35\,A_1A_2\right.$$
$$\left.- 15\,A_0A_3\right\}\alpha_6 - 48\left(47\,A_0A_1|A_1|^2 + 12\,A_0^2\alpha_4\right)\alpha_7 - 48\left(4\,A_0^2\alpha_1 - 3\,A_1^2 - 4\,A_0A_2\right)\alpha_9$$
$$+ 8\left(386\,A_0^2\alpha_1|A_1|^2 - 145\,A_1^2|A_1|^2 - 230\,A_0A_2|A_1|^2\right)\overline{\alpha_2}$$

Then we have

$$- K_g\,\vec{h}_0 \dot{\otimes} \vec{h}_0 =$$

$$
\begin{pmatrix}
16\,A_1^2|A_1|^2 & 3 & -3 & 0\\
-64\,A_0A_1\alpha_1|A_1|^2+64\,A_1A_2|A_1|^2+16\,A_1^2\alpha_4 & 4 & -3 & 0\\
\kappa_1 & 5 & -3 & 0\\
\kappa_2 & 6 & -3 & 0\\
-4\,A_1C_1|A_1|^2-80\,A_1^2\alpha_2 & 0 & 1 & 0\\
8\,A_0C_1\alpha_1|A_1|^2+320\,A_0A_1\alpha_1\alpha_2-8\,A_2C_1|A_1|^2-2\,A_1C_2|A_1|^2-320\,A_1A_2\alpha_2-4\,A_1C_1\alpha_4-40\,A_1^2\overline{\alpha_6} & 1 & 1 & 0\\
8\,A_0C_1|A_1|^4+384\,A_0A_1\alpha_2|A_1|^2-\frac{16}{5}\,A_1B_1|A_1|^2-4\,A_1C_1\overline{\alpha_4}-96\,A_1^2\overline{\alpha_7} & 0 & 2 & 0\\
\kappa_3 & 0 & 3 & 0\\
\kappa_4 & 1 & 2 & 0\\
-40\,A_0C_1\alpha_1\alpha_2+\frac{1}{4}\,C_1C_2|A_1|^2-4\,A_1E_1|A_1|^2+40\,A_2C_1\alpha_2+10\,A_1C_2\alpha_2+\frac{1}{4}\,C_1^2\alpha_4+10\,A_1C_1\overline{\alpha_6} & -2 & 5 & 0\\
\kappa_5 & 2 & 1 & 0\\
64\,A_0^2|A_1|^6-16\,A_1^2|A_1|^2\overline{\alpha_1}-96\,A_0A_1|A_1|^2\overline{\alpha_4}+4\,A_1|A_1|^2\overline{B_1}-24\left(2\,|A_1|^2\overline{\alpha_1}-\overline{\alpha_5}\right)A_1^2 & 3 & -1 & 0\\
\kappa_6 & 4 & -1 & 0\\
\kappa_7 & 3 & 0 & 0\\
-128\,A_0A_1\zeta_0|A_1|^2+64\,A_1|A_1|^2\overline{E_1} & 6 & -3 & 1\\
-64\,A_0A_1|A_1|^4+16\,A_1^2\overline{\alpha_4} & 3 & -2 & 0\\
\kappa_8 & 4 & -2 & 0\\
\kappa_9 & 5 & -2 & 0\\
-64\,A_0A_1\alpha_1\zeta_0+80\,A_1^2\overline{\alpha_1\alpha_2}+64\,A_1A_2\zeta_0+20\,A_1^2\zeta_1-20\,A_1\overline{B_1}\overline{\alpha_2} & 7 & -4 & 0\\
\kappa_{10} & 8 & -5 & 0\\
160\,A_0A_1|A_1|^2\overline{\alpha_2}-40\,A_1^2\alpha_6 & 7 & -5 & 0\\
\frac{1}{4}\,C_1^2|A_1|^2+20\,A_1C_1\alpha_2 & -3 & 5 & 0\\
-48\,A_0C_1\alpha_2|A_1|^2-320\,A_0A_1\alpha_2^2+\frac{2}{5}\,B_1C_1|A_1|^2+16\,A_1B_1\alpha_2+\frac{1}{4}\,C_1^2\overline{\alpha_4}+24\,A_1C_1\overline{\alpha_7} & -3 & 6 & 0\\
-\frac{5}{4}\,C_1^2\alpha_2 & -6 & 9 & 0\\
320\,A_0A_1\alpha_1\overline{\alpha_2}-96\,A_1^2\alpha_7-320\,A_1A_2\overline{\alpha_2} & 8 & -6 & 0\\
\kappa_{11} & 9 & -6 & 0\\
16\,A_1^2\zeta_0 & 6 & -4 & 0\\
-80\,A_1^2\overline{\alpha_2} & 7 & -6 & 0\\
800\,A_0A_1\overline{\alpha_2}^2 & 11 & -8 & 0
\end{pmatrix}
$$

where

$$
\begin{aligned}
\kappa_1 &= 64\,A_0^2\alpha_1^2|A_1|^2-80\,A_1^2\alpha_1|A_1|^2-128\,A_0A_2\alpha_1|A_1|^2-96\,A_0A_1\alpha_3|A_1|^2\\
&\quad-64\,A_0A_1\alpha_1\alpha_4+64\,A_2^2|A_1|^2+96\,A_1A_3|A_1|^2-24\left(2\,\alpha_1|A_1|^2-\alpha_5\right)A_1^2+64\,A_1A_2\alpha_4\\
\kappa_2 &= 192\,A_0A_1\alpha_1^2|A_1|^2+192\,A_0^2\alpha_1\alpha_3|A_1|^2+64\,A_0^2\alpha_1^2\alpha_4+16\left(4\,\alpha_1^2-\zeta_0\right)A_0A_1|A_1|^2-256\,A_1A_2\alpha_1|A_1|^2\\
&\quad-192\,A_0A_3\alpha_1|A_1|^2-112\,A_1^2\alpha_3|A_1|^2-192\,A_0A_2\alpha_3|A_1|^2-64\,A_0A_1\zeta_0|A_1|^2+96\left(2\,\alpha_1|A_1|^2-\alpha_5\right)A_0A_1\alpha_1\\
&\quad-80\,A_1^2\alpha_1\alpha_4-128\,A_0A_2\alpha_1\alpha_4-96\,A_0A_1\alpha_3\alpha_4+192\,A_2A_3|A_1|^2+128\,A_1A_4|A_1|^2+8\,A_1|A_1|^2\overline{E_1}\\
&\quad-32\left(2\,\alpha_3|A_1|^2+\alpha_1\alpha_4-\alpha_{11}\right)A_1^2-96\left(2\,\alpha_1|A_1|^2-\alpha_5\right)A_1A_2+64\,A_2^2\alpha_4+96\,A_1A_3\alpha_4
\end{aligned}
$$

$$
\begin{aligned}
\kappa_3 &= -448\,A_0^2\alpha_2|A_1|^4 + \frac{32}{5}\,A_0B_1|A_1|^4 + \frac{4}{3}\,A_1C_1|A_1|^2\overline{\alpha_1} + 12\,A_0C_1|A_1|^2\overline{\alpha_4} + 448\,A_0A_1|A_1|^2\overline{\alpha_7} - \frac{8}{3}\,A_1B_2|A_1|^2\\
&\quad - \frac{1}{2}\,C_1|A_1|^2\overline{B_1} + 80\,A_1^2\alpha_2\overline{\alpha_1} + 224\,A_0A_1\alpha_2\overline{\alpha_4} + 112\,(\alpha_2\overline{\alpha_1} - \overline{\alpha_8})A_1^2 + 6\,\big(2\,|A_1|^2\overline{\alpha_1} - \overline{\alpha_5}\big)A_1C_1 - 20\,A_1\alpha_2\overline{B_1}\\
&\quad - \frac{16}{5}\,A_1B_1\overline{\alpha_4}\\
\kappa_4 &= -768\,A_0^2\alpha_1\alpha_2|A_1|^2 + \frac{44}{5}\,A_1C_1|A_1|^4 + 4\,A_0C_2|A_1|^4 + \frac{32}{5}\,A_0B_1\alpha_1|A_1|^2 + 528\,A_1^2\alpha_2|A_1|^2 + 768\,A_0A_2\alpha_2|A_1|^2\\
&\quad + 16\,A_0C_1\alpha_4|A_1|^2 + 192\,A_0A_1|A_1|^2\overline{\alpha_6} + 384\,A_0A_1\alpha_2\alpha_4 - \frac{32}{5}\,A_2B_1|A_1|^2 - \frac{8}{5}\,A_1B_3|A_1|^2 + 8\,A_0C_1\alpha_1\overline{\alpha_4}\\
&\quad + 384\,A_0A_1\alpha_1\overline{\alpha_7} + 48\,\big(2\,\alpha_2|A_1|^2 - \overline{\alpha_9}\big)A_1^2 + 8\,\big(2\,|A_1|^4 + \alpha_1\overline{\alpha_1} - \beta\big)A_1C_1 - \frac{16}{5}\,A_1B_1\alpha_4 - \frac{5}{4}\,C_1^2\overline{\alpha_2} - 8\,A_2C_1\overline{\alpha_4}\\
&\quad - 2\,A_1C_2\overline{\alpha_4} - 384\,A_1A_2\overline{\alpha_7}\\
\kappa_5 &= -320\,A_0^2\alpha_1^2\alpha_2 + 8\,A_1C_1\alpha_1|A_1|^2 + 4\,A_0C_2\alpha_1|A_1|^2 + 12\,A_0C_1\alpha_3|A_1|^2 - 16\,A_0A_1|A_1|^2\overline{\zeta_0} + 400\,A_1^2\alpha_1\alpha_2\\
&\quad + 640\,A_0A_2\alpha_1\alpha_2 + 480\,A_0A_1\alpha_2\alpha_3 + 8\,A_0C_1\alpha_1\alpha_4 - 12\,A_3C_1|A_1|^2 - 4\,A_2C_2|A_1|^2 + 2\,A_1\gamma_1|A_1|^2 + 160\,A_0A_1\alpha_1\overline{\alpha_6}\\
&\quad + 6\,\big(2\,\alpha_1|A_1|^2 - \alpha_5\big)A_1C_1 - 320\,A_2^2\alpha_2 - 480\,A_1A_3\alpha_2 - 8\,A_2C_1\alpha_4 - 2\,A_1C_2\alpha_4 - 160\,A_1A_2\overline{\alpha_6}\\
\kappa_6 &= 256\,A_0A_1|A_1|^6 + 192\,A_0^2\alpha_4|A_1|^4 + 64\,A_0A_1\alpha_1|A_1|^2\overline{\alpha_1} + 192\,A_0^2\alpha_1|A_1|^2\overline{\alpha_4} + 192\,\big(2\,|A_1|^4 + \alpha_1\overline{\alpha_1} - \beta\big)A_0A_1|A_1|^2\\
&\quad - 8\,A_0\alpha_1|A_1|^2\overline{B_1} - 64\,A_1A_2|A_1|^2\overline{\alpha_1} - 20\,A_0C_1|A_1|^2\overline{\alpha_2} - 144\,A_1^2|A_1|^2\overline{\alpha_4} - 192\,A_0A_2|A_1|^2\overline{\alpha_4}\\
&\quad + 96\,\big(2\,|A_1|^2\overline{\alpha_1} - \overline{\alpha_5}\big)A_0A_1\alpha_1 + 8\,A_2|A_1|^2\overline{B_1} + 8\,A_1|A_1|^2\overline{B_2} - 16\,A_1^2\alpha_4\overline{\alpha_1} + 480\,A_0A_1\alpha_2\overline{\alpha_2} - 96\,A_0A_1\alpha_4\overline{\alpha_4}\\
&\quad - 48\,\big(2\,|A_1|^2\overline{\alpha_4} + \alpha_4\overline{\alpha_1} + \alpha_1\overline{\alpha_3} - \overline{\alpha_{12}}\big)A_1^2 - 96\,\big(2\,|A_1|^2\overline{\alpha_1} - \overline{\alpha_5}\big)A_1A_2 + 10\,A_1C_1\alpha_6 + 4\,A_1\alpha_4\overline{B_1} + 16\,A_1B_1\overline{\alpha_2}\\
\kappa_7 &= 64\,A_0A_1|A_1|^4\overline{\alpha_1} + 128\,A_0^2|A_1|^4\overline{\alpha_4} - 8\,A_0|A_1|^4\overline{B_1} - \frac{8}{3}\,A_1|A_1|^4\overline{C_1} + 128\,\big(2\,|A_1|^2\overline{\alpha_1} - \overline{\alpha_5}\big)A_0A_1|A_1|^2 - 16\,A_1^2|A_1|^2\overline{\alpha_3}\\
&\quad + \frac{8}{3}\,A_1|A_1|^2\overline{B_3} - 16\,A_1^2\overline{\alpha_1\alpha_4} - 32\,A_0A_1\overline{\alpha_4}^2 - 32\,\big(2\,|A_1|^2\overline{\alpha_3} + \overline{\alpha_1\alpha_4} - \overline{\alpha_{11}}\big)A_1^2 - 4\,A_1C_1\zeta_0 + 4\,A_1\overline{B_1}\overline{\alpha_4}\\
\kappa_8 &= 128\,A_0^2\alpha_1|A_1|^4 - 96\,A_1^2|A_1|^4 - 128\,A_0A_2|A_1|^4 - 128\,A_0A_1\alpha_4|A_1|^2 - 64\,A_0A_1\alpha_1\overline{\alpha_4} - 32\,\big(2\,|A_1|^4 + \alpha_1\overline{\alpha_1} - \beta\big)A_1^2\\
&\quad + 20\,A_1C_1\overline{\alpha_2} + 64\,A_1A_2\overline{\alpha_4}\\
\kappa_9 &= 448\,A_0A_1\alpha_1|A_1|^4 + 192\,A_0^2\alpha_3|A_1|^4 + 256\,A_0^2\alpha_1\alpha_4|A_1|^2 - 320\,A_1A_2|A_1|^4 - 192\,A_0A_3|A_1|^4\\
&\quad + 192\,\big(2\,\alpha_1|A_1|^2 - \alpha_5\big)A_0A_1|A_1|^2 - 176\,A_1^2\alpha_4|A_1|^2 - 256\,A_0A_2\alpha_4|A_1|^2 + 64\,A_0^2\alpha_1^2\overline{\alpha_4} + 128\,\big(2\,|A_1|^4 + \alpha_1\overline{\alpha_1} - \beta\big)A_0A_1\alpha_1\\
&\quad - 64\,A_0A_1\alpha_4^2 - 40\,A_0C_1\alpha_1\overline{\alpha_2} - 80\,A_1^2\alpha_1\overline{\alpha_4} - 128\,A_0A_2\alpha_1\overline{\alpha_4} - 96\,A_0A_1\alpha_3\overline{\alpha_4} - 48\,\big(2\,\alpha_4|A_1|^2 + \alpha_3\overline{\alpha_1} + \alpha_1\overline{\alpha_4} - \alpha_{12}\big)A_1^2\\
&\quad - 128\,\big(2\,|A_1|^4 + \alpha_1\overline{\alpha_1} - \beta\big)A_1A_2 + 24\,A_1C_1\alpha_7 + 40\,A_2C_1\overline{\alpha_2} + 10\,A_1C_2\overline{\alpha_2} + 64\,A_2^2\overline{\alpha_4} + 96\,A_1A_3\overline{\alpha_4}\\
\kappa_{10} &= -320\,A_0^2\alpha_1|A_1|^2\overline{\alpha_2} + 192\,A_0A_1\alpha_7|A_1|^2 + 304\,A_1^2|A_1|^2\overline{\alpha_2} + 320\,A_0A_2|A_1|^2\overline{\alpha_2} + 160\,A_0A_1\alpha_1\alpha_6 + 160\,A_0A_1\alpha_4\overline{\alpha_2}\\
&\quad + 48\,\big(2\,|A_1|^2\overline{\alpha_2} - \alpha_9\big)A_1^2 - 160\,A_1A_2\alpha_6\\
\kappa_{11} &= -320\,A_0^2\alpha_1^2\overline{\alpha_2} + 384\,A_0A_1\alpha_1\alpha_7 + 400\,A_1^2\alpha_1\overline{\alpha_2} + 640\,A_0A_2\alpha_1\overline{\alpha_2} + 480\,A_0A_1\alpha_3\overline{\alpha_2} + 112\,(\alpha_1\overline{\alpha_2} - \alpha_8)A_1^2 - 384\,A_1A_2\alpha_7\\
&\quad - 320\,A_2^2\overline{\alpha_2} - 480\,A_1A_3\overline{\alpha_2}
\end{aligned}
$$

Gathering these two terms, we obtain

$$
\mathscr{Q}_{\vec{\Phi}} - \left(\frac{5}{4}|\vec{H}|^2\vec{h}_0\,\dot{\otimes}\,\vec{h}_0 + \langle\vec{H},\vec{h}_0\rangle^2\right) = g^{-1}\otimes Q(\vec{h}_0) - K_g\,\vec{h}_0\,\dot{\otimes}\,\vec{h}_0 =
$$

$$
\begin{pmatrix}
-16\,A_0C_1\alpha_1 + 16\,A_2C_1 + 2\,A_1C_2 & 0 & 0 & 0 \\
-30\,A_0C_1\alpha_3 + 8\,A_0A_1\overline{\zeta_0} + 30\,A_3C_1 + 6\,A_2C_2 - 6\,(4\,A_1C_1 + A_0C_2)\alpha_1 - A_1\gamma_1 & 1 & 0 & 0 \\
\pi_1 & 2 & 0 & 0 \\
\pi_2 & 3 & 0 & 0 \\
20\,A_1E_1 & -3 & 4 & 0 \\
-48\,A_0E_1\alpha_1 + 48\,A_2E_1 + 12\,A_1E_3 & -2 & 4 & 0 \\
\frac{29}{80}\,C_1^2|A_1|^2 - 35\,A_0E_1|A_1|^2 - \frac{1}{2}\,A_0C_1\overline{\alpha_6} + \frac{1}{40}\,B_3C_1 - \frac{1}{40}\,B_1C_2 + 20\,A_1E_2 + \frac{1}{2}\,(21\,A_1C_1 + A_0C_2)\alpha_2 & -3 & 5 & 0 \\
\pi_3 & -3 & 6 & 0 \\
\pi_4 & -2 & 5 & 0 \\
-\frac{3}{8}\,C_2E_1 - \frac{3}{16}\,C_1E_3 & -5 & 8 & 0 \\
\pi_5 & -1 & 4 & 0 \\
\pi_6 & 0 & 2 & 0 \\
\pi_7 & 1 & 2 & 0 \\
\pi_8 & 0 & 3 & 0 \\
\pi_9 & 3 & 0 & 1 \\
\pi_{10} & 0 & 1 & 0 \\
\pi_{11} & 1 & 1 & 0 \\
\pi_{12} & 2 & 1 & 0 \\
\pi_{13} & 4 & -1 & 0 \\
\pi_{14} & 5 & -2 & 0 \\
\pi_{15} & 4 & -2 & 0 \\
-\frac{1}{2}\,C_1E_1 & -6 & 8 & 0 \\
-\frac{3}{10}\,B_1E_1 - \frac{5}{9}\,C_1E_2 - \frac{1}{72}\,\left(85\,C_1^2 - 432\,A_0E_1\right)\alpha_2 & -6 & 9 & 0 \\
-48\,A_0C_1\zeta_0 + 24\,C_1\overline{E_1} & 2 & 0 & 1 \\
-480\,A_0\alpha_2\overline{E_1} + 48\,\left(20\,A_0^2\alpha_2 - A_0B_1\right)\zeta_0 + 24\,B_1\overline{E_1} & 2 & 1 & 1 \\
\pi_{16} & 3 & -2 & 0 \\
192\,A_0^2\alpha_2|A_1|^2 - \frac{48}{5}\,A_0B_1|A_1|^2 - 3\,A_0C_1\overline{\alpha_4} - 144\,A_0A_1\overline{\alpha_7} + 6\,A_1B_2 + \frac{3}{8}\,C_1\overline{B_1} & -1 & 2 & 0 \\
\pi_{17} & -1 & 3 & 0 \\
-96\,A_0A_1\zeta_4 + 48\,A_0\overline{E_1}\overline{\alpha_4} + 12\,\left(8\,A_0A_1\overline{\alpha_1} - 8\,A_0^2\overline{\alpha_4} + A_0\overline{B_1}\right)\zeta_0 - 6\,\overline{B_1E_1} & 5 & -2 & 1 \\
\pi_{18} & 3 & -1 & 0 \\
\pi_{19} & 6 & -3 & 0 \\
\pi_{20} & 7 & -4 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
\pi_{21} & 6 & -4 & 0 \\
-24\,A_0A_1\zeta_0 + 12\,A_1\overline{E_1} & 5 & -4 & 0 \\
\pi_{22} & 5 & -3 & 0 \\
\pi_{23} & 9 & -6 & 0 \\
-16\,A_0^2\alpha_1|A_1|^2 + 8\,A_1^2|A_1|^2 + 16\,A_0A_2|A_1|^2 - 8\,A_0A_1\alpha_4 & 3 & -3 & 0 \\
-48\,A_0A_1\alpha_1|A_1|^2 - 48\,A_0^2\alpha_3|A_1|^2 + 48\,A_1A_2|A_1|^2 + 48\,A_0A_3|A_1|^2 - 24\,A_0A_1\alpha_5 & 4 & -3 & 0 \\
-9\,A_0C_1|A_1|^2 - 120\,A_0A_1\alpha_2 + 6\,A_1B_1 & -1 & 1 & 0 \\
-96\,A_0^2\zeta_0|A_1|^2 + 48\,A_0|A_1|^2\overline{E_1} & 5 & -3 & 1 \\
\pi_{24} & 6 & -3 & 1 \\
-720\,A_0^2\alpha_7|A_1|^2 - 1360\,A_0A_1|A_1|^2\overline{\alpha_2} - 360\,A_0^2\alpha_4\overline{\alpha_2} + 120\,A_0A_1\alpha_9 - 60\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\alpha_6 & 7 & -5 & 0 \\
\pi_{25} & 8 & -5 & 0 \\
-480\,A_0^2|A_1|^2\overline{\alpha_2} + 80\,A_0A_1\alpha_6 & 6 & -5 & 0 \\
864\,A_0A_1\alpha_{14} & 10 & -7 & 0 \\
240\,A_0A_1\alpha_7 - 120\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\overline{\alpha_2} & 7 & -6 & 0 \\
-240\,A_0^2\alpha_3\overline{\alpha_2} + 336\,A_0A_1\alpha_8 - 192\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\alpha_7 - 16\,(41\,A_0A_1\alpha_1 - 15\,A_1A_2 - 15\,A_0A_3)\overline{\alpha_2} & 8 & -6 & 0 \\
-80\,A_0A_1\overline{\alpha_2}^2 + 1440\,A_0A_1\alpha_{13} & 11 & -8 & 0 \\
160\,A_0A_1\overline{\alpha_2} & 6 & -6 & 0 \\
-160\,A_0^2\zeta_0\overline{\alpha_2} + 80\,A_0\overline{E_1}\overline{\alpha_2} & 9 & -6 & 1 \\
6\,A_1C_1 & -1 & 0 & 0 \\
-6\,A_1C_1\overline{\zeta_0} & -1 & 4 & 1
\end{pmatrix}
$$

where

$$
\begin{aligned}
\pi_1 &= 32\,A_0C_1\alpha_1^2 - 4\,A_0C_1\zeta_0 + 48\,A_4C_1 + 12\,A_3C_2 - 2\,(16\,A_2C_1 + 5\,A_1C_2)\alpha_1 - 6\,(7\,A_1C_1 + 2\,A_0C_2)\alpha_3 + 4\,(A_0\alpha_1 - A_2)\gamma_1 \\
&\quad + \frac{1}{2}\,A_1\gamma_2 + 2\,C_1\overline{E_1} - 32\left(A_0^2\alpha_1 - A_0A_2\right)\overline{\zeta_0} \\
\pi_2 &= -\frac{28}{3}\,A_0|A_1|^4\overline{B_1} - \frac{4}{3}\,A_1|A_1|^4\overline{C_1} - 336\,A_0A_1\alpha_1\overline{\alpha_1}^2 + 32\,A_0^2|A_1|^2\overline{\alpha_{12}} + \frac{16}{3}\,A_1|A_1|^2\overline{B_3} - \frac{8}{3}\,A_0|A_1|^2\overline{B_5} + \frac{4}{3}\,A_2|A_1|^2\overline{C_2} \\
&\quad - 16\,A_0A_1\overline{\alpha_4}^2 + 2240\,A_0^2\overline{\alpha_2\alpha_7} + 2\,(20\,A_1C_1 + 7\,A_0C_2)\alpha_1^2 - 70\,A_0C_1\alpha_{10} - \frac{13}{4}\,A_0C_1\zeta_1 - 32\,A_0A_1\overline{\alpha_{19}} - 28\,A_0A_1\overline{\zeta_4} \\
&\quad + 70\,A_5C_1 + 20\,A_4C_2 - 2\,A_2C_4 + 2\,A_1C_5 - \frac{2}{3}\left(2\,A_0|A_1|^2\overline{C_2} + 60\,A_3C_1 + 21\,A_2C_2 - 3\,A_0C_4\right)\alpha_1 \\
&\quad + 2\,(47\,A_0C_1\alpha_1 - 27\,A_2C_1 - 9\,A_1C_2)\alpha_3 + \frac{2}{3}\left(60\,A_0A_1\overline{\alpha_3} + 2\,A_0\overline{B_3} - A_1\overline{C_2}\right)\alpha_4 + 84\left(20\,A_0^2\alpha_2 - A_0B_1\right)\alpha_6 \\
&\quad + 2\left(24\,A_0A_1\overline{\alpha_1} - A_1\overline{C_1}\right)\beta + (8\,A_1\alpha_1 + 9\,A_0\alpha_3 - 9\,A_3)\gamma_1 + \frac{3}{2}\,A_1\gamma_3 - \frac{1}{2}\,(19\,A_1C_1 + 4\,A_0C_2)\zeta_0 + C_2\overline{E_1} + \frac{13}{8}\,C_1\overline{E_2} \\
&\quad - 2\left(496\,A_0A_1|A_1|^4 - A_1\alpha_1\overline{C_1}\right)\overline{\alpha_1} - 16\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\overline{\alpha_{11}} + \frac{1}{24}\left(1960\,A_0C_1\overline{\alpha_1} - 2240\,A_0B_2 - 171\,C_1\overline{C_1}\right)\overline{\alpha_2} \\
&\quad + 8\left(6\,A_0^2\alpha_1|A_1|^2 - 5\,A_1^2|A_1|^2 - 10\,A_0A_2|A_1|^2\right)\overline{\alpha_3} + 2\left\{32\,A_0^2|A_1|^4 - A_0\alpha_1\overline{C_1} + 2\,A_1\overline{B_1} + A_2\overline{C_1}\right. \\
&\quad \left. + 12\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\overline{\alpha_1}\right\}\overline{\alpha_4} - 16\left(4\,A_0A_1|A_1|^2 + A_0^2\alpha_4\right)\overline{\alpha_5} - 4\left(11\,A_0A_1\alpha_1 + 18\,A_0^2\alpha_3 - 2\,A_1A_2 - 18\,A_0A_3\right)\overline{\zeta_0} \\
\pi_3 &= 200\,A_0A_1\alpha_2^2 - \frac{320}{9}\,A_0E_2|A_1|^2 - 800\,A_0A_1\overline{\alpha_{13}} - A_0C_1\overline{\alpha_9} + \frac{1}{24}\left(14\,B_1|A_1|^2 + B_5\right)C_1 - \frac{1}{24}\,B_2C_2 + 20\,A_1E_4 \\
&\quad - \frac{1}{9}\left(293\,A_0C_1|A_1|^2 - 90\,A_1B_1\right)\alpha_2 + \frac{15}{8}\,E_1\overline{B_1} + \frac{1}{48}\left(17\,C_1^2 - 720\,A_0E_1\right)\overline{\alpha_4} + (11\,A_1C_1 + A_0C_2)\overline{\alpha_7}
\end{aligned}
$$

$$
\begin{aligned}
\pi_4 = {} & \frac{3}{8}\,C_1C_2|A_1|^2 - 53\,A_1E_1|A_1|^2 - 21\,A_0E_3|A_1|^2 - 48\,A_0E_2\alpha_1 - 432\,A_0A_1\overline{\alpha_{14}} + 6\,A_1C_1\overline{\alpha_6} + \frac{1}{2}\,A_0C_1\overline{\zeta_1} + 48\,A_2E_2 \\
& + 12\,A_1E_5 - 2\,(8\,A_0C_1\alpha_1 - 8\,A_2C_1 - A_1C_2)\alpha_2 + \frac{3}{8}\left(C_1^2 - 112\,A_0E_1\right)\alpha_4 - \frac{1}{20}\,(20\,A_0\alpha_2 - B_1)\gamma_1 - \frac{1}{40}\,C_1\overline{\gamma_2} \\
& + \frac{2}{5}\left(20\,A_0^2\alpha_2 - A_0B_1\right)\overline{\zeta_0}
\end{aligned}
$$

$$
\begin{aligned}
\pi_5 = {} & 288\,A_0^2|A_1|^2\overline{\alpha_8} - \frac{72}{7}\,A_0B_4|A_1|^2 - \frac{1}{5}\,B_1|A_1|^2\overline{C_1} - 84\,A_0E_1\alpha_3 - 192\,A_0A_1\overline{\alpha_{15}} + \frac{1}{2}\,A_1C_1\overline{\zeta_0} + 6\,A_1B_6 + 84\,A_3E_1 \\
& + 30\,A_2E_3 - 2\,(32\,A_1E_1 + 15\,A_0E_3)\alpha_1 - \frac{2}{7}\left(2016\,A_0^2|A_1|^2\overline{\alpha_1} - 104\,A_0|A_1|^2\overline{C_1} + 14\,A_0\overline{B_3} + 35\,A_1\overline{C_2}\right)\alpha_2 - \frac{3}{64}\,C_1\gamma_2 \\
& + \frac{1}{2}\,B_2\overline{B_1} + \frac{1}{5}\,B_1\overline{B_3} + \frac{1}{280}\left(1152\,A_0B_1|A_1|^2 + 35\,C_1\overline{B_1}\right)\overline{\alpha_1} - \frac{3}{7}\left(3\,A_0C_1|A_1|^2 - 728\,A_0A_1\alpha_2\right)\overline{\alpha_3} \\
& - (A_0C_1\overline{\alpha_1} + 4\,A_0B_2)\overline{\alpha_4} + \frac{12}{5}\left(20\,A_0^2\alpha_2 - A_0B_1\right)\overline{\alpha_5} + 6\left(56\,A_0A_1\overline{\alpha_1} + 16\,A_0^2\overline{\alpha_4} - 2\,A_0\overline{B_1} - 3\,A_1\overline{C_1}\right)\overline{\alpha_7}
\end{aligned}
$$

$$
\begin{aligned}
\pi_6 = {} & 1104\,A_0A_1\alpha_2|A_1|^2 + 64\,A_0^2|A_1|^2\overline{\alpha_6} - \frac{96}{5}\,A_1B_1|A_1|^2 - \frac{16}{5}\,A_0B_3|A_1|^2 + 8\,A_0C_1\alpha_1\overline{\alpha_1} - 16\,A_0B_2\alpha_1 - 8\,A_0C_1\beta \\
& - 48\,A_0A_1\overline{\alpha_9} + 16\,A_2B_2 + 2\,A_1B_5 + \frac{1}{5}\left(138\,A_0|A_1|^4 + 5\,\overline{B_2}\right)C_1 + \frac{64}{5}\left(20\,A_0^2\alpha_2 - A_0B_1\right)\alpha_4 + \frac{1}{8}\,C_2\overline{B_1} \\
& - (9\,A_1C_1 + A_0C_2)\overline{\alpha_4} + 192\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\overline{\alpha_7}
\end{aligned}
$$

$$
\begin{aligned}
\pi_7 = {} & 144\,A_0A_1\overline{\alpha_1}^3 - 16\,A_0^2|A_1|^2\overline{\zeta_1} - 28\,A_2B_1|A_1|^2 - 8\,A_1B_3|A_1|^2 + \frac{4}{5}\,A_0|A_1|^2\overline{\gamma_2} - 15\,A_0C_1\alpha_{12} \\
& + 3\,(7\,A_1C_1 + A_0C_2)\alpha_1\overline{\alpha_1} + 12\,A_0A_1\overline{\zeta_2} + 30\,A_3B_2 + 6\,A_2B_5 + \frac{1}{8}\left(416\,A_1|A_1|^4 + 15\,\overline{B_4}\right)C_1 + \frac{3}{8}\left(32\,A_0|A_1|^4 + \overline{B_2}\right)C_2 \\
& + \frac{1}{40}\left(2656\,A_0B_1|A_1|^2 - 960\,A_1B_2 - 240\,A_0B_5 + 15\,C_1\overline{B_1}\right)\alpha_1 - 24\left(118\,A_0^2\alpha_1|A_1|^2 - 43\,A_1^2|A_1|^2 - 86\,A_0A_2|A_1|^2\right)\alpha_2 \\
& + 15\,(A_0C_1\overline{\alpha_1} - 2\,A_0B_2)\alpha_3 + \frac{1}{5}\left(267\,A_0C_1|A_1|^2 + 5120\,A_0A_1\alpha_2 - 114\,A_1B_1 - 24\,A_0B_3\right)\alpha_4 + 24\left(20\,A_0^2\alpha_2 - A_0B_1\right)\alpha_5 \\
& - 3\,(7\,A_1C_1 + A_0C_2)\beta + \frac{1}{16}\left(8\,A_0\overline{\alpha_4} - \overline{B_1}\right)\gamma_1 - \frac{15}{16}\left(C_1^2 + 240\,A_0E_1\right)\overline{\alpha_2} + 2\,(13\,A_0C_1\alpha_1 - 7\,A_2C_1 - 2\,A_1C_2)\overline{\alpha_4} \\
& + 16\left(25\,A_0A_1|A_1|^2 + 6\,A_0^2\alpha_4\right)\overline{\alpha_6} + 144\left(9\,A_0A_1\alpha_1 + 5\,A_0^2\alpha_3 - 5\,A_1A_2 - 5\,A_0A_3\right)\overline{\alpha_7} + 72\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\overline{\alpha_9} \\
& - \frac{1}{2}\,A_1\overline{\gamma_3} - \frac{1}{2}\left(40\,A_0A_1\overline{\alpha_1} + 8\,A_0^2\overline{\alpha_4} - A_0\overline{B_1} - 2\,A_1\overline{C_1}\right)\overline{\zeta_0}
\end{aligned}
$$

$$
\begin{aligned}
\pi_8 = {} & \frac{452}{15}\,A_0B_1|A_1|^4 + 80\,A_0^2|A_1|^2\overline{\alpha_9} - \frac{58}{3}\,A_1B_2|A_1|^2 - \frac{10}{3}\,A_0B_5|A_1|^2 + 4\,A_0C_1\alpha_1\overline{\alpha_3} - 16\,A_0B_4\alpha_1 - 4\,A_0C_1\overline{\alpha_{12}} \\
& - 56\,A_0A_1\overline{\alpha_{16}} + 16\,A_2B_4 + 2\,A_1B_7 - \frac{1}{120}\left(49\,|A_1|^2\overline{B_1} - 40\,\overline{B_5}\right)C_1 - \frac{1}{24}\left(|A_1|^2\overline{C_1} - \overline{B_3}\right)C_2 \\
& - 2\left(688\,A_0^2|A_1|^4 - 20\,A_0\alpha_1\overline{C_1} + 19\,A_1\overline{B_1} + 12\,A_0\overline{B_2} + 20\,A_2\overline{C_1} + 96\left(5\,A_0^2\alpha_1 - 2\,A_1^2 - 4\,A_0A_2\right)\overline{\alpha_1}\right)\alpha_2 \\
& + \frac{1}{6}\left(16\,A_0C_1\overline{\alpha_1} - 80\,A_0B_2 - C_1\overline{C_1}\right)\alpha_4 + \frac{48}{5}\left(20\,A_0^2\alpha_2 - A_0B_1\right)\beta + \frac{3}{20}\,B_3\overline{B_1} + \frac{6}{5}\,B_1\overline{B_2} \\
& + \frac{2}{15}\left(125\,A_1C_1|A_1|^2 + 5\,A_0C_2|A_1|^2 + 72\,A_0B_1\alpha_1\right)\overline{\alpha_1} + \frac{1}{15}\left(319\,A_0C_1|A_1|^2 + 8160\,A_0A_1\alpha_2 - 138\,A_1B_1 - 18\,A_0B_3\right)\overline{\alpha_4} \\
& - \frac{1}{2}\,(17\,A_1C_1 + A_0C_2)\overline{\alpha_5} + \left(96\,A_0A_1\overline{\alpha_1} + 24\,A_0^2\overline{\alpha_4} - 3\,A_0\overline{B_1} - 5\,A_1\overline{C_1}\right)\overline{\alpha_6} + 16\left(83\,A_0A_1|A_1|^2 + 20\,A_0^2\alpha_4\right)\overline{\alpha_7} \\
& + 224\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\overline{\alpha_8}
\end{aligned}
$$

