Error analysis for the Willmore-Helfrich Functional.

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I The Willmore Equation in Divergence Form.

The Willmore energy of an oriented surface Σ immersed in $\mathbb{R}^3$ reads

$$W(\vec{\Phi}) = \int_\Sigma |H|^2 \, dvol_g \quad , \tag{I.1}$$

where $\vec{\Phi}$ is the immersion, $\vec{H}$ the mean curvature of the surface and g is the induced metric on the surface from the flat metric of $\mathbb{R}^3$. The complete Helfrich energy introduced by Wolfgang Helfrich in the early seventies for the modelization of cell membranes, see [Hef], involves the substraction to H of a spontaneous curvature H_0 and the addition of further terms proportional to the area A of the surface and the enclosed volume V of the immersion in $\mathbb{R}^3$:

$$Hef(\vec{\Phi}) = \int_\Sigma |H - H_0|^2 \, dvol_g + (\mu - H_0^2)\ A + \lambda\ V \quad , \tag{I.2}$$

where λ and μ are constants. These additional terms are sub-critical in comparison to the L^2 norm of the mean curvature and will be ignored in our presentation since they have no essential influence on the nature of the error estimates we present below.

The Euler Lagrange Equation of the Willmore Functional (I.1) has been written, maybe for the first time, by the student of Wilhelm Blaschke, Gerhardt Thomsen, in his dissertation defended in 1923 : $\vec{\Phi}$ is a critical smooth immersion for W if and only if it satisfies

$$(\mathbf{W}) \qquad \Delta_g H + 2H\ (H^2 - K) = 0 \quad ,$$

where Δ_g is the negative Laplace Beltrami operator on the surface Σ for the induced metric g and K is the Gauss curvature of g.

Trying to develop analysis (compactness, regularity properties, error estimates...) with equation $(\mathbf{W})$ is made very delicate in particular by the fact that the non-linearity is <u>cubic</u> in the principal curvatures of the immersed surface whereas, a-priori, the control given by the Lagrangian (I.1) from which the equation is deduced is only <u>quadratic</u> in the principal curvatures.

We first present the following new formulation of Willmore equation in 3 dimension which solves the functional analysis we are raising: $\vec{\Phi}$ is a critical

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immersion to (I.1), i.e. satisfies (**W**), if and only if, in conformal parametrization ($\vec{\Phi}_x \cdot \vec{\Phi}_y = 0$ and $|\vec{\Phi}_x|^2 = |\vec{\Phi}_y|^2$),

$$(\mathbf{WI}) \qquad div\left(-2\nabla\vec{H} + 3H\,\nabla\vec{n} + \vec{H}\times\nabla^{\perp}\vec{n}\right) = 0$$

where div and ∇ are the classical divergence and gradient operators with respect to the flat metric on the $x, y-$plane, $\vec{n}$ is the unit normal vector to the immersion, $\vec{H} = H\,\vec{n}$ and $\nabla^{\perp}$ is the rotation by $\pi/2$ of ∇ : $\nabla^{\perp} = (-\partial_y\ , \partial_x\ )$. Observe that this formulation (**WI**) of Willmore equation is solving the previous mentioned functional analysis paradox : the new dependance of the principal curvatures in the non-linearity $3H\,\nabla\vec{n} + \vec{H}\times\nabla^{\perp}\vec{n}$ is now <u>quadratic</u> and compatible with the control given by the lagrangian W.

II Conservation laws for Willmore equation - the Conformal Willmore Equation.

From the new formulation (**WI**) we deduce the following conservation laws : if $\vec{\Phi}$ is a smooth local conformal parametrization of a Willmore surface then

$$(\mathbf{WII}) \qquad \left\{\begin{array}{l} \text{There exists a vector field locally on } \Sigma,\ \vec{L}\in\mathbb{R}^3,\ \text{ such that} \\[2mm] \nabla\vec{\Phi}\cdot\nabla^{\perp}\vec{L} = \partial_y\vec{\Phi}\cdot\partial_x\vec{L} - \partial_x\vec{\Phi}\cdot\partial_y\vec{L} = 0 \quad \text{ and} \\[2mm] \nabla\vec{\Phi}\times\nabla^{\perp}\vec{L} - 2\nabla^{\perp}H\,\nabla\Phi = \\[2mm] \partial_y\vec{\Phi}\times\partial_x\vec{L} - \partial_x\vec{\Phi}\times\partial_y\vec{L} - 2\left[\partial_y H\,\partial_x\vec{\Phi} - \partial_x H\,\partial_y\vec{\Phi}\right] = 0 \end{array}\right.$$

When $\vec{\Phi}$ is Willmore - i.e. solves (**WI**) - one takes

$$\nabla^{\perp}\vec{L} = -2\nabla\vec{H} + 3H\,\nabla\vec{n} + \vec{H}\times\nabla^{\perp}\vec{n} \quad , \tag{II.3}$$

but more generally the system (**WII**) can be considered as it is stated, <u>independently</u> of the choice (II.3). The system (**WII**) is made of jacobians and can therefore be written in divergence form again. (**WII**) is stable in the sense that solutions to (**WII**) of uniformly bounded Willmore energy converge weakly, modulo extraction of a subsequence, to a solution of (**WII**). It also passes to the limit for Palais-Smale sequences to Willmore. This leads to new proofs of existence results for minimizers of the Willmore functional under various constraints (for a fixed Σ, for a fixed Σ and a fixed conformal class for g, for a fixed Σ and fixed area of g...etc).

