Variational Principles for immersed Surfaces with L^2-bounded Second Fundamental Form.

Tristan Rivière*

Abstract : *In this work we present new tools for studying the variations of the Willmore functional of immersed surfaces into $\mathbb{R}^m$. This approach gives for instance a new proof of the existence of a Willmore minimizing embedding of an arbitrary closed surface in arbitrary codimension. We explain how the same approach can solve constraint minimization problems for the Willmore functional. We show in particular that, for a given closed surface and a given conformal class for this surface, there is an immersion in $\mathbb{R}^m$, away possibly from isolated branched points, which minimizes the Willmore energy among all possible Lipschitz immersions in $\mathbb{R}^m$ having an L^2-bounded second fundamental form and realizing this conformal class. This branched immersion is either a smooth Conformal Willmore branched immersion or an isothermic branched immersion. We show that branched points do not exist whenever the minimal energy in the conformal class is less than 8π and that these immersions extend to smooth conformal Willmore embeddings or global isothermic embeddings of the surface in that case. Finally, as a by-product of our analysis, we establish that inside a compact subspace of the moduli space the following holds : weak limit of Palais Smale Willmore sequences are Conformal Willmore, that weak limits of Palais Smale sequences of Conformal Willmore are either Conformal Willmore or Global Isothermic and finally we observe also that weakly converging Palais Smale sequences of Global Isothermic Immersions are Global Isothermic. The analysis developed along the paper - in particular these last results - opens the door to the possibility of constructing new critical saddle points of the Willmore functional without or with constraints using min max methods.*

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I   Introduction

The goal of the present paper is to present a suitable framework to proceed to the calculus of variation of the Willmore functional for immersions.

Let $\vec{\Phi}$ be a smooth immersion (rank $d\vec{\Phi}$ is equal to two at every point) from a closed oriented smooth 2-manifold Σ into an euclidian space $\mathbb{R}^m$. The *first fundamental form $g_{\vec{\Phi}}$* defined by this immersion on Σ is the pull-back by $\vec{\Phi}$ of the metric induced by the restriction of the canonical metric of $\mathbb{R}^m$, $g_{\mathbb{R}^m}$, to the tangent planes $\vec{\Phi}_* T\Sigma$ of the immersed surface :

$$g_{\vec{\Phi}} := \vec{\Phi}^* g_{\mathbb{R}^m} \qquad .$$

If there is no ambiguity we simply write g instead of $g_{\vec{\Phi}}$. We shall denote by $\vec{n}_{\vec{\Phi}}$ the *Gauss map* of the immersion $\vec{\Phi}$ which to a point $p \in \Sigma$ assigns the oriented orthonormal $(m-2)-$plane to the tangent plane $\vec{\Phi}_* T_p \Sigma$ of the immersed surface at $\vec{\Phi}(p)$. $\vec{n}_{\vec{\Phi}}$ will be seen as a map into the Grassmanian $\tilde{G}_{m-2}(\mathbb{R}^m)$ (or equivalently $\tilde{G}_2(\mathbb{R}^m)$) of oriented $m-2-$planes (resp. $2-$planes) of $\mathbb{R}^m$. $\vec{n}_{\vec{\Phi}}$ is also a map into the unit

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*Department of Mathematics, ETH Zentrum, CH-8093 Zürich, Switzerland.

simple [1] 2-vectors in $\mathbb{R}^m : \vec{n}_{\vec{\Phi}} \in \wedge^2 \mathbb{R}^m$. We also denote by $\pi_{\vec{n}_{\vec{\Phi}}}$ the orthonormal projections of vectors in $\mathbb{R}^m$ onto the $m-2$-plane given by $\vec{n}_{\vec{\Phi}}$. With these notations the *second fundamental form*

$$\forall X, Y \in T_p \Sigma \qquad \vec{\mathbb{I}}_p(X,Y) := \pi_{\vec{n}_{\vec{\Phi}}} d^2 \vec{\Phi}(X,Y)$$

[2] The *mean curvature vector* of the immersion at p is given by

$$\vec{H} := \frac{1}{2} tr_g(\vec{\mathbb{I}}) = \frac{1}{2} \left[\vec{\mathbb{I}}(\varepsilon_1, \varepsilon_1) + \vec{\mathbb{I}}(\varepsilon_2, \varepsilon_2) \right] \quad ,$$

where $(\varepsilon_1, \varepsilon_2)$ is an orthonormal basis of $T_p \Sigma$ for the metric $g_{\vec{\Phi}}$.

