Sub-criticality of Schrödinger Systems with Antisymmetric Potentials.

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Résumé : Soit m un entier supérieur ou égal à 3 et n un entier positif arbitraire. Nous démontrons que les systèmes de Schrödinger sur la boule unité B^m de la forme

$$-\Delta v = \Omega\ v \quad ,$$

où Ω est un potentiel antisymétrique dans $L^{m/2}(B^m, so(n))$, peuvent être écrits sous forme divergence. Nous démontrons par ailleurs que toute solution v dans $L^{m/(m-2)}(B^m, \mathbb{R}^n)$ est en fait dans L^∞_{loc} et par conséquent aussi dans $W^{2,m/2}_{loc}(B^m)$.

Abstract : Let m be an integer larger or equal to 3 and n an arbitrary positive integer. We prove that Schrödinger systems on B^m with an antisymmetric potential $\Omega \in L^{m/2}(B^m, \mathbb{R}^n)$ of the form

$$-\Delta v = \Omega\ v$$

can be written in divergence form. We prove moreover that solutions v in $L^{m/(m-2)}(B^m, \mathbb{R}^n)$ are in fact in $L^\infty_{loc}(B^m)$ which also implies the membership of v to $W^{2,m/2}_{loc}(B^m, \mathbb{R}^n)$.

Key words : Linear elliptic systems, Schrödinger systems, Regularity, Conservation laws.

AMS MSC 2010 : 35J47, 35J10, 35J20, 70S10

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I Introduction

In [Ri1] the author proved the sub-criticality of the following linear systems in 2 Dimension

$$-\Delta u = \Omega \cdot \nabla u \quad , \tag{I.1}$$

where $u \in W^{1,2}(D^2, \mathbb{R}^n)$ and $\Omega \in L^2(D^2, \mathbb{R}^2 \otimes so(n))$ (n is an arbitrary integer, $so(n)$ is the subspace of $M_n(\mathbb{R})$, the space of $n \times n$ square matrices, made of antisymmetric matrices) and we have using the matrix multiplication : in coordinates (I.1) reads

$$\forall i = 1 \cdots n \qquad -\Delta u^i = \sum_{j=1}^n \Omega^i_j \cdot \nabla u^j \quad .$$

Precisely, it is proved in [Ri1] that such a u is in fact in $W^{2,p}_{loc}(D^2, \mathbb{R}^n)$ for every $p < 2$. This result has been obtained by writing (I.1) in conservative form. This was possible due to the following result

Theorem I.1 *[Ri1] There exists a map, in a neighborhood of the origin, of the form*

$$\begin{array}{rcl} \mathcal{L} \ : \quad L^2(D^2, \mathbb{R}^2 \otimes so(n)) & \longrightarrow & L^\infty \cap W^{1,2}(D^2, Gl_n(\mathbb{R})) \\ & & \\ \Omega & \longrightarrow & A \end{array} \tag{I.2}$$

such that

$$div(\nabla_\Omega A) := div(\nabla A - A\Omega) = 0 \quad , \tag{I.3}$$

and with the following controls

$$\|dist(A, SO(n))\|_\infty + \|A\|_{W^{1,2}} \le C \|\Omega\|_{L^2} \quad , \tag{I.4}$$

where C is a positive constant independent of Ω. □

Once A is constructed one easily see that

$$-\Delta u = \Omega \cdot \nabla u \qquad \Longleftrightarrow \qquad div(A\nabla u + B\nabla^\perp u) = 0 \quad . \tag{I.5}$$

where $\nabla^\perp B := (-\partial_y B, \partial_x B) = \nabla A - A\Omega$. The higher integrability of ∇u is then a direct consequence of this conservative form of the system by applying Wente's estimates (see [Ri1] and [Ri2]). This result has lead in particular to a proof of the Heinz-Hildebrandt's regularity conjecture for critical points to conformally invariant problems in two dimension.

In this paper we will study this time Schrödinger systems of the form

$$-\Delta v = \Omega\ v \quad , \tag{I.6}$$

where $v \in L^{m/(m-2)}(B^m, \mathbb{R}^n)$ and $\Omega \in L^{m/2}(B^m, so(n))$, n is an arbitrary integer and m is an arbitrary integer larger or equal to 3. B^m_r denotes the $m-$dimensional ball centered at the origin of $\mathbb{R}^m$ and when we don't write the subscript it implicitly means that $r = 1$ (i.e B^m denotes the unit ball). In coordinates (I.6) means

$$\forall i = 1 \cdots n \qquad -\Delta v^i = \sum_{j=1}^{n} \Omega^i_j\ v^j \quad .$$

Like (I.1) in 2-dimension, the system (I.6) is also a-priori critical for $v \in L^{m/(m-2)}$ in m dimension. Indeed, under these assumptions $v \in L^{m/(m-2)}$ and $\Omega \in L^{m/2}$ we obtain that the r.h.s. of (I.6) and hence Δv is in L^1 and, using classical singular integral theory, we deduce in return that $v \in L^{m/(m-2),\infty}_{loc}$ which is "almost" the information we started from. Such a structure in general situations offers no hope for having any of the properties that characterize sub-critical problems such as better integrability of v, local uniqueness of the solutions...etc. It is *a-priori* simply critical. However, here again, the antisymmetry of Ω will imply that sub-criticality in fact holds.

Our main result is the following.

Theorem I.2 *Let $m \geq 3$ and $n \in \mathbb{N}^*$. For any $r < +\infty$ there exists a map, in a neighborhood of the origin, of the form*

$$\begin{array}{rccc} \mathcal{S}\ : & L^{m/2}(B^m, so(n)) & \longrightarrow & L^\infty \cap W^{2,m/2}(B^m, Gl_n(\mathbb{R})) \\ & & & \\ & \Omega & \longrightarrow & A \end{array} \tag{I.7}$$

such that

$$\Delta A + A\Omega = 0 \quad . \tag{I.8}$$

with the following controls

$$\|A\|_{L^\infty(B^m)} = \sup_{x\in B^m,\ X\in S^{m-1}} |A(x)\, X| \leq 1 \quad , \tag{I.9}$$

A is moreover invertible almost everywhere and $A^{-1} \in L^r(B^m)$ and there exists $C > 0$, independent of Ω, such that

$$\|A^{-1}\nabla A\|_{L^m(B^m)} + \|\nabla A\|_{W^{1,m/2}(B^m)} \leq C\|\Omega\|_{L^{m/2}(B^m)} \quad . \tag{I.10}$$

□

Remark I.1 *In dimension less or equal to 4 -i.e.* $(m < 5)$ *- one can even prove the following inequality*

$$\|dist(A, SO(n))\|_{L^\infty(B^m)} \leq C\ \|\Omega\|^2_{L^{m/2}(B^m)} \quad . \tag{I.11}$$

It is natural to conjecture that this inequality holds true in higher dimension. □

Once A is constructed one proves the following result.

Theorem I.3 *Let* $\Omega \in L^{m/2}(B^m, so(n))$ *and* A *given by the previous theorem for some* $r < +\infty$*. For any* $v \in L^{m/(m-2)}$*, assuming either* $m < 5$ *or* $\Delta v \in L^1_{loc}$*, then the following equivalence holds*

$$-\Delta v = \Omega\ v \qquad \Longleftrightarrow \qquad div(A\ \nabla v - \nabla A\ v) = 0 \quad . \tag{I.12}$$

□

We have then been able to write Schrödinger Systems with antisymmetric potential in conservative form[1].

Remark I.2 *It would be nice to drop the technical assumption* $\Delta v \in L^1_{loc}$ *whenever* $m \geq 5$. □

Remark I.3 *Results like (I.5) or (I.12) can be interpreted as a PDE version of the constant variation method. Indeed, it suffices to know* <u>*one*</u> *solution of the auxiliary equations (I.3) resp. (I.8) in order to be able to "factorize" the derivative (the divergence operator) for* <u>*any*</u> *solution to the linear PDE (I.1) resp. (I.6). This is exactly what the constant variation Method does for ODE.* □

A corollary of the existence of such conservation law for Schrödinger Systems with anti-symmetric potential is the sub-criticality of such systems. In fact we can even prove the following

Theorem I.4 *Let* $n \in \mathbb{N}^*$ *and* $m \geq 3$*. Let* $v \in L^{m/(m-2)}(B^m, \mathbb{R}^n)$ *satisfying*

$$-\Delta v = \Omega\ v \quad ,$$

where $\Omega \in L^{m/2}(B^m, so(n))$*, then* $|v|$ *is a subharmonic function and hence* $v \in L^\infty_{loc} \cap W^{2,m/2}_{loc}(B^m, \mathbb{R}^n)$*.* □

[1] Observe that the product $A\ \nabla v$ makes sense since $A \in W^{1,m}$, by Sobolev embeddings, and we have $A\ \nabla v := \nabla(A\ v) - \nabla A\ v$.

Our results and their proofs take their source jointly in [Ri1] but also in [DR2] where F. Da Lio and the author were studying the regularity of 1/2-harmonic maps from the real line into manifolds - see also [DR1]. They reduced the original problem to the one of proving that the following equation is sub-critical in one dimension

$$\Delta^{1/4} v = \Omega\, v \quad ,$$

where $v \in L^2(\mathbb{R}, \mathbb{R}^n)$ and $\Omega \in L^2(\mathbb{R}, so(n))$.