The system (**WII**) is equivalent to the conformal Willmore equation - critical points to (I.1) with prescribed conformal structure - : in local conformal coordinates there exists an holomorphic function $f(z)$ such that

$$(\mathbf{CW}) \qquad \Delta_g H + 2H\,(H^2 - K) = e^{2\lambda}\ < h^0, f(z) > \quad ,$$

where $h^0 = h^0_{11} + ih^0_{12}$ is the Weingarten operator in the conformal coordinates $\vec{\Phi}$ - $(h^0_{ij})_{ij}$ is the trace free second fundamental form of Σ - and e^{λ} is the conformal factor $e^\lambda = |\vec{\Phi}_x| = |\vec{\Phi}_y|$. In fact f is not an arbitrary holomorphic function , it is the local expression of a global holomorphic section of $(T^*_{0,1}\Sigma)^{-1}\otimes(T^*_{0,1}\Sigma)^{-1}$ where $T^*_{0,1}\Sigma$ is the canonical bundle of Σ viewed as a riemann surface with the conformal structure induced by g. Hence f belongs to a complex finite dimensional space. In other words our conservation law system (**WII**) is equivalent to Willmore equation (**W**) modulo a Lagrange multiplier belonging to a finite dimensional complex space - its dimension is 1 for instance when Σ is a torus.

III Error Estimates for Willmore Equation.

We address in this section the following question : Let $\vec{\Phi}_k$ be a sequence of immersion satisfying "more and more" the Willmore equation, does $\vec{\Phi}_k$ converge to a solution to Willmore equation and "how fast" does this convergence happen ? We illustrate how the system (**WII**) can be helpful to treat such a question by looking at the simplest possible setting :

Let $\vec{\Phi}_k$ be an embedding of $\mathbb{C}$ into $\mathbb{R}^3$ such that $\int_{\mathbb{C}}|\nabla\vec{n}_k|^2\ dvol_{g_k} < \varepsilon_0$ for some $\varepsilon_0 > 0$ that will be chosed small enough later. By a result of S.Müller and V.Sverak and by F.Hélein we can (see [Hel]), modulo a change of parametrization, choose $\vec{\Phi}_k$ to be a sequence of conformal bilipschitz $W^{2,2}$ embedding such that $\|\nabla^2\vec{\Phi}_k\|_{L^2(\mathbb{C})} + \|\nabla\vec{\Phi}_k\|_{L^\infty(\mathbb{C})}$ is uniformly bounded. Assume that the Willmore equation (**WI**) is satisfied modulo an error e_k which goes to zero strongly in H^{-2} :

$$div\left(-2\nabla\vec{H}_k + 3H_k\,\nabla\vec{n}_k + \vec{H}_k\times\nabla^\perp\vec{n}_k\right) = e_k \longrightarrow 0 \text{ in } H^{-2} \quad .$$

We proceed to the following Hodge decompositions. First, in one hand, there exist a sequence $\vec{L}_k$ uniformly bounded in the weak L^2 space, $L^{2,\infty}$, and a sequence $\vec{E}_k$ converging strongly to zero in L^2 such that $\|\vec{E}_k\|_{L^2} \le C\ \|e_k\|_{H^{-1}}$ and

$$-2\nabla\vec{H}_k + 3H_k\,\nabla\vec{n}_k + \vec{H}_k\times\nabla^\perp\vec{n}_k = \nabla^\perp\vec{L}_k + \nabla\vec{E}_k \quad , \tag{III.4}$$

in the other hand, we have the existence of S_k, T_k, $\vec{R}_k$ and $\vec{Q}_k$ such that

$$\left\{\begin{array}{l} \nabla\vec{\Phi}_k\cdot\vec{L}_k = \nabla S_k + \nabla^\perp T_k \quad , \\[3mm] \nabla\vec{\Phi}_k\times\vec{L}_k - 2H_k\,\nabla\vec{\Phi}_k = \nabla\vec{R}_k + \nabla^\perp\vec{Q}_k \quad . \end{array}\right. \tag{III.5}$$