$$
\begin{aligned}
\pi_9 = {} & 32\,A_0A_1\alpha_1\overline{\zeta_0} - 70\,A_0C_1\zeta_1 + 35\,C_1\overline{C_1}\overline{\alpha_2} - 32\,A_0A_1\overline{\zeta_4} + 2\,(A_0\alpha_1 - A_2)\gamma_2 + 2\,A_1\gamma_3 - 4\,(16\,A_1C_1 + 5\,A_0C_2)\zeta_0 \\
& + 10\,C_2\overline{E_1} + 35\,C_1\overline{E_2}
\end{aligned}
$$

$$
\begin{aligned}
\pi_{10} = {} & -19\,A_1C_1|A_1|^2 - 3\,A_0C_2|A_1|^2 - 16\,A_0B_1\alpha_1 - 12\,A_0C_1\alpha_4 - 40\,A_0A_1\overline{\alpha_6} + 16\,A_2B_1 + 2\,A_1B_3 \\
& + 160\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\alpha_2
\end{aligned}
$$

$$
\begin{aligned}
\pi_{11} = {} & -12\,A_0^2|A_1|^2\overline{\zeta_0} - 28\,A_2C_1|A_1|^2 - 8\,A_1C_2|A_1|^2 + \frac{3}{2}\,A_0\gamma_1|A_1|^2 - \frac{45}{2}\,A_0C_1\alpha_5 + 10\,A_0A_1\overline{\zeta_1} + 30\,A_3B_1 + 6\,A_2B_3 \\
& + 2\left(32\,A_0C_1|A_1|^2 - 12\,A_1B_1 - 3\,A_0B_3\right)\alpha_1 + 120\,(9\,A_0A_1\alpha_1 - 5\,A_1A_2 - 5\,A_0A_3)\alpha_2 + 30\left(20\,A_0^2\alpha_2 - A_0B_1\right)\alpha_3 \\
& - \frac{9}{2}\,(5\,A_1C_1 + A_0C_2)\alpha_4 + 60\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\overline{\alpha_6} - \frac{1}{2}\,A_1\overline{\gamma_2}
\end{aligned}
$$

$$
\begin{aligned}
\pi_{12} = {} & 32\,A_0B_1\alpha_1^2 - 36\,A_3C_1|A_1|^2 - 12\,A_2C_2|A_1|^2 - \frac{3}{4}\,A_0\gamma_2|A_1|^2 - 36\,A_0C_1\alpha_{11} - 10\,A_0A_1\overline{\zeta_3} + 48\,A_4B_1 + 12\,A_3B_3 \\
& + \left(95\,A_1C_1|A_1|^2 + 27\,A_0C_2|A_1|^2 - 32\,A_2B_1 - 10\,A_1B_3\right)\alpha_1 - 4\left\{280\,A_0^2\alpha_1^2 + 120\,A_2^2 + 240\,A_1A_3 + 240\,A_0A_4\right.
\end{aligned}
$$

$$- 200\left(A_1^2+2A_0A_2\right)\alpha_1+13A_0\overline{E_1}\Big\}\alpha_2+3\left(33A_0C_1|A_1|^2+600A_0A_1\alpha_2-14A_1B_1-4A_0B_3\right)\alpha_3$$
$$+2\left(28A_0C_1\alpha_1-16A_2C_1-5A_1C_2\right)\alpha_4-3\left(13A_1C_1+3A_0C_2\right)\alpha_5+\frac{1}{2}\left(13A_1|A_1|^2+6A_0\alpha_4\right)\gamma_1+A_1\gamma_4$$
$$+\frac{26}{5}\left(20A_0^2\alpha_2-A_0B_1\right)\zeta_0+\frac{13}{5}B_1\overline{E_1}+9C_1\overline{E_3}+40\left(11A_0A_1\alpha_1+6A_0^2\alpha_3-6A_1A_2-6A_0A_3\right)\overline{\alpha_6}$$
$$+2\left(A_0\alpha_1-A_2\right)\overline{\gamma_2}-12\left(3A_0A_1|A_1|^2+2A_0^2\alpha_4\right)\overline{\zeta_0}-40\left(A_0^2\alpha_1-A_0A_2\right)\overline{\zeta_1}$$
$$\pi_{13}=-832A_0A_1|A_1|^6+48A_0^2\alpha_{12}|A_1|^2+16A_0A_1\beta|A_1|^2+14A_2|A_1|^2\overline{B_1}+10A_1|A_1|^2\overline{B_2}-6A_0|A_1|^2\overline{B_4}+6A_3|A_1|^2\overline{C_1}$$
$$+72A_0A_1\alpha_3\overline{\alpha_3}+840A_0^2\overline{\alpha_2\alpha_6}-120A_0C_1\alpha_9-72A_0A_1\delta-2\left(10A_0|A_1|^2\overline{B_1}-3A_1\overline{B_3}\right)\alpha_1$$
$$+6\left(24A_0^2|A_1|^2\overline{\alpha_1}-A_0|A_1|^2\overline{C_1}+A_0\overline{B_3}\right)\alpha_3+6\left(16A_0^2|A_1|^4+A_1\overline{B_1}\right)\alpha_4+3\left(32A_0A_1\overline{\alpha_1}-8A_0^2\overline{\alpha_4}+A_0\overline{B_1}-A_1\overline{C_1}\right)\alpha_5$$
$$-\frac{15}{2}\left(13A_1C_1+5A_0C_2\right)\alpha_6+\frac{672}{5}\left(20A_0^2\alpha_2-A_0B_1\right)\alpha_7-6A_3\overline{B_3}+6A_1\overline{B_7}$$
$$-16\left(97A_0A_1\alpha_1|A_1|^2+12A_1A_2|A_1|^2+12A_0A_3|A_1|^2\right)\overline{\alpha_1}+\left(249A_0C_1|A_1|^2+4432A_0A_1\alpha_2-102A_1B_1-42A_0B_3\right)\overline{\alpha_2}$$
$$+8\left(30A_0^2\alpha_1|A_1|^2-15A_1^2|A_1|^2-30A_0A_2|A_1|^2-4A_0A_1\alpha_4\right)\overline{\alpha_4}-72\left(A_0A_1\alpha_1+A_0^2\alpha_3-A_1A_2-A_0A_3\right)\overline{\alpha_5}$$
$$\pi_{14}=-384A_1A_2|A_1|^4-384A_0A_3|A_1|^4+48A_0A_1\alpha_5|A_1|^2-8A_0A_1\alpha_4^2-10A_0\alpha_1^2\overline{B_1}-300A_0^2\overline{\alpha_2\zeta_0}-96A_0A_1\alpha_{19}$$
$$-189A_0C_1\alpha_8-52A_0A_1\zeta_4+\frac{75}{2}A_0\gamma_1\overline{\alpha_2}-2\left(544A_0A_1|A_1|^4-5A_2\overline{B_1}-4A_1\overline{B_2}+3A_0\overline{B_4}\right)\alpha_1$$
$$+24\left(2A_0^2\alpha_1-A_1^2-2A_0A_2\right)\alpha_{12}+6\left(64A_0^2|A_1|^4+2A_1\overline{B_1}+A_0\overline{B_2}+4\left(A_1^2+2A_0A_2\right)\overline{\alpha_1}\right)\alpha_3$$
$$+88\left(2A_0^2\alpha_1|A_1|^2-A_1^2|A_1|^2-2A_0A_2|A_1|^2\right)\alpha_4-3\left(47A_1C_1+21A_0C_2\right)\alpha_7$$
$$-16\left(A_0A_1\alpha_1+3A_0^2\alpha_3-3A_1A_2-3A_0A_3\right)\beta+\frac{1}{2}\left(152A_0A_1\overline{\alpha_1}-8A_0^2\overline{\alpha_4}+A_0\overline{B_1}-6A_1\overline{C_1}\right)\zeta_0-12A_4\overline{B_1}$$
$$-6A_3\overline{B_2}+6A_2\overline{B_4}+12A_1\overline{B_6}-\frac{1}{4}\overline{B_1E_1}-16\left(26A_0A_1\alpha_1^2+3\left(A_1A_2+A_0A_3\right)\alpha_1\right)\overline{\alpha_1}$$
$$+2\left(97A_0C_1\alpha_1-55A_2C_1-20A_1C_2\right)\overline{\alpha_2}+2\Big\{40A_0^2\alpha_1^2-48A_0A_1\alpha_3+24A_2^2+48A_1A_3+48A_0A_4$$
$$-32\left(A_1^2+2A_0A_2\right)\alpha_1+A_0\overline{E_1}\Big\}\overline{\alpha_4}$$
$$\pi_{15}=-80A_1^2|A_1|^4-160A_0A_2|A_1|^4-16A_0A_1\alpha_4|A_1|^2-48A_0A_1\alpha_{12}-144A_0C_1\alpha_7+2\left(80A_0^2|A_1|^4+3A_1\overline{B_1}\right)\alpha_1$$
$$+6\left(8A_0A_1\overline{\alpha_1}+A_0\overline{B_1}\right)\alpha_3-6A_3\overline{B_1}+6A_1\overline{B_4}-15\left(7A_1C_1+3A_0C_2\right)\overline{\alpha_2}-48\left(A_0A_1\alpha_1+A_0^2\alpha_3-A_1A_2-A_0A_3\right)\overline{\alpha_4}$$
$$\pi_{16}=-32A_0A_1|A_1|^4+16A_0A_1\alpha_1\overline{\alpha_1}-16A_0A_1\beta+2A_0\alpha_1\overline{B_1}-105A_0C_1\overline{\alpha_2}-2A_2\overline{B_1}+2A_1\overline{B_2}$$
$$-8\left(2A_0^2\alpha_1-A_1^2-2A_0A_2\right)\overline{\alpha_4}$$
$$\pi_{17}=2A_0C_1|A_1|^2\overline{\alpha_1}+240A_0^2|A_1|^2\overline{\alpha_7}-10A_0B_2|A_1|^2-\frac{1}{8}C_1|A_1|^2\overline{C_1}-\frac{3}{2}A_0C_1\overline{\alpha_5}-168A_0A_1\overline{\alpha_8}+6A_1B_4$$
$$+3\left(96A_0A_1\overline{\alpha_1}-3A_0\overline{B_1}-5A_1\overline{C_1}\right)\alpha_2+\frac{9}{20}B_1\overline{B_1}+\frac{1}{8}C_1\overline{B_3}+\frac{18}{5}\left(20A_0^2\alpha_2-A_0B_1\right)\overline{\alpha_4}$$
$$\pi_{18}=32A_0^2|A_1|^6+16A_0^2\beta|A_1|^2+6A_1|A_1|^2\overline{B_1}-2A_0|A_1|^2\overline{B_2}+2A_2|A_1|^2\overline{C_1}+24A_0A_1\alpha_1\overline{\alpha_3}-\frac{175}{2}A_0C_1\alpha_6$$
$$-24A_0A_1\overline{\alpha_{12}}-2\left(A_0|A_1|^2\overline{C_1}-A_0\overline{B_3}\right)\alpha_1+\left(32A_0A_1\overline{\alpha_1}+A_0\overline{B_1}-A_1\overline{C_1}\right)\alpha_4-2A_2\overline{B_3}+2A_1\overline{B_5}$$
$$+16\left(3A_0^2\alpha_1|A_1|^2-2A_1^2|A_1|^2-4A_0A_2|A_1|^2\right)\overline{\alpha_1}+98\left(20A_0^2\alpha_2-A_0B_1\right)\overline{\alpha_2}-8\left(6A_0A_1|A_1|^2+A_0^2\alpha_4\right)\overline{\alpha_4}$$
$$-12\left(2A_0^2\alpha_1-A_1^2-2A_0A_2\right)\overline{\alpha_5}$$
$$\pi_{19}=-304A_0A_1\alpha_1^2|A_1|^2-160A_0^2\alpha_{10}|A_1|^2+44A_0^2\zeta_1|A_1|^2+56A_0A_1\alpha_1\alpha_5+160A_2A_3|A_1|^2+160A_1A_4|A_1|^2$$
$$+160A_0A_5|A_1|^2+10A_1|A_1|^2\overline{E_1}-22A_0|A_1|^2\overline{E_2}-80A_0A_1\overline{\alpha_2\alpha_3}-400A_0^2\overline{\alpha_2\alpha_5}-80A_0A_1\alpha_{18}-58A_0A_1\zeta_3$$
$$-16\left(17A_1A_2|A_1|^2+17A_0A_3|A_1|^2+A_0\overline{E_3}\right)\alpha_1+32\left(2A_0^2\alpha_1-A_1^2-2A_0A_2\right)\alpha_{11}$$
$$+8\left(36A_0^2\alpha_1|A_1|^2-11A_1^2|A_1|^2-22A_0A_2|A_1|^2\right)\alpha_3+4\Big\{8A_0^2\alpha_1^2-14A_0A_1\alpha_3+8A_2^2+16A_1A_3+16A_0A_4$$
$$-8\left(A_1^2+2A_0A_2\right)\alpha_1-A_0\overline{E_1}\Big\}\alpha_4-10\left(24A_0^2\overline{\alpha_4}-3A_0\overline{B_1}-A_1\overline{C_1}\right)\alpha_6+4\left(31A_0A_1|A_1|^2+2A_0^2\alpha_4\right)\zeta_0$$
$$+16A_2\overline{E_3}+20A_1\overline{E_5}+\frac{2}{3}\left(2160A_0^2|A_1|^2\overline{\alpha_1}-113A_0|A_1|^2\overline{C_1}+50A_0\overline{B_3}+20A_1\overline{C_2}\right)\overline{\alpha_2}$$
$$\pi_{20}=-480A_0^2\alpha_9|A_1|^2-480A_0^2\beta\overline{\alpha_2}-800A_0A_1\overline{\alpha_2\alpha_4}-60A_0A_1\zeta_2-12A_0\alpha_3\overline{E_1}-30A_0\alpha_1\overline{E_2}$$
$$-80\left(10A_0A_1|A_1|^2+3A_0^2\alpha_4\right)\alpha_6-30\left(8A_0A_1\overline{\alpha_1}+16A_0^2\overline{\alpha_4}-2A_0\overline{B_1}-A_1\overline{C_1}\right)\alpha_7$$

$$+\,12\left(7\,A_0A_1\alpha_1+2\,A_0^2\alpha_3-2\,A_1A_2-2\,A_0A_3\right)\zeta_0+30\left(2\,A_0^2\alpha_1-A_1^2-2\,A_0A_2\right)\zeta_1+12\,A_3\overline{E_1}+30\,A_2\overline{E_2}+30\,A_1\overline{E_4}$$
$$+\,30\left(96\,A_0^2|A_1|^4-A_0\alpha_1\overline{C_1}+2\,A_1\overline{B_1}+2\,A_0\overline{B_2}+A_2\overline{C_1}+4\left(6\,A_0^2\alpha_1-A_1^2-2\,A_0A_2\right)\overline{\alpha_1}\right)\overline{\alpha_2}$$
$$\pi_{21}=-320\,A_0^2\alpha_6|A_1|^2-320\,A_0^2\overline{\alpha_2\alpha_4}-40\,A_0A_1\zeta_1-16\,A_0\alpha_1\overline{E_1}+16\left(2\,A_0^2\alpha_1-A_1^2-2\,A_0A_2\right)\zeta_0+16\,A_2\overline{E_1}$$
$$+\,20\,A_1\overline{E_2}-20\left(8\,A_0A_1\overline{\alpha_1}-2\,A_0\overline{B_1}-A_1\overline{C_1}\right)\overline{\alpha_2}$$
$$\pi_{22}=80\,A_0^2\alpha_1^2|A_1|^2-96\,A_0A_1\alpha_3|A_1|^2+20\,A_0^2\zeta_0|A_1|^2+48\,A_2^2|A_1|^2+96\,A_1A_3|A_1|^2+96\,A_0A_4|A_1|^2-10\,A_0|A_1|^2\overline{E_1}$$
$$-\,48\,A_0A_1\alpha_{11}-64\left(A_1^2|A_1|^2+2\,A_0A_2|A_1|^2\right)\alpha_1-8\left(A_0A_1\alpha_1+3\,A_0^2\alpha_3-3\,A_1A_2-3\,A_0A_3\right)\alpha_4$$
$$+\,12\left(2\,A_0^2\alpha_1-A_1^2-2\,A_0A_2\right)\alpha_5+12\,A_1\overline{E_3}$$
$$\pi_{23}=-608\,A_0A_1\alpha_3\overline{\alpha_2}+20\,A_0^2\zeta_0\overline{\alpha_2}+448\,A_0A_1\alpha_{15}-16\left(67\,A_0A_1\alpha_1+27\,A_0^2\alpha_3-27\,A_1A_2-27\,A_0A_3\right)\alpha_7$$
$$-\,280\left(2\,A_0^2\alpha_1-A_1^2-2\,A_0A_2\right)\alpha_8+10\left(56\,A_0^2\alpha_1^2+8\,A_2^2+16\,A_1A_3+16\,A_0A_4-32\left(A_1^2+2\,A_0A_2\right)\alpha_1-A_0\overline{E_1}\right)\overline{\alpha_2}$$
$$\pi_{24}=-160\,A_0^2\zeta_1|A_1|^2+80\,A_0|A_1|^2\overline{C_1}\overline{\alpha_2}+80\,A_1|A_1|^2\overline{E_1}+80\,A_0|A_1|^2\overline{E_2}-80\,A_0A_1\zeta_3+32\,A_0\alpha_4\overline{E_1}$$
$$-\,32\left(5\,A_0A_1|A_1|^2+2\,A_0^2\alpha_4\right)\zeta_0$$
$$\pi_{25}=-1008\,A_0^2\alpha_8|A_1|^2-880\,A_0A_1\alpha_4\overline{\alpha_2}-360\,A_0^2\alpha_5\overline{\alpha_2}+168\,A_0A_1\alpha_{16}-8\left(41\,A_0A_1\alpha_1+15\,A_0^2\alpha_3-15\,A_1A_2-15\,A_0A_3\right)\alpha_6$$
$$-\,48\left(43\,A_0A_1|A_1|^2+12\,A_0^2\alpha_4\right)\alpha_7-96\left(2\,A_0^2\alpha_1-A_1^2-2\,A_0A_2\right)\alpha_9+8\left(346\,A_0^2\alpha_1|A_1|^2-95\,A_1^2|A_1|^2-190\,A_0A_2|A_1|^2\right)\overline{\alpha_2}$$

Finally, the extra terms are (we have two expressions in order to determine in which order we take scalar products)

$$\frac{5}{4}|\vec{H}|^2\vec{h}_0\,\dot{\otimes}\,\vec{h}_0=$$
$$\begin{pmatrix}
\frac{5}{4}\left(4\,A_1^2\right)\left(\frac{1}{4}\,C_1^2\right) & 2 & 0 & 0\\
\frac{5}{2}\left(-8\,A_0A_1\alpha_1\right)\left(\frac{1}{4}\,C_1^2\right)+\frac{5}{2}\left(8\,A_1A_2\right)\left(\frac{1}{4}\,C_1^2\right)+\frac{5}{2}\left(4\,A_1^2\right)\left(\frac{1}{4}\,C_1C_2\right)+\frac{5}{2}\left(-\frac{1}{2}\,A_1C_1\right)\left(\frac{1}{4}\,\overline{C_1}^2\right) & 3 & 0 & 0\\
\frac{5}{2}\left(-\frac{1}{2}\,A_1C_1\right)\left(\frac{1}{4}\,C_1^2\right) & -1 & 4 & 0\\
\frac{5}{2}\left(4\,A_1^2\right)\left(\frac{1}{4}\,B_1C_1\right)+\frac{5}{2}\left(-8\,A_0A_1|A_1|^2\right)\left(\frac{1}{4}\,C_1^2\right) & 2 & 1 & 0\\
\frac{5}{2}\left(4\,A_1^2\right)\left(\frac{1}{4}\,C_1\overline{C_1}\right) & 4 & -2 & 0\\
\frac{5}{2}\left(4\,A_1^2\right)\left(\frac{1}{4}\,C_1\overline{B_1}\right)+5\left(-8\,A_0A_1\alpha_1\right)\left(\frac{1}{4}\,C_1\overline{C_1}\right)+5\left(8\,A_1A_2\right)\left(\frac{1}{4}\,C_1\overline{C_1}\right)+\frac{5}{2}\left(4\,A_1^2\right)\left(\frac{1}{4}\,C_2\overline{C_1}\right) & 5 & -2 & 0\\
5\left(-\frac{1}{2}\,A_1C_1\right)\left(\frac{1}{4}\,C_1\overline{C_1}\right) & 1 & 2 & 0\\
\frac{5}{2}\left(4\,A_1^2\right)\left(\frac{1}{4}\,B_1\overline{C_1}\right)+5\left(-8\,A_0A_1|A_1|^2\right)\left(\frac{1}{4}\,C_1\overline{C_1}\right)+\frac{5}{2}\left(4\,A_1^2\right)\left(\frac{1}{4}\,C_1\overline{C_2}\right) & 4 & -1 & 0\\
\frac{5}{4}\left(4\,A_1^2\right)\left(\frac{1}{4}\,\overline{C_1}^2\right) & 6 & -4 & 0\\
\frac{5}{2}\left(4\,A_1^2\right)\left(\frac{1}{4}\,\overline{B_1C_1}\right)+\frac{5}{2}\left(-8\,A_0A_1\alpha_1\right)\left(\frac{1}{4}\,\overline{C_1}^2\right)+\frac{5}{2}\left(8\,A_1A_2\right)\left(\frac{1}{4}\,\overline{C_1}^2\right) & 7 & -4 & 0\\
\frac{5}{2}\left(-8\,A_0A_1|A_1|^2\right)\left(\frac{1}{4}\,\overline{C_1}^2\right)+\frac{5}{2}\left(4\,A_1^2\right)\left(\frac{1}{4}\,\overline{C_1C_2}\right) & 6 & -3 & 0
\end{pmatrix}$$

$$
= \begin{pmatrix}
\frac{5}{4}\, A_1^2 C_1^2 & 2 & 0 & 0 \\
-5\, A_0 A_1 C_1^2 \alpha_1 + 5\, A_1 A_2 C_1^2 + \frac{5}{2}\, A_1^2 C_1 C_2 - \frac{5}{16}\, A_1 C_1 \overline{C_1}^2 & 3 & 0 & 0 \\
-\frac{5}{16}\, A_1 C_1^3 & -1 & 4 & 0 \\
-5\, A_0 A_1 C_1^2 |A_1|^2 + \frac{5}{2}\, A_1^2 B_1 C_1 & 2 & 1 & 0 \\
\frac{5}{2}\, A_1^2 C_1 \overline{C_1} & 4 & -2 & 0 \\
-10\, A_0 A_1 C_1 \alpha_1 \overline{C_1} + \frac{5}{2}\, A_1^2 C_1 \overline{B_1} + 10\, A_1 A_2 C_1 \overline{C_1} + \frac{5}{2}\, A_1^2 C_2 \overline{C_1} & 5 & -2 & 0 \\
-\frac{5}{8}\, A_1 C_1^2 \overline{C_1} & 1 & 2 & 0 \\
-10\, A_0 A_1 C_1 |A_1|^2 \overline{C_1} + \frac{5}{2}\, A_1^2 B_1 \overline{C_1} + \frac{5}{2}\, A_1^2 C_1 \overline{C_2} & 4 & -1 & 0 \\
\frac{5}{4}\, A_1^2 \overline{C_1}^2 & 6 & -4 & 0 \\
-5\, A_0 A_1 \alpha_1 \overline{C_1}^2 + \frac{5}{2}\, A_1^2 \overline{B_1 C_1} + 5\, A_1 A_2 \overline{C_1}^2 & 7 & -4 & 0 \\
-5\, A_0 A_1 |A_1|^2 \overline{C_1}^2 + \frac{5}{2}\, A_1^2 \overline{C_1 C_2} & 6 & -3 & 0
\end{pmatrix} \tag{6.1.49}
$$

and

$\langle \vec{H}, \vec{h}_0 \rangle^2 =$

$$
\begin{pmatrix}
(A_1 C_1)\,(A_1 C_1) & 2 & 0 & 0 \\
\chi_1 & 3 & 0 & 0 \\
(A_1 C_1)\left(-\frac{1}{8}\, C_1^2\right) + (A_1 C_1)\left(-\frac{1}{8}\, C_1^2\right) & -1 & 4 & 0 \\
\left(-2\, A_0 C_1 |A_1|^2\right)(A_1 C_1) + \left(-2\, A_0 C_1 |A_1|^2\right)(A_1 C_1) + (A_1 B_1)\,(A_1 C_1) + (A_1 B_1)\,(A_1 C_1) & 2 & 1 & 0 \\
(A_1 C_1)\left(A_1 \overline{C_1}\right) + (A_1 C_1)\left(A_1 \overline{C_1}\right) & 4 & -2 & 0 \\
\chi_2 & 5 & -2 & 0 \\
\left(-\frac{1}{8}\, C_1^2\right)\left(A_1 \overline{C_1}\right) + \left(-\frac{1}{8}\, C_1^2\right)\left(A_1 \overline{C_1}\right) + (A_1 C_1)\left(-\frac{1}{8}\, C_1 \overline{C_1}\right) + (A_1 C_1)\left(-\frac{1}{8}\, C_1 \overline{C_1}\right) & 1 & 2 & 0 \\
\chi_3 & 4 & -1 & 0 \\
\left(A_1 \overline{C_1}\right)\left(A_1 \overline{C_1}\right) & 6 & -4 & 0 \\
\chi_4 & 7 & -4 & 0 \\
\left(-2\, A_0 |A_1|^2 \overline{C_1}\right)\left(A_1 \overline{C_1}\right) + \left(-2\, A_0 |A_1|^2 \overline{C_1}\right)\left(A_1 \overline{C_1}\right) + \left(A_1 \overline{C_1}\right)\left(A_1 \overline{C_2}\right) + \left(A_1 \overline{C_1}\right)\left(A_1 \overline{C_2}\right) & 6 & -3 & 0
\end{pmatrix}
$$

$$
= \begin{pmatrix}
A_1^2 C_1^2 & 2 & 0 & 0 \\
-4\,A_0 A_1 C_1^2 \alpha_1 + 4\,A_1 A_2 C_1^2 + 2\,A_1^2 C_1 C_2 - \frac{1}{4}\,A_1 C_1 \overline{C_1}^2 & 3 & 0 & 0 \\
-\frac{1}{4}\,A_1 C_1^3 & -1 & 4 & 0 \\
-4\,A_0 A_1 C_1^2 |A_1|^2 + 2\,A_1^2 B_1 C_1 & 2 & 1 & 0 \\
2\,A_1^2 C_1 \overline{C_1} & 4 & -2 & 0 \\
-8\,A_0 A_1 C_1 \alpha_1 \overline{C_1} + 2\,A_1^2 C_1 \overline{B_1} + 8\,A_1 A_2 C_1 \overline{C_1} + 2\,A_1^2 C_2 \overline{C_1} & 5 & -2 & 0 \\
-\frac{1}{2}\,A_1 C_1^2 \overline{C_1} & 1 & 2 & 0 \\
-8\,A_0 A_1 C_1 |A_1|^2 \overline{C_1} + 2\,A_1^2 B_1 \overline{C_1} + 2\,A_1^2 C_1 \overline{C_2} & 4 & -1 & 0 \\
A_1^2 \overline{C_1}^2 & 6 & -4 & 0 \\
-4\,A_0 A_1 \alpha_1 \overline{C_1}^2 + 2\,A_1^2 \overline{B_1 C_1} + 4\,A_1 A_2 \overline{C_1}^2 & 7 & -4 & 0 \\
-4\,A_0 A_1 |A_1|^2 \overline{C_1}^2 + 2\,A_1^2 \overline{C_1 C_2} & 6 & -3 & 0
\end{pmatrix} \tag{6.1.50}
$$

where

$$
\begin{aligned}
\chi_1 &= (-2\,A_0 C_1 \alpha_1)(A_1 C_1) + (-2\,A_0 C_1 \alpha_1)(A_1 C_1) + (A_1 C_1)(2\,A_2 C_1) + (A_1 C_1)(2\,A_2 C_1) + (A_1 C_1)(A_1 C_2) \\
&\quad + (A_1 C_1)(A_1 C_2) + \left(A_1 \overline{C_1}\right)\left(-\frac{1}{8}\,C_1 \overline{C_1}\right) + \left(A_1 \overline{C_1}\right)\left(-\frac{1}{8}\,C_1 \overline{C_1}\right) \\
\chi_2 &= \left(-2\,A_0 \alpha_1 \overline{C_1}\right)(A_1 C_1) + \left(-2\,A_0 \alpha_1 \overline{C_1}\right)(A_1 C_1) + (A_1 C_1)\left(A_1 \overline{B_1}\right) + (A_1 C_1)\left(A_1 \overline{B_1}\right) + (-2\,A_0 C_1 \alpha_1)\left(A_1 \overline{C_1}\right) \\
&\quad + (-2\,A_0 C_1 \alpha_1)\left(A_1 \overline{C_1}\right) + (2\,A_2 C_1)\left(A_1 \overline{C_1}\right) + (2\,A_2 C_1)\left(A_1 \overline{C_1}\right) + (A_1 C_2)\left(A_1 \overline{C_1}\right) + (A_1 C_2)\left(A_1 \overline{C_1}\right) \\
&\quad + (A_1 C_1)\left(2\,A_2 \overline{C_1}\right) + (A_1 C_1)\left(2\,A_2 \overline{C_1}\right) \\
\chi_3 &= \left(-2\,A_0 |A_1|^2 \overline{C_1}\right)(A_1 C_1) + \left(-2\,A_0 |A_1|^2 \overline{C_1}\right)(A_1 C_1) + \left(-2\,A_0 C_1 |A_1|^2\right)\left(A_1 \overline{C_1}\right) \\
&\quad + \left(-2\,A_0 C_1 |A_1|^2\right)\left(A_1 \overline{C_1}\right) + (A_1 B_1)\left(A_1 \overline{C_1}\right) + (A_1 B_1)\left(A_1 \overline{C_1}\right) + (A_1 C_1)\left(A_1 \overline{C_2}\right) + (A_1 C_1)\left(A_1 \overline{C_2}\right) \\
\chi_4 &= \left(-2\,A_0 \alpha_1 \overline{C_1}\right)\left(A_1 \overline{C_1}\right) + \left(-2\,A_0 \alpha_1 \overline{C_1}\right)\left(A_1 \overline{C_1}\right) + \left(A_1 \overline{B_1}\right)\left(A_1 \overline{C_1}\right) + \left(A_1 \overline{B_1}\right)\left(A_1 \overline{C_1}\right) \\
&\quad + \left(A_1 \overline{C_1}\right)\left(2\,A_2 \overline{C_1}\right) + \left(A_1 \overline{C_1}\right)\left(2\,A_2 \overline{C_1}\right)
\end{aligned}
$$

We need only π_{15} for the coefficient in

$$
z^{\theta_0} \overline{z}^{2-\theta_0}\, dz^4 = z^4 \overline{z}^{-2}\, dz^4
$$

which is

$$
\begin{aligned}
\pi_{15} &= -80\,A_1^2 |A_1|^4 - 160\,A_0 A_2 |A_1|^4 - 16\,A_0 A_1 \alpha_4 |A_1|^2 - 48\,A_0 A_1 \alpha_{12} - 144\,A_0 C_1 \alpha_7 + 2\left(80\,A_0^2 |A_1|^4 + 3\,A_1 \overline{B_1}\right)\alpha_1 \\
&\quad + 6\left(8\,A_0 A_1 \overline{\alpha_1} + A_0 \overline{B_1}\right)\alpha_3 - 6\,A_3 \overline{B_1} + 6\,A_1 \overline{B_4} - 15\,(7\,A_1 C_1 + 3\,A_0 C_2)\overline{\alpha_2} - 48\left(A_0 A_1 \alpha_1 + A_0^2 \alpha_3 - A_1 A_2 - A_0 A_3\right)\overline{\alpha_4} \\
&= 6\langle \alpha_3 \vec{A}_0 - \vec{A}_3, \overline{\vec{B}_1} \rangle + 6\langle \vec{A}_1, \overline{\vec{B}_4} \rangle - 60\overline{\alpha_2}\langle \vec{A}_1, \vec{C}_1 \rangle
\end{aligned}
$$

Now, recall that

$$
\alpha_3 = \frac{1}{12}\langle \vec{A}_1, \vec{C}_1 \rangle + 2\langle \overline{\vec{A}_0}, \vec{A}_3 \rangle
$$

and

$$
|\vec{A}_0|^2 = \frac{1}{2}, \qquad \vec{B}_1 = -2\langle \overline{\vec{A}_1}, \vec{C}_1 \rangle \vec{A}_0
$$

so

$$\begin{aligned}
&\alpha_2=\frac{1}{10}\,B_1\overline{A_0}+\frac{1}{8}\,C_1\overline{A_1}=-\frac{1}{10}\langle\overline{\vec{A}_1},\vec{C}_1\rangle+\frac{1}{8}\langle\overline{\vec{A}_1},\vec{C}_1\rangle=\frac{1}{40}\langle\overline{\vec{A}_1},\vec{C}_1\rangle\\
&\langle\alpha_3\vec{A}_0-\vec{A}_3,\overline{\vec{A}_0}\rangle=\frac{\alpha_3}{2}-\langle\overline{\vec{A}_0},\vec{A}_3\rangle=\frac{1}{24}\langle\vec{A}_1,\vec{C}_1\rangle\\
&\langle\alpha_3\vec{A}_0-\vec{A}_3,\overline{\vec{B}_1}\rangle=-\frac{1}{12}\langle\vec{A}_1,\vec{C}_1\rangle\langle\vec{A}_1,\overline{\vec{C}_1}\rangle.
\end{aligned}$$

Now, we have

$$\frac{1}{2}\vec{B}_4=\left(\begin{array}{cccccc}
-\dfrac{7}{48}\,C_1\overline{C_1} & \overline{A_1} & -2 & 3 & 0 & \textbf{(23)}\\
A_0\alpha_2\overline{C_1}-\dfrac{2}{15}\,B_1\overline{C_1}-\dfrac{7}{48}\,C_1\overline{C_2} & \overline{A_0} & -2 & 3 & 0 & \textbf{(24)}\\
-\dfrac{1}{48}\,\overline{A_1C_1} & C_1 & -2 & 3 & 0 & \textbf{(25)}\\
\dfrac{5}{3}\,\alpha_2\overline{A_0C_1}+C_1\overline{A_0}\overline{\alpha_3}-\dfrac{1}{3}\,B_2\overline{A_1}-\dfrac{2}{3}\,B_1\overline{A_2}-C_1\overline{A_3}+\dfrac{1}{3}\left(2\,B_1\overline{A_0}+3\,C_1\overline{A_1}\right)\overline{\alpha_1} & A_0 & -2 & 3 & 0 & \textbf{(26)}\\
-\dfrac{1}{24}\,C_1\overline{A_1} & \overline{C_1} & -2 & 3 & 0 & \textbf{(27)}.
\end{array}\right)$$

so there exists $\lambda_1,\lambda_2\in\mathbb{C}$ such that

$$\vec{B}_4=-\frac{7}{24}|\vec{C}_1|^2\overline{\vec{A}_1}-\frac{1}{24}\overline{\langle\vec{A}_1,\vec{C}_1\rangle}\vec{C}_1-\frac{1}{12}\langle\overline{\vec{A}_1},\vec{C}_1\rangle\overline{\vec{C}_1}+\lambda_1\vec{A}_0+\lambda_2\overline{\vec{A}_0}.$$

and we have

$$\begin{aligned}
\langle\overline{\vec{A}_1},\vec{B}_4\rangle&=-\frac{7}{24}|\vec{C}_1|^2\overline{\langle\vec{A}_1,\vec{A}_1\rangle}-\frac{1}{24}\overline{\langle\vec{A}_1,\vec{C}_1\rangle}\langle\overline{\vec{A}_1},\vec{C}_1\rangle-\frac{1}{12}\langle\overline{\vec{A}_1},\vec{C}_1\rangle\overline{\langle\vec{A}_1,\vec{C}_1\rangle}\\
&=-\frac{7}{24}|\vec{C}_1|^2\overline{\langle\vec{A}_1,\vec{A}_1\rangle}-\frac{1}{8}\langle\overline{\vec{A}_1},\vec{C}_1\rangle\overline{\langle\vec{A}_1,\vec{C}_1\rangle}.
\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}
\pi_{15}&=6\langle\alpha_3\vec{A}_0-\vec{A}_3,\overline{\vec{B}_1}\rangle+6\langle\vec{A}_1,\overline{\vec{B}_4}\rangle-60\overline{\alpha_2}\langle\vec{A}_1,\vec{C}_1\rangle\\
&=-\frac{1}{2}\langle\vec{A}_1,\vec{C}_1\rangle\langle\vec{A}_1,\overline{\vec{C}_1}\rangle-\frac{7}{4}|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle-\frac{3}{4}\langle\vec{A}_1,\vec{C}_1\rangle\langle\vec{A}_1,\overline{\vec{C}_1}\rangle-\frac{3}{2}\langle\vec{A}_1,\vec{C}_1\rangle\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\\
&=-\frac{7}{4}|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle-\frac{11}{4}\langle\vec{A}_1,\vec{C}_1\rangle\langle\vec{A}_1,\overline{\vec{C}_1}\rangle.
\end{aligned}\tag{6.1.51}$$

Finally, thanks of (6.1.49) and (6.1.50), the coefficient in $z^4\overline{z}^{-2}dz^4$ is

$$\frac{5}{2}|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_1,\vec{C}_1\rangle\langle\vec{A}_1,\overline{\vec{C}_1}\rangle$$

and finally the coefficient in $z^4\overline{z}^{-2}\,dz^4$ in $\mathscr{Q}_{\vec{\Phi}}$ is

$$\Omega_0=\frac{3}{4}\left(|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle-\langle\vec{A}_1,\vec{C}_1\rangle\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\right)=0.\tag{6.1.52}$$

As the coefficient in $\dfrac{\overline{z}^4}{z}\log|z|$ in $\mathscr{Q}_{\vec{\Phi}}$ is

$$\Omega_1=-6\,A_1C_1\overline{\zeta_0}$$

As $\langle\vec{A}_0,\vec{\gamma}_1\rangle=0$, and

$$\vec{E}_1=-\frac{1}{8}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_0}$$

so

$$\zeta_0=\frac{1}{4}\,A_0\gamma_1+\overline{A_0E_1}=-\frac{1}{16}\langle\vec{C}_1,\vec{C}_1\rangle.$$

so we finally obtain

$$\Omega_1=\frac{3}{8}\langle\vec{A}_1,\vec{C}_1\rangle\overline{\langle\vec{C}_1,\vec{C}_1\rangle}=0 \tag{6.1.53}$$

Thanks of the argument given at the beginning at the section, we are done.