We end-up this introduction by making the following remarks.

Remark I.4 *It is important to insist on the fact that, a-priori, from the way we construct them, both the mappings $\mathcal{L}$ and $\mathcal{S}$ are not continuous between, respectively, $L^2(D^2, \mathbb{R}^2 \otimes so(n))$ and $L^\infty \cap W^{1,2}(D^2, Gl_n(\mathbb{R}))$ and between $L^{m/2}(B^m, so(n))$ and $L^\infty \cap W^{2,m/2}(B^m, Gl_n(\mathbb{R}))$. Our constructions both in [Ri1] and in the present paper are realized by the application of successively local inversion theorem and continuity argument like the construction of Coulomb Gauges for $L^{m/2}-$curvatures in [Uh]. Recently a construction of $\mathcal{L}$ using a more direct variational method has been proposed by A.Schikorra in [Sc]. He was following an approach introduced by F.Hélein in order to construct "Coulomb Moving Frames" (see [He] lemma 4.1.3). A construction of $\mathcal{S}$ using such a variational argument might a-priori be possible and would be interesting in itself.* $\Box$

Remark I.5 *In [RS], M. Struwe and the author established the sub-criticality of (I.1) in arbitrary dimension in Morrey spaces. This was motivated by applications to the partial regularity of stationary critical points to conformally invariant Lagrangians in higher dimension. However the existence of the Matrix valued map A in $L^\infty(B^m, Gl_n(\mathbb{R}))$ satisfying*

$$div(\nabla_\Omega A) = 0$$

was problematic due to the fact that Wente integrability by compensation does not provide L^∞ bounds in the classical Morrey spaces but only in their Littlewood-Paley counterpart (see [Ke]). Here however, since the L^∞ control of A in theorem I.2 is obtained by the application of the Maximum principle, the chances are high that theorem I.2 extends to higher dimension for the ad-hoc Morrey spaces which make system (I.6) a-priori critical.

The paper is organized as follows. In section 2 we construct the map $\mathcal{S}$, proving then theorem-I.2, and using an intermediate construction of a solution $P \in W^{2,m/2}(B^m, SO(n))$ solving

$$\frac{1}{2}\left[\Delta P\; P^{-1} - P\; \Delta P^{-1}\right] + P\; \Omega\; P^{-1} = 0$$

that we postpone in the appendix. In section 3 we prove remark I.1. In section 4 we establish theorem I.3 and in section 5 we prove theorem I.4.

II Proof of theorem I.2.

Let $\Omega \in L^{m/2}(B^m, so(n))$ and $v \in L^{m/m-2}(B^m, \mathbb{R}^n)$ satisfying (I.6). Consider $P \in W^{2,m/2}(B^m, SO(n))$ given by lemma A.1. We compute

$$-\Delta(P\ v) = \Delta P\ v - P\ \Delta v - 2\ div(\nabla P\ v) \quad .$$

Introducing $w := P\ v$, the equation (I.6) is then equivalent to

$$-\Delta w = \left[\Delta P\ P^{-1} + P\,\Omega\,P^{-1}\right]\ w - 2\ div(\nabla P\ P^{-1}\ w) \quad .$$

Taking into account this special choice of P we have made and satisfying (A.1), with our notations the system (I.6) becomes equivalent to

$$-\Delta w - \frac{1}{2}\left[\Delta P\ P^{-1} + P\ \Delta P^{-1}\right]\ w + 2\ div(\nabla P\ P^{-1}\ w) = 0 \quad . \tag{II.1}$$

Observe that

$$\begin{aligned} -\left[\Delta P\ P^{-1} + P\ \Delta P^{-1}\right] &= -div(\nabla P\ P^{-1} + P\ \nabla P^{-1}) + 2\nabla P\cdot\nabla P^{-1} \\ &= -2(\nabla P\ P^{-1})^2 \end{aligned}$$

where we have used twice that $\nabla P\ P^{-1} = -P\ \nabla P^{-1}$. The notation for the r.h.s $-2(\nabla P\ P^{-1})^2$ has to be understood as follows

$$-2(\nabla P\ P^{-1})^2 := -2\sum_{j=1}^{m}(\partial_{x_j} P\ P^{-1})^2$$

where the squares in the r.h.s refer to Matrix multiplication. Observe that each $\partial_{x_j} P\,P^{-1}$ is an L^m map taking values into $so(n)$ therefore each $-(\partial_{x_j} P\ P^{-1})^2$ is an $L^{m/2}$ map taking values into the space $Sym_n^+(\mathbb{R})$ of symmetric non-negative $n\times n-$matrices[2] . Hence

$$-(\nabla P\ P^{-1})^2 \in L^{m/2}(B^m, Sym_n^+(\mathbb{R})) \quad .$$

Combining (II.1) with the previous observations, the Schrödinger system (I.6) becomes equivalent to

$$-\Delta w - (\nabla P\ P^{-1})^2\ w + 2\ div(\nabla P\ P^{-1}\ w) = 0 \quad . \tag{II.2}$$

[2]Indeed if a is a real antisymmetric matrix we have that $(a^2)^t = a^t a^t = a^2$ and for every x in $\mathbb{R}^n$ $< x, -(a)^2 x >= -x^t a^2 x = x^t a^t a x = (ax)^t a x \geq 0$

Standard elliptic estimates gives that for any given $r < m/2$, if $\|\nabla P\|_{L^m}$ is small enough - depending on r a-priori -, then there exists a unique solution $Q \in W^{2,r}(B^m, M_n(\mathbb{R}))$ of the following problem

$$\left\{\begin{array}{l} -\Delta Q - 2\nabla Q\cdot\nabla P\ P^{-1} - Q\ (\nabla P\ P^{-1})^2 = 0 \qquad \text{in } B^m \\ \\ \quad Q = Id \qquad \text{on } \partial B^m \end{array}\right. \tag{II.3}$$

This comes from the following a-priori estimates

$$\|\nabla Q\cdot\nabla P\ P^{-1}\|_{L^r} \le \|\nabla Q\|_{L^{rm/m-r}}\ \|\nabla P\|_{L^m} \le C_r\ \|Q - Id\|_{W^{2,r}_0}\ \|\nabla P\|_{L^m} \tag{II.4}$$

and

$$\begin{aligned} \|(Q-Id)\ (\nabla P\ P^{-1})^2\|_{L^r} &\le \|(Q-id)\|_{L^{rm/m-2r}}\ \|(\nabla P\ P^{-1})^2\|_{L^{m/2}} \\ &\le C_r\ \|Q-Id\|_{W^{2,r}_0}\ \|\nabla P\|^2_{L^m} \quad . \end{aligned} \tag{II.5}$$

We establish now the following lemma.

Lemma II.1 *Let $m \ge 3$ and $n \in \mathbb{N}^*$. There exists $\varepsilon_0 > 0$ such that for any $P \in W^{1,m}(B^m, SO(n))$ satisfying*

$$\int_{B^m} |\nabla P|^m < \varepsilon_0 \quad ,$$

and any $Q \in W^{2,2m/(m+2)}(B^m, M_n(\mathbb{R}))$ solving

$$\left\{\begin{array}{l} -\Delta Q - 2\nabla Q\cdot\nabla P\ P^{-1} - Q\ (\nabla P\ P^{-1})^2 = 0 \qquad \textit{in } B^m \\ \\ \quad Q = Id \qquad \textit{on } \partial B^m \quad . \end{array}\right.$$

Then $Q \in L^\infty \cap W^{2,m/2}(B^m, M_n(\mathbb{R}))$ and

$$\sup_{X\in\mathbb{R}^n} \|Q\,X\|^2_{L^\infty(B^m)} \le 1 \quad . \tag{II.6}$$

□

Proof of Lemma II.1.

We first show that for any $X \in \mathbb{R}^n$ the following inequality holds :

$$\Delta(X^t\,Q\,Q^t\,X) \ge 0 \quad . \tag{II.7}$$

We have

$$\begin{aligned}
\Delta(X^t\,Q\,Q^t\,X) &= X^t\,\Delta Q\,Q^t\,X + X^t\,Q\,\Delta Q^t\,X + 2X^t\,\nabla Q\cdot\nabla Q^t\,X\\
&= -2\,X^t\,\nabla Q\cdot(\nabla P\ P^{-1})\,Q^t\,X - X^t\,Q(\nabla P\ P^{-1})^2\,Q^t\,X\\
&\quad +2\,X^t\,Q\ (\nabla P\ P^{-1})\cdot\nabla Q^t\,X - X^t\,Q\,(\nabla P\ P^{-1})^2\,Q^t\,X\\
&\quad +2X^t\,\nabla Q\cdot\nabla Q^t\,X
\end{aligned}$$

where all this above operations make a distributional sense (Leibnitz rule) as long as $Q\in W^{2,2m/(m+2)}(B^m)$, which is our assumption. Observe that[3]

$$\begin{aligned}
-2\,X^t\,\nabla Q\cdot(\nabla P\ P^{-1})\,Q^t\,X &= -2\,((\nabla P\ P^{-1})\,Q^t\,X)^t\cdot(X^t\,\nabla Q)^t\\
&= 2\,X^t\,Q\,(\nabla P\ P^{-1})\cdot\nabla Q^t\,X \quad .
\end{aligned}$$