Some computation - the same as the one which proves (**WI**) $\Longrightarrow$ (**WII**) - leads to the identities $\nabla^\perp\vec{L}_k\cdot\nabla\vec{\Phi}_k = -\nabla\vec{\Phi}_k\cdot\nabla\vec{E}_k = -\Delta T_k$ and $\nabla\vec{\Phi}_k\times\nabla^\perp\vec{L}_k - 2\nabla^\perp H_k\cdot\nabla\vec{\Phi}_k = -\nabla\vec{\Phi}_k\times\nabla\vec{E}_k = \Delta\vec{Q}_k$ We deduce from classical elliptic theory the following estimates

$$\left\{\begin{array}{l} \|\nabla T_k\|_{L^{2,\infty}} \le C\left[\|\nabla\vec{\Phi}_k\|_{L^\infty} + \|\Delta\vec{\Phi}\|_{L^2}\right]\ \|e_k\|_{H^{-1}} \quad , \\[3mm] \|\nabla\vec{Q}_k\|_{L^{2,\infty}} \le C\left[\|\nabla\vec{\Phi}_k\|_{L^\infty} + \|\Delta\vec{\Phi}\|_{L^2}\right]\ \|e_k\|_{H^{-1}} \quad . \end{array}\right. \tag{III.6}$$

Moreover the pair $(\vec{R}_k, S_k)$ satisfy the system on $\mathbb{C}$

$$\left\{\begin{array}{l}\Delta\vec{R}_k = \nabla^\perp\vec{R}_k\times\nabla\vec{n}_k + \nabla S_k\cdot\nabla^\perp\vec{n}_k + div(\nabla T_k\ \vec{n}_k) \quad ,\\[3mm] \Delta S_k = \nabla^\perp\vec{R}_k\cdot\nabla\vec{n}_k - div(\nabla\vec{Q}_k\cdot\vec{n}_k) \quad .\end{array}\right. \tag{III.7}$$

Observe that beside the *error terms* $div(\nabla T_k\ \vec{n}_k)$ and $div(\nabla\vec{Q}_k\cdot\vec{n}_k)$, there are only jacobians in the right-hand-sides of (III.7). We deduce from this last crucial fact the following estimates, for ε_0 chosen small enough, using a Wente type inequality due to F.Bethuel (theorem 3.4.5 in [Hel]) and standard elliptic theory,

$$\|\nabla\vec{R}_k\|_{L^{2,\infty}} + \|\nabla S_k\|_{L^{2,\infty}} \le C\left[\|\nabla\vec{\Phi}_k\|_{L^\infty} + \|\Delta\vec{\Phi}\|_{L^2}\right]\ \|e_k\|_{H^{-1}} \quad . \tag{III.8}$$

A further independent computation gives

$$2\Delta\vec{\Phi} = \left[\nabla^\perp S - \nabla T\right]\cdot\nabla\vec{\Phi} - \left[\nabla\vec{R} + \nabla^\perp\vec{Q}\right]\times\nabla^\perp\vec{\Phi} \quad . \tag{III.9}$$

Combining (III.6), (III.8) and (III.9) we obtain

$$\|\Delta\vec{\Phi}\|_{L^{2,\infty}} \le C\left[\|\nabla\vec{\Phi}_k\|^2_{L^\infty} + \|\Delta\vec{\Phi}\|^2_{L^2}\right]\ \|e_k\|_{H^{-1}} \quad . \tag{III.10}$$

We can normalize the sequence $\nabla\vec{\Phi}_k$ in such a way that $\|\nabla\vec{\Phi}_k\|_\infty = \|\nabla id_{\mathbb{C}}\|_\infty$. Thus we conclude for every $p < 2$, that

$$\|\nabla(\vec{\Phi}_k - id_{\mathbb{C}})\|_{W^{1,p}_{loc}} \le C\left\|div\left(2\nabla\vec{H}_k - 3H_k\,\nabla\vec{n}_k - \vec{H}_k\times\nabla^\perp\vec{n}_k\right)\right\|_{H^{-2}} \tag{III.11}$$

We were here considering the simplest framework of the embedding of a plane with little Willmore energy. In the general situation, even for the flow, such an estimate can be established except that the limiting immersion is only a-priori Conformal Willmore (satisfy (**CW**)). Then, the possible cancelation of the holomorphic Lagrange multiplier f , that would make the equation satisfied by the limiting map being exactly Willmore, has to be further understood.

Such an error control estimate of the form (III.11) has been established when the error was converging to zero <u>in the space L^2</u> by E.Kuwert and R.Schätzle (see [KS]). Here we have gained 2 derivatives by requiring the equation to be solved modulo an error controled only <u>in the space H^{-2}</u> which is critical for Willmore Euler Lagrange equation (WI) - one cannot afford less.

The results presented in this talk have been established in the following two works [Ri] and [BR], the last one being a collaboration in preparation with Yann Bernard.

References

[BR] Bernard, Yann and Rivière, Tristan "Palais Smale Sequences for the Conformal Willmore Equation." in preparation (2008).

[Hel] Hélein, Frédéric "Harmonic maps, conservation laws and moving frames". Cambridge Tracts in Mathematics, 150. Cambridge University Press, Cambridge, 2002.

[Hef] Helfrich, W. Z. Naturforsch., C28 (1973), 693-703.

[KS] Kuwert, Ernst; Schätzle, Reiner "The Willmore flow with small initial energy". J. Differential Geom. 57 (2001), no. 3, 409–441.

[Ri] Rivière, Tristan "Analysis aspects of Willmore Surfaces." Invent. Math., 174 (2008), no 1, 1-45.