6.2 The case where $\theta_0=3$

Recall the following development valid for all $\theta_0\geq 3$

$$\begin{cases}\partial_z\vec{\Phi}=\vec{A}_0z^{\theta_0-1}+\vec{A}_1z^{\theta_0}+\vec{A}_2z^{\theta_0+1}+\dfrac{1}{4\theta_0}\vec{C}_1z\,\overline{z}^{\theta_0}+\dfrac{1}{8}\overline{\vec{C}_1}z^{\theta_0-1}\overline{z}^2+O(|z|^{\theta_0+2-\varepsilon})\\ e^{2\lambda}=e^{2\lambda}=|z|^{2\theta_0-2}\left(1+2|\vec{A}_1|^2|z|^2+2\,\mathrm{Re}\,\left(\alpha_0z+\alpha_1z^2\right)+O(|z|^{3-\varepsilon})\right)\\ \vec{h}_0=2\left(\vec{A}_1-\alpha_0\vec{A}_0-\left(2|\vec{A}_1|^2-|\alpha_0|^2\right)\vec{A}_0\overline{z}\right)z^{\theta_0-1}+2\left(2\vec{A}_2-\alpha_0\vec{A}_1-(2\alpha_1-\alpha_0^2)\vec{A}_0\right)z^{\theta_0}\\ \qquad-\dfrac{(\theta_0-2)}{2\theta_0}\vec{C}_1\overline{z}^{\theta_0}+O(|z|^{\theta_0+1-\varepsilon})\\ \vec{H}=\mathrm{Re}\left(\dfrac{\vec{C}_1}{z^{\theta_0-2}}\right)+O(|z|^{3-\theta_0-\varepsilon})\\ \mathscr{Q}_{\vec{\Phi}}=(\theta_0-1)(\theta_0-2)\langle\vec{A}_1,\vec{C}_1\rangle\dfrac{dz^4}{z}+O(|z|^{-\varepsilon}).\end{cases} \tag{6.2.1}$$

Also, recall that by conformality of $\vec{\Phi}$, we have

$$\begin{aligned}0=\langle\partial_z\vec{\Phi},\partial_z\vec{\Phi}\rangle=\langle\vec{A}_0,\vec{A}_0\rangle z^{2\theta_0-2}+2\langle\vec{A}_0,\vec{A}_1\rangle z^{2\theta_0-1}+\left(\langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_0,\vec{A}_2\rangle\right)z^{2\theta_0}+\frac{1}{2\theta_0}\langle\vec{A}_0,\vec{C}_1\rangle|z|^{2\theta_0}\\ +\frac{1}{4}\langle\vec{A}_0,\overline{\vec{C}_1}\rangle z^{2\theta_0-2}\overline{z}^2+O(|z|^{2\theta_0+1-\varepsilon})\end{aligned}$$

so that

$$\langle\vec{A}_0,\vec{A}_0\rangle=\langle\vec{A}_0,\vec{A}_1\rangle=\langle\vec{A}_0,\vec{C}_1\rangle=\langle\vec{A}_0,\overline{\vec{C}_1}\rangle=0. \tag{6.2.2}$$

Now, we take $\theta_0=3$ in (6.2.1), we obtain

$$\begin{cases}\partial_z\vec{\Phi}=\vec{A}_0z^2+\vec{A}_1z^3+\vec{A}_2z^4+\dfrac{1}{12}\vec{C}_1z\overline{z}^3+\dfrac{1}{8}\overline{\vec{C}_1}|z|^4+O(|z|^{5-\varepsilon})\\ e^{2\lambda}=|z|^4+2|\vec{A}_1|^2|z|^6+2\,\mathrm{Re}\,(\alpha_0z^3\overline{z}^2+\alpha_1z^4\overline{z}^2)+O(|z|^{7-\varepsilon})\\ \vec{h}_0=2\left(\vec{A}_1-\alpha_0\vec{A}_0-\left(2|\vec{A}_1|^2-|\alpha_0|^2\right)\vec{A}_0\overline{z}\right)z^2+2\left(2\vec{A}_2-\alpha_0\vec{A}_1-(2\alpha_1-\alpha_0^2)\vec{A}_0\right)z^3-\dfrac{1}{6}\vec{C}_1\overline{z}^3+O(|z|^{4-\varepsilon})\\ \vec{H}=\mathrm{Re}\left(\dfrac{\vec{C}_1}{z}\right)+O(|z|^{-\varepsilon})\\ \mathscr{Q}_{\vec{\Phi}}=2\langle\vec{A}_1,\vec{C}_1\rangle\dfrac{dz^4}{z}+O(|z|^{-\varepsilon}).\end{cases}$$

And as logarithm will appear in the forthcoming computations, the last column will indicate the power of the logarithm. Furthermore, as this is a new source of possible mistakes (the algorithms will change to include logarithms), we will only start from

$$\begin{cases}\partial_z\vec{\Phi}=\vec{A}_0z^2+\vec{A}_1z^3+\vec{A}_2z^4+\dfrac{1}{12}\vec{C}_1z\overline{z}^3+\dfrac{1}{8}\overline{\vec{C}_1}|z|^4+O(|z|^{5-\varepsilon})\\ e^{2\lambda}=|z|^4+2|\vec{A}_1|^2|z|^6+2\,\mathrm{Re}\,(\alpha_0z^3\overline{z}^2+\alpha_1z^4\overline{z}^2)+O(|z|^{7-\varepsilon})\\ \vec{H}=\mathrm{Re}\left(\dfrac{\vec{C}_1}{z}\right)+O(|z|^{-\varepsilon})\end{cases} \tag{6.2.3}$$

and recompute $\vec{h}_0$. First, we check that there is no discrepancy in our transcription.

$$\partial_z\vec{\Phi} = \begin{pmatrix} 1 & A_0 & 2 & 0 & 0 \\ 1 & A_1 & 3 & 0 & 0 \\ 1 & A_2 & 4 & 0 & 0 \\ \frac{1}{12} & C_1 & 1 & 3 & 0 \\ \frac{1}{8} & \overline{C_1} & 2 & 2 & 0 \end{pmatrix}, \quad e^{2\lambda} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 2\,|A_1|^2 & 3 & 3 & 0 \\ \alpha_0 & 3 & 2 & 0 \\ \overline{\alpha_0} & 2 & 3 & 0 \\ \alpha_1 & 4 & 2 & 0 \\ \overline{\alpha_1} & 2 & 4 & 0 \end{pmatrix}, \quad \vec{H} = \begin{pmatrix} \frac{1}{2} & C_1 & -1 & 0 & 0 \\ \frac{1}{2} & \overline{C_1} & 0 & -1 & 0 \end{pmatrix} \tag{6.2.4}$$

which checks with (6.2.3). Now, we compute the new expression of $\vec{h}_0$ as

$$\vec{h}_0 = \begin{pmatrix} 2 & A_1 & 2 & 0 & 0 \\ 4 & A_2 & 3 & 0 & 0 \\ -\frac{1}{6} & C_1 & 0 & 3 & 0 \\ -4\,|A_1|^2 + 2\,\alpha_0\overline{\alpha_0} & A_0 & 2 & 1 & 0 \\ -2\,\alpha_0 & A_0 & 2 & 0 & 0 \\ -2\,\alpha_0 & A_1 & 3 & 0 & 0 \\ 2\,\alpha_0^2 - 4\,\alpha_1 & A_0 & 3 & 0 & 0 \end{pmatrix} \tag{6.2.5}$$

to be compared with

$$\vec{h}_0 = 2\left(\vec{A}_1 - \alpha_0\vec{A}_0 - \left(2|\vec{A}_1|^2 - |\alpha_0|^2\right)\vec{A}_0\overline{z}\right)z^2 + 2\left(2\vec{A}_2 - \alpha_0\vec{A}_1 - (2\alpha_1 - \alpha_0^2)\vec{A}_0\right)z^3 - \frac{1}{6}\vec{C}_1\overline{z}^3 + O(|z|^{4-\varepsilon}). \tag{6.2.6}$$

First, there are 7 different terms (of the form $\lambda\vec{\Lambda}z^\alpha\overline{z}^\beta$ for some $\lambda \in \mathbb{C}, \vec{\Lambda} \in \vec{C^n}, \alpha \in \mathbb{Z}, \beta \in \mathbb{Z}$) in the two expressions, and we easily check that each coefficient coincide between (6.2.5) and (6.2.6). Now, recall the fundamental equation

$$\partial\left(\vec{H} - 2i\vec{L} + \vec{\gamma}_0\log|z|\right) = -|\vec{H}|^2\partial\vec{\Phi} - 2\,g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi} \tag{6.2.7}$$

If $\vec{Q} \in C^\infty(D^2\setminus\{0\}, \vec{C^n})$ is the unique anti-holomorphic free of the equation

$$\partial\vec{Q} = -|\vec{H}|^2\partial\vec{\Phi} - 2\,g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}$$

we deduce that

$$\partial\left(\vec{H} - 2i\vec{L} - \vec{Q}\right) = 0,$$

so there exists some $\overline{\vec{D}_2} \in \mathbb{C}^n$ such that

$$\vec{H} - 2i\vec{L} = \frac{\overline{\vec{C}_1}}{z} + \overline{\vec{D}_2} + \vec{Q} + O(|z|^{1-\varepsilon}).$$

so that as $\vec{L}$ is *real*, one obtains

$$\vec{H} = \mathrm{Re}\left(\frac{\vec{C}_1}{z}\right) + \mathrm{Re}\left(\vec{D}_2\right) + \mathrm{Re}\left(\vec{Q}\right) + O(|z|^{1-\varepsilon}) \tag{6.2.8}$$

and we have

$$\begin{aligned}\mathrm{Re}\,(\vec{Q}) &= \begin{pmatrix} 2\,(\cancel{A_0\alpha_0} - A_1)C_1 & \overline{A_0} & 0 & 0 & 1\\ 2\left(\cancel{\overline{A_0}\overline{\alpha_0}} - \overline{A_1}\right)\overline{C_1} & A_0 & 0 & 0 & 1\\ (\cancel{A_0\alpha_0} - A_1)\overline{C_1} & \overline{A_0} & 1 & -1 & 0\\ \left(\cancel{\overline{A_0}\overline{\alpha_0}} - \overline{A_1}\right)C_1 & A_0 & -1 & 1 & 0 \end{pmatrix} + O(|z|^{1-\varepsilon}) = \begin{pmatrix} -2\,A_1C_1 & \overline{A_0} & 0 & 0 & 1\\ -2\,\overline{A_1C_1} & A_0 & 0 & 0 & 1\\ -A_1\overline{C_1} & \overline{A_0} & 1 & -1 & 0\\ -\overline{A_1}C_1 & A_0 & -1 & 1 & 0 \end{pmatrix} + O(|z|^{1-\varepsilon})\\ &= -2\,\mathrm{Re}\left(\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \vec{A}_0 \frac{\overline{z}}{z}\right) - 4\,\mathrm{Re}\left(\langle \vec{A}_1, \vec{C}_1\rangle \overline{\vec{A}_0}\right)\log|z| + O(|z|^{1-\varepsilon}) \end{aligned} \tag{6.2.9}$$

as $\langle \vec{A}_0, \vec{C}_1\rangle = \langle \overline{\vec{A}_0}, \vec{C}_1\rangle = 0$ by (6.2.2). Therefore, if we define

$$\begin{cases} \vec{C}_2 = \mathrm{Re}\left(\vec{D}_2\right) \in \mathbb{R}^n\\ \vec{B}_1 = -2\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \vec{A}_0 \in \mathbb{C}^n\\ \vec{\gamma}_1 = -\vec{\gamma}_0 - 4\,\mathrm{Re}\left(\langle \vec{A}_1, \vec{C}_1\rangle \overline{\vec{A}_0}\right) \in \mathbb{R}^n \end{cases} \tag{6.2.10}$$

we obtain by (6.2.8) and (6.2.9)

$$\vec{H} = \mathrm{Re}\left(\frac{\vec{C}_1}{z} + \vec{B}_1\frac{\overline{z}}{z}\right) + \vec{C}_2 + \vec{\gamma}_1 \log|z| + O(|z|^{1-\varepsilon}). \tag{6.2.11}$$

This is also something that we can easily obtain by hand, as we only need the first order development of the tensors. Let us check this.

First, as for all $\theta_0 \geq 3$, we have $\vec{H} = O(|z|^{2-\theta_0})$, and $\partial_z\vec{\Phi} = O(|z|^{\theta_0-1})$, we have

$$|\vec{H}|^2\partial_z\vec{\Phi} = O(|z|^{3-\theta_0}). \tag{6.2.12}$$

Furthermore, we have

$$\vec{h}_0 = 2\left(\vec{A}_1 - \alpha_0\vec{A}_0\right) z^{\theta_0-1}dz^2 + O(|z|^{\theta_0}), \qquad \vec{H} = \frac{1}{2}\frac{\vec{C}_1}{z^{\theta_0-2}} + \frac{1}{2}\frac{\overline{\vec{C}_1}}{\overline{z}^{\theta_0-2}} + O(|z|^{3-\theta_0-\varepsilon})$$

so (as $\langle \vec{A}_0, \vec{C}_1\rangle = \langle \overline{\vec{A}_0}, \vec{C}_1\rangle = 0$)

$$\langle \vec{H}, \vec{h}_0\rangle = \langle \vec{A}_1, \vec{C}_1\rangle z\, dz + \langle \vec{A}_1, \overline{\vec{C}_1}\rangle z^{\theta_0-1}\overline{z}^{2-\theta_0}\, dz + O(|z|^2) \tag{6.2.13}$$

Now, as

$$\partial_z\vec{\Phi} = \vec{A}_0 z^{\theta_0-1} + O(|z|^{\theta_0}), \quad e^{2\lambda} = |z|^{2\theta_0-2} + O(|z|^{2\theta_0-1})$$

we trivially have

$$e^{-2\lambda}\partial_{\overline{z}}\vec{\Phi} = \overline{\vec{A}_0} z^{1-\theta_0} + O(|z|^{2-\theta_0-\varepsilon}). \tag{6.2.14}$$

Finally, by (6.2.13) and (6.2.14), we have

$$g^{-1}\otimes\langle \vec{H}, \vec{h}_0\rangle \otimes \overline{\partial}\vec{\Phi} = \langle \vec{A}_1, \vec{C}_1\rangle \overline{\vec{A}_0} z^{2-\theta_0}\, dz + \langle \vec{A}_1, \overline{\vec{C}_1}\rangle \overline{\vec{A}_0}\overline{z}^{2-\theta_0}\, dz + O(|z|^{3-\theta_0})$$

so we obtain by (6.2.12) the equation

$$\partial\left(\vec{H}-2i\vec{L}+\vec{\gamma}_0\log|z|\right)=-|\vec{H}|^2\partial\vec{\Phi}-2\,g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi} \tag{6.2.15}$$

$$=-2\langle\vec{A}_1,\vec{C}_1\rangle z^{2-\theta_0}-2\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\overline{\vec{A}_0}\overline{z}^{2-\theta_0}+O(|z|^{3-\theta_0-\varepsilon}). \tag{6.2.16}$$

Taking $\theta_0=3$ in (6.2.15) yields

$$\partial\left(\vec{H}-2i\vec{L}+\vec{\gamma}_0\log|z|\right)=-2\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_0}\frac{dz}{z}-2\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\frac{dz}{\overline{z}}+O(|z|^{-\varepsilon}).$$

so for some $\overline{\vec{D}_2}\in\mathbb{C}^n$, we have

$$\vec{H}-2i\vec{L}+\vec{\gamma}_0\log|z|=-4\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_0}\log|z|-2\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\overline{\vec{A}_0}\frac{z}{\overline{z}}+O(|z|^{1-\varepsilon})$$

which immediately gives by taking the real part the development in (6.2.11) thanks of (6.2.10).

Now we check that the transcription was correct :

$$\vec{H}=\begin{pmatrix}\frac{1}{2} & C_1 & -1 & 0 & 0\\ \frac{1}{2} & B_1 & -1 & 1 & 0\\ 1 & C_2 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & \gamma_1 & 0 & 0 & 1\\ \frac{1}{2} & \overline{C_1} & 0 & -1 & 0\\ \frac{1}{2} & \overline{B_1} & 1 & -1 & 0\end{pmatrix}$$

We know from the previous step that we do not have to develop to the next order $\vec{H}$ to obtain $\vec{A}_1=0$, and by integrating the equation

$$\partial_{\overline{z}}\left(\partial_z\vec{\Phi}\right)=\frac{e^{2\lambda}}{2}\vec{H}$$

we obtain for some $\vec{A}_3\in\mathbb{C}^n$

$$\partial_z\vec{\Phi}=\begin{pmatrix}1 & A_0 & 2 & 0 & 0\\ 1 & A_1 & 3 & 0 & 0\\ 1 & A_2 & 4 & 0 & 0\\ 1 & A_3 & 5 & 0 & 0\\ \frac{1}{12} & C_1 & 1 & 3 & 0\\ \frac{1}{16} & B_1 & 1 & 4 & 0\\ \frac{1}{16}\overline{\alpha_0} & C_1 & 1 & 4 & 0\\ \frac{1}{8} & \overline{C_1} & 2 & 2 & 0\end{pmatrix}\begin{pmatrix}-\frac{1}{36} & \gamma_1 & 2 & 3 & 0\\ \frac{1}{6} & C_2 & 2 & 3 & 0\\ \frac{1}{12}\overline{\alpha_0} & \overline{C_1} & 2 & 3 & 0\\ \frac{1}{12}\alpha_0 & C_1 & 2 & 3 & 0\\ \frac{1}{6} & \gamma_1 & 2 & 3 & 1\\ \frac{1}{8} & \overline{B_1} & 3 & 2 & 0\\ \frac{1}{8}\alpha_0 & \overline{C_1} & 3 & 2 & 0.\end{pmatrix}+O(|z|^{6-\varepsilon})$$

and by cutting this development to one order less we recover

$$\partial_z\vec{\Phi}=\begin{pmatrix}1 & A_0 & 2 & 0 & 0\\ 1 & A_1 & 3 & 0 & 0\\ 1 & A_2 & 4 & 0 & 0\\ \frac{1}{12} & C_1 & 1 & 3 & 0\\ \frac{1}{8} & \overline{C_1} & 2 & 2 & 0\end{pmatrix}$$

as expected (see (6.2.4)). As $\vec{\Phi}$ is real, we also have by direct integration

$$\vec{\Phi}(z) = \begin{pmatrix} \frac{1}{3} & \overline{A_0} & 0 & 3 & 0 \\ \frac{1}{4} & \overline{A_1} & 0 & 4 & 0 \\ \frac{1}{5} & \overline{A_2} & 0 & 5 & 0 \\ \frac{1}{6} & \overline{A_3} & 0 & 6 & 0 \\ \frac{1}{3} & A_0 & 3 & 0 & 0 \\ \frac{1}{4} & A_1 & 4 & 0 & 0 \\ \frac{1}{5} & A_2 & 5 & 0 & 0 \\ \frac{1}{6} & A_3 & 6 & 0 & 0 \\ \frac{1}{24} & C_1 & 2 & 3 & 0 \\ \frac{1}{32} & B_1 & 2 & 4 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{32}\,\overline{\alpha_0} & C_1 & 2 & 4 & 0 \\ \frac{1}{24} & \overline{C_1} & 3 & 2 & 0 \\ -\frac{1}{54} & \gamma_1 & 3 & 3 & 0 \\ \frac{1}{18} & C_2 & 3 & 3 & 0 \\ \frac{1}{36}\,\overline{\alpha_0} & \overline{C_1} & 3 & 3 & 0 \\ \frac{1}{36}\,\alpha_0 & C_1 & 3 & 3 & 0 \\ \frac{1}{18} & \gamma_1 & 3 & 3 & 1 \\ \frac{1}{32} & \overline{B_1} & 4 & 2 & 0 \\ \frac{1}{32}\,\alpha_0 & \overline{C_1} & 4 & 2 & 0 \end{pmatrix} \tag{6.2.17}$$

By conformality, we have

$$0 = \langle \partial_z\vec{\Phi}, \partial_z\vec{\Phi}\rangle = \begin{pmatrix} A_0^2 & 4 & 0 & 0 \\ 2\,A_0A_1 & 5 & 0 & 0 \\ A_1^2 + 2\,A_0A_2 & 6 & 0 & 0 \\ 2\,A_1A_2 + 2\,A_0A_3 & 7 & 0 & 0 \\ \frac{1}{6}\,A_0C_1 & 3 & 3 & 0 \\ \frac{1}{8}\,A_0C_1\overline{\alpha_0} + \frac{1}{8}\,A_0B_1 & 3 & 4 & 0 \\ \frac{1}{4}\,A_0\overline{C_1} & 4 & 2 & 0 \\ \frac{1}{6}\,A_0\overline{C_1}\overline{\alpha_0} + \frac{1}{6}\,(A_0\alpha_0 + A_1)C_1 + \frac{1}{3}\,A_0C_2 - \frac{1}{18}\,A_0\gamma_1 & 4 & 3 & 0 \\ \frac{1}{3}\,A_0\gamma_1 & 4 & 3 & 1 \\ \frac{1}{4}\,A_0\overline{B_1} + \frac{1}{4}\,(A_0\alpha_0 + A_1)\overline{C_1} & 5 & 2 & 0 \end{pmatrix}$$

In particular, we deduce that

$$\begin{cases} \langle \vec{A}_0, \vec{A}_0\rangle = \langle \vec{A}_0, \vec{A}_1\rangle = \langle \vec{A}_0, \vec{C}_1\rangle = \langle \vec{A}_0, \overline{\vec{C}_1}\rangle = \langle \vec{A}_0, \vec{\gamma}_1\rangle = 0 \\ \langle \vec{A}_1, \vec{A}_1\rangle + 2\langle \vec{A}_0, \vec{A}_2\rangle = 0, \quad \langle \vec{A}_1, \vec{C}_1\rangle + 2\langle \vec{A}_0, \vec{C}_2\rangle = 0. \end{cases} \tag{6.2.18}$$

Remark as $\langle \vec{A}_0, \vec{A}_0\rangle = 0$ that

$$0 = \langle \vec{A}_0, \vec{\gamma}_1\rangle = -\left\langle \vec{A}_0, \vec{\gamma}_0 + 4\,\mathrm{Re}\left(\langle \vec{A}_1, \vec{C}_1\rangle\overline{\vec{A}_0}\right)\right\rangle = -\left(\langle \vec{A}_0, \vec{\gamma}_0\rangle + \langle \vec{A}_1, \vec{C}_1\rangle\right) \tag{6.2.19}$$

so for a *true* Willmore sphere, we have $\vec{\gamma}_0 = 0$, and

$$\langle \vec{A}_1, \vec{C}_1\rangle = 0,$$

which this proves the holomorphy of the quartic form. Now, let us come back to the general case where no hypothesis is made on $\vec{\gamma}_0$.

Now, the expansion of the metric is by (6.2.2) (notice that $\vec{\gamma}_1$ and $\vec{C}_2$ are *real* and $\vec{B}_1 \in \mathrm{Span}(\vec{A}_0)$)

$$
e^{2\lambda} = \begin{pmatrix}
2\,A_0\overline{A_0} & 2 & 2 & 0 \\
2\,A_0\overline{A_1} & 2 & 3 & 0 \\
2\,A_0\overline{A_2} + \frac{1}{4}\,\cancel{\overline{A_0 C_1}} & 2 & 4 & 0 \\
\frac{1}{6}\,\cancel{C_1\alpha_0\overline{A_0}} + \frac{1}{3}\,C_2\overline{A_0} - \frac{1}{18}\,\cancel{\gamma_1\overline{A_0}} + 2\,A_0\overline{A_3} + \frac{1}{12}\,(\cancel{2\,\overline{A_0}\overline{\alpha_0}} + 3\,\overline{A_1})\overline{C_1} & 2 & 5 & 0 \\
\frac{1}{6}\,\cancel{A_0\overline{C_1}} & 5 & 1 & 0 \\
\frac{1}{8}\,A_0\overline{B_1} + \frac{1}{24}\,(\cancel{3\,A_0\alpha_0} + 4\,A_1)\overline{C_1} & 6 & 1 & 0 \\
\frac{1}{4}\,\cancel{A_0 C_1} + 2\,A_2\overline{A_0} & 4 & 2 & 0 \\
\frac{1}{6}\,\cancel{A_0\overline{C_1}\overline{\alpha_0}} + \frac{1}{12}\,(\cancel{2\,A_0\alpha_0} + 3\,A_1)C_1 + \frac{1}{3}\,A_0 C_2 - \frac{1}{18}\,\cancel{A_0\gamma_1} + 2\,A_3\overline{A_0} & 5 & 2 & 0 \\
\frac{1}{3}\,\cancel{A_0\gamma_1} & 5 & 2 & 1 \\
\frac{1}{4}\,\cancel{A_0 C_1\overline{\alpha_0}} + \frac{1}{4}\,\cancel{A_0 B_1} + 2\,A_2\overline{A_1} & 4 & 3 & 0 \\
2\,A_1\overline{A_0} & 3 & 2 & 0 \\
2\,A_1\overline{A_1} & 3 & 3 & 0 \\
\frac{1}{4}\,\alpha_0\cancel{\overline{A_0 C_1}} + 2\,A_1\overline{A_2} + \frac{1}{4}\,\cancel{\overline{A_0 B_1}} & 3 & 4 & 0 \\
\frac{1}{6}\,\cancel{C_1\overline{A_0}} & 1 & 5 & 0 \\
\frac{1}{24}\,(\cancel{3\,\overline{A_0}\overline{\alpha_0}} + 4\,\overline{A_1})C_1 + \frac{1}{8}\,B_1\overline{A_0} & 1 & 6 & 0 \\
\frac{1}{3}\,\cancel{\gamma_1\overline{A_0}} & 2 & 5 & 1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
2\,A_0\overline{A_0} & 2 & 2 & 0\\
2\,A_0\overline{A_1} & 2 & 3 & 0\\
2\,A_0\overline{A_2} & 2 & 4 & 0\\
\dfrac{1}{3}\,C_2\overline{A_0}+2\,A_0\overline{A_3}+\dfrac{1}{12}\,(3\,\overline{A_1})\overline{C_1} & 2 & 5 & 0\\
\dfrac{1}{8}\,A_0\overline{B_1}+\dfrac{1}{24}\,(4\,A_1)\overline{C_1} & 6 & 1 & 0\\
2\,A_2\overline{A_0} & 4 & 2 & 0\\
\dfrac{1}{12}\,(3\,A_1)C_1+\dfrac{1}{3}\,A_0C_2+2\,A_3\overline{A_0} & 5 & 2 & 0\\
2\,A_2\overline{A_1} & 4 & 3 & 0\\
2\,A_1\overline{A_0} & 3 & 2 & 0\\
2\,A_1\overline{A_1} & 3 & 3 & 0\\
2\,A_1\overline{A_2} & 3 & 4 & 0\\
\dfrac{1}{24}\,(4\,\overline{A_1})C_1+\dfrac{1}{8}\,B_1\overline{A_0} & 1 & 6 & 0
\end{pmatrix}
= \begin{pmatrix}
2\,A_0\overline{A_0} & 2 & 2 & 0\\
2\,A_0\overline{A_1} & 2 & 3 & 0\\
2\,A_0\overline{A_2} & 2 & 4 & 0\\
\dfrac{1}{12}\overline{A_1C_1}+2\,A_0\overline{A_3} & 2 & 5 & 0\\
\dfrac{1}{24}\,A_1\overline{C_1} & 6 & 1 & 0\\
2\,A_2\overline{A_0} & 4 & 2 & 0\\
\dfrac{1}{12}\,A_1C_1+2\,A_3\overline{A_0} & 5 & 2 & 0\\
2\,A_2\overline{A_1} & 4 & 3 & 0\\
2\,A_1\overline{A_0} & 3 & 2 & 0\\
2\,A_1\overline{A_1} & 3 & 3 & 0\\
2\,A_1\overline{A_2} & 3 & 4 & 0\\
\dfrac{1}{24}\,\overline{A_1}C_1 & 1 & 6 & 0
\end{pmatrix}.
$$

as by (6.2.2)

$$
\begin{cases}
\dfrac{1}{12}\,(3\,A_1)C_1+\dfrac{1}{3}\,A_0C_2\\
\dfrac{1}{4}\langle\vec{A}_1,\vec{C}_1\rangle-\dfrac{1}{6}\langle\vec{A}_1,\vec{C}_1\rangle=\dfrac{1}{12}\langle\vec{A}_1,\vec{C}_1\rangle\\
\langle\overline{\vec{A}_0},\vec{B}_1\rangle=\langle\overline{\vec{A}_0},-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\rangle=-\langle\overline{\vec{A}_1},\vec{C}_1\rangle\\
\dfrac{1}{24}\,(4\,\overline{A_1})C_1+\dfrac{1}{8}\,B_1\overline{A_0}=\dfrac{1}{6}\langle\overline{\vec{A}_1},\vec{C}_1\rangle-\dfrac{1}{8}\langle\overline{\vec{A}_1},\vec{C}_1\rangle.
\end{cases}
$$

Therefore, if $\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{C}$ are defined by

$$
\begin{cases}
\alpha_0=2\langle\overline{\vec{A}_0},\vec{A}_1\rangle\\
\alpha_1=2\langle\overline{\vec{A}_0},\vec{A}_2\rangle\\
\alpha_2=\dfrac{1}{24}\langle\overline{\vec{A}_1},\vec{C}_1\rangle\\
\alpha_3=\dfrac{1}{12}\langle\vec{A}_1,\vec{C}_1\rangle+2\langle\overline{\vec{A}_0},\vec{A}_3\rangle\\
\alpha_4=2\langle\overline{\vec{A}_1},\vec{A}_2\rangle,
\end{cases}
\tag{6.2.20}
$$

we obtain

$$
\begin{aligned}
e^{2\lambda}&=|z|^4+2|\vec{A}_1|^2|z|^6+2\,\mathrm{Re}\left(\alpha_0z^3\overline{z}^2+\alpha_1z^4\overline{z}^2+\alpha_2z\overline{z}^6+\alpha_3z^5\overline{z}^2+\alpha_4z^4\overline{z}^3\right)+O(|z|^{8-\varepsilon})\\
&=|z|^4\left(1+2|\vec{A}_1|^2|z|^2+2\,\mathrm{Re}\left(\alpha_0z+\alpha_1z^2+\alpha_2z^{-1}\overline{z}^3+\alpha_3z^3+\alpha_4z^2\overline{z}\right)+O(|z|^{4-\varepsilon})\right)\\
&=|z|^{2\theta_0-2}\left(1+2|\vec{A}_1|^2|z|^2+2\,\mathrm{Re}\left(\alpha_0z+\alpha_1z^2+\alpha_2z^{2-\theta_0}\overline{z}^{\theta_0}+\alpha_3z^3+\alpha_4z^2\overline{z}\right)+O(|z|^{4-\varepsilon})\right)
\end{aligned}
$$

which is translated as

$$e^{2\lambda} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 2\left|A_1\right|^2 & 3 & 3 & 0 \\ \alpha_0 & 3 & 2 & 0 \\ \alpha_1 & 4 & 2 & 0 \\ \alpha_2 & 1 & 6 & 0 \\ \alpha_3 & 5 & 2 & 0 \\ \alpha_4 & 4 & 3 & 0 \end{pmatrix} \begin{pmatrix} \overline{\alpha_0} & 2 & 3 & 0 \\ \overline{\alpha_1} & 2 & 4 & 0 \\ \overline{\alpha_2} & 6 & 1 & 0 \\ \overline{\alpha_3} & 2 & 5 & 0 \\ \overline{\alpha_4} & 3 & 4 & 0. \end{pmatrix}$$

and we obtain

$$\vec{h}_0 = \begin{pmatrix} 2 & A_1 & 2 & 0 & 0 \\ 4 & A_2 & 3 & 0 & 0 \\ 6 & A_3 & 4 & 0 & 0 \\ -\frac{1}{6} & C_1 & 0 & 3 & 0 \\ -\frac{1}{8} & B_1 & 0 & 4 & 0 \\ -\frac{1}{8}\,\overline{\alpha_0} & C_1 & 0 & 4 & 0 \\ \frac{1}{6} & \gamma_1 & 1 & 3 & 0 \\ -\frac{1}{6}\,\alpha_0 & C_1 & 1 & 3 & 0 \\ \frac{1}{4} & \overline{B_1} & 2 & 2 & 0 \\ -4\,|A_1|^2 + 2\,\alpha_0\overline{\alpha_0} & A_0 & 2 & 1 & 0 \\ -4\,|A_1|^2 + 2\,\alpha_0\overline{\alpha_0} & A_1 & 3 & 1 & 0 \\ -2\,\alpha_0 & A_0 & 2 & 0 & 0 \\ -2\,\alpha_0 & A_1 & 3 & 0 & 0 \\ -2\,\alpha_0 & A_2 & 4 & 0 & 0 \\ 2\,\alpha_0^2 - 4\,\alpha_1 & A_0 & 3 & 0 & 0 \\ 2\,\alpha_0^2 - 4\,\alpha_1 & A_1 & 4 & 0 & 0 \\ 2\,\alpha_2 & A_0 & 0 & 4 & 0 \\ -10\,\alpha_0^3 + 6\,\alpha_0\alpha_1 - 6\,\alpha_3 & A_0 & 4 & 0 & 0 \\ 8\,\alpha_0|A_1|^2 - 28\,\alpha_0^2\overline{\alpha_0} + 4\,\alpha_1\overline{\alpha_0} - 4\,\alpha_4 & A_0 & 3 & 1 & 0 \\ -8\,\overline{\alpha_2} & A_0 & 5 & -1 & 0 \\ -8\,\overline{\alpha_0}^3 & A_0 & 1 & 3 & 0 \\ 4\,|A_1|^2\overline{\alpha_0} - 26\,\alpha_0\overline{\alpha_0}^2 + 2\,\alpha_0\overline{\alpha_1} - 2\,\overline{\alpha_4} & A_0 & 2 & 2 & 0 \end{pmatrix}$$