Hence we have

$$\begin{aligned}
\Delta(X^t\,Q\,Q^t\,X) &= +4\,X^t\,Q\ (\nabla P\ P^{-1})\cdot\nabla Q^t\,X\\
&\quad -2X^t\,Q\,(\nabla P\ P^{-1})^2\,Q^t\,X + 2X^t\,\nabla Q\cdot\nabla Q^t\,X \quad .
\end{aligned} \tag{II.8}$$

Cauchy-Schwartz inequality tells that

$$\begin{aligned}
-2\,X^t\,Q\ (\nabla P\ P^{-1})\cdot\nabla Q^t\,X &\le X^t\,Q\ (\nabla P\ P^{-1})\cdot(\nabla P\ P^{-1})^t\,Q^t\,X\\
&\quad +X^t\,\nabla Q\cdot\nabla Q^t\,X \quad .
\end{aligned}$$

Since again $(\nabla P\ P^{-1})^t = -(\nabla P\ P^{-1})$, the previous inequality implies

$$\begin{aligned}
4\,X^t\,Q\ (\nabla P\ P^{-1})\cdot\nabla Q^t\,X &\ge 2X^t\,Q\,(\nabla P\ P^{-1})^2\,Q^t\,X\\
&\quad -2X^t\,\nabla Q\cdot\nabla Q^t\,X \quad .
\end{aligned} \tag{II.9}$$

Combining (II.8) and (II.9) we obtain (II.7). Applying the Maximum Principle we obtain[4] (II.6). This implies that $Q\in L^\infty(B^m)$. Hence $Q\ (\nabla P\ P^{-1})^2\in L^{m/2}(B^m)$. Since we have the *a-priori* estimate (for any $1<r<m$)

$$\begin{aligned}
\|\nabla Q\cdot\nabla P\ P^{-1}\|_{L^r} &\le \|\nabla P\|_{L^m}\ \|\nabla Q\|_{L^{rm/m-r}}\\
&\le C_r\epsilon_0\ \|Q-Id\|_{W^{2,r}_0(B^m)} \quad ,
\end{aligned}$$

[3]Since for Y and Z in $\mathbb{R}^n$ we have $Y^t\,Z = Z^t\,Y$

[4]Since $|Q^t X|^2 = X^t\,Q\,Q^t\,X$.

Applying it successively for $r = 2m/m+2$ and $r = m/2$ we deduce that, for ϵ_0 chosen small enough, the operator

$$\begin{array}{rcl} K_P \ : \ W^{2,r}_0(B^m, M_n(\mathbb{R})) & \longrightarrow & L^r(B^m, M_n(\mathbb{R})) \\ & & \\ \eta & \longrightarrow & -\Delta\eta - 2\nabla\eta \cdot \nabla P \ P^{-1} \end{array}$$

is an isomorphism for both $r = 2m/m+2$ and $r = m/2$. Applying it to $\eta = Q - Id$ we obtain, since $Q\ (\nabla P\ P^{-1})^2 \in L^{m/2}(B^m)$, that $Q \in W^{2,m/2}(B^m, M_n(\mathbb{R}))$ and the following estimate holds

$$\|Q - Id\|_{W^{2,m/2}_0(B^m)} \le C_m \ \left[\int_{B^m} |\nabla P|^m\right]^{2/m} \quad . \tag{II.10}$$

This ends the proof of lemma II.1. □

We shall now combine the construction of P (lemma A.1), the estimates on Q (lemma II.1) and the a-priori estimates on $A = QP$ (lemma A.4) in order to construct A, assuming first that $\Omega \in L^q$ for some $m/2 < q < m$. We prove the following lemma.

Lemma II.2 *Let $m > q > m/2$ and $1 < r < +\infty$. There exists $\varepsilon_0 > 0$ and $C > 0$ such that for any $\Omega \in L^q(B^m, so(n))$ satisfying*

$$\|\Omega\|_{L^{m/2}(B^m)} \le \varepsilon_0 \quad , \tag{II.11}$$

there exists $A \in W^{2,q}(B^m, Gl_n(\mathbb{R}))$ with $A^{-1} \in L^\infty(B^m)$ such that

$$\left\{ \begin{array}{l} \Delta A + A\,\Omega = 0 \qquad \text{in } B^m \\ \\ A = I_n \qquad \text{on } \partial B^m \end{array} \right. \tag{II.12}$$

and the following inequalities hold

i)

$$\|A^{-1}\nabla A\|_{L^m(B^m)} \le C\ \|\Omega\|_{L^{m/2}(B^m)} \quad , \tag{II.13}$$

ii)

$$\|A^{-1} - I_n\|_{L^r(B^m)} \le C\ \|\Omega\|_{L^{m/2}(B^m)} \quad , \tag{II.14}$$

iii)

$$\|A^{-1}\nabla A\|_{L^{qm/(m-q)}(B^m)} \le C\ \|\Omega\|_{L^q(B^m)} \quad , \tag{II.15}$$

iv)

$$\|A^{-1} - I_n\|_{L^\infty(B^m)} \le C\ \|\Omega\|_{L^q(B^m)} \quad . \tag{II.16}$$

$\square$

Proof of lemma II.2. The construction of P given by lemma A.1

$$\mathcal{J}\ :\ \mathcal{U}^q_{\varepsilon_0}\ \longrightarrow W^{2,q}(B^m,SO(n))$$

$$\Omega\ \longrightarrow P$$

is continuous for $m > q > m/2$ (though it is not necessarily continuous for $q = m/2$ which is the main difficulty in this lemma). We prove now that the map which to $\eta \in L^{qm/(m-q)}(B^m,\mathbb{R}^m\otimes so(n))$ assigns $Q \in W^{2,q}(B^m,M_n(\mathbb{R}))$ satisfying

$$\left\{\begin{array}{l} -\Delta Q - 2\nabla Q\cdot\eta - Q\ (\eta)^2 = 0 \qquad \text{in } B^m \\ \\ \quad Q = Id \qquad \text{on } \partial B^m \end{array}\right. \tag{II.17}$$

is also continuous for $m > q > m/2$ and $\int_{B^m}|\eta|^m < \varepsilon_0$ for some ε_0 small enough. This comes from the following : Let

$$L_\eta\ :\ W^{2,q}_0(B^m,M_n(\mathbb{R})) \longrightarrow L^q(B^m,M_n(\mathbb{R}))$$

$$u \longrightarrow -\Delta u - 2\nabla u\cdot\eta - u(\eta)^2$$

We claim that, for ε_0 small enough L_η is continuous and invertible from $W^{2,q}_0(B^m,M_n(\mathbb{R}))$ into $L^q(B^m,M_n(\mathbb{R}))$. Indeed, using the estimates (II.4) and (II.5) we have that for any fixed $s < m/2$ and ε_0 small enough -depending on the choice of s - L_η realizes an isomorphism from $W^{2,s}_0(B^m,M_n)$ into $L^s(B^m,M_n)$. We choose then $s < m/2$ such that $s^{-1} - 2m^{-1} + q^{-1} < 2m^{-1}$. Let $f \in L^q(B^m,M_n(\mathbb{R}))$ and u be the unique solution in $W^{2,s}_0(B^m,M_n(\mathbb{R}))$ solving $L_\eta u = f$. We have that

$$-tr[\Delta u\, u^t] - 2tr[\nabla u\cdot\eta\, u^t] - tr[u(\eta)^2u^t] = tr[f\, u^t]$$

Denote $<\cdot,\cdot>$ the scalar product on $M_n(\mathbb{R})$ given by $<A,B>= tr(A\,B^t)$. We have then

$$-\Delta\frac{|u|^2}{2} + |\nabla u|^2 + 2 < \nabla u, u\,\eta > + |u\eta|^2 =< f,u > \quad . \tag{II.18}$$

where we have used that $\eta^t = -\eta$. Which implies, by Cauchy Schwartz inequality

$$\Delta\frac{|u|^2}{2} + < f,u > \geq 0 \quad . \tag{II.19}$$

Let φ solving

$$\left\{\begin{array}{ll}\Delta\varphi=<f,u> & \text{in } B^m\\ \\ \varphi=0 & \text{in } \partial B^m\end{array}\right. .$$

Since $s^{-1}-2\,m^{-1}+q^{-1}<2\,m^{-1}$ we have that

$$\|\phi\|_\infty\le C_q\,\|f\|_q\ \|u\|_{sm/(m-2s)}\le C_{q,s}\,\|f\|_q\ \|f\|_s\le C\,\|f\|_q^2\quad . \tag{II.20}$$

Since $u=0$ on ∂B^m, the combination of (II.19) and (II.20) together with the maximum principle gives

$$-\|\varphi\|_\infty\le\frac{|u|^2}{2}+\varphi\le 0\quad ,$$

from which we deduce

$$\|u\|^2_\infty\le C_{q,s}\ \|f\|_q^2\quad . \tag{II.21}$$

The equation $L_\eta u=f$ implies then

$$\begin{aligned}\|-\Delta u-2\nabla u\cdot\eta\|_{L^q}&\le C\ (1+\|\eta^2\|_q)\ \|f\|_q\\ &\le C\ (1+\|\eta\|_m\ \|\eta\|_{qm/(m-q)})\ \|f\|_q\quad .\end{aligned}$$

Combining this fact with the a-priori estimate

$$\|2\nabla u\cdot\eta\|_{L^q}\le C\|\nabla u\|_{qm/(m-q)}\ \|\eta\|_m\le C\ \varepsilon_0^{1/m}\ \|u\|_{W^{2,q}_0}\quad ,$$

we obtain that, for ε_0 small enough, the solution u of $L_\eta u=f$ in $W^{2,s}_0$ is in fact in $W^{2,q}_0$. This proves the invertibility of L_η from $W^{2,q}_0(B^m,M_n(\mathbb{R}))$ into $L^q(B^m,M_n(\mathbb{R}))$.

Having established the invertibility of L_η in these spaces, the continuity of the map which to η in $L^{qm/(m-q)}(B^m,\mathbb{R}^m\otimes so(n))$ assigns $Q\in W^{2,q}(B^m,M_n(\mathbb{R}))$ solving (II.17) can now be proved as follows :

Consider a perturbation $\delta\in L^{qm/(m-q)}(B^m,\mathbb{R}^m\otimes so(n))$ such that we still have $\|\eta+\delta\|_m^m<\varepsilon_0$ and denote $Q+q$ the solution of $L_{\eta+\delta}(Q+q)=0$ equal to the identity matrix I_n on ∂B^m. Hence q satisfies

$$\left\{\begin{array}{ll}L_{\eta+\delta}q=-L_{\eta+\delta}(Q)+L_\eta Q=2\nabla Q\cdot\delta+Q\ [(\eta+\delta)^2-(\eta)^2] & \text{in } B^m\\ \\ q=0\qquad \text{on } \partial B^m & \end{array}\right. .$$

The inversibility of $L_{\eta+\delta}$ we established previously implies

$$\|q\|_{W^{2,q}_0}\le C\ \|\nabla Q\|_{\frac{qm}{m-q}}\ \|\delta\|_m+C\ \|Q\|_\infty\ \left[\|\eta\|_{m)}+\|\delta\|_m\right]\ \|\delta\|_{\frac{qm}{m-q}}\quad ,$$

which gives the continuity of the map which to $\eta \in L^{qm/(m-q)}$ satisfying $\int_{B^m} |\eta|_m < \varepsilon_0$ assigns $Q \in W^{2,q}$ satisfying (II.17).

Hence, the map we have constructed

$$\mathcal{K} \ : \ \mathcal{U}^q_{\varepsilon_0} \ \longrightarrow W^{2,q}(B^m, M_n(\mathbb{R})))$$

$$\Omega \ \longrightarrow A := QP$$

is continuous for $m > q > m/2$.

Let $r \in (1, +\infty)$, $\varepsilon_0 > 0$, $C > 0$ and denote

$$\mathcal{W}^{q,r}_{\varepsilon_0,C} := \left\{\Omega \in \mathcal{U}^q_{\varepsilon_0} \quad ; \quad A := \mathcal{K}(\Omega) \text{ satisfies (II.12)} \cdots \text{(II.16)} \ \right\} \quad .$$

We claim that for any $m > q > m/2$ and $r \in (1, +\infty)$ there exists $\varepsilon_0 > 0$ and $C > 0$ such that $\mathcal{U}^q_{\varepsilon_0} = \mathcal{W}^{q,r}_{\varepsilon_0,C}$. This will prove the lemma.

We fix $m > q > m/2$ and $r \in (1, +\infty)$. Similarly as above in the construction of P we shall prove that there exists ε_0 and $C > 0$ such that $\mathcal{W}^{q,r}_{\varepsilon_0,C}$ is non empty, open and closed in $\mathcal{U}^q_{\varepsilon_0}$ which is clearly arc connected. This will imply the claim.

First we can show that $\mathcal{W}^{q,r}_{\varepsilon_0,C} \neq \emptyset$ for $C > 0$ and $\epsilon := \delta/2$ given by lemma A.4 : in the L^q neighborhood of zero, since $\mathcal{K}$ is continuous both $\|nablaA\|_m$ and $\|A - I_n\|_\infty$ are small[5], therefore $\|A^{-1} - I_n\|_\infty$ is also small and hence we have in a L^q neighborhood of 0 that $\|A^{-1}\nabla A\|_m < \delta/2$ which implies that the conditions (A.17) is satisfied and we deduce (II.12) $\cdots$ (II.16) for the constant C.

We prove now the closedness of $\mathcal{W}^{q,r}_{\varepsilon_0,C}$ for the L^q distance. Let $\Omega_k \in \mathcal{W}^{q,r}_{\varepsilon_0,C}$ converging strongly to Ω_∞ in L^q. By the continuity of $\mathcal{K}$ we have that $A_k := \mathcal{K}(\Omega_k)$ converges stongly to the limit $A_\infty := \mathcal{K}(\Omega_\infty)$ in $W^{2,q}$. Our assumptions, $\Omega_k \in \mathcal{W}^{q,r}_{\varepsilon_0,C}$ implies that

$$\begin{aligned} &\|A_k^{-1}\nabla A_k\|_m \leq C \ \|\Omega_k\|_m \leq C \ \varepsilon_0 \\ \text{and} \quad &\|A_k^{-1} - I_n\|_\infty \leq C \ \|\Omega_k\|_q \quad . \end{aligned} \tag{II.22}$$

Hence $\|A_k^{-1}\|_\infty$ and $\|\nabla A_k^{-1} \, A_k\|_m = \|A_k^{-1}\nabla A_k\|_m$ are uniformly bounded. We deduce that $\|\nabla A_k^{-1}\|_m$ is uniformly bounded and therefore A_k^{-1} converges strongly to a limit in L^s ($\forall s < +\infty$). Since A_k also strongly converges in L^∞ and since $A_k A_k^{-1} = A_k^{-1} A_k = I_n$ the limit of A_k^{-1} has to be A_∞^{-1} and then inequalities (II.12) $\cdots$ (II.16) hold for A_∞ which implies that $\Omega_\infty \in \mathcal{W}^{q,r}_{\varepsilon_0,C}$.

We prove now that $\mathcal{W}^{q,r}_{\varepsilon_0,C}$ is open in $\mathcal{U}^q_{\varepsilon_0}$ for the L^q distance if we have taken ε_0 small enough and the constant C given by lemma A.4.

[5]Using the fact that $W^{2,q}$ embeds in L^∞ for $q > m/2$.

Let $\Omega \in \mathcal{W}^{q,r}_{\varepsilon_0,C}$ and denote $A := \mathcal{K}(\Omega)$. Since $\Omega \in \mathcal{W}^{q,r}_{\varepsilon_0,C}$, we have that $\|A^{-1}\|_\infty < +\infty$. Let $a \in W^{2,q}_0(B^m, M_n(\mathbb{R}))$, we write $A+a = A\ (I_n + A^{-1}a)$. Hence, Since $W^{2,q}_0$ embedds in L^∞, for $\|a\|_{W^{2,q}}$ small enough, we have

$$\|(A+a)^{-1} - A^{-1}\|_\infty \le C\|A^{-1}\|_\infty\ \|a\|_{W^{2,q}_0} \quad . \tag{II.23}$$

We have

$$\|(A+a)^{-1}\nabla(A+a) - A^{-1}\nabla A\|_m \le C\ \left[\|A^{-1}\|_\infty + \|\nabla A\|_m\right]\ \|a\|_{W^{2,q}_0} \quad . \tag{II.24}$$

Hence, since $\mathcal{K}$ is continuous and since $\|A^{-1}\nabla A\|_m < C\ \varepsilon_0$, there exists a radius $\rho_\Omega > 0$ such that for any $\omega \in L^q(B^m)$ such that $\|\omega\|_q < \rho_\Omega$ one has

$$\|\mathcal{K}(\Omega+\omega)^{-1}\nabla(\mathcal{K}(\Omega+\omega)\|_m \le 2C\ \varepsilon_0 \quad .$$

Having chosen ε_0 small enough in such a way that $\varepsilon_0 + 2C\ \varepsilon_0 < \delta$, we can apply lemma A.4 and we obtain that $\mathcal{K}(\Omega+\omega)$ satisfies (II.12) $\cdots$ (II.16) for $\|\omega\|_q < \rho_\Omega$ and for the constant C given by lemma A.4. This proves that $\mathcal{W}^{q,r}_{\varepsilon_0,C}$ is open in $\mathcal{U}^q_{\varepsilon_0}$ for the L^q distance if we have taken ε_0 small enough and the constant C given by lemma A.4. We have then concluded the proof of lemma II.2. □

We shall now deduce the following lemma which implies theorem I.2 :

Lemma II.3 *Let $1 < r < +\infty$. There exists $\varepsilon_0 > 0$ and $C > 0$ such that for any $\Omega \in L^{m/2}(B^m, so(n))$ satisfying*

$$\|\Omega\|_{L^{m/2}(B^m)} \le \varepsilon_0 \quad , \tag{II.25}$$

there exists $A \in L^\infty \cap W^{2,m/2}(B^m, Gl_n(\mathbb{R}))$ with $A^{-1} \in L^r(B^m)$ such that

$$\left\{\begin{array}{l} \Delta A + A\,\Omega = 0 \qquad \text{in } B^m \\[2mm] A = I_n \qquad \text{on } \partial B^m \end{array}\right. \tag{II.26}$$

and the following inequalities hold

i)

$$\|A^{-1}\nabla A\|_{L^m(B^m)} \le C\ \|\Omega\|_{L^{m/2}(B^m)} \quad , \tag{II.27}$$

ii)

$$\|A^{-1} - I_n\|_{L^r(B^m)} \le C\ \|\Omega\|_{L^{m/2}(B^m)} \quad , \tag{II.28}$$

□

Proof of lemma II.3.