Notice that there is no logarithm in this expansion, so we can replace in the case $\theta_0 = 3$

$$\mathscr{Q}_{\vec{\Phi}} = (\theta_0 - 1)(\theta_0 - 2)\langle \vec{A}_1, \vec{C}_1\rangle \frac{dz^4}{z} + O(|z|^{-\varepsilon}) = 2\langle \vec{A}_1, \vec{C}_1\rangle \frac{dz^4}{z} + O(|z|^{-\varepsilon})$$

by

$$\mathscr{Q}_{\vec{\Phi}} = 2\langle \vec{A}_1, \vec{C}_1\rangle \frac{dz^4}{z} + O(1) \tag{6.2.21}$$

Now, the conservation law associated to the invariance by inversions yields if

$$\vec{\alpha} = \partial\vec{H} + |\vec{H}|^2\partial\vec{\Phi} + 2\,g^{-1}\otimes\langle \vec{H}, \vec{h}_0\rangle \otimes \overline{\partial}\vec{\Phi}$$

$$\vec{\beta} = |\vec{\Phi}|^2\vec{\alpha} - 2\langle\vec{\Phi},\vec{\alpha}\rangle\vec{\Phi} - g^{-1}\otimes\left(\overline{\partial}|\vec{\Phi}|^2\otimes\vec{h}_0 - 2\langle\vec{\Phi},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}\right)$$

the identity

$$d\,\mathrm{Im}\,(\vec{\beta}) = d\,\mathrm{Im}\,\left(|\vec{\Phi}|^2\vec{\alpha} - 2\langle\vec{\Phi},\vec{\alpha}\rangle\vec{\Phi} - g^{-1}\otimes\left(\overline{\partial}|\vec{\Phi}|^2\otimes\vec{h}_0 - 2\langle\vec{\Phi},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}\right)\right) = 0$$

which is equivalent if $\vec{F}\in C^\infty(D^2\setminus\{0\}\,,\mathbb{C}^n)$ is such that

$$|\vec{\Phi}|^2\vec{\alpha} - 2\langle\vec{\Phi},\vec{\alpha}\rangle\vec{\Phi} - g^{-1}\otimes\left(\overline{\partial}|\vec{\Phi}|^2\otimes\vec{h}_0 - 2\langle\vec{\Phi},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}\right) = \vec{F}(z)dz$$

to

$$d\,\mathrm{Im}\,(\vec{\beta}) = \mathrm{Im}\,\left(\overline{\partial}\vec{\beta}\right) = \mathrm{Im}\,\left(\partial_{\overline{z}}\vec{F}(z)d\overline{z}\wedge dz\right) = -2\,\mathrm{Re}\,\left(\partial_{\overline{z}}\vec{F}(z)\right)dx_1\wedge dx_2.$$

Then, we compute

$$\mathrm{Re}\,(\partial_{\overline{z}}\vec{F}(z)) =$$

$$
\begin{pmatrix}
\frac{1}{6}\,\overline{A_0}^2 & C_1 & -2 & 5 & 0 \\
\frac{5}{72}\,\overline{A_0}^2 & B_1 & -2 & 6 & 0 \\
-\frac{1}{24}\,\overline{A_0}^2\overline{\alpha_0} + \frac{1}{3}\,\overline{A_0 A_1} & C_1 & -2 & 6 & 0 \\
\frac{1}{9}\,C_1\overline{A_0}^3\overline{\alpha_0} - \frac{1}{9}\,C_1\overline{A_0}^2\overline{A_1} - 2\,\alpha_2\overline{A_0}^2 & A_0 & -2 & 6 & 0 \\
-\frac{1}{24}\,C_1\overline{A_0} & \overline{A_1} & -2 & 6 & 0 \\
\frac{2}{9}\,A_0 C_1\overline{A_0 A_1} + 2\,A_0\alpha_2\overline{A_0} - \frac{1}{72}\left(16\,A_0\overline{A_0}^2 - 3\,\overline{A_0}\right)C_1\overline{\alpha_0} + \frac{11}{72}\,B_1\overline{A_0} & \overline{A_0} & -2 & 6 & 0 \\
-\frac{1}{6}\,\overline{A_0}^2 & \gamma_1 & -1 & 5 & 0 \\
\frac{2}{3}\,A_0 C_1\alpha_0\overline{A_0}^2 - 16\,A_0\overline{A_0}\,\overline{\alpha_0}^3 - \frac{2}{3}\,A_1 C_1\overline{A_0}^2 & \overline{A_0} & -1 & 5 & 0 \\
16\,\overline{A_0}^2\overline{\alpha_0}^3 & A_0 & -1 & 5 & 0 \\
-2\,A_0\alpha_0\overline{A_0} + 2\,A_1\overline{A_0} & \overline{A_0} & 0 & 2 & 0 \\
-2\,\overline{A_0}^2 & A_1 & 0 & 2 & 0 \\
2\,\alpha_0\overline{A_0}^2 & A_0 & 0 & 2 & 0 \\
-2\,A_0\alpha_0\overline{A_0} + 2\,A_1\overline{A_0} & \overline{A_1} & 0 & 3 & 0 \\
-4\,A_0|A_1|^2\overline{A_0} - 2\,A_0\alpha_0\overline{A_1} + 2\,A_1\overline{A_1} + 2\left(2\,A_0\alpha_0\overline{A_0} - A_1\overline{A_0}\right)\overline{\alpha_0} & \overline{A_0} & 0 & 3 & 0 \\
2\,\overline{A_0}^2\overline{\alpha_0} - 4\,\overline{A_0 A_1} & A_1 & 0 & 3 & 0 \\
4\,|A_1|^2\overline{A_0}^2 - 4\,\alpha_0\overline{A_0}^2\overline{\alpha_0} + 4\,\alpha_0\overline{A_0 A_1} & A_0 & 0 & 3 & 0 \\
-\frac{11}{36}\,\overline{A_0}^2 & \overline{B_1} & 0 & 4 & 0 \\
\lambda_1 & \overline{A_0} & 0 & 4 & 0 \\
-2\,A_0\alpha_0\overline{A_0} + 2\,A_1\overline{A_0} & \overline{A_2} & 0 & 4 & 0 \\
-4\,A_0|A_1|^2\overline{A_0} - 2\,A_0\alpha_0\overline{A_1} + 2\,A_1\overline{A_1} + 2\left(2\,A_0\alpha_0\overline{A_0} - A_1\overline{A_0}\right)\overline{\alpha_0} & \overline{A_1} & 0 & 4 & 0 \\
\lambda_2 & A_0 & 0 & 4 & 0 \\
-2\,\overline{A_0}^2\overline{\alpha_0}^2 + 4\,\overline{A_0 A_1}\,\overline{\alpha_0} + 2\,\overline{A_0}^2\overline{\alpha_1} - 2\,\overline{A_1}^2 - 4\,\overline{A_0 A_2} & A_1 & 0 & 4 & 0 \\
-4\,\alpha_0^2\overline{A_0}^2 + 4\,\alpha_1\overline{A_0}^2 + \frac{1}{6}\,C_1\overline{A_0} & A_0 & 1 & 2 & 0 \\
4\,A_0\alpha_0^2\overline{A_0} - 4\,A_1\alpha_0\overline{A_0} - 4\,A_0\alpha_1\overline{A_0} + 4\,A_2\overline{A_0} & \overline{A_0} & 1 & 2 & 0 \\
-4\,\overline{A_0}^2 & A_2 & 1 & 2 & 0 \\
4\,\alpha_0\overline{A_0}^2 & A_1 & 1 & 2 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{1}{9}\,A_0\overline{A_0} & B_1 & 1 & 3 & 0 \\
\lambda_3 & A_0 & 1 & 3 & 0 \\
-\frac{1}{6}\,A_0\overline{A_0}\,\overline{\alpha_0} + \frac{1}{6}\,A_0\overline{A_1} & C_1 & 1 & 3 & 0 \\
\lambda_4 & \overline{A_0} & 1 & 3 & 0 \\
4\,A_0\alpha_0^2\overline{A_0} - 4\,A_1\alpha_0\overline{A_0} - 4\,A_0\alpha_1\overline{A_0} - \frac{1}{3}\,A_0C_1 + 4\,A_2\overline{A_0} & \overline{A_1} & 1 & 3 & 0 \\
4\,\overline{A_0}^2\overline{\alpha_0} - 8\,\overline{A_0A_1} & A_2 & 1 & 3 & 0 \\
8\,|A_1|^2\overline{A_0}^2 - 8\,\alpha_0\overline{A_0}^2\overline{\alpha_0} + 8\,\alpha_0\overline{A_0A_1} & A_1 & 1 & 3 & 0 \\
-2\,A_0\overline{A_0}\,\overline{\alpha_0} + 2\,A_0\overline{A_1} & A_0 & 2 & 0 & 0 \\
-2\,A_0^2 & \overline{A_1} & 2 & 0 & 0 \\
2\,A_0^2\overline{\alpha_0} & \overline{A_0} & 2 & 0 & 0 \\
-4\,A_0^2\overline{\alpha_0}^2 + 4\,A_0^2\overline{\alpha_1} + \frac{1}{6}\,A_0\overline{C_1} & \overline{A_0} & 2 & 1 & 0 \\
4\,A_0\overline{A_0}\,\overline{\alpha_0}^2 - 4\,A_0\overline{A_1}\,\overline{\alpha_0} - 4\,A_0\overline{A_0}\,\overline{\alpha_1} + 4\,A_0\overline{A_2} & A_0 & 2 & 1 & 0 \\
-4\,A_0^2 & \overline{A_2} & 2 & 1 & 0 \\
4\,A_0^2\overline{\alpha_0} & \overline{A_1} & 2 & 1 & 0 \\
\lambda_5 & \overline{A_0} & 2 & 2 & 0 \\
\lambda_6 & A_0 & 2 & 2 & 0 \\
-\frac{1}{4}\,A_0\alpha_0\overline{A_0} + \frac{1}{4}\,A_1\overline{A_0} & C_1 & 2 & 2 & 0 \\
-\frac{1}{4}\,A_0\overline{A_0}\,\overline{\alpha_0} + \frac{1}{4}\,A_0\overline{A_1} & \overline{C_1} & 2 & 2 & 0 \\
-6\,\alpha_0^2\overline{A_0}^2 + 6\,\alpha_1\overline{A_0}^2 - \frac{3}{8}\,C_1\overline{A_0} & A_1 & 2 & 2 & 0 \\
-6\,A_0^2\overline{\alpha_0}^2 + 6\,A_0^2\overline{\alpha_1} - \frac{3}{8}\,A_0\overline{C_1} & \overline{A_1} & 2 & 2 & 0 \\
-6\,\overline{A_0}^2 & A_3 & 2 & 2 & 0 \\
-6\,A_0^2 & \overline{A_3} & 2 & 2 & 0 \\
6\,\alpha_0\overline{A_0}^2 & A_2 & 2 & 2 & 0 \\
6\,A_0^2\overline{\alpha_0} & \overline{A_2} & 2 & 2 & 0 \\
-2\,A_0\overline{A_0}\,\overline{\alpha_0} + 2\,A_0\overline{A_1} & A_1 & 3 & 0 & 0 \\
-4\,A_0|A_1|^2\overline{A_0} - 2\,A_0\alpha_0\overline{A_1} + 2\,A_1\overline{A_1} + 2\left(2\,A_0\alpha_0\overline{A_0} - A_1\overline{A_0}\right)\overline{\alpha_0} & A_0 & 3 & 0 & 0 \\
2\,A_0^2\alpha_0 - 4\,A_0A_1 & \overline{A_1} & 3 & 0 & 0 \\
4\,A_0^2|A_1|^2 - 4\left(A_0^2\alpha_0 - A_0A_1\right)\overline{\alpha_0} & \overline{A_0} & 3 & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
-\frac{1}{9}\, A_0\overline{A_0} & \overline{B_1} & 3 & 1 & 0 \\
\lambda_7 & \overline{A_0} & 3 & 1 & 0 \\
-\frac{1}{6}\, A_0\alpha_0\overline{A_0} + \frac{1}{6}\, A_1\overline{A_0} & \overline{C_1} & 3 & 1 & 0 \\
\lambda_8 & A_0 & 3 & 1 & 0 \\
4\, A_0\overline{A_0}\,\overline{\alpha_0}^2 - 4\, A_0\overline{A_1}\,\overline{\alpha_0} - 4\, A_0\overline{A_0}\,\overline{\alpha_1} + 4\, A_0\overline{A_2} - \frac{1}{3}\, \overline{A_0C_1} & A_1 & 3 & 1 & 0 \\
4\, A_0^2\alpha_0 - 8\, A_0A_1 & \overline{A_2} & 3 & 1 & 0 \\
8\, A_0^2|A_1|^2 - 8\left(A_0^2\alpha_0 - A_0A_1\right)\overline{\alpha_0} & \overline{A_1} & 3 & 1 & 0 \\
-\frac{11}{36}\, A_0^2 & B_1 & 4 & 0 & 0 \\
\lambda_9 & A_0 & 4 & 0 & 0 \\
-2\, A_0\overline{A_0}\,\overline{\alpha_0} + 2\, A_0\overline{A_1} & A_2 & 4 & 0 & 0 \\
-4\, A_0|A_1|^2\overline{A_0} - 2\, A_0\alpha_0\overline{A_1} + 2\, A_1\overline{A_1} + 2\left(2\, A_0\alpha_0\overline{A_0} - A_1\overline{A_0}\right)\overline{\alpha_0} & A_1 & 4 & 0 & 0 \\
-8\, A_0^2\alpha_0|A_1|^2 + 8\, A_0A_1|A_1|^2 - 4\, A_0^2\alpha_1\overline{\alpha_0} + 2\, A_0^2\alpha_4 + 2\left(19\, A_0^2\alpha_0^2 - 4\, A_0A_1\alpha_0 + A_1^2 + 2\, A_0A_2\right)\overline{\alpha_0} & \overline{A_0} & 4 & 0 & 0 \\
-2\, A_0^2\alpha_0^2 + 4\, A_0A_1\alpha_0 + 2\, A_0^2\alpha_1 - 2\, A_1^2 - 4\, A_0A_2 & \overline{A_1} & 4 & 0 & 0 \\
\frac{1}{6}\, A_0^2 & \overline{C_1} & 5 & -2 & 0 \\
-\frac{1}{6}\, A_0^2 & \gamma_1 & 5 & -1 & 0 \\
-16\, A_0\alpha_0^3\overline{A_0} + \frac{2}{3}\, A_0^2\overline{A_0C_1}\,\overline{\alpha_0} - \frac{2}{3}\, A_0^2\overline{A_1C_1} & A_0 & 5 & -1 & 0 \\
16\, A_0^2\alpha_0^3 & \overline{A_0} & 5 & -1 & 0 \\
\frac{5}{72}\, A_0^2 & \overline{B_1} & 6 & -2 & 0 \\
-\frac{1}{24}\, A_0^2\alpha_0 + \frac{1}{3}\, A_0A_1 & \overline{C_1} & 6 & -2 & 0 \\
\frac{1}{9}\, A_0^3\alpha_0\overline{C_1} - \frac{1}{9}\, A_0^2A_1\overline{C_1} - 2\, A_0^2\overline{\alpha_2} & \overline{A_0} & 6 & -2 & 0 \\
-\frac{1}{24}\, A_0\overline{C_1} & A_1 & 6 & -2 & 0 \\
\frac{2}{9}\, A_0A_1\overline{A_0C_1} - \frac{1}{72}\left(16\, A_0^2\overline{A_0} - 3\, A_0\right)\alpha_0\overline{C_1} + 2\, A_0\overline{A_0}\,\overline{\alpha_2} + \frac{11}{72}\, A_0\overline{B_1} & A_0 & 6 & -2 & 0
\end{pmatrix}
$$

As $\langle \vec{A}_0, \vec{A}_0\rangle = \langle \vec{A}_0, \vec{C}_1\rangle = \langle \overline{\vec{A}_0}, \vec{C}_1\rangle = 0$, we have

$$
\begin{aligned}
A_0\overline{A_0}^2 C_1 &\in \left\{ \langle \vec{A}_0, \overline{\vec{A}_0}\rangle \langle \overline{\vec{A}_0}, \vec{C}_1\rangle, \langle \overline{\vec{A}_0}, \overline{\vec{A}_0}\rangle \langle \vec{A}_0, \vec{C}_1\rangle \right\} = \{0\} \\
A_1\overline{A_0}^2 C_1 &\in \left\{ \langle \vec{A}_1, \overline{\vec{A}_0}\rangle \langle \overline{\vec{A}_0}, \vec{C}_1\rangle, \langle \vec{A}_1, \vec{C}_1\rangle \overline{\langle \vec{A}_0, \vec{A}_0\rangle} \right\} = \{0\}\,,
\end{aligned}
$$

and we obtain

$$
\left(A_0\alpha_0\overline{A_0}^2 - A_1\overline{A_0}^2\right)C_1 = 0,
$$

so the coefficient in $\dfrac{\overline{z}^{\theta_0+2}}{z} = \dfrac{\overline{z}^5}{z}$ in the Taylor expansion is

$$\begin{pmatrix} \cancel{-\frac{1}{6}\,\overline{A_0}^2} & \gamma_1 & -1 & 5 & 0 \\ \cancel{\frac{2}{3}\,A_0 C_1 \alpha_0 \overline{A_0}^2} - 16\,A_0\overline{A_0}\,\overline{\alpha_0}^3 - \cancel{\frac{2}{3}\,A_1 C_1 \overline{A_0}^2} & \overline{A_0} & -1 & 5 & 0 \\ \cancel{16\,\overline{A_0}^2\,\overline{\alpha_0}^3} & A_0 & -1 & 5 & 0 \end{pmatrix} = -16|\vec{A}_0|^2\overline{\alpha_0}^3\overline{\vec{A}_0} = -8\overline{\alpha_0}^3\overline{\vec{A}_0} = 0 \tag{6.2.22}$$

as $|\vec{A}_0|^2 = \dfrac{1}{2}$. Therefore, we obtain by (6.2.22)

$$\alpha_0 = 2\langle\overline{\vec{A}_0}, \vec{A}_1\rangle = 0 \tag{6.2.23}$$

At this point, we will rerun our previous computations by plugging $\alpha_0 = 0$ from the stage when we integrate

$$\partial_{\overline{z}}\left(\partial_z\vec{\Phi}\right) = \frac{e^{2\lambda}}{2}\vec{H}.$$

We obtain

$$\partial_z\vec{\Phi} = \begin{pmatrix} 1 & A_0 & 2 & 0 & 0 \\ 1 & A_1 & 3 & 0 & 0 \\ 1 & A_2 & 4 & 0 & 0 \\ 1 & A_3 & 5 & 0 & 0 \\ \frac{1}{12} & C_1 & 1 & 3 & 0 \\ \frac{1}{16} & B_1 & 1 & 4 & 0 \\ \frac{1}{8} & \overline{C_1} & 2 & 2 & 0 \\ -\frac{1}{36} & \gamma_1 & 2 & 3 & 0 \\ \frac{1}{6} & C_2 & 2 & 3 & 0 \\ \frac{1}{6} & \gamma_1 & 2 & 3 & 1 \\ \frac{1}{8} & \overline{B_1} & 3 & 2 & 0 \end{pmatrix} \quad \vec{\Phi}(z) = \begin{pmatrix} \frac{1}{3} & \overline{A_0} & 0 & 3 & 0 \\ \frac{1}{4} & \overline{A_1} & 0 & 4 & 0 \\ \frac{1}{5} & \overline{A_2} & 0 & 5 & 0 \\ \frac{1}{6} & \overline{A_3} & 0 & 6 & 0 \\ \frac{1}{3} & A_0 & 3 & 0 & 0 \\ \frac{1}{4} & A_1 & 4 & 0 & 0 \\ \frac{1}{5} & A_2 & 5 & 0 & 0 \\ \frac{1}{6} & A_3 & 6 & 0 & 0 \\ \frac{1}{24} & C_1 & 2 & 3 & 0 \\ \frac{1}{32} & B_1 & 2 & 4 & 0 \\ \frac{1}{24} & \overline{C_1} & 3 & 2 & 0 \\ -\frac{1}{54} & \gamma_1 & 3 & 3 & 0 \\ \frac{1}{18} & C_2 & 3 & 3 & 0 \\ \frac{1}{18} & \gamma_1 & 3 & 3 & 1 \\ \frac{1}{32} & \overline{B_1} & 4 & 2 & 0 \end{pmatrix}, \quad e^{2\lambda} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 2\,|A_1|^2 & 3 & 3 & 0 \\ \alpha_1 & 4 & 2 & 0 \\ \alpha_2 & 1 & 6 & 0 \\ \alpha_3 & 5 & 2 & 0 \\ \alpha_4 & 4 & 3 & 0 \\ \overline{\alpha_1} & 2 & 4 & 0 \\ \overline{\alpha_2} & 6 & 1 & 0 \\ \overline{\alpha_3} & 2 & 5 & 0 \\ \overline{\alpha_4} & 3 & 4 & 0 \end{pmatrix} \tag{6.2.24}$$

$$\vec{h}_0 = \begin{pmatrix} 2 & A_1 & 2 & 0 & 0 \\ 4 & A_2 & 3 & 0 & 0 \\ 6 & A_3 & 4 & 0 & 0 \\ -\frac{1}{6} & C_1 & 0 & 3 & 0 \\ -\frac{1}{8} & B_1 & 0 & 4 & 0 \\ \frac{1}{6} & \gamma_1 & 1 & 3 & 0 \\ \frac{1}{4} & \overline{B_1} & 2 & 2 & 0 \\ -4\,|A_1|^2 & A_0 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -4\,|A_1|^2 & A_1 & 3 & 1 & 0 \\ -4\,\alpha_1 & A_0 & 3 & 0 & 0 \\ -4\,\alpha_1 & A_1 & 4 & 0 & 0 \\ 2\,\alpha_2 & A_0 & 0 & 4 & 0 \\ -6\,\alpha_3 & A_0 & 4 & 0 & 0 \\ -4\,\alpha_4 & A_0 & 3 & 1 & 0 \\ -8\,\overline{\alpha_2} & A_0 & 5 & -1 & 0 \\ -2\,\overline{\alpha_4} & A_0 & 2 & 2 & 0 \end{pmatrix} \tag{6.2.25}$$

$$\vec{\alpha} = \begin{pmatrix} -\frac{1}{6}\,C_1^2 & \overline{A_0} & -3 & 3 & 0 \\ -\frac{1}{2} & C_1 & -2 & 0 & 0 \\ -\frac{1}{2} & B_1 & -2 & 1 & 0 \\ -\frac{1}{6}\,C_1\overline{C_1} & \overline{A_0} & -2 & 2 & 0 \\ \frac{1}{2} & \gamma_1 & -1 & 0 & 0 \\ 2\,A_1C_1 & \overline{A_0} & -1 & 0 & 0 \\ 2\,A_1C_1 & \overline{A_1} & -1 & 1 & 0 \\ -4\,A_0C_1|A_1|^2 + 2\,A_1B_1 & \overline{A_0} & -1 & 1 & 0 \\ \frac{1}{2} & \overline{B_1} & 0 & -1 & 0 \\ 2\,A_1\overline{C_1} & \overline{A_0} & 0 & -1 & 0 \\ 2\,A_1\overline{C_1} & \overline{A_1} & 0 & 0 & 0 \\ \frac{1}{4}\,C_1^2 & A_0 & 0 & 0 & 0 \\ -4\,A_0|A_1|^2\overline{C_1} - 4\,A_0C_1\alpha_1 + 4\,A_2C_1 + 4\,A_1C_2 & \overline{A_0} & 0 & 0 & 0 \\ 4\,A_1\gamma_1 & \overline{A_0} & 0 & 0 & 1 \\ \frac{1}{2}\,C_1\overline{C_1} & A_0 & 1 & -1 & 0 \\ -4\,A_0\alpha_1\overline{C_1} + 2\,A_1\overline{B_1} + 4\,A_2\overline{C_1} & \overline{A_0} & 1 & -1 & 0 \\ \frac{1}{4}\,\overline{C_1}^2 & A_0 & 2 & -2 & 0 \end{pmatrix} \tag{6.2.26}$$

Then, we have for some $\vec{D}_3 \in \mathbb{C}^n$

$$\vec{H} + \vec{\gamma}_0 \log|z| = \vec{C}_2 + \mathrm{Re}\left(\vec{D}_3 z\right)$$

$$
+\begin{pmatrix}
-\frac{1}{12}\,C_1^2 & \overline{A_0} & -2 & 3 & 0 & \mathbf{(1)}\\
-C_1\overline{A_1} & A_0 & -1 & 1 & 0 & \mathbf{(2)}\\
-\frac{5}{24}\,C_1\overline{C_1} & \overline{A_0} & -1 & 2 & 0 & \mathbf{(3)}\\
\cancel{C_1\overline{A_0}\overline{\alpha_1}} - \frac{1}{2}\cancel{B_1\overline{A_1}} - C_1\overline{A_2} & A_0 & -1 & 2 & 0 & \mathbf{(4)}\\
-2\,A_1C_1 & \overline{A_0} & 0 & 0 & 1 & \mathbf{(5)}\\
-2\,\overline{A_1C_1} & A_0 & 0 & 0 & 1 & \mathbf{(6)}\\
-\frac{1}{8}\,\overline{C_1}^2 & \overline{A_0} & 0 & 1 & 0 & \mathbf{(7)}\\
\cancel{2\,C_1|A_1|^2\overline{A_0}} + \cancel{2\,\overline{A_0C_1\alpha_1}} - 2\,C_2\overline{A_1} + \gamma_1\overline{A_1} - 2\,\overline{A_2C_1} & A_0 & 0 & 1 & 0 & \mathbf{(8)}\\
-C_1\overline{A_1} & A_1 & 0 & 1 & 0 & \mathbf{(9)}\\
-2\,A_1C_1 & \overline{A_1} & 0 & 1 & 1 & \mathbf{(10)}\\
\cancel{4\,A_0C_1|A_1|^2} - \cancel{2\,A_1B_1} & \overline{A_0} & 0 & 1 & 1 & \mathbf{(11)}\\
-2\,\gamma_1\overline{A_1} & A_0 & 0 & 1 & 1 & \mathbf{(12)}\\
-A_1\overline{C_1} & \overline{A_0} & 1 & -1 & 0 & \mathbf{(13)}\\
-\frac{1}{8}\,C_1^2 & A_0 & 1 & 0 & 0 & \mathbf{(14)}\\
\cancel{2\,A_0|A_1|^2\overline{C_1}} + \cancel{2\,A_0C_1\alpha_1} - 2\,A_2C_1 - 2\,A_1C_2 + A_1\gamma_1 & \overline{A_0} & 1 & 0 & 0 & \mathbf{(15)}\\
-A_1\overline{C_1} & \overline{A_1} & 1 & 0 & 0 & \mathbf{(16)}\\
-2\,A_1\gamma_1 & \overline{A_0} & 1 & 0 & 1 & \mathbf{(17)}\\
-2\,\overline{A_1C_1} & A_1 & 1 & 0 & 1 & \mathbf{(18)}\\
\cancel{4\,|A_1|^2\overline{A_0C_1}} - \cancel{2\,\overline{A_1B_1}} & A_0 & 1 & 0 & 1 & \mathbf{(19)}\\
-\frac{5}{24}\,C_1\overline{C_1} & A_0 & 2 & -1 & 0 & \mathbf{(20)}\\
\cancel{A_0\alpha_1\overline{C_1}} - \frac{1}{2}\cancel{A_1\overline{B_1}} - A_2\overline{C_1} & \overline{A_0} & 2 & -1 & 0 & \mathbf{(21)}\\
-\frac{1}{12}\,\overline{C_1}^2 & A_0 & 3 & -2 & 0 & \mathbf{(22)}
\end{pmatrix}
$$

Now, recall that

$$
\vec{H} = \begin{pmatrix}
\frac{1}{2} & C_1 & -1 & 0 & 0\\
\frac{1}{2} & B_1 & -1 & 1 & 0\\
1 & C_2 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & \gamma_1 & 0 & 0 & 1\\
\frac{1}{2} & \overline{C_1} & 0 & -1 & 0\\
\frac{1}{2} & \overline{B_1} & 1 & -1 & 0
\end{pmatrix}
$$

and we see that there exists $\vec{C}_3, \vec{B}_2, \vec{E}_1, \vec{\gamma}_2 \in \mathbb{C}^n$ such that

$$
\vec{H} = \mathrm{Re}\left(\frac{\vec{C}_1}{z} + \vec{C}_3 z + \vec{B}_1\frac{\overline{z}}{z} + \vec{B}_2\frac{\overline{z}^2}{z} + \vec{E}_1\frac{\overline{z}^3}{z^2}\right) + \vec{C}_2 + \vec{\gamma}_1\log|z| + \mathrm{Re}\left(\vec{\gamma}_2 z\right)\log|z| + O(|z|^{2-\varepsilon}).
$$

Then, we have

$$\vec{H} = \begin{pmatrix} \frac{1}{2} & C_1 & -1 & 0 & 0 \\ 1 & C_2 & 0 & 0 & 0 \\ \frac{1}{2} & C_3 & 1 & 0 & 0 \\ \frac{1}{2} & B_1 & -1 & 1 & 0 \\ \frac{1}{2} & B_2 & -1 & 2 & 0 \\ \frac{1}{2} & E_1 & -2 & 3 & 0 \\ 1 & \gamma_1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \gamma_2 & 1 & 0 & 1 \\ \frac{1}{2} & \overline{C_1} & 0 & -1 & 0 \\ \frac{1}{2} & \overline{C_3} & 0 & 1 & 0 \\ \frac{1}{2} & \overline{B_1} & 1 & -1 & 0 \\ \frac{1}{2} & \overline{B_2} & 2 & -1 & 0 \\ \frac{1}{2} & \overline{E_1} & 3 & -2 & 0 \\ \frac{1}{2} & \overline{\gamma_2} & 0 & 1 & 1 \end{pmatrix}$$

and we obtain immediately

$$\frac{1}{2}\vec{B}_2 = \begin{pmatrix} -\frac{5}{24}\, C_1\overline{C_1} & \overline{A_0} & -1 & 2 & 0 & \mathbf{(3)} \\ C_1\overline{A_0}\alpha_1 - \frac{1}{2}B_1\overline{A_1} - C_1\overline{A_2} & A_0 & -1 & 2 & 0 & \mathbf{(4)} \end{pmatrix} = -\frac{5}{24}|\vec{C}_1|^2\overline{\vec{A}_0} - \langle \overline{\vec{A}_2}, \vec{C}_1\rangle \vec{A}_0$$

so

$$\vec{B}_2 = -\frac{5}{12}|\vec{C}_1|^2\overline{\vec{A}_0} - 2\langle \overline{\vec{A}_2}, \vec{C}_1\rangle \vec{A}_0. \tag{6.2.27}$$

Then, we have

$$\frac{1}{2}\vec{E}_1 = \begin{pmatrix} -\frac{1}{12}\, C_1^2 & \overline{A_0} & -2 & 3 & 0 & \mathbf{(1)} \end{pmatrix} \tag{6.2.28}$$

so

$$\vec{E}_1 = -\frac{1}{6}\langle \vec{C}_1, \vec{C}_1\rangle \overline{\vec{A}_0}. \tag{6.2.29}$$

Finally, we have

$$\frac{1}{2}\vec{\gamma}_2 = \begin{pmatrix} -2\, A_1\gamma_1 & \overline{A_0} & 1 & 0 & 1 & \mathbf{(17)} \\ -2\,\overline{A_1 C_1} & A_1 & 1 & 0 & 1 & \mathbf{(18)} \\ 4\,|A_1|^2\overline{A_0 C_1} - 2\,\overline{A_1 B_1} & A_0 & 1 & 0 & 1 & \mathbf{(19)} \end{pmatrix}$$
$$= -2\langle \vec{A}_1, \vec{\gamma}_1\rangle \overline{\vec{A}_0} - 2\overline{\langle \vec{A}_1, \vec{C}_1\rangle} \vec{A}_1$$

so

$$\vec{\gamma}_2 = -4\langle \vec{A}_1, \vec{\gamma}_1\rangle \overline{\vec{A}_0} - 4\overline{\langle \vec{A}_1, \vec{C}_1\rangle} \vec{A}_1. \tag{6.2.30}$$

Finally, we obtain by (6.2.27), (6.2.29) and (6.2.30)

$$
\begin{cases}
\vec{C}_2 = \mathrm{Re}\left(\vec{D}_2\right) \in \mathbb{R}^n \\
\vec{C}_3 = \vec{D}_3 - \dfrac{1}{4}\langle \vec{C}_1, \vec{C}_1\rangle \vec{A}_0 + \left(-4(\langle \vec{A}_2, \vec{C}_1\rangle + \langle \vec{A}_1, \vec{C}_2\rangle) + 2\langle \vec{A}_1, \vec{\gamma}_1\rangle\right)\overline{\vec{A}_0} - 2\langle \vec{A}_1, \overline{\vec{C}_1}\rangle \overline{\vec{A}_1} \\
\vec{B}_1 = -2\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \vec{A}_0 \\
\vec{B}_2 = -\dfrac{5}{12}|\vec{C}_1|^2\overline{\vec{A}_0} - 2\langle \overline{\vec{A}_2}, \vec{C}_1\rangle \vec{A}_0. \\
\vec{E}_1 = -\dfrac{1}{6}\langle \vec{C}_1, \vec{C}_1\rangle \overline{\vec{A}_0} \\
\vec{\gamma}_1 = -\vec{\gamma}_0 - 4\,\mathrm{Re}\left(\langle \vec{A}_1, \vec{C}_1\rangle \overline{\vec{A}_0}\right) \in \mathbb{R}^n \\
\vec{\gamma}_2 = -4\langle \vec{A}_1, \vec{\gamma}_1\rangle \overline{\vec{A}_0} - 4\overline{\langle \vec{A}_1, \vec{C}_1\rangle}\vec{A}_1.
\end{cases}
\tag{6.2.31}
$$

and

$$
\partial_z\vec{\Phi} = \begin{pmatrix}
1 & A_0 & 2 & 0 & 0 \\
1 & A_1 & 3 & 0 & 0 \\
1 & A_2 & 4 & 0 & 0 \\
1 & A_3 & 5 & 0 & 0 \\
1 & A_4 & 6 & 0 & 0 \\
\dfrac{1}{24} & E_1 & 0 & 6 & 0 \\
\dfrac{1}{12} & C_1 & 1 & 3 & 0 \\
\dfrac{1}{16} & B_1 & 1 & 4 & 0 \\
\dfrac{1}{20} & B_2 & 1 & 5 & 0 \\
\dfrac{1}{20}\,\overline{\alpha_1} & C_1 & 1 & 5 & 0 \\
\dfrac{1}{8} & \overline{C_1} & 2 & 2 & 0 \\
-\dfrac{1}{36} & \gamma_1 & 2 & 3 & 0 \\
\dfrac{1}{6} & C_2 & 2 & 3 & 0 \\
\dfrac{1}{6} & \gamma_1 & 2 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
-\dfrac{1}{128} & \overline{\gamma_2} & 2 & 4 & 0 \\
\dfrac{1}{16} & \overline{C_3} & 2 & 4 & 0 \\
\dfrac{1}{8}\,|A_1|^2 & C_1 & 2 & 4 & 0 \\
\dfrac{1}{16}\,\overline{\alpha_1} & \overline{C_1} & 2 & 4 & 0 \\
\dfrac{1}{16} & \overline{\gamma_2} & 2 & 4 & 1 \\
\dfrac{1}{8} & \overline{B_1} & 3 & 2 & 0 \\
-\dfrac{1}{72} & \gamma_2 & 3 & 3 & 0 \\
\dfrac{1}{12} & C_3 & 3 & 3 & 0 \\
\dfrac{1}{6}\,|A_1|^2 & \overline{C_1} & 3 & 3 & 0 \\
\dfrac{1}{12}\,\alpha_1 & C_1 & 3 & 3 & 0 \\
\dfrac{1}{12} & \gamma_2 & 3 & 3 & 1 \\
\dfrac{1}{8} & \overline{B_2} & 4 & 2 & 0 \\
\dfrac{1}{8}\,\alpha_1 & \overline{C_1} & 4 & 2 & 0 \\
\dfrac{1}{4} & \overline{E_1} & 5 & 1 & 0
\end{pmatrix}
$$