Let $m/2 < q < m$ and let $\Omega \in L^{m/2}(B^m, so(n))$ satisfying $\|\Omega_k\|_{L^{m/2}} < \varepsilon_0$, where ε_0 is given by lemma II.2. We take a sequence $\Omega_k \in L^q(B^m, so(n))$ converging to $\Omega \in L^{m/2}(B^m, so(n))$ and also satisfying $\|\Omega_k\|_{L^{m/2}} < \varepsilon_0$.

Consider A_k given by lemma II.2. We know that $\|A_k\|_\infty \leq 1$ and that $\|\nabla A_k\|_m$, $\|A_k^{-1}\nabla A_k\|_m$ and $\|A_k^{-1} - I_n\|_r$ are uniformly bounded. We can then extract a subsequence $A_{k'}$ which weakly converges to some A in $W^{1,m}$ and clearly (A, Ω) satisfies (II.26).

Since $\|A_k^{-1}\nabla A_k\|_m = \|\nabla A_k^{-1}\, A_k\|_m$ and since, together with $\|A_k^{-1}\|_r$, these sequences are uniformly bounded, we have that $\|\nabla A_k^{-1}\|_{rm/(m+r)}$ is uniformly bounded. Hence we can then extract our subsequence $A_{k'}$ in such a way that $A_{k'}^{-1}$ weakly converges in $W^{1,rm/(m+r)}$. Therefore $A_{k'}^{-1}$ strongly converges in L^s for any $s < r$.

Since $A_{k'}^{-1}A_{k'} = A_{k'}A_{k'}^{-1} = I_n$ and since $A_{k'}$ strongly converges in L^p for any $p < +\infty$, we can pass to the limit in this identities and we deduce that the strong limit of $A_{k'}^{-1}$ is A^{-1}.

We can now pass to the limit in the estimates (II.13) and (II.14) and we obtain (II.27) and (II.28) which ends the proof of lemma II.3. $\Box$

III Proof of remark I.1.

In this part we restrict to dimensions $m = 3$ and $m = 4$. We prove that there exists $C_m > 0$ such that

$$\|Q - I_n\|_{L^\infty(B^m)} \leq C_m \left[\int_{B^m} |\nabla P|^m\right]^{4/m} \quad . \tag{III.1}$$

Where Q is the $L^\infty \cap W^{2,m/2}$ map given by lemma II.1. This last estimate, by taking $A := QP$, implies theorem I.1 directly and permits hence to skip the use of lemmas A.4 and lemma II.2.

(III.1) can be proved as follows : Let P given by lemma A.1. From estimate (A.2) and Sobolev-Lorentz estimates (see for instance [Ta]) we deduce that

$$\|\nabla P\|_{L^{m,m/2}(B^m)} \leq C\ \|\Omega\|_{L^{m/2}(B^m)} \quad , \tag{III.2}$$

where $L^{m,m/2}$ is the Lorentz Space of measurable functions satisfying

$$\int_{\mathbb{R}_+} t^{-1/2} f^*(t)^{m/2}\ dt < +\infty$$

(f^* denotes here the decreasing rearangement of $|f|$). Since the product of two $L^{m,m/2}$ function is in $L^{m/2,m/4}$ and since we are working in this section under the assumption $m \leq 4$, we deduce the following estimates for any $u \in W_0^{2,m/2}(B^m, M_n(\mathbb{R}))$ such that $\Delta u \in L^{m/2,m/4}(B^m)$

$$\begin{aligned} \|u\,(\nabla P\,P^{-1})^2\|_{L^{m/2,m/4}} &\leq C\ \|u\|_\infty\ \|\nabla P\|^2_{L^{m,m/2}(B^m)} \\ &\leq C\ \|\Delta u\|_{L^{m/2,m/4}(B^m)}\ \|\nabla P\|^2_{L^{m,m/2}(B^m)} \quad , \end{aligned} \tag{III.3}$$

where we used the fact that, under the assumption $m \leq 4$, a function having two derivatives in $L^{m/2,m/4}$ is bounded (see again Lorentz-Sobolev embeddings in [Ta]). We have moreover

$$\begin{aligned} \|\nabla u\cdot\nabla P\,P^{-1}\|_{L^{m/2,m/4}(B^m)} &\leq C\ \|\nabla u\|_{L^{m,m/2}(B^m)}\ \|\nabla P\|_{L^{m,m/2}(B^m)} \\ &\|\Delta u\|_{L^{m/2,m/4}(B^m)}\ \|\nabla P\|_{L^{m,m/2}(B^m)} \end{aligned} \tag{III.4}$$

Hence under the assumption that $\|\Omega\|_{m/2}$ is bellow a sufficiently small constant (which implies that $\|\nabla P\|_{L^{m,m/2}(B^m)}$ is small) we deduce that there exists a unique u with 2 derivatives in $L^{m/2,m/4}$ satisfying

$$\left\{ \begin{array}{l} -\Delta u - 2\nabla u\cdot\nabla P\,P^{-1} - u(\nabla P\,P^{-1})^2 = (\nabla P\,P^{-1})^2 \\ \\ u = 0 \qquad\quad \text{on} \quad \partial B^m \quad , \end{array} \right. \tag{III.5}$$

and u satisfies in particular

$$\|u\|_\infty \leq C\ \|\nabla P\|^2_{L^{m,m/2}(B^m)} \leq C\ \|\Omega\|^2_{m/2} \quad . \tag{III.6}$$

As we have seen in lemma II.1, $Q - I_n$ is the unique solution to (III.5) in $W_0^{2,m/2}(B^m, M_n(\mathbb{R}))$. Hence (III.6) holds for $u = Q - I_n$ which implies (III.1). $\Box$

IV Proof of theorem I.3.

Let $v \in L^{\frac{m}{m-2}}$, since $A \in W^{2,m/2}$ one has

$$div(A\nabla v - \nabla A v) = A\Delta v - \Delta A\ v \quad .$$

This comes simply from a density argument. Hence we have

$$div(A\nabla v - \nabla A v) = A\Delta v + A\Omega\ v \quad . \tag{IV.1}$$

Hence, if $\Delta v = -\Omega\ v$, we have that $div(A\nabla v - \nabla A v) = 0$.

Assuming now that $div(A\nabla v - \nabla A v) = 0$. If the dimension $m < 5$ one has from remark I.1 that $A^{-1} \in L^\infty \cap W^{2,m/2}$. We can then multiply (IV.1) by A^{-1} and one obtains that $\Delta v = -\Omega\, v$. If now $\Delta v \in L^1_{loc}$ we interpret the identity $0 = A\Delta v + A\Omega\, v$ in the almost everywhere sense and since A is invertible almost everywhere, we obtain that $\Delta v = -\Omega\, v$ a.e. which implies the same identity in the distributional sense and the result is proved. □

V Proof of theorem I.4.

We prove that $|v|$ is a subharmonic function : $\Delta|v| \geq 0$. This fact implies[6] that $|v|$ is in $L^\infty_{loc}(B^m)$ and theorem I.4 will be proved.

Let $\varepsilon > 0$. Since $\Delta v = -\Omega\, v \in L^1(B^m)$ we can consider the scalar product between Δv and the L^∞ map given by $v/(\varepsilon + |v|)$. This gives

$$\frac{v}{\varepsilon + |v|} \cdot \Delta v = \frac{v^t}{\varepsilon + |v|} \Omega\, v = 0 \qquad a.e. \tag{V.1}$$

where we are using the fact that for almost every point $x \in B^m$ and any vector $X \in \mathbb{R}^n$ $X^t \Omega(x)\, X = 0$ since Ω is antisymmetric almost everywhere.

Let $\phi_\delta = \delta^{-m}\phi(\cdot/\delta)$ where $\phi \in C^\infty_0(B^m_1)$ and $\int_{B^m} \phi = 1$. Denote by v_δ the convolution between v and ϕ_δ. We clearly have that Δv_δ converges strongly in L^1 to Δv and that, moreover, v_δ converges almost everywhere to v. Writting

$$\left| \frac{v}{\varepsilon + |v|} \cdot \Delta v - \frac{v_\delta}{\varepsilon + |v_\delta|} \cdot \Delta v_\delta \right| \leq \left| \frac{v}{\varepsilon + |v|} - \frac{v_\delta}{\varepsilon + |v_\delta|} \right| \, |\Delta v| + |\Delta v - \Delta v_\delta|$$

Hence, using dominated convergence, we deduce that

$$\frac{v_\delta}{\varepsilon + |v_\delta|} \cdot \Delta v_\delta \longrightarrow \frac{v}{\varepsilon + |v|} \cdot \Delta v \qquad \text{strongly in } L^1 \quad . \tag{V.2}$$

A short computation gives

$$\begin{aligned} \frac{v_\delta}{\varepsilon + |v_\delta|} \cdot \Delta v_\delta &= div \left[\frac{|v_\delta|}{\varepsilon + |v_\delta|} \ \nabla |v_\delta| \right] \\ &\quad -(\varepsilon + |v_\delta|)^{-2} \ \left[(\varepsilon + |v_\delta|) \ |\nabla v_\delta|^2 - |v_\delta| \ |\nabla |v_\delta||^2 \right] \end{aligned} \tag{V.3}$$

Using Kato inequality : $|\nabla v| \geq |\nabla |v||$ we deduce that

$$\frac{v_\delta}{\varepsilon + |v_\delta|} \cdot \Delta v_\delta - div \left[\frac{|v_\delta|}{\varepsilon + |v_\delta|} \ \nabla |v_\delta| \right] \leq 0 \tag{V.4}$$

[6] For a subharmonic function f the map which to r assigns $|\partial B_r(x)|^{-1} \int_{\partial B_r(x)} f$ is increasing.

For $t \geq 0$ we denote $f_\varepsilon(t) := t - \varepsilon\ \log[(t+\varepsilon)/\varepsilon]$. Observe that $f'_\varepsilon(t) = t/(t+\varepsilon)$. We the have that

$$\frac{v_\delta}{\varepsilon + |v_\delta|} \cdot \Delta v_\delta - \Delta f_\varepsilon(|v_\delta|) \leq 0 \tag{V.5}$$

We have that $f_\varepsilon(|v_\delta|)$ converges to $f_\varepsilon(|v|)$ in $L^{\frac{m}{m-2}}$ as δ converges to 0. Hence, using also (V.2) we deduce that

$$\frac{v_\delta}{\varepsilon + |v_\delta|} \cdot \Delta v_\delta - \Delta f_\varepsilon(|v_\delta|) \longrightarrow \frac{v}{\varepsilon + |v|} \cdot \Delta v - \Delta f_\varepsilon(|v|) \quad \text{in } \mathcal{D}'(B^m) \quad . \tag{V.6}$$

Combining (V.1) (V.5) and (V.6) we deduce that

$$-\Delta f_\varepsilon(|v|) \leq 0 \tag{V.7}$$

Since $f_\varepsilon(|v|)$ converges towards $|v|$ in $L^{\frac{m}{m-2}}$ as ε goes to zero, we deduce that $\Delta|v| \geq 0$ and the theorem I.4 is proved. □

A Appendix

The appendix is devoted to the proof of the following lemma.

Lemma A.1 *Let $m \geq 3$ and $n \in \mathbb{N}^*$. There exists $\varepsilon_0 > 0$ and $C > 0$ such that, for any $\Omega \in L^{m/2}(B^m, so(n))$ satisfying*

$$\|\Omega\|_{L^{m/2}(B^m, so(n))} \leq \varepsilon_0 \quad ,$$

there exists $P \in W^{2,m/2}(B^m, SO(n))$ satisfying

$$\left\{ \begin{array}{l} \frac{1}{2}\left[\Delta P\ P^{-1} - P\ \Delta P^{-1}\right] + P\ \Omega\ P^{-1} = 0 \qquad \textit{in} \quad \mathcal{D}'(B^m) \\[4mm] P = Id_{SO(n)} \qquad \textit{on} \quad \mathcal{D}'(B^m) \end{array} \right. \tag{A.1}$$

and

$$\|P - Id\|_{W^{2,m/2}_0(B^m)} \leq C\ \|\Omega\|_{L^{m/2}} \quad . \tag{A.2}$$

□

Proof of lemma A.1. We follow a similar approach to the one introduced in the appendix of [Ri1] which was itself inspired by the work of K.Uhlenbeck [Uh]. Let $q > m/2$ and $\varepsilon > 0$. Consider

$$\mathcal{U}^q_\varepsilon = \left\{ \Omega \in L^q(B^m, so(n)) \ : \ \int_{\mathbb{R}} |\Omega|^{m/2} dx < \varepsilon \right\} .$$

Claim: *There exist $\varepsilon_0 > 0$ and $C > 0$ such that*

$$\mathcal{V}^q_{\epsilon_0,C} := \left\{ \begin{array}{c} \Omega \in \mathcal{U}^q_{\varepsilon_0} \ : \text{there exits } P \text{ satisfying (A.1) and (A.2)} \\ \\ \text{and } P = exp(U) \text{ with } ||U||_{W^{2,q}_0(B^m)} \leq C\|\Omega\|_{L^q(B^m)} \end{array} \right\}$$

is open and closed in $\mathcal{U}^q_{\varepsilon_0}$ for the L^q-norm and thus $\mathcal{V}^q_{\varepsilon_0,C} \equiv \mathcal{U}^q_{\varepsilon_0}$ (since $\mathcal{U}^q_{\varepsilon_0}$ is clearly path connected).

This claim implies lemma A.1. Indeed, for this ε_0 we consider $\Omega \in L^{m/2}(B^m, so(n))$ such that $\|\Omega\|_{L^{m/2}} < \varepsilon_0$. By convolutions one gets a sequence of maps $\Omega_k \in \mathcal{U}^q_\varepsilon$ converging strongly to Ω in $L^{m/2}$.

Let $P_k \in W^{2,q}(B^m, SO(n))$ given by the claim and satisfying both (A.1) and (A.2) for Ω_k. We can extract a subsequence that weakly converges in $W^{2,m/2}(B^m, SO(n))$ to a limit P in $W^{2,m/2}(B^m, M_n(\mathbb{R}))$.

By lower semicontinuity of the $W^{2,m/2}-$norm under weak convergence and by Rellich compactness embedding, we deduce that P satisfies (A.2) and that P takes values into the rotations $SO(n)$. Again by compactness embedding we have that P_k converges strongly to P in every L^q for $q < +\infty$ and since ΔP_k converges weakly to ΔP in $L^{m/2}$ we pass easily to the limit in the equation (A.1) and lemma-A.1 is proved.

It then remains to prove the claim.

Step 1 : *For any $\varepsilon_0 > 0$ and $C > 0$ $\mathcal{V}^q_{\varepsilon_0}$ is closed in $\mathcal{U}^q_{\varepsilon_0}$* . The proof of this step follows one by one the argument we just used to prove that the claim implies lemma A.1.

It then remains to establish the following.

Step 2 : *There exists $\varepsilon_0 > 0$ and $C > 0$ such that $\mathcal{V}^q_{\varepsilon_0,C}$ is open in $\mathcal{U}^q_{\varepsilon_0}$.*

Before to establish the step 2, we will prove a lemma that roughly tells us that as soon as $\|P - Id\|_{W^{2,m/2}}$ is small enough then (A.2) automatically holds. Precisely we have.

Lemma A.2 *Let $m \geq 3$ and $n \in \mathbb{N}^*$. There exists $\varepsilon_1 > 0$ and $C_1 > 0$ such that for any $P \in W^{2,m/2}(B^m, SO(m))$ sucht that $P = Id$ on ∂B^m, if*

$$\|P - Id\|_{W^{2,m/2}_0(B^m)} \leq \varepsilon_1 \tag{A.3}$$

then

$$\|P - Id\|_{W^{2,m/2}_0(B^m)} \leq C_1 \ \|P^{-1} \ \Delta P - \Delta P^{-1} \ P\|_{L^{m/2}(B^m)} \quad , \tag{A.4}$$

and such that for any $P \in W^{2,q}(B^m, SO(m))$ satisfying $P = Id$ on ∂B^m and (A.3) we have also

$$\|P - Id\|_{W^{2,q}_0(B^m)} \leq C_1 \ \|P^{-1} \ \Delta P - \Delta P^{-1} \ P\|_{L^q(B^m)} \quad . \tag{A.5}$$

$\square$

Proof of lemma A.2. We write

$$
\begin{aligned}
P^{-1}\Delta P \ &= \frac{1}{2}\left[P^{-1}\Delta P - \Delta P^{-1}\ P\right] \\
&+ \frac{1}{2}\left[P^{-1}\ \Delta P + \Delta P^{-1}\ P\right]
\end{aligned}
\tag{A.6}
$$

Moreover we have

$$
\begin{aligned}
P^{-1}\ \Delta P + \Delta P^{-1}\ P \ &= div\left(P^{-1}\nabla P + \nabla P^{-1}\ P\right) - 2\nabla\, P^{-1}\cdot\nabla P \\
&= -2\nabla\, P^{-1}\cdot\nabla P
\end{aligned}
\tag{A.7}
$$

Hence, by assumption, we have

$$
\begin{aligned}
\|P^{-1}\ \Delta P + \Delta P^{-1}\ P\|_{L^{m/2}(B^m)} \ &\leq 2\|\nabla P\|_{L^m(B^m)}\ \|\nabla P\|_{L^m(B^m)} \\
&\leq 2\varepsilon_1\ \|\nabla P\|_{L^m(B^m)}
\end{aligned}
\tag{A.8}
$$

Since $P - Id = 0$ on ∂B^m, standard elliptic estimates give

$$
\|\nabla P\|_{L^m(B^m)} \leq C_m\ \|\Delta P\|_{L^{m/2}(B^m)} \quad .
$$

This last fact combined with (A.7) and (A.8) give for $2\varepsilon_1\ C_m < 1/2$

$$
\|\Delta P\|_{L^{m/2}(B^m)} \leq \frac{2}{3}\|P^{-1}\Delta P - \Delta P^{-1}\ P\|_{L^{m/2}(B^m)} \quad .
$$

Using again the fact that $P - Id = 0$ on ∂B^m, standard elliptic estimates combined with the previous inequality gives (A.4).