By conformality, we obtain

$$
0 = \langle \partial_z\vec{\Phi}, \partial_z\vec{\Phi}\rangle =
$$

$$
\begin{pmatrix}
A_0^2 & 4 & 0 & 0 \\
2\,A_0A_1 & 5 & 0 & 0 \\
A_1^2 + 2\,A_0A_2 & 6 & 0 & 0 \\
2\,A_1A_2 + 2\,A_0A_3 & 7 & 0 & 0 \\
A_2^2 + 2\,A_1A_3 + 2\,A_0A_4 & 8 & 0 & 0 \\
\frac{1}{144}\,C_1^2 + \frac{1}{12}\,A_0E_1 & 2 & 6 & 0 \\
\frac{1}{6}\,A_0C_1 & 3 & 3 & 0 \\
\frac{1}{8}\,A_0B_1 & 3 & 4 & 0 \\
\frac{1}{10}\,A_0C_1\overline{\alpha_1} + \frac{1}{10}\,A_0B_2 + \frac{1}{48}\,C_1\overline{C_1} & 3 & 5 & 0 \\
\frac{1}{4}\,A_0\overline{C_1} & 4 & 2 & 0 \\
\frac{1}{6}\,A_1C_1 + \frac{1}{3}\,A_0C_2 - \frac{1}{18}\,A_0\gamma_1 & 4 & 3 & 0 \\
\frac{1}{3}\,A_0\gamma_1 & 4 & 3 & 1 \\
\frac{1}{4}\,A_0C_1|A_1|^2 + \frac{1}{8}\,A_0\overline{C_1}\overline{\alpha_1} + \frac{1}{8}\,A_1B_1 + \frac{1}{64}\,\overline{C_1}^2 + \frac{1}{8}\,A_0\overline{C_3} - \frac{1}{64}\,A_0\overline{\gamma_2} & 4 & 4 & 0 \\
\frac{1}{8}\,A_0\overline{\gamma_2} & 4 & 4 & 1 \\
\frac{1}{4}\,A_0\overline{B_1} + \frac{1}{4}\,A_1\overline{C_1} & 5 & 2 & 0 \\
\frac{1}{3}\,A_0|A_1|^2\overline{C_1} + \frac{1}{6}\,A_0C_1\alpha_1 + \frac{1}{6}\,A_2C_1 + \frac{1}{3}\,A_1C_2 + \frac{1}{6}\,A_0C_3 - \frac{1}{18}\,A_1\gamma_1 - \frac{1}{36}\,A_0\gamma_2 & 5 & 3 & 0 \\
\frac{1}{3}\,A_1\gamma_1 + \frac{1}{6}\,A_0\gamma_2 & 5 & 3 & 1 \\
\frac{1}{4}\,A_0\alpha_1\overline{C_1} + \frac{1}{4}\,A_1\overline{B_1} + \frac{1}{4}\,A_0\overline{B_2} + \frac{1}{4}\,A_2\overline{C_1} & 6 & 2 & 0 \\
\frac{1}{2}\,A_0\overline{E_1} & 7 & 1 & 0
\end{pmatrix}
$$

Now, we obtain

$$
e^{2\lambda} = \begin{pmatrix}
2\,A_0\overline{A_0} & 2 & 2 & 0 \\
2\,A_0\overline{A_1} & 2 & 3 & 0 \\
2\,A_0\overline{A_2} + \frac{1}{4}\,\overline{A_0C_1} & 2 & 4 & 0 \\
\frac{1}{3}\,C_2\overline{A_0} - \frac{1}{18}\,\gamma_1\overline{A_0} + 2\,A_0\overline{A_3} + \frac{1}{4}\,\overline{A_1C_1} & 2 & 5 & 0 \\
\frac{1}{4}\,C_1|A_1|^2\overline{A_0} + \frac{1}{8}\,\overline{A_0C_1}\overline{\alpha_1} + \frac{1}{3}\,C_2\overline{A_1} - \frac{1}{18}\,\gamma_1\overline{A_1} + 2\,A_0\overline{A_4} + \frac{1}{4}\,\overline{A_2C_1} + \frac{1}{8}\,\overline{A_0C_3} - \frac{1}{64}\,\overline{A_0}\overline{\gamma_2} & 2 & 6 & 0 \\
\frac{1}{12}\,A_0\overline{E_1} & 8 & 0 & 0 \\
\frac{1}{6}\,A_0\overline{C_1} & 5 & 1 & 0 \\
\frac{1}{8}\,A_0\overline{B_1} + \frac{1}{6}\,A_1\overline{C_1} & 6 & 1 & 0 \\
\frac{1}{10}\,A_0\alpha_1\overline{C_1} + \frac{1}{8}\,A_1\overline{B_1} + \frac{1}{10}\,A_0\overline{B_2} + \frac{1}{6}\,A_2\overline{C_1} & 7 & 1 & 0 \\
\frac{1}{4}\,A_0C_1 + 2\,A_2\overline{A_0} & 4 & 2 & 0 \\
\frac{1}{4}\,A_1C_1 + \frac{1}{3}\,A_0C_2 - \frac{1}{18}\,A_0\gamma_1 + 2\,A_3\overline{A_0} & 5 & 2 & 0 \\
\frac{1}{3}\,A_0\gamma_1 & 5 & 2 & 1 \\
\frac{1}{4}\,A_0|A_1|^2\overline{C_1} + \frac{1}{8}\,A_0C_1\alpha_1 + \frac{1}{4}\,A_2C_1 + \frac{1}{3}\,A_1C_2 + \frac{1}{8}\,A_0C_3 - \frac{1}{18}\,A_1\gamma_1 - \frac{1}{64}\,A_0\gamma_2 + 2\,A_4\overline{A_0} & 6 & 2 & 0 \\
\frac{1}{3}\,A_1\gamma_1 + \frac{1}{8}\,A_0\gamma_2 & 6 & 2 & 1 \\
\frac{1}{4}\,A_0B_1 + 2\,A_2\overline{A_1} & 4 & 3 & 0 \\
\frac{1}{3}\,A_0C_1|A_1|^2 + \frac{1}{6}\,A_0\overline{C_1}\overline{\alpha_1} + \frac{1}{4}\,A_1B_1 + 2\,A_3\overline{A_1} + \frac{1}{48}\,\overline{C_1}^2 + \frac{1}{6}\,A_0\overline{C_3} + \frac{1}{2}\,\overline{A_0E_1} - \frac{1}{36}\,A_0\overline{\gamma_2} & 5 & 3 & 0 \\
\frac{1}{6}\,A_0\overline{\gamma_2} & 5 & 3 & 1 \\
\frac{1}{4}\,\alpha_1\overline{A_0C_1} + \frac{1}{4}\,A_0C_1\overline{\alpha_1} + \frac{1}{4}\,A_0B_2 + 2\,A_2\overline{A_2} + \frac{1}{4}\,\overline{A_0B_2} + \frac{13}{288}\,C_1\overline{C_1} & 4 & 4 & 0 \\
\frac{1}{3}\,|A_1|^2\overline{A_0C_1} + \frac{1}{6}\,C_1\alpha_1\overline{A_0} + \frac{1}{48}\,C_1^2 + \frac{1}{2}\,A_0E_1 + \frac{1}{6}\,C_3\overline{A_0} - \frac{1}{36}\,\gamma_2\overline{A_0} + 2\,A_1\overline{A_3} + \frac{1}{4}\,\overline{A_1B_1} & 3 & 5 & 0 \\
2\,A_1\overline{A_0} & 3 & 2 & 0 \\
2\,A_1\overline{A_1} & 3 & 3 & 0 \\
2\,A_1\overline{A_2} + \frac{1}{4}\,\overline{A_0B_1} & 3 & 4 & 0 \\
\frac{1}{12}\,E_1\overline{A_0} & 0 & 8 & 0 \\
\frac{1}{6}\,C_1\overline{A_0} & 1 & 5 & 0 \\
\frac{1}{8}\,B_1\overline{A_0} + \frac{1}{6}\,C_1\overline{A_1} & 1 & 6 & 0 \\
\frac{1}{10}\,C_1\overline{A_0}\overline{\alpha_1} + \frac{1}{10}\,B_2\overline{A_0} + \frac{1}{8}\,B_1\overline{A_1} + \frac{1}{6}\,C_1\overline{A_2} & 1 & 7 & 0 \\
\frac{1}{3}\,\gamma_1\overline{A_0} & 2 & 5 & 1 \\
\frac{1}{3}\,\gamma_1\overline{A_1} + \frac{1}{8}\,\overline{A_0}\overline{\gamma_2} & 2 & 6 & 1 \\
\frac{1}{6}\,\gamma_2\overline{A_0} & 3 & 5 & 1
\end{pmatrix}
$$

$e^{2\lambda} =$

$$
\begin{pmatrix}
2\,A_0\overline{A_0} & 2 & 2 & 0 & \mathbf{(1)} \\
2\,A_0\overline{A_1} & 2 & 3 & 0 & \mathbf{(2)} \\
2\,A_0\overline{A_2} + \frac{1}{4}\,\overline{A_0C_1} & 2 & 4 & 0 & \mathbf{(3)} \\
\frac{1}{3}\,C_2\overline{A_0} - \frac{1}{18}\,\gamma_1\overline{A_0} + 2\,A_0\overline{A_3} + \frac{1}{4}\,\overline{A_1C_1} & 2 & 5 & 0 & \mathbf{(4)} \\
\frac{1}{4}\,C_1|A_1|^2\overline{A_0} + \frac{1}{8}\,\overline{A_0C_1\alpha_1} + \frac{1}{3}\,C_2\overline{A_1} - \frac{1}{18}\,\gamma_1\overline{A_1} + 2\,A_0\overline{A_4} + \frac{1}{4}\,\overline{A_2C_1} + \frac{1}{8}\,\overline{A_0C_3} - \frac{1}{64}\,\overline{A_0\gamma_2} & 2 & 6 & 0 & \mathbf{(5)} \\
\frac{1}{12}\,A_0\overline{E_1} & 8 & 0 & 0 & \mathbf{(6)} \\
\frac{1}{6}\,A_0\overline{C_1} & 5 & 1 & 0 & \mathbf{(7)} \\
\frac{1}{8}\,A_0\overline{B_1} + \frac{1}{6}\,A_1\overline{C_1} & 6 & 1 & 0 & \mathbf{(8)} \\
\frac{1}{10}\,A_0\alpha_1\overline{C_1} + \frac{1}{8}\,A_1\overline{B_1} + \frac{1}{10}\,A_0\overline{B_2} + \frac{1}{6}\,A_2\overline{C_1} & 7 & 1 & 0 & \mathbf{(9)} \\
\frac{1}{4}\,A_0C_1 + 2\,A_2\overline{A_0} & 4 & 2 & 0 & \mathbf{(10)} \\
\frac{1}{4}\,A_1C_1 + \frac{1}{3}\,A_0C_2 - \frac{1}{18}\,A_0\gamma_1 + 2\,A_3\overline{A_0} & 5 & 2 & 0 & \mathbf{(11)} \\
\frac{1}{3}\,A_0\gamma_1 & 5 & 2 & 1 & \mathbf{(12)} \\
\frac{1}{4}\,A_0|A_1|^2\overline{C_1} + \frac{1}{8}\,A_0C_1\alpha_1 + \frac{1}{4}\,A_2C_1 + \frac{1}{3}\,A_1C_2 + \frac{1}{8}\,A_0C_3 - \frac{1}{18}\,A_1\gamma_1 - \frac{1}{64}\,A_0\gamma_2 + 2\,A_4\overline{A_0} & 6 & 2 & 0 & \mathbf{(13)} \\
\frac{1}{3}\,A_1\gamma_1 + \frac{1}{8}\,A_0\gamma_2 & 6 & 2 & 1 & \mathbf{(14)} \\
\frac{1}{4}\,A_0B_1 + 2\,A_2\overline{A_1} & 4 & 3 & 0 & \mathbf{(15)} \\
\frac{1}{3}\,A_0C_1|A_1|^2 + \frac{1}{6}\,A_0\overline{C_1}\overline{\alpha_1} + \frac{1}{4}\,A_1B_1 + 2\,A_3\overline{A_1} + \frac{1}{48}\,\overline{C_1}^2 + \frac{1}{6}\,A_0\overline{C_3} + \frac{1}{2}\,\overline{A_0E_1} - \frac{1}{36}\,A_0\overline{\gamma_2} & 5 & 3 & 0 & \mathbf{(16)} \\
\frac{1}{6}\,A_0\overline{\gamma_2} & 5 & 3 & 1 & \mathbf{(17)} \\
\frac{1}{4}\,\alpha_1\overline{A_0C_1} + \frac{1}{4}\,A_0C_1\overline{\alpha_1} + \frac{1}{4}\,A_0B_2 + 2\,A_2\overline{A_2} + \frac{1}{4}\,\overline{A_0B_2} + \frac{13}{288}\,C_1\overline{C_1} & 4 & 4 & 0 & \mathbf{(18)} \\
\frac{1}{3}\,|A_1|^2\overline{A_0C_1} + \frac{1}{6}\,C_1\alpha_1\overline{A_0} + \frac{1}{48}\,C_1^2 + \frac{1}{2}\,A_0E_1 + \frac{1}{6}\,C_3\overline{A_0} - \frac{1}{36}\,\gamma_2\overline{A_0} + 2\,A_1\overline{A_3} + \frac{1}{4}\,\overline{A_1B_1} & 3 & 5 & 0 & \mathbf{(19)} \\
2\,A_1\overline{A_0} & 3 & 2 & 0 & \mathbf{(20)} \\
2\,A_1\overline{A_1} & 3 & 3 & 0 & \mathbf{(21)} \\
2\,A_1\overline{A_2} + \frac{1}{4}\,\overline{A_0B_1} & 3 & 4 & 0 & \mathbf{(22)} \\
\frac{1}{12}\,E_1\overline{A_0} & 0 & 8 & 0 & \mathbf{(23)} \\
\frac{1}{6}\,C_1\overline{A_0} & 1 & 5 & 0 & \mathbf{(24)} \\
\frac{1}{8}\,B_1\overline{A_0} + \frac{1}{6}\,C_1\overline{A_1} & 1 & 6 & 0 & \mathbf{(25)} \\
\frac{1}{10}\,C_1\overline{A_0\alpha_1} + \frac{1}{10}\,B_2\overline{A_0} + \frac{1}{8}\,B_1\overline{A_1} + \frac{1}{6}\,C_1\overline{A_2} & 1 & 7 & 0 & \mathbf{(26)} \\
\frac{1}{3}\,\gamma_1\overline{A_0} & 2 & 5 & 1 & \mathbf{(27)} \\
\frac{1}{3}\,\gamma_1\overline{A_1} + \frac{1}{8}\,\overline{A_0\gamma_2} & 2 & 6 & 1 & \mathbf{(28)} \\
\frac{1}{6}\,\gamma_2\overline{A_0} & 3 & 5 & 1 & \mathbf{(29)}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
2\,A_0\overline{A_0} & 2 & 2 & 0 & \mathbf{(1)} \\
2\,A_0\overline{A_1} & 2 & 3 & 0 & \mathbf{(2)} \\
2\,A_0\overline{A_2} + \frac{1}{4}\,\overline{A_0C_1} & 2 & 4 & 0 & \mathbf{(3)} \\
\frac{1}{3}\,C_2\overline{A_0} - \frac{1}{18}\,\gamma_1\overline{A_0} + 2\,A_0\overline{A_3} + \frac{1}{4}\,\overline{A_1C_1} & 2 & 5 & 0 & \mathbf{(4)} \\
\frac{1}{3}\,C_2\overline{A_1} - \frac{1}{18}\,\gamma_1\overline{A_1} + 2\,A_0\overline{A_4} + \frac{1}{4}\,\overline{A_2C_1} + \frac{1}{8}\,\overline{A_0C_3} - \frac{1}{64}\,\overline{A_0}\,\overline{\gamma_2} & 2 & 6 & 0 & \mathbf{(5)} \\
\frac{1}{6}\,A_0\overline{C_1} & 5 & 1 & 0 & \mathbf{(7)} \\
\frac{1}{8}\,A_0\overline{B_1} + \frac{1}{6}\,A_1\overline{C_1} & 6 & 1 & 0 & \mathbf{(8)} \\
\frac{1}{10}\,A_0\overline{B_2} + \frac{1}{6}\,A_2\overline{C_1} & 7 & 1 & 0 & \mathbf{(9)} \\
\frac{1}{4}\,A_0C_1 + 2\,A_2\overline{A_0} & 4 & 2 & 0 & \mathbf{(10)} \\
\frac{1}{4}\,A_1C_1 + \frac{1}{3}\,A_0C_2 - \frac{1}{18}\,A_0\gamma_1 + 2\,A_3\overline{A_0} & 5 & 2 & 0 & \mathbf{(11)} \\
\frac{1}{3}\,A_0\gamma_1 & 5 & 2 & 1 & \mathbf{(12)} \\
\frac{1}{4}\,A_2C_1 + \frac{1}{3}\,A_1C_2 + \frac{1}{8}\,A_0C_3 - \frac{1}{18}\,A_1\gamma_1 - \frac{1}{64}\,A_0\gamma_2 + 2\,A_4\overline{A_0} & 6 & 2 & 0 & \mathbf{(13)} \\
\frac{1}{3}\,A_1\gamma_1 + \frac{1}{8}\,A_0\gamma_2 & 6 & 2 & 1 & \mathbf{(14)} \\
\frac{1}{4}\,A_0B_1 + 2\,A_2\overline{A_1} & 4 & 3 & 0 & \mathbf{(15)} \\
2\,A_3\overline{A_1} + \frac{1}{48}\,\overline{C_1}^2 + + \frac{1}{6}\,A_0\overline{C_3} + \frac{1}{2}\,\overline{A_0E_1} & 5 & 3 & 0 & \mathbf{(16)} \\
\frac{1}{4}\,A_0B_2 + 2\,A_2\overline{A_2} + \frac{1}{4}\,\overline{A_0B_2} + \frac{13}{288}\,C_1\overline{C_1} & 4 & 4 & 0 & \mathbf{(18)} \\
\frac{1}{48}\,C_1^2 + \frac{1}{2}\,A_0E_1 + \frac{1}{6}\,C_3\overline{A_0} + 2\,A_1\overline{A_3} & 3 & 5 & 0 & \mathbf{(19)} \\
2\,A_1\overline{A_0} & 3 & 2 & 0 & \mathbf{(20)} \\
2\,A_1\overline{A_1} & 3 & 3 & 0 & \mathbf{(21)} \\
2\,A_1\overline{A_2} + \frac{1}{4}\,\overline{A_0B_1} & 3 & 4 & 0 & \mathbf{(22)} \\
\frac{1}{6}\,C_1\overline{A_0} & 1 & 5 & 0 & \mathbf{(24)} \\
\frac{1}{8}\,B_1\overline{A_0} + \frac{1}{6}\,C_1\overline{A_1} & 1 & 6 & 0 & \mathbf{(25)} \\
\frac{1}{10}\,B_2\overline{A_0} + \frac{1}{6}\,C_1\overline{A_2} & 1 & 7 & 0 & \mathbf{(26)} \\
\frac{1}{3}\,\gamma_1\overline{A_0} & 2 & 5 & 1 & \mathbf{(27)} \\
\frac{1}{3}\,\gamma_1\overline{A_1} + \frac{1}{8}\,\overline{A_0}\,\overline{\gamma_2} & 2 & 6 & 1 & \mathbf{(28)}
\end{pmatrix}
$$

Then, recall that

$$e^{2\lambda} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 2\,|A_1|^2 & 3 & 3 & 0 \\ \alpha_1 & 4 & 2 & 0 \\ \alpha_2 & 1 & 6 & 0 \\ \alpha_3 & 5 & 2 & 0 \\ \alpha_4 & 4 & 3 & 0 \end{pmatrix} \begin{pmatrix} \overline{\alpha_1} & 2 & 4 & 0 \\ \overline{\alpha_2} & 6 & 1 & 0 \\ \overline{\alpha_3} & 2 & 5 & 0 \\ \overline{\alpha_4} & 3 & 4 & 0 \end{pmatrix}$$

so we need only look at powers of order $2\theta_0 + 2 = 8$, which are

$$(1\ 7),\ (7\ 1),\ (2\ 6),\ (6\ 2),\ (2\ 6\ 1),\ (6\ 2\ 1),\ (3\ 5),\ (5\ 3),\ (4\ 4).$$

Therefore, there exists $\alpha_5, \alpha_6, \alpha_7, \zeta_0 \in \mathbb{C}$ and $\beta \in \mathbb{R}$ such that (left $\TeX$ and right Sage)

$$e^{2\lambda} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 2\,|A_1|^2 & 3 & 3 & 0 \\ \beta & 4 & 4 & 0 \\ \alpha_1 & 4 & 2 & 0 \\ \alpha_2 & 1 & 6 & 0 \\ \alpha_3 & 5 & 2 & 0 \\ \alpha_4 & 4 & 3 & 0 \\ \alpha_5 & 7 & 1 & 0 \\ \alpha_6 & 6 & 2 & 0 \\ \alpha_7 & 3 & 5 & 0 \\ \zeta_0 & 6 & 2 & 1 \end{pmatrix} \begin{pmatrix} \overline{\alpha_1} & 2 & 4 & 0 \\ \overline{\alpha_2} & 6 & 1 & 0 \\ \overline{\alpha_3} & 2 & 5 & 0 \\ \overline{\alpha_4} & 3 & 4 & 0 \\ \overline{\alpha_5} & 1 & 7 & 0 \\ \overline{\alpha_6} & 2 & 6 & 0 \\ \overline{\alpha_7} & 5 & 3 & 0 \\ \overline{\zeta_0} & 2 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 0 \\ 2\,|A_1|^2 & 3 & 3 & 0 \\ \beta & 4 & 4 & 0 \\ \alpha_1 & 4 & 2 & 0 \\ \alpha_2 & 1 & 6 & 0 \\ \alpha_3 & 5 & 2 & 0 \\ \alpha_4 & 4 & 3 & 0 \\ \alpha_5 & 7 & 1 & 0 \\ \alpha_6 & 6 & 2 & 0 \\ \alpha_7 & 3 & 5 & 0 \\ \zeta_0 & 6 & 2 & 1 \end{pmatrix} \begin{pmatrix} \overline{\alpha_1} & 2 & 4 & 0 \\ \overline{\alpha_2} & 6 & 1 & 0 \\ \overline{\alpha_3} & 2 & 5 & 0 \\ \overline{\alpha_4} & 3 & 4 & 0 \\ \overline{\alpha_5} & 1 & 7 & 0 \\ \overline{\alpha_6} & 2 & 6 & 0 \\ \overline{\alpha_7} & 5 & 3 & 0 \\ \overline{\zeta_0} & 2 & 6 & 1 \end{pmatrix}$$

Then, by developing the quartic form up to order 3, the only term involving logarithm terms in the Taylor expansion are

$$-2\,\zeta_0\langle \vec{A}_1, \vec{C}_1\rangle z^3 \log|z| dz^4 - 2\,\overline{\zeta_0}\langle \vec{A}_1, \vec{C}_1\rangle \overline{z}^3 \log|z| dz^4$$

so we obtain

$$\zeta_0\langle \vec{A}_1, \vec{C}_1\rangle = 0 \tag{6.2.32}$$

and as

$$\langle \vec{A}_1, \vec{\gamma}_1\rangle + 2\langle \vec{A}_0, \vec{\gamma}_2\rangle = 0$$

Then, we have

$$\zeta_0 = \frac{1}{3}\,A_1\gamma_1 + \frac{1}{8}\,A_0\gamma_2 = \frac{1}{12}\langle \vec{A}_1, \vec{\gamma}_1\rangle = -\frac{1}{12}\langle \vec{A}_1, \vec{\gamma}_0\rangle$$

so we obtain

$$\langle \vec{A}_1, \vec{\gamma}_0\rangle\langle \vec{A}_1, \vec{C}_1\rangle = 0$$

so $\langle \vec{A}_1, \vec{C}_1\rangle = 0$ or $\langle \vec{A}_1, \vec{\gamma}_0\rangle = 0$, but the last relation does not imply anything useful, as we have

$$\langle \vec{A}_0, \vec{\gamma}_0\rangle + \langle \vec{A}_1, \vec{C}_1\rangle = 0.$$

Then, we have

$$\vec{h}_0 = \begin{pmatrix} -\dfrac{1}{6} & E_1 & -1 & 6 & 0 \\ -\dfrac{1}{6} & C_1 & 0 & 3 & 0 \\ -\dfrac{1}{8} & B_1 & 0 & 4 & 0 \\ 2\,\alpha_2 & A_0 & 0 & 4 & 0 \\ -\dfrac{1}{10} & B_2 & 0 & 5 & 0 \\ -\dfrac{1}{10}\,\overline{\alpha_1} & C_1 & 0 & 5 & 0 \\ 2\,\overline{\alpha_5} & A_0 & 0 & 5 & 0 \\ \dfrac{1}{6} & \gamma_1 & 1 & 3 & 0 \\ -\dfrac{1}{3}\,|A_1|^2 & C_1 & 1 & 4 & 0 \\ \dfrac{1}{16} & \overline{\gamma_2} & 1 & 4 & 0 \\ 2\,\alpha_2 & A_1 & 1 & 4 & 0 \\ -\overline{\zeta_0} & A_0 & 1 & 4 & 0 \\ 2 & A_1 & 2 & 0 & 0 \\ -4\,|A_1|^2 & A_0 & 2 & 1 & 0 \\ \dfrac{1}{4} & \overline{B_1} & 2 & 2 & 0 \\ -2\,\overline{\alpha_4} & A_0 & 2 & 2 & 0 \\ -\dfrac{1}{6}\,|A_1|^2 & \overline{C_1} & 2 & 3 & 0 \\ \dfrac{1}{18} & \gamma_2 & 2 & 3 & 0 \\ \dfrac{1}{6} & C_3 & 2 & 3 & 0 \\ -\dfrac{1}{6}\,\alpha_1 & C_1 & 2 & 3 & 0 \\ 4\,|A_1|^2\overline{\alpha_1} - 2\,\alpha_7 & A_0 & 2 & 3 & 0 \\ \dfrac{1}{6} & \gamma_2 & 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & A_2 & 3 & 0 & 0 \\ -4\,\alpha_1 & A_0 & 3 & 0 & 0 \\ -4\,|A_1|^2 & A_1 & 3 & 1 & 0 \\ -4\,\alpha_4 & A_0 & 3 & 1 & 0 \\ \dfrac{1}{2} & \overline{B_2} & 3 & 2 & 0 \\ 8\,|A_1|^4 + 4\,\alpha_1\overline{\alpha_1} - 4\,\beta & A_0 & 3 & 2 & 0 \\ -2\,\overline{\alpha_4} & A_1 & 3 & 2 & 0 \\ 6 & A_3 & 4 & 0 & 0 \\ -4\,\alpha_1 & A_1 & 4 & 0 & 0 \\ -6\,\alpha_3 & A_0 & 4 & 0 & 0 \\ -4\,|A_1|^2 & A_2 & 4 & 1 & 0 \\ \dfrac{3}{2} & \overline{E_1} & 4 & 1 & 0 \\ -4\,\alpha_4 & A_1 & 4 & 1 & 0 \\ 12\,\alpha_1|A_1|^2 - 6\,\overline{\alpha_7} & A_0 & 4 & 1 & 0 \\ -8\,\overline{\alpha_2} & A_0 & 5 & -1 & 0 \\ 8 & A_4 & 5 & 0 & 0 \\ -4\,\alpha_1 & A_2 & 5 & 0 & 0 \\ -6\,\alpha_3 & A_1 & 5 & 0 & 0 \\ 4\,\alpha_1^2 - 8\,\alpha_6 - \zeta_0 & A_0 & 5 & 0 & 0 \\ -8\,\zeta_0 & A_0 & 5 & 0 & 1 \\ -10\,\alpha_5 & A_0 & 6 & -1 & 0 \\ -8\,\overline{\alpha_2} & A_1 & 6 & -1 & 0 \end{pmatrix}$$

Now, we need the next order development of the tensors, and we first have for some $\vec{D}_4 \in \mathbb{C}^n$

$\vec{H} + \vec{\gamma}_0 \log|z| = \vec{C}_2 + \mathrm{Re}\left(\vec{D}_3 z + \vec{D}_4 z^2\right) +$

$$
\begin{pmatrix}
-\frac{1}{12}\,C_1^2 & \overline{A_0} & -2 & 3 & 0 & \mathbf{(1)} \\
-\frac{7}{96}\,C_1^2 & \overline{A_1} & -2 & 4 & 0 & \mathbf{(2)} \\
\frac{1}{2}\,A_0C_1\alpha_2 - \frac{13}{96}\,B_1C_1 & \overline{A_0} & -2 & 4 & 0 & \mathbf{(3)} \\
-\frac{1}{48}\,C_1\overline{A_1} & C_1 & -2 & 4 & 0 & \mathbf{(4)} \\
C_1\alpha_2\overline{A_0} - \frac{1}{4}\,E_1\overline{A_1} & A_0 & -2 & 4 & 0 & \mathbf{(5)} \\
-C_1\overline{A_1} & A_0 & -1 & 1 & 0 & \mathbf{(6)} \\
-\frac{5}{24}\,C_1\overline{C_1} & \overline{A_0} & -1 & 2 & 0 & \mathbf{(7)} \\
C_1\overline{A_0}\overline{\alpha_1} - \frac{1}{2}\,B_1\overline{A_1} - C_1\overline{A_2} & A_0 & -1 & 2 & 0 & \mathbf{(8)} \\
A_0\alpha_2\overline{C_1} - \frac{1}{3}\,C_1C_2 + A_1E_1 + \frac{1}{36}\,C_1\gamma_1 - \frac{7}{48}\,B_1\overline{C_1} & \overline{A_0} & -1 & 3 & 0 & \mathbf{(9)} \\
-\frac{1}{6}\,C_1\overline{C_1} & \overline{A_1} & -1 & 3 & 0 & \mathbf{(10)} \\
-\frac{1}{36}\,\overline{A_1C_1} & C_1 & -1 & 3 & 0 & \mathbf{(11)} \\
\frac{4}{3}\,\alpha_2\overline{A_0C_1} + C_1\overline{A_0}\overline{\alpha_3} - \frac{1}{3}\,B_2\overline{A_1} - \frac{2}{3}\,B_1\overline{A_2} - C_1\overline{A_3} + \frac{1}{3}\left(2\,B_1\overline{A_0} + 3\,C_1\overline{A_1}\right)\overline{\alpha_1} & A_0 & -1 & 3 & 0 & \mathbf{(12)} \\
-\frac{1}{24}\,C_1\overline{A_1} & \overline{C_1} & -1 & 3 & 0 & \mathbf{(13)} \\
-\frac{1}{3}\,C_1\gamma_1 & \overline{A_0} & -1 & 3 & 1 & \mathbf{(14)} \\
-2\,A_1C_1 & \overline{A_0} & 0 & 0 & 1 & \mathbf{(15)} \\
-2\,\overline{A_1C_1} & A_0 & 0 & 0 & 1 & \mathbf{(16)} \\
-\frac{1}{8}\,\overline{C_1}^2 & \overline{A_0} & 0 & 1 & 0 & \mathbf{(17)} \\
2\,C_1|A_1|^2\overline{A_0} + 2\,\overline{A_0C_1}\overline{\alpha_1} - 2\,C_2\overline{A_1} + \gamma_1\overline{A_1} - 2\,\overline{A_2C_1} & A_0 & 0 & 1 & 0 & \mathbf{(18)} \\
-C_1\overline{A_1} & A_1 & 0 & 1 & 0 & \mathbf{(19)} \\
-2\,A_1C_1 & \overline{A_1} & 0 & 1 & 1 & \mathbf{(20)} \\
4\,A_0C_1|A_1|^2 - 2\,A_1B_1 & \overline{A_0} & 0 & 1 & 1 & \mathbf{(21)} \\
-2\,\gamma_1\overline{A_1} & A_0 & 0 & 1 & 1 & \mathbf{(22)} \\
-\frac{1}{16}\,\overline{C_1}^2 & \overline{A_1} & 0 & 2 & 0 & \mathbf{(23)} \\
-\frac{1}{8}\,C_1\overline{B_1} - \frac{1}{4}\,C_2\overline{C_1} + \frac{1}{16}\,\gamma_1\overline{C_1} & \overline{A_0} & 0 & 2 & 0 & \mathbf{(24)} \\
-\frac{1}{16}\,\overline{A_1C_1} & \overline{C_1} & 0 & 2 & 0 & \mathbf{(25)} \\
\lambda_1 & A_0 & 0 & 2 & 0 & \mathbf{(26)} \\
C_1\overline{A_0}\overline{\alpha_1} - \frac{1}{2}\,B_1\overline{A_1} - C_1\overline{A_2} & A_1 & 0 & 2 & 0 & \mathbf{(27)}
\end{pmatrix}
$$