(A.5) is proved in a similar way. Observe that

$$
\begin{aligned}
\|P^{-1}\ \Delta P + \Delta P^{-1}\ P\|_{L^q(B^m)} \ &\leq 2\|\nabla P\|_{L^m(B^m)}\ \|\nabla P\|_{L^{qm/m-q}(B^m)} \\
&\leq 2\varepsilon_1\ \|\nabla P\|_{L^{qm/m-q}(B^m)}
\end{aligned}
\tag{A.9}
$$

Since $P - Id = 0$ on ∂B^m, standard elliptic estimates give

$$
\|\nabla P\|_{L^{qm/m-q}(B^m)} \leq C_m\ \|\Delta P\|_{L^q(B^m)} \quad .
$$

and we finish the argument as in the case $q = m/2$ in order to get (A.5) this completes the proof of lemma A.2. $\square$

We start now the proof of step 2. For any $P_0 \in W^{2,q}(B^m, SO(n))$ we introduce the map F^{P_0} defined as follows

$$F^{P_0} \ : \ W_0^{2,q}(B^m, so(n)) \longrightarrow L^q(B^m, so(n))$$

$$V \ \longrightarrow (P_0 \exp(V))^{-1}\Delta(P_0 \exp(V)) - \Delta(P_0 \exp(V))^{-1} \ P_0 \exp(V)$$

We first prove that the map F^{P_0} is C^1. This comes from the following facts

i) Since $W^{2,q}$ for $q > m/2$ embedds continuously in C^0, the map $V \rightarrow exp\,(V)$ is clearly smooth from $W_0^{2,q}(B^m, so(n))$ into $W^{2,q}(B^m, SO(n))$.

ii) The operator Δ is a smooth linear map from $W^{2,q}(B^m, M_n(\mathbb{R}))$ into $L^q(B^m, M_n(\mathbb{R}))$.

iii) Since again $W^{2,q}$ embedds continuously in L^∞ - $W^{2,q}$ is an algebra - the following map

$$\begin{array}{rcl} \Pi \ : \ W_0^{2,q}(B^m, M_n(\mathbb{R})) \times L^q(B^m, M_n(\mathbb{R})) & \longrightarrow & L^q(B^m, M_n(\mathbb{R})) \\ (A, B) & \longrightarrow & A\,B \end{array}$$

is also smooth.

Observe that for any $\zeta \in W_0^{2,q}(B^m, so(n))$

$$\frac{1}{2} dF_0^{P_0} \cdot \zeta = L^{P_0} \cdot \zeta := \Delta\zeta + [P_0^{-1}\nabla P_0, \nabla\zeta] + [\Omega_0, \zeta] \tag{A.10}$$

where $2\Omega_0 := P_0^{-1} \ \Delta P_0 - \Delta P_0^{-1} \ P_0$ We now establish the following lemma

Lemma A.3 *There exists $\varepsilon_2 > 0$ such that for any $U_0 \in W_0^{2,q}(B^m, so(n))$ satisfying*

$$\|exp(U_0) - Id\|_{W^{2,m/2}} \le \varepsilon_2 \quad , \tag{A.11}$$

then $dF_0^{P_0}$ is invertible between $W_0^{2,q}(B^m, so(n))$ and $L^q(B^m, so(n))$. □

Proof of Lemma A.3. We aim to prove that there exists $\varepsilon > 0$ such that whenever $\|exp(U_0) - Id\|_{W^{2,m/2}} \le \varepsilon$, there exists $C_{U_0} > 0$, such that for any $\omega \in L^q(B^m, so(n))$ there exists a unique $\zeta \in W_0^{2,q}(B^m, so(n))$ for which

$$\left\{ \begin{array}{l} L^{P_0}\zeta = \omega \quad , \\ \\ \|\zeta\|_{W_0^{2,q}(B^m, so(n))} \le C_0 \ \|\omega\|_{L^q(B^m, so(n))} \end{array} \right. \tag{A.12}$$

Since $W_0^{2,q}(B^m)$ embedds continuously in $L^\infty(B^m)$ it is clear that $[\Omega_0,\zeta]\in L^q$. Moreover

$$\|[P_0^{-1}\nabla P_0,\nabla\zeta]\|_{L^q}\le 2\|\nabla P_0\|_{L^m}\ \|\nabla\zeta\|_{L^{qm/m-q}}\le C\|P_0-id\|_{W_0^{2,m/2}}\ \|\zeta\|_{W_0^{2,q}}\quad .$$

Hence L^{P_0} is sending continuously $W_0^{2,q}(B^m,so(n))$ into $L^q(B^m,so(n))$. Since $m>q>m/2$ we have that $4/m-1/q>2/m$. We can hence choose r such that $4/m-1/q>1/r>2/m$ (for instance $1/r:=3/m-1/2q$). For such a r we have

$$\|[\Omega_0,\zeta]\|_{L^r}\le 2\ \|\Omega_0\|_{L^{m/2}}\ \|\zeta\|_{L^{rm/m-2r}}\le C_q\ \|\Omega_0\|_{L^{m/2}}\ \|\zeta\|_{W_0^{2,r}}\tag{A.13}$$

and

$$\|[P_0^{-1}\nabla P_0,\nabla\zeta]\|_{L^r}\le 2\|\nabla P_0\|_{L^m}\ \|\nabla\zeta\|_{L^{rm/m-r}}\le C\|P_0-id\|_{W_0^{2,m/2}}\ \|\zeta\|_{W_0^{2,r}}\quad .\tag{A.14}$$

Hence using standard elliptic theory, we obtain that for $\|P_0-id\|_{W_0^{2,m/2}}$ small enough, for any $\omega\in L^r(B^m,M_n(\mathbb{R}))$ there exists a unique solution ζ in $W_0^{2,r}(B^m,M_n(\mathbb{R})$ of $L_{P_0}\zeta=\omega$. Assume moreover that ω takes values into $so(n)$ then we have, since $(P_0^{-1}\ \nabla P_0)^t=-P_0^{-1}\ \nabla P_0$ and $\Omega_0^t=-\Omega_0$,

$$L^{P_0}\cdot(\zeta+\zeta^t)=0\quad .$$

The uniqueness result we just proved gives then $\zeta^t=-\zeta$.

Hence we have established that

$$\begin{array}{rcl} L^{P_0}\ :\ W_0^{2,r}(B^m,so(n)) & \longrightarrow & L^r(B^m,so(n))\\[2ex] \zeta & \longrightarrow & \Delta\zeta+[P_0^{-1}\nabla P_0,\nabla\zeta]+[\Omega_0,\zeta]\end{array}$$

is an isomorphism.

Let $1/s:=1/q+1/r-2/m$. Our assumption on r gives $1/s<2/m$. Denoting Δ_0^{-1} the Inverse of the laplacian on B^m for the zero Dirichlet boundary data, we have

$$\|\Delta_0^{-1}([\Omega_0,\zeta])\|_\infty\le C\ \|[\Omega_0,\zeta]\|_{L^s}\le C\ \|\Omega_0\|_{L^q}\ \|\zeta\|_{L^{mr/m-2r}}\le C\ \|\Omega_0\|_{L^q}\ \|\zeta\|_{W_0^{2,r}}\quad .$$

Moreover

$$\begin{aligned}\|\Delta_0^{-1}([P_0^{-1}\nabla P_0,\nabla\zeta])\|_\infty &\le C\ \|[P_0^{-1}\nabla P_0,\nabla\zeta]\|_{L^s}\\ &\le C\ \|\nabla P_0\|_{L^{qm/m-q}}\ \|\nabla\zeta\|_{L^{rm/m-r}}\\ &\le C\ \|P_0-Id\|_{W_0^{2,q}}\ \|\zeta\|_{W_0^{2,r}}\end{aligned}$$

From the two previous estimates we deduce that for any $\omega \in L^q(B^m, so(n))$, the unique solution $\zeta \in W_0^{2,r}(B^m, so(n))$ of $L_{P_0}\zeta = \omega$ is in fact in L^∞ and the following estimate holds

$$
\begin{aligned}
\|\zeta\|_{L^\infty(B^m)} &\le C_q\ \|P_0 - Id\|_{W_0^{2,q}(B^m)}\ \|\zeta\|_{W_0^{2,r}(B^m)} + C_q\ \|\omega\|_{L^q(B^m)} \\
&\le C_q\ \left[1 + \|P_0 - Id\|_{W_0^{2,q}(B^m)}\right]\ \|\omega\|_{L^q(B^m)}
\end{aligned}
\tag{A.15}
$$

We then obtain that

$$
\|[\Omega_0, \zeta]\|_{L^q(B^m)} \le C_q\ \|\Delta P_0\|_{L^q}\ \left[1 + \|P_0 - Id\|_{W_0^{2,q}(B^m)}\right]\ \|\omega\|_{L^q(B^m)} \tag{A.16}
$$

Observe that inequality (A.14) is valid for any $r < m$ and hence in particular it holds for q : we have for any ξ in $W_0^{2,q}$

$$
\|[P_0^{-1}\nabla P_0, \nabla\xi]\|_{L^q} \le 2\|\nabla P_0\|_{L^m}\ \|\nabla\zeta\|_{L^{qm/m-q}} \le C\|P_0 - id\|_{W_0^{2,m/2}}\ \|\zeta\|_{W_0^{2,q}} \quad .
$$

Hence for $\|P_0 - id\|_{W_0^{2,m/2}}$ having been chosen small enough, by standard elliptic estimates, the following map

$$
\begin{aligned}
H^{P_0}\ :\ W_0^{2,q}(B^m, so(n)) &\longrightarrow L^q(B^m, so(n)) \\
\xi &\longrightarrow \Delta\xi + [P_0^{-1}\nabla P_0, \nabla\xi]
\end{aligned}
$$

is an isomorphism. Let $\xi := (H^{P_0})^{-1}[\omega - [\Omega, \zeta]]$. The argumentation we followed above for L^{P_0} applies to H^{P_0} in order to show that it realizes an isomorphism between $W_0^{2,r}(B^m, so(n))$ and $L^r(B^m, so(n))$. Hence since $H^{P_0}(\xi - \zeta) = 0$ we deduce that $\zeta = \xi$ and hence we have proved that $\zeta \in W_0^{2,q}(B^m, so(n))$ and the following estimate holds :

$$
\|\zeta\|_{W_0^{2,q}(B^m, so(n))} \le C_q\ \left[1 + \|\Delta P_0\|_{L^q}\ \left[1 + \|P_0 - Id\|_{W_0^{2,q}(B^m)}\right]\right]\ \|\omega\|_{L^q(B^m)}
$$

We have then established (A.12) and we have proved lemma A.3. $\Box$.