$-2\,A_1C_1$	$\overline{A_2}$	0	2	1	**(28)**
$4\,A_0B_1\lvert A_1\rvert^2 + 2\,A_1C_1\overline{\alpha_1} + 2\,A_0C_1\overline{\alpha_4} - 2\,A_1B_2 - \frac{1}{12}\,C_1\overline{B_1} - \frac{5}{12}\,\gamma_1\overline{C_1}$	$\overline{A_0}$	0	2	1	**(29)**
$4\,A_0C_1\lvert A_1\rvert^2 - 2\,A_1B_1$	$\overline{A_1}$	0	2	1	**(30)**
$2\left(\overline{A_0}\overline{\alpha_1} - \overline{A_2}\right)\gamma_1 - \frac{1}{2}\,\overline{A_1}\overline{\gamma_2}$	A_0	0	2	1	**(31)**
$-A_1\overline{C_1}$	$\overline{A_0}$	1	-1	0	**(32)**
$-\frac{1}{8}\,C_1^2$	A_0	1	0	0	**(33)**
$2\,A_0\lvert A_1\rvert^2\overline{C_1} + 2\,A_0C_1\alpha_1 - 2\,A_2C_1 - 2\,A_1C_2 + A_1\gamma_1$	$\overline{A_0}$	1	0	0	**(34)**
$-A_1\overline{C_1}$	$\overline{A_1}$	1	0	0	**(35)**
$-2\,A_1\gamma_1$	$\overline{A_0}$	1	0	1	**(36)**
$-2\,\overline{A_1C_1}$	A_1	1	0	1	**(37)**
$4\,\lvert A_1\rvert^2\overline{A_0C_1} - 2\,\overline{A_1B_1}$	A_0	1	0	1	**(38)**
λ_2	A_0	1	1	0	**(39)**
λ_3	$\overline{A_0}$	1	1	0	**(40)**
$2\,A_0\lvert A_1\rvert^2\overline{C_1} + 2\,A_0C_1\alpha_1 - 2\,A_2C_1 - 2\,A_1C_2 + A_1\gamma_1$	$\overline{A_1}$	1	1	0	**(41)**
$-A_1\overline{C_1}$	$\overline{A_2}$	1	1	0	**(42)**
$2\,C_1\lvert A_1\rvert^2\overline{A_0} + 2\,\overline{A_0C_1}\overline{\alpha_1} - 2\,C_2\overline{A_1} + \gamma_1\overline{A_1} - 2\,\overline{A_2C_1}$	A_1	1	1	0	**(43)**
$-C_1\overline{A_1}$	A_2	1	1	0	**(44)**
$-2\,A_1\gamma_1$	$\overline{A_1}$	1	1	1	**(45)**
$4\,A_0\gamma_1\lvert A_1\rvert^2 - A_1\overline{\gamma_2}$	$\overline{A_0}$	1	1	1	**(46)**
$-2\,\gamma_1\overline{A_1}$	A_1	1	1	1	**(47)**
$4\,\gamma_1\lvert A_1\rvert^2\overline{A_0} - \gamma_2\overline{A_1}$	A_0	1	1	1	**(48)**
$-\frac{5}{24}\,C_1\overline{C_1}$	A_0	2	-1	0	**(49)**
$A_0\alpha_1\overline{C_1} - \frac{1}{2}\,A_1\overline{B_1} - A_2\overline{C_1}$	$\overline{A_0}$	2	-1	0	**(50)**
$-\frac{1}{16}\,C_1^2$	A_1	2	0	0	**(51)**
$-\frac{1}{4}\,C_1C_2 + \frac{1}{16}\,C_1\gamma_1 - \frac{1}{8}\,B_1\overline{C_1}$	A_0	2	0	0	**(52)**
$-\frac{1}{16}\,A_1C_1$	C_1	2	0	0	**(53)**
λ_4	$\overline{A_0}$	2	0	0	**(54)**
$A_0\alpha_1\overline{C_1} - \frac{1}{2}\,A_1\overline{B_1} - A_2\overline{C_1}$	$\overline{A_1}$	2	0	0	**(55)**

$$\begin{pmatrix}
4\,|A_1|^2\overline{A_0B_1}+2\,\alpha_4\overline{A_0C_1}+2\,\alpha_1\overline{A_1C_1}-\dfrac{5}{12}\,C_1\gamma_1-2\,\overline{A_1B_2}-\dfrac{1}{12}\,B_1\overline{C_1} & A_0 & 2 & 0 & 1 & \textbf{(56)}\\
2\,(A_0\alpha_1-A_2)\gamma_1-\dfrac{1}{2}\,A_1\gamma_2 & \overline{A_0} & 2 & 0 & 1 & \textbf{(57)}\\
-2\,\overline{A_1C_1} & A_2 & 2 & 0 & 1 & \textbf{(58)}\\
4\,|A_1|^2\overline{A_0C_1}-2\,\overline{A_1B_1} & A_1 & 2 & 0 & 1 & \textbf{(59)}\\
-\dfrac{1}{12}\,\overline{C_1}^2 & A_0 & 3 & -2 & 0 & \textbf{(60)}\\
-\dfrac{1}{6}\,C_1\overline{C_1} & A_1 & 3 & -1 & 0 & \textbf{(61)}\\
C_1\overline{A_0}\overline{\alpha_2}-\dfrac{7}{48}\,C_1\overline{B_1}-\dfrac{1}{3}\,C_2\overline{C_1}+\dfrac{1}{36}\,\gamma_1\overline{C_1}+\overline{A_1E_1} & A_0 & 3 & -1 & 0 & \textbf{(62)}\\
-\dfrac{1}{36}\,A_1C_1 & \overline{C_1} & 3 & -1 & 0 & \textbf{(63)}\\
A_0\alpha_3\overline{C_1}+\dfrac{4}{3}\,A_0C_1\overline{\alpha_2}+\dfrac{1}{3}\left(2\,A_0\overline{B_1}+3\,A_1\overline{C_1}\right)\alpha_1-\dfrac{2}{3}\,A_2\overline{B_1}-\dfrac{1}{3}\,A_1\overline{B_2}-A_3\overline{C_1} & \overline{A_0} & 3 & -1 & 0 & \textbf{(64)}\\
-\dfrac{1}{24}\,A_1\overline{C_1} & C_1 & 3 & -1 & 0 & \textbf{(65)}\\
-\dfrac{1}{3}\,\gamma_1\overline{C_1} & A_0 & 3 & -1 & 1 & \textbf{(66)}\\
-\dfrac{7}{96}\,\overline{C_1}^2 & A_1 & 4 & -2 & 0 & \textbf{(67)}\\
\dfrac{1}{2}\,\overline{A_0C_1}\overline{\alpha_2}-\dfrac{13}{96}\,\overline{B_1C_1} & A_0 & 4 & -2 & 0 & \textbf{(68)}\\
-\dfrac{1}{48}\,A_1\overline{C_1} & \overline{C_1} & 4 & -2 & 0 & \textbf{(69)}\\
A_0\overline{C_1}\overline{\alpha_2}-\dfrac{1}{4}\,A_1\overline{E_1} & \overline{A_0} & 4 & -2 & 0 & \textbf{(70)}
\end{pmatrix}\tag{6.2.33}$$

where

$$\begin{aligned}
\lambda_1 &= B_1|A_1|^2\overline{A_0}+2\,C_1|A_1|^2\overline{A_1}+\frac{3}{2}\,\overline{A_0C_1}\overline{\alpha_3}+C_1\overline{A_0}\overline{\alpha_4}-\frac{1}{2}\left(\overline{A_0}\overline{\alpha_1}-\overline{A_2}\right)\gamma_1-2\,C_2\overline{A_2}-\frac{3}{2}\,\overline{A_3C_1}-\frac{1}{2}\,\overline{A_1C_3}\\
&\quad+\frac{1}{2}\left(4\,C_2\overline{A_0}+3\,\overline{A_1C_1}\right)\overline{\alpha_1}+\frac{1}{8}\,\overline{A_1}\overline{\gamma_2}\\
\lambda_2 &= 4\,C_2|A_1|^2\overline{A_0}-2\,\gamma_1|A_1|^2\overline{A_0}+4\,|A_1|^2\overline{A_1C_1}+C_1\alpha_4\overline{A_0}+C_1\alpha_1\overline{A_1}+2\,\overline{A_0B_1}\overline{\alpha_1}+2\,\overline{A_0C_1}\overline{\alpha_4}-\frac{3}{8}\,B_1C_1\\
&\quad-C_3\overline{A_1}+\frac{1}{2}\,\gamma_2\overline{A_1}-2\,\overline{A_2B_1}\\
\lambda_3 &= 4\,A_1C_1|A_1|^2+4\,A_0C_2|A_1|^2-2\,A_0\gamma_1|A_1|^2+2\,A_0B_1\alpha_1+2\,A_0C_1\alpha_4+A_1\overline{C_1}\overline{\alpha_1}+A_0\overline{C_1}\overline{\alpha_4}-2\,A_2B_1-\frac{3}{8}\,\overline{B_1C_1}\\
&\quad-A_1\overline{C_3}+\frac{1}{2}\,A_1\overline{\gamma_2}\\
\lambda_4 &= A_0|A_1|^2\overline{B_1}+2\,A_1|A_1|^2\overline{C_1}+\frac{3}{2}\,A_0C_1\alpha_3+A_0\alpha_4\overline{C_1}-\frac{3}{2}\,A_3C_1-2\,A_2C_2-\frac{1}{2}\,A_1C_3+\frac{1}{2}\,(3\,A_1C_1+4\,A_0C_2)\alpha_1\\
&\quad-\frac{1}{2}\,(A_0\alpha_1-A_2)\gamma_1+\frac{1}{8}\,A_1\gamma_2
\end{aligned}$$

We see that the new powers are

$$\mathrm{Re}\,(z^2),\qquad \mathrm{Re}\,(z^2)\log|z|,\ \ \mathrm{Re}\left(\frac{\overline{z}^3}{z}\right),\ \ \mathrm{Re}\left(\frac{\overline{z}^3}{z}\right)\log|z|,\ \ \mathrm{Re}\left(\frac{\overline{z}^4}{z^2}\right),\ \ |z|^2,\ \ |z|^2\log|z|$$

so there exists $\vec{C}_4,\vec{B}_3,\vec{E}_2,\vec{\gamma}_3,\vec{\gamma}_4\in\mathbb{C}^n$ and $\vec{E}_4,\vec{\gamma}_5\in\mathbb{R}^n$ such that

$$\begin{aligned}
\vec{H} &= \mathrm{Re}\left(\frac{\vec{C}_1}{z}+\vec{C}_3z+\vec{C}_4+\vec{B}_1\frac{\overline{z}}{z}+\vec{B}_2\frac{\overline{z}^2}{z}+\vec{B}_3\frac{\overline{z}^3}{z}+\vec{E}_1\frac{\overline{z}^3}{z^2}+\vec{E}_2\frac{\overline{z}^4}{z^2}\right)+\vec{C}_2+\vec{B}_4|z|^2\\
&\quad+\vec{\gamma}_1\log|z|+\mathrm{Re}\left(\vec{\gamma}_2z+\vec{\gamma}_3z^2+\frac{\overline{z}^3}{z}\vec{\gamma}_4\right)\log|z|+\vec{\gamma}_5|z|^2\log|z|+O(|z|^{2-\varepsilon}).
\end{aligned}$$

which gives, once translated in Sage

$$\begin{pmatrix} \frac{1}{2} & C_1 & -1 & 0 & 0 \\ 1 & C_2 & 0 & 0 & 0 \\ \frac{1}{2} & C_3 & 1 & 0 & 0 \\ \frac{1}{2} & C_4 & 2 & 0 & 0 \\ \frac{1}{2} & B_1 & -1 & 1 & 0 \\ \frac{1}{2} & B_2 & -1 & 2 & 0 \\ \frac{1}{2} & B_3 & -1 & 3 & 0 \\ 1 & B_4 & 1 & 1 & 0 \\ \frac{1}{2} & E_1 & -2 & 3 & 0 \\ \frac{1}{2} & E_2 & -2 & 4 & 0 \\ 1 & \gamma_1 & 0 & 0 & 1 \\ \frac{1}{2} & \gamma_2 & 1 & 0 & 1 \\ \frac{1}{2} & \gamma_3 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \gamma_4 & -1 & 3 & 0 \\ 1 & \gamma_5 & 1 & 1 & 1 \\ \frac{1}{2} & \overline{C_1} & 0 & -1 & 0 \\ \frac{1}{2} & \overline{C_3} & 0 & 1 & 0 \\ \frac{1}{2} & \overline{C_4} & 0 & 2 & 0 \\ \frac{1}{2} & \overline{B_1} & 1 & -1 & 0 \\ \frac{1}{2} & \overline{B_2} & 2 & -1 & 0 \\ \frac{1}{2} & \overline{B_3} & 3 & -1 & 0 \\ \frac{1}{2} & \overline{E_1} & 3 & -2 & 0 \\ \frac{1}{2} & \overline{E_2} & 4 & -2 & 0 \\ \frac{1}{2} & \overline{\gamma_2} & 0 & 1 & 1 \\ \frac{1}{2} & \overline{\gamma_3} & 0 & 2 & 1 \\ \frac{1}{2} & \overline{\gamma_4} & 3 & -1 & 0 \end{pmatrix}$$

Now, by integrating

$$\partial_{\overline{z}}\left(\partial_z \vec{\Phi}\right) = \frac{e^{2\lambda}}{2}\vec{H},$$

we obtain for some $\vec{A}_5 \in \mathbb{C}^n$

$$\partial_z\vec{\Phi} = \begin{pmatrix} 1 & A_0 & 2 & 0 & 0 \\ 1 & A_1 & 3 & 0 & 0 \\ 1 & A_2 & 4 & 0 & 0 \\ 1 & A_3 & 5 & 0 & 0 \\ 1 & A_4 & 6 & 0 & 0 \\ 1 & A_5 & 7 & 0 & 0 \\ \frac{1}{24} & E_1 & 0 & 6 & 0 \\ \frac{1}{28} & E_2 & 0 & 7 & 0 \\ \frac{1}{28}\,\alpha_2 & C_1 & 0 & 7 & 0 \\ \frac{1}{12} & C_1 & 1 & 3 & 0 \\ \frac{1}{16} & B_1 & 1 & 4 & 0 \\ \frac{1}{20} & B_2 & 1 & 5 & 0 \\ \frac{1}{20}\,\overline{\alpha_1} & C_1 & 1 & 5 & 0 \\ \frac{1}{24} & B_3 & 1 & 6 & 0 \\ \frac{1}{24} & \gamma_4 & 1 & 6 & 0 \\ \frac{1}{24}\,\alpha_2 & \overline{C_1} & 1 & 6 & 0 \\ \frac{1}{24}\,\overline{\alpha_1} & B_1 & 1 & 6 & 0 \\ \frac{1}{24}\,\overline{\alpha_3} & C_1 & 1 & 6 & 0 \\ \frac{1}{8} & \overline{C_1} & 2 & 2 & 0 \\ -\frac{1}{36} & \gamma_1 & 2 & 3 & 0 \\ \frac{1}{6} & C_2 & 2 & 3 & 0 \\ \frac{1}{6} & \gamma_1 & 2 & 3 & 1 \\ -\frac{1}{128} & \overline{\gamma_2} & 2 & 4 & 0 \\ \frac{1}{16} & \overline{C_3} & 2 & 4 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{8}\,|A_1|^2 & C_1 & 2 & 4 & 0 \\ \frac{1}{16}\,\overline{\alpha_1} & \overline{C_1} & 2 & 4 & 0 \\ \frac{1}{16} & \overline{\gamma_2} & 2 & 4 & 1 \\ -\frac{1}{200} & \overline{\gamma_3} & 2 & 5 & 0 \\ \frac{1}{20} & \overline{C_4} & 2 & 5 & 0 \\ \frac{1}{10}\,|A_1|^2 & B_1 & 2 & 5 & 0 \\ \frac{1}{10}\,\overline{\alpha_1} & C_2 & 2 & 5 & 0 \\ -\frac{1}{100}\,\overline{\alpha_1} & \gamma_1 & 2 & 5 & 0 \\ \frac{1}{20}\,\overline{\alpha_3} & \overline{C_1} & 2 & 5 & 0 \\ \frac{1}{20}\,\overline{\alpha_4} & C_1 & 2 & 5 & 0 \\ \frac{1}{20} & \overline{\gamma_3} & 2 & 5 & 1 \\ \frac{1}{10}\,\overline{\alpha_1} & \gamma_1 & 2 & 5 & 1 \\ \frac{1}{8} & \overline{B_1} & 3 & 2 & 0 \\ -\frac{1}{72} & \gamma_2 & 3 & 3 & 0 \\ \frac{1}{12} & C_3 & 3 & 3 & 0 \\ \frac{1}{6}\,|A_1|^2 & \overline{C_1} & 3 & 3 & 0 \\ \frac{1}{12}\,\alpha_1 & C_1 & 3 & 3 & 0 \\ \frac{1}{12} & \gamma_2 & 3 & 3 & 1 \\ -\frac{1}{64} & \gamma_5 & 3 & 4 & 0 \\ \frac{1}{8} & B_4 & 3 & 4 & 0 \\ \frac{1}{4}\,|A_1|^2 & C_2 & 3 & 4 & 0 \\ -\frac{1}{32}\,|A_1|^2 & \gamma_1 & 3 & 4 & 0 \\ \frac{1}{16}\,\alpha_1 & B_1 & 3 & 4 & 0 \\ \frac{1}{16}\,\alpha_4 & C_1 & 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{16}\,\overline{\alpha_1} & \overline{B_1} & 3 & 4 & 0 \\ \frac{1}{16}\,\overline{\alpha_4} & \overline{C_1} & 3 & 4 & 0 \\ \frac{1}{8} & \gamma_5 & 3 & 4 & 1 \\ \frac{1}{4}\,|A_1|^2 & \gamma_1 & 3 & 4 & 1 \\ \frac{1}{8} & \overline{B_2} & 4 & 2 & 0 \\ \frac{1}{8}\,\alpha_1 & \overline{C_1} & 4 & 2 & 0 \\ -\frac{1}{72} & \gamma_3 & 4 & 3 & 0 \\ \frac{1}{12} & C_4 & 4 & 3 & 0 \\ \frac{1}{6}\,|A_1|^2 & \overline{B_1} & 4 & 3 & 0 \\ \frac{1}{6}\,\alpha_1 & C_2 & 4 & 3 & 0 \\ -\frac{1}{36}\,\alpha_1 & \gamma_1 & 4 & 3 & 0 \\ \frac{1}{12}\,\alpha_3 & C_1 & 4 & 3 & 0 \\ \frac{1}{12}\,\alpha_4 & \overline{C_1} & 4 & 3 & 0 \\ \frac{1}{12} & \gamma_3 & 4 & 3 & 1 \\ \frac{1}{6}\,\alpha_1 & \gamma_1 & 4 & 3 & 1 \\ \frac{1}{4} & \overline{E_1} & 5 & 1 & 0 \\ \frac{1}{8} & \overline{B_3} & 5 & 2 & 0 \\ \frac{1}{8} & \overline{\gamma_4} & 5 & 2 & 0 \\ \frac{1}{8}\,\alpha_1 & \overline{B_1} & 5 & 2 & 0 \\ \frac{1}{8}\,\alpha_3 & \overline{C_1} & 5 & 2 & 0 \\ \frac{1}{8}\,\overline{\alpha_2} & C_1 & 5 & 2 & 0 \\ \frac{1}{4} & \overline{E_2} & 6 & 1 & 0 \\ \frac{1}{4}\,\overline{\alpha_2} & \overline{C_1} & 6 & 1 & 0 \end{pmatrix}$$

Then, we obtain by conformality of the immersion $\vec{\Phi}$

$$0 = \langle \partial_z\vec{\Phi}, \partial_z\vec{\Phi}\rangle \tag{6.2.34}$$

$$
= \begin{pmatrix}
A_0^2 & 4 & 0 & 0 \\
2\,A_0A_1 & 5 & 0 & 0 \\
A_1^2 + 2\,A_0A_2 & 6 & 0 & 0 \\
2\,A_1A_2 + 2\,A_0A_3 & 7 & 0 & 0 \\
A_2^2 + 2\,A_1A_3 + 2\,A_0A_4 & 8 & 0 & 0 \\
2\,A_2A_3 + 2\,A_1A_4 + 2\,A_0A_5 & 9 & 0 & 0 \\
\frac{1}{2}\,A_0\overline{E_1} & 7 & 1 & 0 \\
\frac{1}{2}\,A_0\overline{C_1}\,\overline{\alpha_2} + \frac{1}{2}\,A_1\overline{E_1} + \frac{1}{2}\,A_0\overline{E_2} & 8 & 1 & 0 \\
\frac{1}{4}\,A_0\overline{C_1} & 4 & 2 & 0 \\
\frac{1}{4}\,A_0\overline{B_1} + \frac{1}{4}\,A_1\overline{C_1} & 5 & 2 & 0 \\
\frac{1}{4}\,A_0\alpha_1\overline{C_1} + \frac{1}{4}\,A_1\overline{B_1} + \frac{1}{4}\,A_0\overline{B_2} + \frac{1}{4}\,A_2\overline{C_1} & 6 & 2 & 0 \\
\frac{1}{4}\,A_0\alpha_3\overline{C_1} + \frac{1}{4}\,A_0C_1\overline{\alpha_2} + \frac{1}{4}\,\left(A_0\overline{B_1} + A_1\overline{C_1}\right)\alpha_1 + \frac{1}{4}\,A_2\overline{B_1} + \frac{1}{4}\,A_1\overline{B_2} + \frac{1}{4}\,A_0\overline{B_3} + \frac{1}{4}\,A_3\overline{C_1} + \frac{1}{4}\,A_0\overline{\gamma_4} & 7 & 2 & 0 \\
\frac{1}{6}\,A_0C_1 & 3 & 3 & 0 \\
\frac{1}{6}\,A_1C_1 + \frac{1}{3}\,A_0C_2 - \frac{1}{18}\,A_0\gamma_1 & 4 & 3 & 0 \\
\frac{1}{3}\,A_0|A_1|^2\overline{C_1} + \frac{1}{6}\,A_0C_1\alpha_1 + \frac{1}{6}\,A_2C_1 + \frac{1}{3}\,A_1C_2 + \frac{1}{6}\,A_0C_3 - \frac{1}{18}\,A_1\gamma_1 - \frac{1}{36}\,A_0\gamma_2 & 5 & 3 & 0 \\
\lambda_1 & 6 & 3 & 0 \\
\frac{1}{3}\,A_0\gamma_1 & 4 & 3 & 1 \\
\frac{1}{3}\,A_1\gamma_1 + \frac{1}{6}\,A_0\gamma_2 & 5 & 3 & 1 \\
\frac{1}{3}\,(A_0\alpha_1 + A_2)\gamma_1 + \frac{1}{6}\,A_1\gamma_2 + \frac{1}{6}\,A_0\gamma_3 & 6 & 3 & 1 \\
\frac{1}{8}\,A_0B_1 & 3 & 4 & 0 \\
\frac{1}{4}\,A_0C_1|A_1|^2 + \frac{1}{8}\,A_0\overline{C_1}\,\overline{\alpha_1} + \frac{1}{8}\,A_1B_1 + \frac{1}{64}\,\overline{C_1}^2 + \frac{1}{8}\,A_0\overline{C_3} - \frac{1}{64}\,A_0\overline{\gamma_2} & 4 & 4 & 0 \\
\lambda_2 & 5 & 4 & 0 \\
\frac{1}{8}\,A_0\overline{\gamma_2} & 4 & 4 & 1 \\
\frac{1}{2}\,A_0\gamma_1|A_1|^2 + \frac{1}{4}\,A_0\gamma_5 + \frac{1}{8}\,A_1\overline{\gamma_2} & 5 & 4 & 1 \\
\frac{1}{10}\,A_0C_1\overline{\alpha_1} + \frac{1}{10}\,A_0B_2 + \frac{1}{48}\,C_1\overline{C_1} & 3 & 5 & 0 \\
\lambda_3 & 4 & 5 & 0 \\
\frac{1}{120}\,\left(24\,A_0\overline{\alpha_1} + 5\,\overline{C_1}\right)\gamma_1 + \frac{1}{10}\,A_0\overline{\gamma_3} & 4 & 5 & 1 \\
\frac{1}{144}\,C_1^2 + \frac{1}{12}\,A_0E_1 & 2 & 6 & 0 \\
\frac{1}{12}\,A_0\alpha_2\overline{C_1} + \frac{1}{12}\,A_0B_1\overline{\alpha_1} + \frac{1}{12}\,A_0C_1\overline{\alpha_3} + \frac{1}{12}\,A_0B_3 + \frac{1}{36}\,C_1C_2 + \frac{1}{12}\,A_1E_1 - \frac{1}{216}\,C_1\gamma_1 + \frac{1}{12}\,A_0\gamma_4 + \frac{1}{64}\,B_1\overline{C_1} & 3 & 6 & 0 \\
\frac{1}{36}\,C_1\gamma_1 & 3 & 6 & 1 \\
\frac{1}{14}\,A_0C_1\alpha_2 + \frac{1}{96}\,B_1C_1 + \frac{1}{14}\,A_0E_2 & 2 & 7 & 0
\end{pmatrix} \tag{6.2.35}
$$

where

$$
\lambda_1 = \frac{1}{3}\,A_0|A_1|^2\overline{B_1} + \frac{1}{3}\,A_1|A_1|^2\overline{C_1} + \frac{1}{6}\,A_0C_1\alpha_3 + \frac{1}{6}\,A_0\alpha_4\overline{C_1} + \frac{1}{6}\,A_3C_1 + \frac{1}{3}\,A_2C_2 + \frac{1}{6}\,A_1C_3 + \frac{1}{6}\,A_0C_4
$$
$$
+ \frac{1}{6}\,(A_1C_1 + 2\,A_0C_2)\alpha_1 - \frac{1}{18}\,(A_0\alpha_1 + A_2)\gamma_1 - \frac{1}{36}\,A_1\gamma_2 - \frac{1}{36}\,A_0\gamma_3
$$

$$\lambda_2 = \frac{1}{4}\,A_1C_1|A_1|^2 + \frac{1}{2}\,A_0C_2|A_1|^2 - \frac{1}{16}\,A_0\gamma_1|A_1|^2 + \frac{1}{8}\,A_0B_1\alpha_1 + \frac{1}{8}\,A_0C_1\alpha_4 + \frac{1}{8}\,A_0\overline{C_1}\overline{\alpha_4} + \frac{1}{8}\,A_2B_1 + \frac{1}{4}\,A_0B_4$$
$$- \frac{1}{32}\,A_0\gamma_5 + \frac{1}{32}\,\overline{B_1C_1} + \frac{1}{8}\,A_1\overline{C_3} + \frac{1}{8}\left(A_0\overline{B_1} + A_1\overline{C_1}\right)\overline{\alpha_1} - \frac{1}{64}\,A_1\overline{\gamma_2}$$
$$\lambda_3 = \frac{1}{5}\,A_0B_1|A_1|^2 + \frac{1}{10}\,A_0\overline{C_1}\overline{\alpha_3} + \frac{1}{10}\,A_0C_1\overline{\alpha_4} + \frac{1}{10}\,A_1B_2 - \frac{1}{3600}\left(72\,A_0\overline{\alpha_1} + 25\,\overline{C_1}\right)\gamma_1 + \frac{1}{48}\,C_1\overline{B_1} + \frac{1}{24}\,C_2\overline{C_1} + \frac{1}{10}\,A_0\overline{C_4}$$
$$+ \frac{1}{10}\,(A_1C_1 + 2\,A_0C_2)\overline{\alpha_1} - \frac{1}{100}\,A_0\overline{\gamma_3}$$

while

$$
e^{2\lambda} = \begin{pmatrix}
\frac{1}{12}\,A_0\overline{E_1} & 8 & 0 & 0 \\
\frac{1}{14}\,A_0\overline{C_1}\,\overline{\alpha_2} + \frac{1}{12}\,A_1\overline{E_1} + \frac{1}{14}\,A_0\overline{E_2} & 9 & 0 & 0 \\
\frac{1}{6}\,A_0\overline{C_1} & 5 & 1 & 0 \\
\frac{1}{8}\,A_0\overline{B_1} + \frac{1}{6}\,A_1\overline{C_1} & 6 & 1 & 0 \\
\frac{1}{10}\,A_0\alpha_1\overline{C_1} + \frac{1}{8}\,A_1\overline{B_1} + \frac{1}{10}\,A_0\overline{B_2} + \frac{1}{6}\,A_2\overline{C_1} & 7 & 1 & 0 \\
\frac{1}{12}\,A_0\alpha_3\overline{C_1} + \frac{1}{12}\,A_0C_1\overline{\alpha_2} + \frac{1}{60}\left(5\,A_0\overline{B_1} + 6\,A_1\overline{C_1}\right)\alpha_1 + \frac{1}{8}\,A_2\overline{B_1} + \frac{1}{10}\,A_1\overline{B_2} + \frac{1}{12}\,A_0\overline{B_3} + \frac{1}{6}\,A_3\overline{C_1} + \frac{1}{12}\,A_0\overline{\gamma_4} & 8 & 1 & 0 \\
2\,A_0\overline{A_0} & 2 & 2 & 0 \\
2\,A_1\overline{A_0} & 3 & 2 & 0 \\
\frac{1}{4}\,A_0C_1 + 2\,A_2\overline{A_0} & 4 & 2 & 0 \\
\frac{1}{4}\,A_1C_1 + \frac{1}{3}\,A_0C_2 - \frac{1}{18}\,A_0\gamma_1 + 2\,A_3\overline{A_0} & 5 & 2 & 0 \\
\frac{1}{4}\,A_0|A_1|^2\overline{C_1} + \frac{1}{8}\,A_0C_1\alpha_1 + \frac{1}{4}\,A_2C_1 + \frac{1}{3}\,A_1C_2 + \frac{1}{8}\,A_0C_3 - \frac{1}{18}\,A_1\gamma_1 - \frac{1}{64}\,A_0\gamma_2 + 2\,A_4\overline{A_0} & 6 & 2 & 0 \\
\mu_1 & 7 & 2 & 0 \\
\frac{1}{3}\,A_0\gamma_1 & 5 & 2 & 1 \\
\frac{1}{3}\,A_1\gamma_1 + \frac{1}{8}\,A_0\gamma_2 & 6 & 2 & 1 \\
\frac{1}{15}\,(3\,A_0\alpha_1 + 5\,A_2)\gamma_1 + \frac{1}{8}\,A_1\gamma_2 + \frac{1}{10}\,A_0\gamma_3 & 7 & 2 & 1 \\
2\,A_0\overline{A_1} & 2 & 3 & 0 \\
2\,A_1\overline{A_1} & 3 & 3 & 0 \\
\frac{1}{4}\,A_0B_1 + 2\,A_2\overline{A_1} & 4 & 3 & 0 \\
\frac{1}{3}\,A_0C_1|A_1|^2 + \frac{1}{6}\,A_0\overline{C_1}\,\overline{\alpha_1} + \frac{1}{4}\,A_1B_1 + 2\,A_3\overline{A_1} + \frac{1}{48}\,\overline{C_1}^2 + \frac{1}{6}\,A_0\overline{C_3} + \frac{1}{2}\,\overline{A_0E_1} - \frac{1}{36}\,A_0\overline{\gamma_2} & 5 & 3 & 0 \\
\mu_2 & 6 & 3 & 0 \\
\frac{1}{6}\,A_0\overline{\gamma_2} & 5 & 3 & 1 \\
\frac{1}{2}\,A_0\gamma_1|A_1|^2 + \frac{1}{4}\,A_0\gamma_5 + \frac{1}{6}\,A_1\overline{\gamma_2} & 6 & 3 & 1 \\
2\,A_0\overline{A_2} + \frac{1}{4}\,\overline{A_0C_1} & 2 & 4 & 0 \\
2\,A_1\overline{A_2} + \frac{1}{4}\,\overline{A_0B_1} & 3 & 4 & 0 \\
\frac{1}{4}\,\alpha_1\overline{A_0C_1} + \frac{1}{4}\,A_0C_1\overline{\alpha_1} + \frac{1}{4}\,A_0B_2 + 2\,A_2\overline{A_2} + \frac{1}{4}\,\overline{A_0B_2} + \frac{13}{288}\,C_1\overline{C_1} & 4 & 4 & 0 \\
\mu_3 & 5 & 4 & 0 \\
\frac{1}{72}\left(24\,A_0\overline{\alpha_1} + 5\,\overline{C_1}\right)\gamma_1 + \frac{1}{6}\,A_0\overline{\gamma_3} & 5 & 4 & 1 \\
\frac{1}{6}\,C_1\overline{A_0} & 1 & 5 & 0 \\
\frac{1}{3}\,C_2\overline{A_0} - \frac{1}{18}\,\gamma_1\overline{A_0} + 2\,A_0\overline{A_3} + \frac{1}{4}\,\overline{A_1C_1} & 2 & 5 & 0 \\
\frac{1}{3}\,|A_1|^2\overline{A_0C_1} + \frac{1}{6}\,C_1\alpha_1\overline{A_0} + \frac{1}{48}\,C_1^2 + \frac{1}{2}\,A_0E_1 + \frac{1}{6}\,C_3\overline{A_0} - \frac{1}{36}\,\gamma_2\overline{A_0} + 2\,A_1\overline{A_3} + \frac{1}{4}\,\overline{A_1B_1} & 3 & 5 & 0 \\
\mu_4 & 4 & 5 & 0 \\
\frac{1}{3}\,\gamma_1\overline{A_0} & 2 & 5 & 1 \\
\frac{1}{6}\,\gamma_2\overline{A_0} & 3 & 5 & 1 \\
\frac{1}{72}\left(24\,\alpha_1\overline{A_0} + 5\,C_1\right)\gamma_1 + \frac{1}{6}\,\gamma_3\overline{A_0} & 4 & 5 & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{1}{8}\,B_1\overline{A_0}+\frac{1}{6}\,C_1\overline{A_1} & 1 & 6 & 0\\
\frac{1}{4}\,C_1|A_1|^2\overline{A_0}+\frac{1}{8}\,\overline{A_0C_1}\,\overline{\alpha_1}+\frac{1}{3}\,C_2\overline{A_1}-\frac{1}{18}\,\gamma_1\overline{A_1}+2\,A_0\overline{A_4}+\frac{1}{4}\,\overline{A_2C_1}+\frac{1}{8}\,\overline{A_0C_3}-\frac{1}{64}\,\overline{A_0}\,\overline{\gamma_2} & 2 & 6 & 0\\
\mu_5 & 3 & 6 & 0\\
\frac{1}{3}\,\gamma_1\overline{A_1}+\frac{1}{8}\,\overline{A_0}\,\overline{\gamma_2} & 2 & 6 & 1\\
\frac{1}{2}\,\gamma_1|A_1|^2\overline{A_0}+\frac{1}{4}\,\gamma_5\overline{A_0}+\frac{1}{6}\,\gamma_2\overline{A_1} & 3 & 6 & 1\\
\frac{1}{10}\,C_1\overline{A_0}\,\overline{\alpha_1}+\frac{1}{10}\,B_2\overline{A_0}+\frac{1}{8}\,B_1\overline{A_1}+\frac{1}{6}\,C_1\overline{A_2} & 1 & 7 & 0\\
\mu_6 & 2 & 7 & 0\\
\frac{1}{15}\left(3\,\overline{A_0}\,\overline{\alpha_1}+5\,\overline{A_2}\right)\gamma_1+\frac{1}{8}\,\overline{A_1}\,\overline{\gamma_2}+\frac{1}{10}\,\overline{A_0}\,\overline{\gamma_3} & 2 & 7 & 1\\
\frac{1}{12}\,E_1\overline{A_0} & 0 & 8 & 0\\
\frac{1}{12}\,\alpha_2\overline{A_0C_1}+\frac{1}{12}\,C_1\overline{A_0}\,\overline{\alpha_3}+\frac{1}{12}\,B_3\overline{A_0}+\frac{1}{12}\,\gamma_4\overline{A_0}+\frac{1}{10}\,B_2\overline{A_1}+\frac{1}{8}\,B_1\overline{A_2}+\frac{1}{6}\,C_1\overline{A_3}+\frac{1}{60}\left(5\,B_1\overline{A_0}+6\,C_1\overline{A_1}\right)\overline{\alpha_1} & 1 & 8 & 0\\
\frac{1}{14}\,C_1\alpha_2\overline{A_0}+\frac{1}{14}\,E_2\overline{A_0}+\frac{1}{12}\,E_1\overline{A_1} & 0 & 9 & 0
\end{pmatrix}
$$

where

$$
\begin{aligned}
\mu_1 &= \frac{1}{5}\,A_0|A_1|^2\overline{B_1}+\frac{1}{4}\,A_1|A_1|^2\overline{C_1}+\frac{1}{10}\,A_0C_1\alpha_3+\frac{1}{10}\,A_0\alpha_4\overline{C_1}+\frac{1}{4}\,A_3C_1+\frac{1}{3}\,A_2C_2+\frac{1}{8}\,A_1C_3+\frac{1}{10}\,A_0C_4\\
&\quad+\frac{1}{40}\,(5\,A_1C_1+8\,A_0C_2)\alpha_1-\frac{1}{450}\,(9\,A_0\alpha_1+25\,A_2)\gamma_1-\frac{1}{64}\,A_1\gamma_2-\frac{1}{100}\,A_0\gamma_3+2\,A_5\overline{A_0}\\
\mu_2 &= \frac{1}{3}\,A_1C_1|A_1|^2+\frac{1}{2}\,A_0C_2|A_1|^2-\frac{1}{16}\,A_0\gamma_1|A_1|^2+\frac{1}{8}\,A_0B_1\alpha_1+\frac{1}{8}\,A_0C_1\alpha_4+\frac{1}{2}\,\overline{A_0C_1}\,\overline{\alpha_2}+\frac{1}{8}\,A_0\overline{C_1}\,\overline{\alpha_4}+\frac{1}{4}\,A_2B_1\\
&\quad+\frac{1}{4}\,A_0B_4-\frac{1}{32}\,A_0\gamma_5+2\,A_4\overline{A_1}+\frac{7}{192}\,\overline{B_1C_1}+\frac{1}{6}\,A_1\overline{C_3}+\frac{1}{2}\,\overline{A_0E_2}+\frac{1}{24}\left(3\,A_0\overline{B_1}+4\,A_1\overline{C_1}\right)\overline{\alpha_1}-\frac{1}{36}\,A_1\overline{\gamma_2}\\
\mu_3 &= \frac{1}{3}\,A_0B_1|A_1|^2+\frac{1}{4}\,\alpha_1\overline{A_0B_1}+\frac{1}{4}\,\alpha_3\overline{A_0C_1}+\frac{1}{4}\,C_1\overline{A_0}\,\overline{\alpha_2}+\frac{1}{6}\,A_0\overline{C_1}\,\overline{\alpha_3}+\frac{1}{6}\,A_0C_1\overline{\alpha_4}+\frac{1}{4}\,A_1B_2-\frac{1}{432}\left(24\,A_0\overline{\alpha_1}+5\,\overline{C_1}\right)\gamma_1\\
&\quad+2\,A_3\overline{A_2}+\frac{1}{24}\,C_1\overline{B_1}+\frac{1}{4}\,\overline{A_0B_3}+\frac{5}{72}\,C_2\overline{C_1}+\frac{1}{6}\,A_0\overline{C_4}+\frac{1}{2}\,\overline{A_1E_1}+\frac{1}{12}\,(3\,A_1C_1+4\,A_0C_2)\overline{\alpha_1}-\frac{1}{36}\,A_0\overline{\gamma_3}+\frac{1}{4}\,\overline{A_0}\,\overline{\gamma_4}\\
\mu_4 &= \frac{1}{3}\,|A_1|^2\overline{A_0B_1}+\frac{1}{6}\,C_1\alpha_3\overline{A_0}+\frac{1}{4}\,A_0\alpha_2\overline{C_1}+\frac{1}{6}\,\alpha_4\overline{A_0C_1}+\frac{1}{4}\,A_0B_1\overline{\alpha_1}+\frac{1}{4}\,A_0C_1\overline{\alpha_3}+\frac{1}{4}\,A_0B_3+\frac{5}{72}\,C_1C_2+\frac{1}{2}\,A_1E_1\\
&\quad+\frac{1}{12}\left(4\,C_2\overline{A_0}+3\,\overline{A_1C_1}\right)\alpha_1-\frac{1}{432}\left(24\,\alpha_1\overline{A_0}+5\,C_1\right)\gamma_1+\frac{1}{4}\,A_0\gamma_4+\frac{1}{6}\,C_4\overline{A_0}-\frac{1}{36}\,\gamma_3\overline{A_0}+2\,A_2\overline{A_3}+\frac{1}{4}\,\overline{A_1B_2}+\frac{1}{24}\,B_1\overline{C_1}\\
\mu_5 &= \frac{1}{2}\,C_2|A_1|^2\overline{A_0}-\frac{1}{16}\,\gamma_1|A_1|^2\overline{A_0}+\frac{1}{3}\,|A_1|^2\overline{A_1C_1}+\frac{1}{2}\,A_0C_1\alpha_2+\frac{1}{8}\,C_1\alpha_4\overline{A_0}+\frac{1}{8}\,\overline{A_0B_1}\,\overline{\alpha_1}+\frac{1}{8}\,\overline{A_0C_1}\,\overline{\alpha_4}+\frac{7}{192}\,B_1C_1\\
&\quad+\frac{1}{2}\,A_0E_2+\frac{1}{24}\left(3\,B_1\overline{A_0}+4\,C_1\overline{A_1}\right)\alpha_1+\frac{1}{4}\,B_4\overline{A_0}-\frac{1}{32}\,\gamma_5\overline{A_0}+\frac{1}{6}\,C_3\overline{A_1}-\frac{1}{36}\,\gamma_2\overline{A_1}+2\,A_1\overline{A_4}+\frac{1}{4}\,\overline{A_2B_1}\\
\mu_6 &= \frac{1}{5}\,B_1|A_1|^2\overline{A_0}+\frac{1}{4}\,C_1|A_1|^2\overline{A_1}+\frac{1}{10}\,\overline{A_0C_1}\,\overline{\alpha_3}+\frac{1}{10}\,C_1\overline{A_0}\,\overline{\alpha_4}-\frac{1}{450}\left(9\,\overline{A_0}\,\overline{\alpha_1}+25\,\overline{A_2}\right)\gamma_1+\frac{1}{3}\,C_2\overline{A_2}+2\,A_0\overline{A_5}\\
&\quad+\frac{1}{4}\,\overline{A_3C_1}+\frac{1}{8}\,\overline{A_1C_3}+\frac{1}{10}\,\overline{A_0C_4}+\frac{1}{40}\left(8\,C_2\overline{A_0}+5\,\overline{A_1C_1}\right)\overline{\alpha_1}-\frac{1}{64}\,\overline{A_1}\,\overline{\gamma_2}-\frac{1}{100}\,\overline{A_0}\,\overline{\gamma_3}
\end{aligned}
$$

as

$$
\frac{1}{2}\vec{E}_2=\begin{pmatrix}
-\frac{7}{96}\,C_1^2 & \overline{A_1} & -2 & 4 & 0 & \mathbf{(2)}\\
\frac{1}{2}\,A_0C_1\alpha_2-\frac{13}{96}\,B_1C_1 & \overline{A_0} & -2 & 4 & 0 & \mathbf{(3)}\\
-\frac{1}{48}\,C_1\overline{A_1} & C_1 & -2 & 4 & 0 & \mathbf{(4)}\\
C_1\alpha_2\overline{A_0}-\frac{1}{4}\,E_1\overline{A_1} & A_0 & -2 & 4 & 0 & \mathbf{(5)}
\end{pmatrix}=-\frac{7}{96}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_1}-\frac{1}{48}\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{C}_1
$$

so

$$
\vec{E}_2=-\frac{7}{48}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_1}-\frac{1}{24}\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{C}_1 \tag{6.2.36}
$$

and

$$
\langle\vec{A}_0,\vec{E}_2\rangle=\langle\overline{\vec{A}_0},\vec{E}_2\rangle=0.
$$

Therefore, recalling that

$$e^{2\lambda} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 2\,|A_1|^2 & 3 & 3 & 0 \\ \beta & 4 & 4 & 0 \\ \alpha_1 & 4 & 2 & 0 \\ \alpha_2 & 1 & 6 & 0 \\ \alpha_3 & 5 & 2 & 0 \\ \alpha_4 & 4 & 3 & 0 \\ \alpha_5 & 7 & 1 & 0 \\ \alpha_6 & 6 & 2 & 0 \\ \alpha_7 & 3 & 5 & 0 \\ \zeta_0 & 6 & 2 & 1 \end{pmatrix} \begin{pmatrix} \overline{\alpha_1} & 2 & 4 & 0 \\ \overline{\alpha_2} & 6 & 1 & 0 \\ \overline{\alpha_3} & 2 & 5 & 0 \\ \overline{\alpha_4} & 3 & 4 & 0 \\ \overline{\alpha_5} & 1 & 7 & 0 \\ \overline{\alpha_6} & 2 & 6 & 0 \\ \overline{\alpha_7} & 5 & 3 & 0 \\ \overline{\zeta_0} & 2 & 6 & 1 \end{pmatrix}$$

we see that the only new powers of degree 9 are

$$\mathrm{Re}\,(z^8\overline{z}),\ \ \mathrm{Re}\,(z^7\overline{z}^2),\ \ \mathrm{Re}\,(z^7\overline{z}^2)\log|z|,\ \ \mathrm{Re}\,(z^6\overline{z}^3),\ \ \mathrm{Re}\,(z^6\overline{z}^3)\log|z|,\ \ \mathrm{Re}\,(z^5\overline{z}^4),\ \ \mathrm{Re}\,(z^5\overline{z}^4)\log|z|,$$

so there exists $\alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \zeta_1, \zeta_2, \zeta_3 \in \mathbb{C}$ such that

$$e^{2\lambda} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 2\,|A_1|^2 & 3 & 3 & 0 \\ \beta & 4 & 4 & 0 \\ \alpha_1 & 4 & 2 & 0 \\ \alpha_2 & 1 & 6 & 0 \\ \alpha_3 & 5 & 2 & 0 \\ \alpha_4 & 4 & 3 & 0 \\ \alpha_5 & 7 & 1 & 0 \\ \alpha_6 & 6 & 2 & 0 \\ \alpha_7 & 3 & 5 & 0 \end{pmatrix} \begin{pmatrix} \alpha_8 & 8 & 1 & 0 \\ \alpha_9 & 7 & 2 & 0 \\ \alpha_{10} & 6 & 3 & 0 \\ \alpha_{11} & 5 & 4 & 0 \\ \zeta_0 & 6 & 2 & 1 \\ \zeta_1 & 7 & 2 & 1 \\ \zeta_2 & 6 & 3 & 1 \\ \zeta_3 & 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} \overline{\alpha_1} & 2 & 4 & 0 \\ \overline{\alpha_2} & 6 & 1 & 0 \\ \overline{\alpha_3} & 2 & 5 & 0 \\ \overline{\alpha_4} & 3 & 4 & 0 \\ \overline{\alpha_5} & 1 & 7 & 0 \\ \overline{\alpha_6} & 2 & 6 & 0 \\ \overline{\alpha_7} & 5 & 3 & 0 \end{pmatrix} \begin{pmatrix} \overline{\alpha_8} & 1 & 8 & 0 \\ \overline{\alpha_9} & 2 & 7 & 0 \\ \overline{\alpha_{10}} & 3 & 6 & 0 \\ \overline{\alpha_{11}} & 4 & 5 & 0 \\ \overline{\zeta_0} & 2 & 6 & 1 \\ \overline{\zeta_1} & 2 & 7 & 1 \\ \overline{\zeta_2} & 3 & 6 & 1 \\ \overline{\zeta_3} & 4 & 5 & 1 \end{pmatrix}.$$

Let us compare this to the TEX version (on the left, while Sage is on the right)

$$e^{2\lambda} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 2\left|A_1\right|^2 & 3 & 3 & 0 \\ \beta & 4 & 4 & 0 \\ \alpha_1 & 4 & 2 & 0 \\ \alpha_2 & 1 & 6 & 0 \\ \alpha_3 & 5 & 2 & 0 \\ \alpha_4 & 4 & 3 & 0 \\ \alpha_5 & 7 & 1 & 0 \\ \alpha_6 & 6 & 2 & 0 \\ \alpha_7 & 3 & 5 & 0 \\ \alpha_8 & 8 & 1 & 0 \\ \alpha_9 & 7 & 2 & 0 \\ \alpha_{10} & 6 & 3 & 0 \\ \alpha_{11} & 5 & 4 & 0 \\ \zeta_0 & 6 & 2 & 1 \\ \zeta_1 & 7 & 2 & 1 \\ \zeta_2 & 6 & 3 & 1 \\ \zeta_3 & 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} \overline{\alpha_1} & 2 & 4 & 0 \\ \overline{\alpha_2} & 6 & 1 & 0 \\ \overline{\alpha_3} & 2 & 5 & 0 \\ \overline{\alpha_4} & 3 & 4 & 0 \\ \overline{\alpha_5} & 1 & 7 & 0 \\ \overline{\alpha_6} & 2 & 6 & 0 \\ \overline{\alpha_7} & 5 & 3 & 0 \\ \overline{\alpha_8} & 1 & 8 & 0 \\ \overline{\alpha_9} & 2 & 7 & 0 \\ \overline{\alpha_{10}} & 3 & 6 & 0 \\ \overline{\alpha_{11}} & 4 & 5 & 0 \\ \overline{\zeta_0} & 2 & 6 & 1 \\ \overline{\zeta_1} & 2 & 7 & 1 \\ \overline{\zeta_2} & 3 & 6 & 1 \\ \overline{\zeta_3} & 4 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 2\left|A_1\right|^2 & 3 & 3 & 0 \\ \beta & 4 & 4 & 0 \\ \alpha_1 & 4 & 2 & 0 \\ \alpha_2 & 1 & 6 & 0 \\ \alpha_3 & 5 & 2 & 0 \\ \alpha_4 & 4 & 3 & 0 \\ \alpha_5 & 7 & 1 & 0 \\ \alpha_6 & 6 & 2 & 0 \\ \alpha_7 & 3 & 5 & 0 \\ \alpha_8 & 8 & 1 & 0 \\ \alpha_9 & 7 & 2 & 0 \\ \alpha_{10} & 6 & 3 & 0 \\ \alpha_{11} & 5 & 4 & 0 \\ \zeta_0 & 6 & 2 & 1 \\ \zeta_1 & 7 & 2 & 1 \\ \zeta_2 & 6 & 3 & 1 \\ \zeta_3 & 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} \overline{\alpha_1} & 2 & 4 & 0 \\ \overline{\alpha_2} & 6 & 1 & 0 \\ \overline{\alpha_3} & 2 & 5 & 0 \\ \overline{\alpha_4} & 3 & 4 & 0 \\ \overline{\alpha_5} & 1 & 7 & 0 \\ \overline{\alpha_6} & 2 & 6 & 0 \\ \overline{\alpha_7} & 5 & 3 & 0 \\ \overline{\alpha_8} & 1 & 8 & 0 \\ \overline{\alpha_9} & 2 & 7 & 0 \\ \overline{\alpha_{10}} & 3 & 6 & 0 \\ \overline{\alpha_{11}} & 4 & 5 & 0 \\ \overline{\zeta_0} & 2 & 6 & 1 \\ \overline{\zeta_1} & 2 & 7 & 1 \\ \overline{\zeta_2} & 3 & 6 & 1 \\ \overline{\zeta_3} & 4 & 5 & 1 \end{pmatrix}$$

Then, we obtain

$$
\vec{h}_0 = \begin{pmatrix}
2 & A_1 & 2 & 0 & 0 \\
4 & A_2 & 3 & 0 & 0 \\
6 & A_3 & 4 & 0 & 0 \\
8 & A_4 & 5 & 0 & 0 \\
10 & A_5 & 6 & 0 & 0 \\
-\frac{1}{6} & C_1 & 0 & 3 & 0 \\
-\frac{1}{8} & B_1 & 0 & 4 & 0 \\
-\frac{1}{10} & B_2 & 0 & 5 & 0 \\
-\frac{1}{10}\,\overline{\alpha_1} & C_1 & 0 & 5 & 0 \\
-\frac{1}{12} & B_3 & 0 & 6 & 0 \\
-\frac{1}{12} & \gamma_4 & 0 & 6 & 0 \\
\frac{1}{6}\,\alpha_2 & \overline{C_1} & 0 & 6 & 0 \\
-\frac{1}{12}\,\overline{\alpha_1} & B_1 & 0 & 6 & 0 \\
-\frac{1}{12}\,\overline{\alpha_3} & C_1 & 0 & 6 & 0 \\
\frac{1}{6} & \gamma_1 & 1 & 3 & 0 \\
\frac{1}{16} & \overline{\gamma_2} & 1 & 4 & 0 \\
-\frac{1}{3}\,|A_1|^2 & C_1 & 1 & 4 & 0 \\
\frac{1}{20} & \overline{\gamma_3} & 1 & 5 & 0 \\
-\frac{1}{4}\,|A_1|^2 & B_1 & 1 & 5 & 0 \\
\frac{1}{10}\,\overline{\alpha_1} & \gamma_1 & 1 & 5 & 0 \\
-\frac{1}{6}\,\overline{\alpha_4} & C_1 & 1 & 5 & 0 \\
\frac{1}{4} & \overline{B_1} & 2 & 2 & 0 \\
\frac{1}{18} & \gamma_2 & 2 & 3 & 0 \\
\frac{1}{6} & C_3 & 2 & 3 & 0 \\
-\frac{1}{6}\,|A_1|^2 & \overline{C_1} & 2 & 3 & 0 \\
-\frac{1}{6}\,\alpha_1 & C_1 & 2 & 3 & 0 \\
\frac{1}{6} & \gamma_2 & 2 & 3 & 1 \\
\frac{3}{32} & \gamma_5 & 2 & 4 & 0 \\
\frac{1}{4} & B_4 & 2 & 4 & 0 \\
-\frac{1}{6}\,|A_1|^2 & C_2 & 2 & 4 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{43}{144}\,|A_1|^2 & \gamma_1 & 2 & 4 & 0 \\
-\frac{1}{8}\,\alpha_1 & B_1 & 2 & 4 & 0 \\
-\frac{5}{24}\,\alpha_4 & C_1 & 2 & 4 & 0 \\
\frac{1}{8}\,\overline{\alpha_1} & \overline{B_1} & 2 & 4 & 0 \\
-\frac{1}{8}\,\overline{\alpha_4} & \overline{C_1} & 2 & 4 & 0 \\
\frac{1}{4} & \gamma_5 & 2 & 4 & 1 \\
-\frac{1}{6}\,|A_1|^2 & \gamma_1 & 2 & 4 & 1 \\
\frac{1}{2} & \overline{B_2} & 3 & 2 & 0 \\
\frac{1}{36} & \gamma_3 & 3 & 3 & 0 \\
\frac{1}{3} & C_4 & 3 & 3 & 0 \\
\frac{1}{6}\,|A_1|^2 & \overline{B_1} & 3 & 3 & 0 \\
\frac{1}{6}\,\alpha_1 & \gamma_1 & 3 & 3 & 0 \\
-\frac{1}{6}\,\alpha_3 & C_1 & 3 & 3 & 0 \\
-\frac{1}{6}\,\alpha_4 & \overline{C_1} & 3 & 3 & 0 \\
\frac{1}{3} & \gamma_3 & 3 & 3 & 1 \\
\frac{3}{2} & \overline{E_1} & 4 & 1 & 0 \\
\frac{3}{4} & \overline{B_3} & 4 & 2 & 0 \\
\frac{3}{4} & \overline{\gamma_4} & 4 & 2 & 0 \\
\frac{1}{4}\,\alpha_1 & \overline{B_1} & 4 & 2 & 0 \\
\frac{1}{12}\,\overline{\alpha_2} & C_1 & 4 & 2 & 0 \\
2 & \overline{E_2} & 5 & 1 & 0 \\
\overline{\alpha_2} & \overline{C_1} & 5 & 1 & 0 \\
-\frac{1}{6} & E_1 & -1 & 6 & 0 \\
-\frac{1}{7} & E_2 & -1 & 7 & 0 \\
\frac{1}{42}\,\alpha_2 & C_1 & -1 & 7 & 0 \\
-4\,|A_1|^2 & A_0 & 2 & 1 & 0 \\
-4\,|A_1|^2 & A_1 & 3 & 1 & 0 \\
-4\,|A_1|^2 & A_2 & 4 & 1 & 0 \\
-4\,|A_1|^2 & A_3 & 5 & 1 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
8\,|A_1|^4 + 4\,\alpha_1\overline{\alpha_1} - 4\,\beta & A_0 & 3 & 2 & 0 \\
8\,|A_1|^4 + 4\,\alpha_1\overline{\alpha_1} - 4\,\beta & A_1 & 4 & 2 & 0 \\
-4\,\alpha_1 & A_0 & 3 & 0 & 0 \\
-4\,\alpha_1 & A_1 & 4 & 0 & 0 \\
-4\,\alpha_1 & A_2 & 5 & 0 & 0 \\
-4\,\alpha_1 & A_3 & 6 & 0 & 0 \\
2\,\alpha_2 & A_0 & 0 & 4 & 0 \\
2\,\alpha_2 & A_1 & 1 & 4 & 0 \\
2\,\alpha_2 & A_2 & 2 & 4 & 0 \\
-6\,\alpha_3 & A_0 & 4 & 0 & 0 \\
-6\,\alpha_3 & A_1 & 5 & 0 & 0 \\
-6\,\alpha_3 & A_2 & 6 & 0 & 0 \\
-4\,\alpha_4 & A_0 & 3 & 1 & 0 \\
-4\,\alpha_4 & A_1 & 4 & 1 & 0 \\
-4\,\alpha_4 & A_2 & 5 & 1 & 0 \\
-10\,\alpha_5 & A_0 & 6 & -1 & 0 \\
-10\,\alpha_5 & A_1 & 7 & -1 & 0 \\
4\,\alpha_1^2 - 8\,\alpha_6 - \zeta_0 & A_0 & 5 & 0 & 0 \\
4\,\alpha_1^2 - 8\,\alpha_6 - \zeta_0 & A_1 & 6 & 0 & 0 \\
4\,|A_1|^2\overline{\alpha_1} - 2\,\alpha_7 & A_0 & 2 & 3 & 0 \\
4\,|A_1|^2\overline{\alpha_1} - 2\,\alpha_7 & A_1 & 3 & 3 & 0 \\
12\,\alpha_1\overline{\alpha_2} - 16\,\alpha_8 & A_0 & 7 & -1 & 0 \\
20\,|A_1|^2\overline{\alpha_2} + 10\,\alpha_1\alpha_3 - 14\,\alpha_9 - \zeta_1 & A_0 & 6 & 0 & 0 \\
16\,\alpha_3|A_1|^2 + 8\,\alpha_1\alpha_4 + 8\,\overline{\alpha_1\alpha_2} - 12\,\alpha_{10} - \zeta_2 & A_0 & 5 & 1 & 0 \\
12\,\alpha_4|A_1|^2 + 6\,\alpha_3\overline{\alpha_1} + 6\,\alpha_1\overline{\alpha_4} - 10\,\alpha_{11} - \zeta_3 & A_0 & 4 & 2 & 0 \\
-8\,\zeta_0 & A_0 & 5 & 0 & 1 \\
-8\,\zeta_0 & A_1 & 6 & 0 & 1 \\
-14\,\zeta_1 & A_0 & 6 & 0 & 1 \\
-12\,\zeta_2 & A_0 & 5 & 1 & 1 \\
-10\,\zeta_3 & A_0 & 4 & 2 & 1
\end{pmatrix}
$$

$$\begin{pmatrix}
-8\,\overline{\alpha_2} & A_0 & 5 & -1 & 0 \\
-8\,\overline{\alpha_2} & A_1 & 6 & -1 & 0 \\
-8\,\overline{\alpha_2} & A_2 & 7 & -1 & 0 \\
-2\,\overline{\alpha_4} & A_0 & 2 & 2 & 0 \\
-2\,\overline{\alpha_4} & A_1 & 3 & 2 & 0 \\
-2\,\overline{\alpha_4} & A_2 & 4 & 2 & 0 \\
2\,\overline{\alpha_5} & A_0 & 0 & 5 & 0 \\
2\,\overline{\alpha_5} & A_1 & 1 & 5 & 0 \\
-\overline{\zeta_0} & A_0 & 1 & 4 & 0 \\
-\overline{\zeta_0} & A_1 & 2 & 4 & 0 \\
12\,\alpha_1|A_1|^2 - 6\,\overline{\alpha_7} & A_0 & 4 & 1 & 0 \\
12\,\alpha_1|A_1|^2 - 6\,\overline{\alpha_7} & A_1 & 5 & 1 & 0 \\
-2\,\alpha_2\overline{\alpha_1} - 2\,\overline{\alpha_8} & A_0 & 0 & 6 & 0 \\
-4\,\overline{\alpha_9} - \overline{\zeta_1} & A_0 & 1 & 5 & 0 \\
4\,|A_1|^2\overline{\alpha_3} + 2\,\alpha_1\alpha_2 + 2\,\overline{\alpha_1\alpha_4} - 6\,\overline{\alpha_{10}} - \overline{\zeta_2} & A_0 & 2 & 4 & 0 \\
8\,|A_1|^2\overline{\alpha_4} + 4\,\alpha_4\overline{\alpha_1} + 4\,\alpha_1\overline{\alpha_3} - 8\,\overline{\alpha_{11}} - \overline{\zeta_3} & A_0 & 3 & 3 & 0 \\
-4\,\overline{\zeta_1} & A_0 & 1 & 5 & 1 \\
-6\,\overline{\zeta_2} & A_0 & 2 & 4 & 1 \\
-8\,\overline{\zeta_3} & A_0 & 3 & 3 & 1
\end{pmatrix}$$

We now come to the last expansion of the quartic form. We first have

$g^{-1} \otimes Q(\vec{h}_0) =$

$$
\begin{pmatrix}
-6\,A_0C_1\alpha_1 + 6\,A_2C_1 - A_1\gamma_1 & 0 & 0 & 0\\
-10\,A_1C_1\alpha_1 - 12\,A_0C_1\alpha_3 + 4\,A_0\alpha_1\gamma_1 + 12\,A_3C_1 - 4\,A_2\gamma_1 + \frac{1}{2}\,A_1\gamma_2 & 1 & 0 & 0\\
\omega_1 & 2 & 0 & 0\\
6\,A_1E_1 & -2 & 3 & 0\\
\frac{1}{18}\,C_1^2|A_1|^2 - 10\,A_0E_1|A_1|^2 - \frac{10}{3}\,A_1C_1\alpha_2 - \frac{1}{3}\,A_0\alpha_2\gamma_1 + \frac{1}{6}\,A_0C_1\overline{\zeta_0} + 6\,A_1E_2 + \frac{1}{48}\,B_1\gamma_1 - \frac{1}{96}\,C_1\overline{\gamma_2} & -2 & 4 & 0\\
\omega_2 & -1 & 3 & 0\\
\omega_3 & 0 & 2 & 0\\
\omega_4 & 0 & 1 & 0\\
\omega_5 & 1 & 1 & 0\\
\omega_6 & 3 & -1 & 0\\
-16\,A_0^2\alpha_1\overline{\alpha_4} + 16\left(2\,|A_1|^4 + \alpha_1\overline{\alpha_1} - \beta\right)A_0A_1 + 2\,A_0\alpha_1\overline{B_1} - \frac{80}{3}\,A_0C_1\overline{\alpha_2} - 8\,A_1^2\overline{\alpha_4} + 16\,A_0A_2\overline{\alpha_4} - 2\,A_2\overline{B_1} + 2\,A_1\overline{B_2} & 2 & -1 & 0\\
48\,A_0^2\alpha_2|A_1|^2 - 3\,A_0B_1|A_1|^2 - \frac{2}{3}\,A_0C_1\overline{\alpha_4} - 40\,A_0A_1\overline{\alpha_5} + 2\,A_1B_2 + \frac{1}{12}\,C_1\overline{B_1} & -1 & 2 & 0\\
\omega_7 & 4 & -2 & 0\\
2\,A_0\alpha_1\gamma_2 - 20\,A_0C_1\zeta_0 - 48\,A_0A_1\overline{\zeta_3} - 2\,A_2\gamma_2 + 2\,A_1\gamma_3 & 2 & 0 & 1\\
-32\,A_0A_1\alpha_1|A_1|^2 - 48\,A_0^2\alpha_3|A_1|^2 - 16\,A_1A_2|A_1|^2 + 48\,A_0A_3|A_1|^2 + 24\left(2\,\alpha_1|A_1|^2 - \overline{\alpha_7}\right)A_0A_1 - 16\,A_1^2\alpha_4 + 6\,A_1\overline{E_1} & 3 & -2 & 0\\
-\frac{1}{12}\,C_1E_1 & -4 & 6 & 0\\
-16\,A_0^2\alpha_1|A_1|^2 - 8\,A_1^2|A_1|^2 + 16\,A_0A_2|A_1|^2 - 8\,A_0A_1\alpha_4 & 2 & -2 & 0\\
-\frac{8}{3}\,A_0C_1|A_1|^2 - 32\,A_0A_1\alpha_2 + 2\,A_1B_1 & -1 & 1 & 0\\
-320\,A_0^2\alpha_5|A_1|^2 - 480\,A_0A_1|A_1|^2\overline{\alpha_2} + 32\,A_0^2\alpha_1\zeta_0 - 128\,A_0^2\alpha_4\overline{\alpha_2} - 32\,A_1^2\zeta_0 - 32\,A_0A_2\zeta_0 - 56\,A_0A_1\zeta_1 & 5 & -3 & 0\\
-96\,A_0^2\zeta_0|A_1|^2 - 72\,A_0A_1\zeta_2 & 4 & -2 & 1\\
-192\,A_0^2|A_1|^2\overline{\alpha_2} - 24\,A_0A_1\zeta_0 & 4 & -3 & 0\\
-64\,A_0^2\alpha_1\overline{\alpha_2} + 80\,A_0A_1\alpha_5 + 64\,A_1^2\overline{\alpha_2} + 64\,A_0A_2\overline{\alpha_2} & 5 & -4 & 0\\
\omega_8 & 6 & -4 & 0\\
-80\,A_0A_1\zeta_3 & 3 & -1 & 1\\
48\,A_0A_1\overline{\alpha_2} & 4 & -4 & 0\\
2\,A_1C_1 & -1 & 0 & 0\\
80\,A_0A_1\overline{\zeta_1} & 0 & 2 & 1
\end{pmatrix}
$$

where

$$
\begin{aligned}
\omega_1 = {} & -16\left(2\,|A_1|^4 + \alpha_1\overline{\alpha_1} - \beta\right)A_0^2|A_1|^2 - 2\,A_0\alpha_1|A_1|^2\overline{C_1} + 24\left(2\,|A_1|^2\overline{\alpha_1} - \alpha_7\right)A_0^2\alpha_1 - 2\,A_0C_1\alpha_1^2 + 2\,A_1|A_1|^2\overline{B_1}\\
& - 2\,A_0|A_1|^2\overline{B_2} + 2\,A_2|A_1|^2\overline{C_1} + 400\,A_0^2\alpha_2\overline{\alpha_2} - 8\,A_0^2\alpha_4\overline{\alpha_4} + 6\left(8\,|A_1|^2\overline{\alpha_4} + 4\,\alpha_4\overline{\alpha_1} + 4\,\alpha_1\overline{\alpha_3} - 8\,\overline{\alpha_{11}} - \overline{\zeta_3}\right)A_0A_1\\
& + 12\left(2\,|A_1|^2\overline{\alpha_1} - \alpha_7\right)A_1^2 - 24\left(2\,|A_1|^2\overline{\alpha_1} - \alpha_7\right)A_0A_2 + \frac{5}{2}\left(4\,\alpha_1^2 - 8\,\alpha_6 - \zeta_0\right)A_0C_1 - 8\,A_2C_1\alpha_1 + 2\,A_0C_3\alpha_1 - 18\,A_1C_1\alpha_3\\
& + 7\,A_1\alpha_1\gamma_1 + 9\,A_0\alpha_3\gamma_1 + \frac{4}{3}\,A_0C_1\zeta_0 + A_0\alpha_4\overline{B_1} - A_1\alpha_4\overline{C_1} - 25\,A_0B_1\overline{\alpha_2} - 32\,A_0A_1\overline{\zeta_3} + 20\,A_4C_1 - 2\,A_2C_3 + 2\,A_1C_4\\
& + (6\,A_0C_1\alpha_1 - 6\,A_2C_1 + A_1\gamma_1)\alpha_1 - 9\,A_3\gamma_1 + \frac{3}{2}\,A_1\gamma_3 + 8\left(2\,A_0^2\alpha_1|A_1|^2 + A_1^2|A_1|^2 - 2\,A_0A_2|A_1|^2 + A_0A_1\alpha_4\right)\overline{\alpha_1}\\
\omega_2 = {} & -\frac{16}{5}\,A_0C_1|A_1|^2\overline{\alpha_1} + 64\,A_0^2|A_1|^2\overline{\alpha_5} - \frac{16}{5}\,A_0B_2|A_1|^2 + 16\,A_0^2\alpha_2\overline{\alpha_4} + 48\,(\alpha_2\overline{\alpha_1} + \overline{\alpha_8})A_0A_1 - 16\,A_0E_1\alpha_1 - 2\,A_0\alpha_2\overline{B_1}\\
& - 4\,A_1\alpha_2\overline{C_1} + 2\,A_1B_1\overline{\alpha_1} - A_0B_1\overline{\alpha_4} + 2\,A_1B_3 + 16\,A_2E_1 - \frac{1}{36}\,C_1\gamma_2 + 2\,A_1\gamma_4 + \frac{1}{8}\,B_1\overline{B_1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{3}\left(4\,A_0C_1|A_1|^2+48\,A_0A_1\alpha_2-3\,A_1B_1\right)\overline{\alpha_1}\\
\omega_3 &= -4\,A_0C_1|A_1|^4+96\,A_0A_1\alpha_2|A_1|^2-12\,A_0^2|A_1|^2\overline{\zeta_0}+72\,A_0^2\alpha_2\alpha_4-2\,A_1B_1|A_1|^2-6\,A_0C_1\alpha_1\overline{\alpha_1}+120\,A_0^2\alpha_1\overline{\alpha_5}\\
&+\frac{3}{4}\,A_0|A_1|^2\overline{\gamma_2}+2\left(2\,|A_1|^4+\alpha_1\overline{\alpha_1}-\beta\right)A_0C_1-6\,A_0B_2\alpha_1-\frac{9}{2}\,A_0B_1\alpha_4+10\,A_0A_1\left(4\,\overline{\alpha_9}+\overline{\zeta_1}\right)\\
&+\frac{4}{3}\left(4\,A_0C_1|A_1|^2+48\,A_0A_1\alpha_2-3\,A_1B_1\right)|A_1|^2+6\,A_2C_1\overline{\alpha_1}-A_1\gamma_1\overline{\alpha_1}-\frac{4}{3}\,A_1C_1\overline{\alpha_4}+\frac{1}{3}\,A_0\gamma_1\overline{\alpha_4}-20\,A_1^2\overline{\alpha_5}-120\,A_0A_2\overline{\alpha_5}\\
&-16\,A_0A_1\overline{\zeta_1}+6\,A_2B_2-\frac{1}{24}\,\gamma_1\overline{B_1}+\frac{1}{4}\,C_1\overline{B_2}+(6\,A_0C_1\alpha_1-6\,A_2C_1+A_1\gamma_1)\overline{\alpha_1}-\frac{1}{2}\,A_1\overline{\gamma_3}\\
\omega_4 &= 96\,A_0^2\alpha_1\alpha_2-\frac{16}{3}\,A_1C_1|A_1|^2+\frac{4}{3}\,A_0\gamma_1|A_1|^2-6\,A_0B_1\alpha_1-16\,A_1^2\alpha_2-96\,A_0A_2\alpha_2-4\,A_0C_1\alpha_4+8\,A_0A_1\overline{\zeta_0}\\
&+6\,A_2B_1-\frac{1}{2}\,A_1\overline{\gamma_2}\\
\omega_5 &= -\frac{32}{3}\,A_0C_1\alpha_1|A_1|^2+192\,A_0A_1\alpha_1\alpha_2+192\,A_0^2\alpha_2\alpha_3+\frac{16}{3}\,A_2C_1|A_1|^2+2\,A_1\gamma_1|A_1|^2-\frac{2}{3}\,A_0\gamma_2|A_1|^2-32\,A_0^2\alpha_1\overline{\zeta_0}\\
&+8\left(2\,\alpha_1|A_1|^2-\overline{\alpha_7}\right)A_0C_1-8\,A_1B_1\alpha_1-64\,A_1A_2\alpha_2-192\,A_0A_3\alpha_2-12\,A_0B_1\alpha_3-\frac{22}{3}\,A_1C_1\alpha_4+\frac{8}{3}\,A_0\alpha_4\gamma_1\\
&+2\,(6\,A_0C_1\alpha_1-6\,A_2C_1+A_1\gamma_1)|A_1|^2+2\,A_0\alpha_1\overline{\gamma_2}+32\,A_0A_2\overline{\zeta_0}-24\,A_0A_1\overline{\zeta_2}+12\,A_3B_1\\
&+\frac{2}{3}\left(4\,A_0C_1|A_1|^2+48\,A_0A_1\alpha_2-3\,A_1B_1\right)\alpha_1+A_1\gamma_5+2\,C_1\overline{E_1}-2\,A_2\overline{\gamma_2}\\
\omega_6 &= -32\,A_0A_1\alpha_1\overline{\alpha_4}-48\,A_0^2\alpha_3\overline{\alpha_4}+8\left(12\,\alpha_4|A_1|^2+6\,\alpha_3\overline{\alpha_1}+6\,\alpha_1\overline{\alpha_4}-10\,\alpha_{11}-\zeta_3\right)A_0A_1+32\left(2\,|A_1|^4+\alpha_1\overline{\alpha_1}-\beta\right)A_1^2\\
&-40\,A_0C_1\alpha_5-40\,A_0A_1\zeta_3+16\left(2\,A_0^2\alpha_1|A_1|^2+A_1^2|A_1|^2-2\,A_0A_2|A_1|^2+A_0A_1\alpha_4\right)|A_1|^2+6\,A_1\alpha_1\overline{B_1}+6\,A_0\alpha_3\overline{B_1}\\
&-\frac{100}{3}\,A_1C_1\overline{\alpha_2}+\frac{64}{3}\,A_0\gamma_1\overline{\alpha_2}-16\,A_1A_2\overline{\alpha_4}+48\,A_0A_3\overline{\alpha_4}-6\,A_3\overline{B_1}+6\,A_1\overline{B_3}+6\,A_1\overline{\gamma_4}\\
\omega_7 &= 12\left(4\,\alpha_1^2-8\,\alpha_6-\zeta_0\right)A_0^2|A_1|^2-16\,A_1^2\alpha_1|A_1|^2-32\,A_0A_2\alpha_1|A_1|^2-96\,A_0A_1\alpha_3|A_1|^2+32\,A_0^2\zeta_0|A_1|^2\\
&-24\left(2\,\alpha_1|A_1|^2-\overline{\alpha_7}\right)A_0^2\alpha_1-24\,A_0^2\alpha_3\alpha_4-16\,A_2^2|A_1|^2+96\,A_0A_4|A_1|^2-48\,A_0A_1\overline{\alpha_1\alpha_2}-144\,A_0^2\overline{\alpha_2\alpha_4}\\
&+6\left(16\,\alpha_3|A_1|^2+8\,\alpha_1\alpha_4+8\,\overline{\alpha_1\alpha_2}-12\,\alpha_{10}-\zeta_2\right)A_0A_1+36\left(2\,\alpha_1|A_1|^2-\overline{\alpha_7}\right)A_1^2+24\left(2\,\alpha_1|A_1|^2-\overline{\alpha_7}\right)A_0A_2\\
&-40\,A_1A_2\alpha_4+24\,A_0A_3\alpha_4-48\,A_0A_1\zeta_2-6\,A_0\alpha_1\overline{E_1}+18\,A_0\overline{B_1}\overline{\alpha_2}+6\,A_1\overline{C_1}\overline{\alpha_2}\\
&+8\left(2\,A_0^2\alpha_1|A_1|^2+A_1^2|A_1|^2-2\,A_0A_2|A_1|^2+A_0A_1\alpha_4\right)\alpha_1+6\,A_2\overline{E_1}+12\,A_1\overline{E_2}\\
\omega_8 &= -120\,A_0^2\alpha_1\alpha_5-176\,A_0A_1\alpha_1\overline{\alpha_2}-48\,A_0^2\alpha_3\overline{\alpha_2}-40\,(3\,\alpha_1\overline{\alpha_2}-4\,\alpha_8)A_0A_1+100\,A_1^2\alpha_5+120\,A_0A_2\alpha_5+176\,A_1A_2\overline{\alpha_2}\\
&+48\,A_0A_3\overline{\alpha_2}
\end{aligned}
$$