End of the proof of step 2. We fix an ε_0 smaller than the ε_1 of lemma A.2 and smaller than the ε_2 of lemma A.3. Consider also C equal to C_1 given by lemma A.2. Let $\Omega_0 \in \mathcal{V}^q_{\varepsilon_0,C}$. According to Lemma A.3 we can apply the local inversion theorem and then there exists a neighborhood of Ω_0 in $L^q(B^m, so(n))$ such that for any Ω in this neighborhood there exists $P \in W^{2,q}(B^m, SO(n))$ such that (A.1) holds. In particular this is true for any Ω in the intersection of this neighborhood with $\mathcal{U}^q_{\varepsilon_0}$. Since $\varepsilon_0 \le \varepsilon_1$, Lemma A.2 applies and we deduce that all these Ω belong to $\mathcal{V}_{\varepsilon_0,C}$. Hence we have proved that there exists a neighborhood of Ω_0 whose intersection with $\mathcal{U}^q_{\varepsilon_0}$ is included in $\mathcal{V}^q_{\varepsilon_0,C}$. This shows that for this choice of ε_0 and C $\mathcal{V}^q_{\varepsilon_0,C}$ is open in $\mathcal{U}^q_{\varepsilon_0}$. We have the proved step 2 and we deduce lemma A.1.$\Box$

Lemma A.4 *Let $m > q > m/2$ and $r \in (1,+\infty)$. There exists $\delta_0 > 0$ and $C > 0$ such that for any $\Omega \in L^q(B^m, M_n(\mathbb{R}))$ and $A \in W^{2,q}(B^m, Gl_n(\mathbb{R}))$ with $A^{-1} \in L^\infty(B^m)$ satisfying*

$$\|\Omega\|_{L^{m/2}(B^m)} + \|A^{-1}\nabla A\|_{L^m(B^m)} \le \delta \quad , \tag{A.17}$$

and solving

$$\begin{cases} \Delta A + A\,\Omega = 0 & \text{in } B^m \\ \\ A = I_n & \text{on } \partial B^m \end{cases} \tag{A.18}$$

then the following inequalities hold

i)

$$\|A^{-1}\nabla A\|_{L^m(B^m)} \le C\ \|\Omega\|_{L^{m/2}(B^m)} \quad , \tag{A.19}$$

ii)

$$\|A^{-1} - I_n\|_{L^r(B^m)} \le C\ \|\Omega\|_{L^{m/2}(B^m)} \quad , \tag{A.20}$$

iii)

$$\|A^{-1}\nabla A\|_{L^q(B^m)} \le C\ \|\Omega\|_{L^q(B^m)} \quad , \tag{A.21}$$

iv)

$$\|A^{-1} - I_n\|_{L^\infty(B^m)} \le C\ \|\Omega\|_{L^q(B^m)} \quad . \tag{A.22}$$

□

Proof of lemma A.4. Equation (A.18) is equivalent to the following elliptic system satisfied by $A^{-1}dA$

$$\begin{cases} d^*(A^{-1}dA) = -\Omega - A^{-1}dA\cdot A^{-1}dA & \text{in } B^m \\ \\ d(A^{-1}dA) = A^{-1}dA \wedge A^{-1}dA & \text{in } B^m \\ \\ \iota_{\partial B^m} A^{-1}dA = 0 \quad , \end{cases} \tag{A.23}$$

where $\iota_{\partial B^m}$ denotes the canonical inclusion of ∂B^m in $\mathbb{R}^m$. Classical elliptic estimates give the existence of a constant C_m, independent of A, such that

$$\|A^{-1}dA\|_{L^m(B^m)} \le C_m\ \|\Omega\|_{L^{m/2}(B^m)} + C_m\ \|A^{-1}dA\|^2_{L^m(B^m)} \quad . \tag{A.24}$$

Choosing then $\delta > 0$ small enough in such a way that $C_m\delta < 1/2$, gives (A.19) with $C = 2\,C_m$. The system (A.18) implies moreover (since we know that A^{-1} is a well defined measurable, $L^\infty-$bounded, Matrix valued function)

$$\left\{\begin{array}{l}\Delta(A^{-1}-I_n)=\left[\Omega+2(A^{-1}dA)^2\right](A^{-1}-I)+\left[\Omega+2(A^{-1}dA)^2\right]\qquad\text{in } B^m\\[5mm] A^{-1}-I_n=0\qquad\quad\text{on }\partial B^m\end{array}\right.\tag{A.25}$$

Let $1<r<+\infty$ chosen in such a way that $r^{-1}<2\,m^{-1}-q^{-1}$. Since $W^{1,m}_0(B^m)$ embeds in every L^p and thus in $L^r(B^m)$, we have

$$\begin{array}{l}\|A^{-1}-I_n\|_{L^r(B^m)}\le C_r\ \|\left[\Omega+2(A^{-1}dA)^2\right](A^{-1}-I)\|_{L^{rm/(m+2r)}}\\[5mm] \quad+C_r\ \|\left[\Omega+2(A^{-1}dA)^2\right]\|_{L^{m/2}}\quad .\end{array}\tag{A.26}$$

which implies that

$$\begin{array}{l}\|A^{-1}-I_n\|_{L^r(B^m)}\le C_r\ \|\left[\Omega+2(A^{-1}dA)^2\right]\|_{L^{m/2}}\ \|A^{-1}-I_n\|_{L^r(B^m)}\\[5mm] \quad+C_r\ \|\left[\Omega+2(A^{-1}dA)^2\right]\|_{L^{m/2}}\quad .\end{array}\tag{A.27}$$

Hence from (A.19) that we just proved, for δ being chosen small enough - once r is fixed - in such a way that $C_r\ \|[\Omega+2(A^{-1}dA)^2]\|_{L^{m/2}}<1/2$ we obtain

$$\|A^{-1}-I_n\|_{L^r(B^m)}\le 2\,C_r\ \|\left[\Omega+2(A^{-1}dA)^2\right]\|_{L^{m/2}}\le C_{m,r}\ \|\Omega\|_{L^{m/2}(B^m)}\quad .\tag{A.28}$$

From the elliptic system (A.23) again we have the existence of a constant $C_{q,m}>0$ such that

$$\begin{array}{l}\|A^{-1}\nabla A\|_{L^{\frac{qm}{m-q}}(B^m)}\le C_{q,m}\ \|\Omega\|_q+C_{q,m}\ \|(A^{-1}\nabla A)^2\|_q\quad .\\[5mm] \quad\le C_{q,m}\ \left[\|\Omega\|_q+\|A^{-1}\nabla A\|_m\ \|A^{-1}\nabla A\|_{L^{\frac{qm}{m-q}}(B^m)}\right]\quad .\end{array}\tag{A.29}$$

For δ chosen small enough in such a way that

$$C_{q,m}\ \|A^{-1}\nabla A\|_m<2\,C_{q,m}\ C_m\,\delta<1/2\quad ,$$

we obtain that

$$\|A^{-1}\nabla A\|_{L^{\frac{qm}{m-q}}(B^m)}\le C_{q,m}\ \|\Omega\|_q\quad .\tag{A.30}$$

We write $\nabla(A^{-1}-I_n)=\nabla A^{-1}\ A\ A^{-1}=-A^{-1}\nabla A\ (A^{-1}-I_n)-A^{-1}\nabla A$. Let $s^{-1}=r^{-1}+q^{-1}-m^{-1}$, we have $s>m$ and hence

$$
\begin{aligned}
\|A^{-1}-I_n\|_{L^\infty(B^m)} &\le C_{r,q,m}\ \|\nabla(A^{-1}-I_n)\|_s\\
&\le C_{r,q,m}\ \|A^{-1}\nabla A\|_{\frac{qm}{m-q}}\ \|A^{-1}-I_n\|_r+C_{r,q,m}\ \|A^{-1}\nabla A\|_s\\
&\le C_{r,q,m}\ \|A^{-1}\nabla A\|_{\frac{qm}{m-q}}\ \|A^{-1}-I_n\|_r+C_{r,q,m}\ \|A^{-1}\nabla A\|_{\frac{qm}{m-q}}\quad .
\end{aligned}
\tag{A.31}
$$

Combining (A.28), (A.30) and (A.31) we obtain (A.22) and lemma A.4 is proved. □

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