Now, we remark as

$$|\vec{H}|^2\vec{h}_0\,\dot{\otimes}\,\vec{h}_0=O(|z|^2),\quad \langle\vec{H},\vec{h}_0\rangle^2=O(|z|^2)$$

that the development of these two tensors $|\vec{H}|^2\vec{h}_0\,\dot{\otimes}\,\vec{h}_0$ and $\langle\vec{H},\vec{h}_0\rangle^2$ up to order $O(|z|^{3-\varepsilon})$ only depends on their first order terms in their Taylor expansion, and as the following expansion is valid at a branch point of multiplicity $\theta_0\geq 3$

$$\vec{H}=\mathrm{Re}\left(\frac{\vec{C}_1}{z^{\theta_0-2}}\right)+O(|z|^{3-\varepsilon}),\quad \vec{h}_0=2\vec{A}_1z^{\theta_0-1}dz^2+O(|z|^{\theta_0})$$

we can use (6.2.37) to obtain

$$\frac{5}{4}|\vec{H}|^2\vec{h}_0\,\dot{\otimes}\,\vec{h}_0=\begin{pmatrix}\left(4\,A_1^2\right)\left(\frac{5}{16}\,C_1^2\right) & 2 & 0\\ \left(4\,A_1^2\right)\left(\frac{5}{16}\,C_1\overline{C_1}\right) & \theta_0 & -\theta_0+2\\ \left(4\,A_1^2\right)\left(\frac{5}{16}\,C_1\overline{C_1}\right) & \theta_0 & -\theta_0+2\\ \left(4\,A_1^2\right)\left(\frac{5}{16}\,\overline{C_1}^2\right) & 2\,\theta_0-2 & -2\,\theta_0+4\end{pmatrix}\tag{6.2.37}$$

$$
\langle\vec{H},\vec{h}_0\rangle^2 = \begin{pmatrix} (A_1C_1)\,(A_1C_1) & 2 & 0 \\ (A_1C_1)\left(A_1\overline{C_1}\right) & \theta_0 & -\theta_0+2 \\ (A_1C_1)\left(A_1\overline{C_1}\right) & \theta_0 & -\theta_0+2 \\ \left(A_1\overline{C_1}\right)\left(A_1\overline{C_1}\right) & 2\,\theta_0-2 & -2\,\theta_0+4 \end{pmatrix} \tag{6.2.38}
$$

which yields for $\theta_0 = 3$

$$
\begin{aligned}
&\frac{5}{4}|\vec{H}|^2\vec{h}_0\,\dot{\otimes}\,\vec{h}_0 = \frac{5}{4}\langle\vec{A}_1,\vec{A}_1\rangle\langle\vec{C}_1,\vec{C}_1\rangle z^2dz^4 + \frac{5}{2}|\vec{C}_1|^2\langle\vec{A}_1,\vec{A}_1\rangle z^3\overline{z}^{-1}dz^4 + \frac{5}{4}\langle\vec{A}_1,\vec{A}_1\rangle\overline{\langle\vec{C}_1,\vec{C}_1\rangle}z^4\overline{z}^{-2}dz^4 \\
&\langle\vec{H},\vec{h}_0\rangle^2 = \langle\vec{A}_1,\vec{C}_1\rangle^2z^2dz^4 + 2\langle\vec{A}_1,\vec{C}_1\rangle\langle\vec{A}_1,\overline{\vec{C}_1}\rangle z^3\overline{z}^{-1}dz^4 + \langle\vec{A}_1,\overline{\vec{C}_1}\rangle^2z^4\overline{z}^{-2}dz^4
\end{aligned} \tag{6.2.39}
$$

Therefore, we need only develop the Gauss curvature, which is

$$
-K_g\vec{h}_0\,\dot{\otimes}\,\vec{h}_0 =
$$

$$
\begin{pmatrix}
16\,A_1^2|A_1|^2 & 2 & -2 & 0 \\
-64\,A_0A_1\alpha_1|A_1|^2 + 64\,A_1A_2|A_1|^2 + 16\,A_1^2\alpha_4 & 3 & -2 & 0 \\
\kappa_1 & 4 & -2 & 0 \\
-\frac{8}{3}\,A_1C_1|A_1|^2 - 32\,A_1^2\alpha_2 & 0 & 1 & 0 \\
\frac{16}{3}\,A_0C_1|A_1|^4 + 160\,A_0A_1\alpha_2|A_1|^2 - 2\,A_1B_1|A_1|^2 - \frac{8}{3}\,A_1C_1\overline{\alpha_4} - 40\,A_1^2\overline{\alpha_5} & 0 & 2 & 0 \\
\frac{16}{3}\,A_0C_1\alpha_1|A_1|^2 + 128\,A_0A_1\alpha_1\alpha_2 - \frac{16}{3}\,A_2C_1|A_1|^2 + \frac{8}{3}\,A_1\gamma_1|A_1|^2 - 128\,A_1A_2\alpha_2 - \frac{8}{3}\,A_1C_1\alpha_4 & 1 & 1 & 0 \\
64\,A_0^2|A_1|^6 - 16\,A_1^2|A_1|^2\overline{\alpha_1} - 96\,A_0A_1|A_1|^2\overline{\alpha_4} + 4\,A_1|A_1|^2\overline{B_1} - 24\left(2\,|A_1|^2\overline{\alpha_1} - \alpha_7\right)A_1^2 & 2 & 0 & 0 \\
-64\,A_0A_1|A_1|^4 + 16\,A_1^2\overline{\alpha_4} & 2 & -1 & 0 \\
\kappa_2 & 3 & -1 & 0 \\
16\,A_1^2\zeta_0 & 5 & -3 & 0 \\
\frac{1}{9}\,C_1^2|A_1|^2 + \frac{16}{3}\,A_1C_1\alpha_2 & -2 & 4 & 0 \\
128\,A_0A_1\alpha_1\overline{\alpha_2} - 40\,A_1^2\alpha_5 - 128\,A_1A_2\overline{\alpha_2} & 6 & -4 & 0 \\
-32\,A_1^2\overline{\alpha_2} & 5 & -4 & 0
\end{pmatrix}
$$

where

$$
\begin{aligned}
\kappa_1 &= 64\,A_0^2\alpha_1^2|A_1|^2 - 80\,A_1^2\alpha_1|A_1|^2 - 128\,A_0A_2\alpha_1|A_1|^2 - 96\,A_0A_1\alpha_3|A_1|^2 - 64\,A_0A_1\alpha_1\alpha_4 + 64\,A_2^2|A_1|^2 + 96\,A_1A_3|A_1|^2 \\
&\quad - 24\left(2\,\alpha_1|A_1|^2 - \overline{\alpha_7}\right)A_1^2 + 64\,A_1A_2\alpha_4 \\
\kappa_2 &= 128\,A_0^2\alpha_1|A_1|^4 - 96\,A_1^2|A_1|^4 - 128\,A_0A_2|A_1|^4 - 128\,A_0A_1\alpha_4|A_1|^2 - 64\,A_0A_1\alpha_1\overline{\alpha_4} - 32\left(2\,|A_1|^4 + \alpha_1\overline{\alpha_1} - \beta\right)A_1^2 \\
&\quad + \frac{16}{3}\,A_1C_1\overline{\alpha_2} + 64\,A_1A_2\overline{\alpha_4}
\end{aligned}
$$

Finally, we have

$$
g^{-1}\otimes Q(\vec{h}_0) - K_g\vec{h}_0\,\dot{\otimes}\,\vec{h}_0 = \tag{6.2.40}
$$

$$
\begin{pmatrix}
-6\,A_0C_1\alpha_1 + 6\,A_2C_1 - A_1\gamma_1 & 0 & 0 & 0 \\
-10\,A_1C_1\alpha_1 - 12\,A_0C_1\alpha_3 + 12\,A_3C_1 + 4\,(A_0\alpha_1 - A_2)\gamma_1 + \frac{1}{2}\,A_1\gamma_2 & 1 & 0 & 0 \\
\pi_1 & 2 & 0 & 0 \\
6\,A_1E_1 & -2 & 3 & 0 \\
\frac{1}{6}\,C_1^2|A_1|^2 - 10\,A_0E_1|A_1|^2 + 2\,A_1C_1\alpha_2 + \frac{1}{6}\,A_0C_1\overline{\zeta_0} + 6\,A_1E_2 - \frac{1}{48}\,(16\,A_0\alpha_2 - B_1)\gamma_1 - \frac{1}{96}\,C_1\overline{\gamma_2} & -2 & 4 & 0 \\
\pi_2 & -1 & 3 & 0 \\
\pi_3 & 0 & 2 & 0 \\
\pi_4 & 0 & 1 & 0 \\
\pi_5 & 1 & 1 & 0 \\
\pi_6 & 3 & -1 & 0 \\
\pi_7 & 2 & -1 & 0 \\
48\,A_0^2\alpha_2|A_1|^2 - 3\,A_0B_1|A_1|^2 - \frac{2}{3}\,A_0C_1\overline{\alpha_4} - 40\,A_0A_1\overline{\alpha_5} + 2\,A_1B_2 + \frac{1}{12}\,C_1\overline{B_1} & -1 & 2 & 0 \\
\pi_8 & 4 & -2 & 0 \\
-20\,A_0C_1\zeta_0 - 48\,A_0A_1\overline{\zeta_3} + 2\,(A_0\alpha_1 - A_2)\gamma_2 + 2\,A_1\gamma_3 & 2 & 0 & 1 \\
-48\,A_0A_1\alpha_1|A_1|^2 - 48\,A_0^2\alpha_3|A_1|^2 + 48\,A_1A_2|A_1|^2 + 48\,A_0A_3|A_1|^2 - 24\,A_0A_1\overline{\alpha_7} + 6\,A_1\overline{E_1} & 3 & -2 & 0 \\
-\frac{1}{12}\,C_1E_1 & -4 & 6 & 0 \\
-16\,A_0^2\alpha_1|A_1|^2 + 8\,A_1^2|A_1|^2 + 16\,A_0A_2|A_1|^2 - 8\,A_0A_1\alpha_4 & 2 & -2 & 0 \\
-\frac{8}{3}\,A_0C_1|A_1|^2 - 32\,A_0A_1\alpha_2 + 2\,A_1B_1 & -1 & 1 & 0 \\
-320\,A_0^2\alpha_5|A_1|^2 - 480\,A_0A_1|A_1|^2\overline{\alpha_2} - 128\,A_0^2\alpha_4\overline{\alpha_2} - 56\,A_0A_1\zeta_1 + 16\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\zeta_0 & 5 & -3 & 0 \\
-96\,A_0^2\zeta_0|A_1|^2 - 72\,A_0A_1\zeta_2 & 4 & -2 & 1 \\
-192\,A_0^2|A_1|^2\overline{\alpha_2} - 24\,A_0A_1\zeta_0 & 4 & -3 & 0 \\
80\,A_0A_1\alpha_5 - 32\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\overline{\alpha_2} & 5 & -4 & 0 \\
-48\,A_0^2\alpha_3\overline{\alpha_2} + 160\,A_0A_1\alpha_8 - 60\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\alpha_5 - 24\,(7\,A_0A_1\alpha_1 - 2\,A_1A_2 - 2\,A_0A_3)\overline{\alpha_2} & 6 & -4 & 0 \\
-80\,A_0A_1\zeta_3 & 3 & -1 & 1 \\
48\,A_0A_1\overline{\alpha_2} & 4 & -4 & 0 \\
2\,A_1C_1 & -1 & 0 & 0 \\
80\,A_0A_1\overline{\zeta_1} & 0 & 2 & 1
\end{pmatrix}
\tag{6.2.41}
$$

where

$$
\begin{aligned}
\pi_1 = {} & 32\,A_0^2|A_1|^6 + 16\,A_0^2\beta|A_1|^2 + 14\,A_0C_1\alpha_1^2 + 6\,A_1|A_1|^2\overline{B_1} - 2\,A_0|A_1|^2\overline{B_2} + 2\,A_2|A_1|^2\overline{C_1} + 24\,A_0A_1\alpha_1\overline{\alpha_3} \\
& - 18\,A_1C_1\alpha_3 - 20\,A_0C_1\alpha_6 - \frac{7}{6}\,A_0C_1\zeta_0 - 48\,A_0A_1\overline{\alpha_{11}} - 38\,A_0A_1\overline{\zeta_3} + 20\,A_4C_1 - 2\,A_2C_3 + 2\,A_1C_4 \\
& - 2\left(A_0|A_1|^2\overline{C_1} + 7\,A_2C_1 - A_0C_3\right)\alpha_1 + \left(32\,A_0A_1\overline{\alpha_1} + A_0\overline{B_1} - A_1\overline{C_1}\right)\alpha_4 - 12\left(2\,A_0^2\alpha_1 - A_1^2 - 2\,A_0A_2\right)\alpha_7 \\
& + (8\,A_1\alpha_1 + 9\,A_0\alpha_3 - 9\,A_3)\gamma_1 + \frac{3}{2}\,A_1\gamma_3 + 16\left(3\,A_0^2\alpha_1|A_1|^2 - 2\,A_1^2|A_1|^2 - 4\,A_0A_2|A_1|^2\right)\overline{\alpha_1} \\
& + 25\left(16\,A_0^2\alpha_2 - A_0B_1\right)\overline{\alpha_2} - 8\left(6\,A_0A_1|A_1|^2 + A_0^2\alpha_4\right)\overline{\alpha_4} \\
\pi_2 = {} & -\frac{8}{15}\,A_0C_1|A_1|^2\overline{\alpha_1} + 64\,A_0^2|A_1|^2\overline{\alpha_5} - \frac{16}{5}\,A_0B_2|A_1|^2 - 16\,A_0E_1\alpha_1 + 48\,A_0A_1\overline{\alpha_8} + 2\,A_1B_3 + 16\,A_2E_1 \\
& + 2\left(40\,A_0A_1\overline{\alpha_1} - A_0\overline{B_1} - 2\,A_1\overline{C_1}\right)\alpha_2 - \frac{1}{36}\,C_1\gamma_2 + 2\,A_1\gamma_4 + \frac{1}{8}\,B_1\overline{B_1} + \left(16\,A_0^2\alpha_2 - A_0B_1\right)\overline{\alpha_4}
\end{aligned}
$$

$$
\begin{aligned}
\pi_3 &= 320\,A_0A_1\alpha_2|A_1|^2-12\,A_0^2|A_1|^2\overline{\zeta_0}-8\,A_1B_1|A_1|^2+2\,A_0C_1\alpha_1\overline{\alpha_1}+\frac{3}{4}\,A_0|A_1|^2\overline{\gamma_2}-6\,A_0B_2\alpha_1-2\,A_0C_1\beta\\
&-4\,A_1C_1\overline{\alpha_4}+40\,A_0A_1\overline{\alpha_9}-6\,A_0A_1\overline{\zeta_1}+6\,A_2B_2+\frac{1}{12}\left(128\,A_0|A_1|^4+3\,\overline{B_2}\right)C_1+\frac{9}{2}\left(16\,A_0^2\alpha_2-A_0B_1\right)\alpha_4\\
&+\frac{1}{24}\left(8\,A_0\overline{\alpha_4}-\overline{B_1}\right)\gamma_1+60\left(2\,A_0^2\alpha_1-A_1^2-2\,A_0A_2\right)\overline{\alpha_5}-\frac{1}{2}\,A_1\overline{\gamma_3}\\
\pi_4 &= -8\,A_1C_1|A_1|^2+\frac{4}{3}\,A_0\gamma_1|A_1|^2-6\,A_0B_1\alpha_1-4\,A_0C_1\alpha_4+8\,A_0A_1\overline{\zeta_0}+6\,A_2B_1+48\left(2\,A_0^2\alpha_1-A_1^2-2\,A_0A_2\right)\alpha_2\\
&-\frac{1}{2}\,A_1\overline{\gamma_2}\\
\pi_5 &= -12\,A_2C_1|A_1|^2-\frac{2}{3}\,A_0\gamma_2|A_1|^2-10\,A_1C_1\alpha_4-8\,A_0C_1\overline{\alpha_7}-24\,A_0A_1\overline{\zeta_2}+12\,A_3B_1+\frac{2}{3}\left(38\,A_0C_1|A_1|^2-15\,A_1B_1\right)\alpha_1\\
&+32\left(11\,A_0A_1\alpha_1-6\,A_1A_2-6\,A_0A_3\right)\alpha_2+12\left(16\,A_0^2\alpha_2-A_0B_1\right)\alpha_3+\frac{4}{3}\left(5\,A_1|A_1|^2+2\,A_0\alpha_4\right)\gamma_1+A_1\gamma_5+2\,C_1\overline{E_1}\\
&+2\left(A_0\alpha_1-A_2\right)\overline{\gamma_2}-32\left(A_0^2\alpha_1-A_0A_2\right)\overline{\zeta_0}\\
\pi_6 &= -80\,A_1^2|A_1|^4-160\,A_0A_2|A_1|^4-16\,A_0A_1\alpha_4|A_1|^2-80\,A_0A_1\alpha_{11}-40\,A_0C_1\alpha_5-48\,A_0A_1\zeta_3-28\,A_1C_1\overline{\alpha_2}\\
&+\frac{64}{3}\,A_0\gamma_1\overline{\alpha_2}+2\left(80\,A_0^2|A_1|^4+3\,A_1\overline{B_1}\right)\alpha_1+6\left(8\,A_0A_1\overline{\alpha_1}+A_0\overline{B_1}\right)\alpha_3-6\,A_3\overline{B_1}+6\,A_1\overline{B_3}\\
&-48\left(A_0A_1\alpha_1+A_0^2\alpha_3-A_1A_2-A_0A_3\right)\overline{\alpha_4}+6\,A_1\overline{\gamma_4}\\
\pi_7 &= -32\,A_0A_1|A_1|^4+16\,A_0A_1\alpha_1\overline{\alpha_1}-16\,A_0A_1\beta+2\,A_0\alpha_1\overline{B_1}-\frac{80}{3}\,A_0C_1\overline{\alpha_2}-2\,A_2\overline{B_1}+2\,A_1\overline{B_2}\\
&-8\left(2\,A_0^2\alpha_1-A_1^2-2\,A_0A_2\right)\overline{\alpha_4}\\
\pi_8 &= 80\,A_0^2\alpha_1^2|A_1|^2-96\,A_0A_1\alpha_3|A_1|^2-96\,A_0^2\alpha_6|A_1|^2+20\,A_0^2\zeta_0|A_1|^2+48\,A_2^2|A_1|^2+96\,A_1A_3|A_1|^2+96\,A_0A_4|A_1|^2\\
&-144\,A_0^2\overline{\alpha_2\alpha_4}-72\,A_0A_1\alpha_{10}-54\,A_0A_1\zeta_2-2\left(32\,A_1^2|A_1|^2+64\,A_0A_2|A_1|^2+3\,A_0\overline{E_1}\right)\alpha_1\\
&-8\left(A_0A_1\alpha_1+3\,A_0^2\alpha_3-3\,A_1A_2-3\,A_0A_3\right)\alpha_4+6\,A_2\overline{E_1}+12\,A_1\overline{E_2}+6\left(3\,A_0\overline{B_1}+A_1\overline{C_1}\right)\overline{\alpha_2}\\
&+12\left(2\,A_0^2\alpha_1-A_1^2-2\,A_0A_2\right)\overline{\alpha_7}
\end{aligned}
$$

As we will be only interested in π_4 and π_6, which correspond respectively to the coefficients

$$
\overline{z}\,dz^4,\quad \text{and } z^{\theta_0}\overline{z}^{2-\theta_0}=z^3\overline{z}^{-1}\,dz^4,
$$

in (6.2.40), we deduce that we only need to compute $\vec{B}_3$ and $\vec{\gamma}_4$. First, recall that by (6.2.31)

$$
\begin{cases}
\vec{C}_2=\mathrm{Re}\left(\vec{D}_2\right)\in\mathbb{R}^n\\
\vec{C}_3=\vec{D}_3-\dfrac{1}{4}\langle\vec{C}_1,\vec{C}_1\rangle\vec{A}_0+\left(-4(\langle\vec{A}_2,\vec{C}_1\rangle+\langle\vec{A}_1,\vec{C}_2\rangle)+2\langle\vec{A}_1,\vec{\gamma}_1\rangle\right)\overline{\vec{A}_0}-2\langle\vec{A}_1,\overline{\vec{C}_1}\rangle\overline{\vec{A}_1}\\
\vec{B}_1=-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\vec{A}_0\\
\vec{B}_2=-\dfrac{5}{12}|\vec{C}_1|^2\overline{\vec{A}_0}-2\langle\overline{\vec{A}_2},\vec{C}_1\rangle\vec{A}_0.\\
\vec{E}_1=-\dfrac{1}{6}\langle\vec{C}_1,\vec{C}_1\rangle\overline{\vec{A}_0}\\
\vec{\gamma}_1=-\vec{\gamma}_0-4\,\mathrm{Re}\left(\langle\vec{A}_1,\vec{C}_1\rangle\overline{\vec{A}_0}\right)\in\mathbb{R}^n\\
\vec{\gamma}_2=-4\langle\vec{A}_1,\vec{\gamma}_1\rangle\overline{\vec{A}_0}-4\overline{\langle\vec{A}_1,\vec{C}_1\rangle}\vec{A}_1.
\end{cases}
\tag{6.2.42}
$$

Therefore, as $\vec{B}_1\in\mathrm{Span}(\vec{A}_0)$, and as by (6.2.34), we have

$$
\langle\vec{A}_0,\vec{A}_0\rangle=\langle\vec{A}_0,\vec{A}_1\rangle=\langle\vec{A}_0,\vec{C}_1\rangle=0,\quad \langle\vec{A}_1,\vec{A}_1\rangle+2\langle\vec{A}_0,\vec{A}_2\rangle=0,
$$

we deduce that

$$
\begin{cases}
\langle\vec{A}_2,\vec{B}_1\rangle=-2\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_0,\vec{A}_2\rangle=\langle\overline{\vec{A}_1},\vec{C}_1\rangle\langle\vec{A}_1,\vec{A}_1\rangle\\
\langle\vec{A}_1,\overline{\vec{\gamma}_2}\rangle=-4|\vec{A}_1|^2\langle\vec{A}_1,\vec{C}_1\rangle
\end{cases}
\tag{6.2.43}
$$

so by (6.2.42) and (6.2.43), we finally obtain

$$
\pi_4=-8\,A_1C_1|A_1|^2+\frac{4}{3}\,A_0\gamma_1|A_1|^2-6\,A_0B_1\alpha_1-4\,A_0C_1\alpha_4+8\,A_0A_1\overline{\zeta_0}+6\,A_2B_1+48\left(2\,A_0^2\alpha_1-A_1^2-2\,A_0A_2\right)\alpha_2
$$

$$
\begin{aligned}
&- \frac{1}{2}\, A_1\overline{\gamma_2} \\
&= -8|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle + 6\langle \overline{\vec{A}_1}, \vec{C}_1\rangle\langle \vec{A}_1, \vec{A}_1\rangle - \frac{1}{2}\left(-4|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle\right) \\
&= -6\left(|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle - \langle \overline{\vec{A}_1}, \vec{C}_1\rangle\langle \vec{A}_1, \vec{A}_1\rangle\right) \\
&= 0
\end{aligned}
$$

and we obtain the first line of the system in $\langle \vec{A}_1, \vec{C}_1\rangle, \langle \vec{A}_1, \vec{A}_1\rangle$

$$
|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle = \langle \overline{\vec{A}_1}, \vec{C}_1\rangle\langle \vec{A}_1, \vec{A}_1\rangle. \tag{6.2.44}
$$

Now, notice that thanks of (6.2.34), we have

$$
\langle \vec{A}_0, \vec{A}_3\rangle + \langle \vec{A}_1, \vec{A}_2\rangle = 0, \tag{6.2.45}
$$

Then, we have thanks of (6.2.33)

$$
\frac{1}{2}\vec{B}_3 = \begin{pmatrix}
A_0\alpha_2\overline{C_1} - \frac{1}{3}\, C_1C_2 + A_1E_1 + \frac{1}{36}\, C_1\gamma_1 - \frac{7}{48}\, B_1\overline{C_1} & \overline{A_0} & -1 & 3 & 0 & \textbf{(9)} \\
-\frac{1}{6}\, C_1\overline{C_1} & \overline{A_1} & -1 & 3 & 0 & \textbf{(10)} \\
-\frac{1}{36}\, \overline{A_1C_1} & C_1 & -1 & 3 & 0 & \textbf{(11)} \\
\frac{4}{3}\, \alpha_2\overline{A_0C_1} + C_1\overline{A_0}\overline{\alpha_3} - \frac{1}{3}\, B_2\overline{A_1} - \frac{2}{3}\, B_1\overline{A_2} - C_1\overline{A_3} + \frac{1}{3}\left(2\, B_1\overline{A_0} + 3\, C_1\overline{A_1}\right)\overline{\alpha_1} & A_0 & -1 & 3 & 0 & \textbf{(12)} \\
-\frac{1}{24}\, C_1\overline{A_1} & \overline{C_1} & -1 & 3 & 0 & \textbf{(13)}
\end{pmatrix}
$$

Therefore, we have

$$
\begin{aligned}
\frac{1}{2}\langle \overline{\vec{A}_1}, \vec{B}_3\rangle &= -\frac{1}{6}|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle - \frac{1}{36}\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle - \frac{1}{24}\langle \overline{\vec{A}_1}, \vec{C}_1\rangle\langle \vec{A}_1, \vec{C}_1\rangle \\
&= -\frac{1}{6}|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle - \frac{5}{72}\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle
\end{aligned}
$$

so

$$
\begin{aligned}
\langle \overline{\vec{A}_1}, \vec{B}_3\rangle &= -\frac{1}{3}|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle - \frac{5}{36}\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle \\
6\langle \overline{\vec{A}_1}, \vec{B}_3\rangle &= -2|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle - \frac{5}{6}\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle
\end{aligned} \tag{6.2.46}
$$

We also have

$$
\frac{1}{2}\vec{\gamma_4} = \begin{pmatrix} -\frac{1}{3}\, C_1\gamma_1 & \overline{A_0} & -1 & 3 & 1 & \textbf{(14)} \end{pmatrix}
$$

so

$$
\vec{\gamma}_4 = -\frac{2}{3}\langle \vec{C}_1, \vec{\gamma}_1\rangle\overline{\vec{A}_0} = \frac{2}{3}\langle \vec{\gamma}_0, \vec{C}_1\rangle\overline{\vec{A}_0} \tag{6.2.47}
$$

and

$$
\langle \vec{A}_1, \vec{\gamma}_4\rangle = \langle \overline{\vec{A}_1}, \vec{\gamma}_4\rangle = 0. \tag{6.2.48}
$$

Also, recall that by (6.2.20), we have

$$
\begin{cases}
\alpha_0 = 2\langle \overline{\vec{A}_0}, \vec{A}_1\rangle \\
\alpha_1 = 2\langle \overline{\vec{A}_0}, \vec{A}_2\rangle \\
\alpha_2 = \frac{1}{24}\langle \overline{\vec{A}_1}, \vec{C}_1\rangle \\
\alpha_3 = \frac{1}{12}\langle \vec{A}_1, \vec{C}_1\rangle + 2\langle \overline{\vec{A}_0}, \vec{A}_3\rangle \\
\alpha_4 = 2\langle \overline{\vec{A}_1}, \vec{A}_2\rangle,
\end{cases} \tag{6.2.49}
$$

In particular, we deduce as $|\vec{A}_0|^2 = \dfrac{1}{2}$ that

$$
\begin{aligned}
&\langle \alpha_3 \vec{A}_0 - \vec{A}_3, \overline{\vec{A}_0}\rangle = \frac{\alpha_3}{2} - \langle \overline{\vec{A}_0}, \vec{A}_3\rangle = \frac{1}{24}\langle \vec{A}_1, \vec{C}_1\rangle \\
&\langle \alpha_3 \vec{A}_0 - \vec{A}_3, \overline{\vec{B}_1}\rangle = -2\langle \vec{A}_1, \overline{\vec{C}_1}\rangle\langle \alpha_3 \vec{A}_0 - \vec{A}_3, \overline{\vec{A}_0}\rangle = -\frac{1}{12}\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle \\
&6\langle \alpha_3 \vec{A}_0 - \vec{A}_3, \overline{\vec{B}_1}\rangle = -\frac{1}{2}\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle
\end{aligned}
\tag{6.2.50}
$$

Now, we have by (6.2.42), (6.2.43), (6.2.45), (6.2.46) and (6.2.50)

$$
\begin{aligned}
\pi_6 &= -\cancel{80\, A_1^2|A_1|^4} - \cancel{160\, A_0 A_2|A_1|^4} - \cancel{16\, A_0 A_1 \alpha_4 |A_1|^2} - \cancel{80\, A_0 A_1 \alpha_{11}} - \cancel{40\, A_0 C_1 \alpha_5} - \cancel{48\, A_0 A_1 \zeta_3} - 28\, A_1 C_1 \overline{\alpha_2} \\
&+ \frac{64}{3}\cancel{A_0 \gamma_1 \overline{\alpha_2}} + 2\left(\cancel{80\, A_0^2 |A_1|^4} + \cancel{3\, A_1 \overline{B_1}}\right)\alpha_1 + 6\left(\cancel{8\, A_0 A_1 \overline{\alpha_1}} + A_0 \overline{B_1}\right)\alpha_3 - 6\, A_3 \overline{B_1} + 6\, A_1 \overline{B_3} \\
&- 48\left(\cancel{A_0 A_1 \alpha_1} + \cancel{A_0^2 \alpha_3} - \cancel{A_1 A_2} - \cancel{A_0 A_3}\right)\overline{\alpha_4} + \cancel{6\, A_1 \overline{\gamma_4}} \\
&= -28\langle \vec{A}_1, \vec{C}_1\rangle\overline{\alpha_2} - \frac{1}{2}\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle - 2|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle - \frac{5}{6}\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle \\
&= -\frac{28}{24}\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle - \frac{1}{2}\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle - 2|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle - \frac{5}{6}\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle \qquad (6.2.51) \\
&= -\frac{5}{2}\langle \vec{A}_1, \vec{C}_1\rangle\langle \overline{\vec{A}_1}, \vec{C}_1\rangle - 2|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle \qquad (6.2.52)
\end{aligned}
$$

Now, thanks of (6.2.39), the coefficient in $z^3\overline{z}^{-1}dz^4$ in the Taylor expansion of

$$
\frac{5}{4}|\vec{H}|^2 \vec{h}_0 \,\dot{\otimes}\, \vec{h}_0 + \langle \vec{H}, \vec{h}_0\rangle^2 \tag{6.2.53}
$$

is

$$
2\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle + \frac{5}{2}|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle, \tag{6.2.54}
$$

so the coefficient in $z^3\overline{z}^{-1}dz^4$ in the Taylor expansion of the *meromorphic* quartic form is

$$
\Omega = \pi_6 + 2\langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle + \frac{5}{2}|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle = \frac{1}{2}\left(|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle - \langle \vec{A}_1, \vec{C}_1\rangle\langle \vec{A}_1, \overline{\vec{C}_1}\rangle\right) = 0,
$$

and we finally recover the system

$$
\begin{cases}
|\vec{A}_1|^2\langle \vec{A}_1, \vec{C}_1\rangle = \langle \overline{\vec{A}_1}, \vec{C}_1\rangle\langle \vec{A}_1, \vec{A}_1\rangle \\
|\vec{C}_1|^2\langle \vec{A}_1, \vec{A}_1\rangle = \langle \vec{A}_1, \overline{\vec{C}_1}\rangle\langle \vec{A}_1, \vec{C}_1\rangle,
\end{cases} \tag{6.2.55}
$$

Now, as by (6.2.19), we have

$$
\langle \vec{A}_0, \vec{\gamma}_0\rangle + \langle \vec{A}_1, \vec{C}_1\rangle = 0, \tag{6.2.56}
$$

and for a *true* Willmore disk, we have $\vec{\gamma}_0 = 0$, so

$$
\langle \vec{A}_1, \vec{C}_1\rangle = 0,
$$

the meromorphic quartic differential

$$
\mathscr{Q}_{\vec{\Phi}} = 2\langle \vec{A}_1, \vec{C}_1\rangle\frac{dz^4}{z} + O(1)
$$

is holomorphic, and thanks of (6.2.55), we also deduce that the octic form $\mathscr{O}_{\vec{\Phi}}$ is holomorphic thanks of the analysis of chapter 5. This concludes the proof of the case $\theta_0 = 3$.

6.3 The case where $\theta_0 = 2$

In this case, as the proof is very short (only two pages), we do not transcript our computer computations here. Notice that we cannot show the holomorphy in general if we assume non-zero first residue $\vec{\gamma}_0$ as such branch points. As for true branch points of multiplicity $\theta_0 = 2$, the Willmore immersion is smooth, the form $\mathscr{Q}_{\vec{\Phi}}$ and $\mathscr{O}_{\vec{\Phi}}$ are trivially holomorphic.

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