In the present paper we are mainly interested with the Lagrangian given by the L^2 norm of the second fundamental form :

$$E(\vec{\Phi}) := \int_\Sigma |\vec{\mathbb{I}}|_g^2 \, dvol_g \quad ,$$

where $dvol_g$ is the volume form induced by the metric $g_{\vec{\Phi}}$. An elementary computation gives

$$E(\vec{\Phi}) := \int_\Sigma |\vec{\mathbb{I}}|_g^2 \, dvol_g = \int_\Sigma |d\vec{n}_{\vec{\Phi}}|_g^2 \, dvol_g \quad .$$

This energy E can be hence seen as being the *Dirichlet Energy* of the Gauss map $\vec{n}_{\vec{\Phi}}$ with respect to the induced metric $g_{\vec{\Phi}}$. The Gauss Bonnet theorem implies that

$$E(\vec{\Phi}) := \int_\Sigma |\vec{\mathbb{I}}|_g^2 \, dvol_g = 4 \int_\Sigma |\vec{H}|^2 \, dvol_g - 4\pi \, \chi(\Sigma) \quad , \tag{I.1}$$

where $\chi(\Sigma)$ is the *Euler characteristic* of the surface Σ. The energy

$$W(\vec{\Phi}) := \int_\Sigma |\vec{H}|^2 \, dvol_g \quad ,$$

is the so called *Willmore energy* and has been extensively studied since the early 20th century due in one hand to it's rich mathematical signification but also to it's importance in other area of science (in general relativity, mechanics, biology, optics...etc). Probably the main property it satisfies which makes this lagrangian so universal is the conformal invariance : For any conformal diffeomorphism $\vec{U}$ of $\mathbb{R}^m$ one has (see [Bla])

$$W(\vec{\Phi}) = W(\vec{U} \circ \vec{\Phi}) \quad . \tag{I.2}$$

For a fixed surface Σ , because of (I.1) studying the variations of the L^2-norm of the second fundamental form or the variations of Willmore energy is identical.

Since the lower bound to $W(\vec{\Phi})$ among all possible immersions of closed surfaces is non zero and equal to 4π (see for instance [Wi]) it is natural to look at the existence of optimal immersions which minimize W for a given surface Σ. When Σ is a sphere it is well known that $W(\vec{\Phi})$ achieves it's minimal value 4π for the standard unit S^2 in $\mathbb{R}^3 \subset \mathbb{R}^m$ and only for this submanifold. When Σ is a genus 1 surfaces the existence of a smooth immersion into $\mathbb{R}^m$ minimizing W was established by L.Simon in [Si]. It has been conjectured by T.J.Willmore that the minimizing configuration should be achieved by the torus of revolution in $\mathbb{R}^3$ obtained by rotating around the $z-$axis the vertical circle included in the Oxz plane, of center $(\sqrt{2}, 0, 0)$ and radius 1 and the minimal energy would then be $2\pi^2$. This conjecture is still open at

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[1] $\vec{n}_{\vec{\Phi}} \wedge \vec{n}_{\vec{\Phi}} = 0$.

[2] In order to define $d^2 \vec{\Phi}(X,Y)$ one has to extend locally the vector X or Y by a vectorfield but it is not difficult to check that $\pi_{\vec{n}_{\vec{\Phi}}} d^2 \vec{\Phi}(X,Y)$ is independent of this extension.

this stage. The existence result of L.Simon has been extended to surfaces of arbitrary genus by M.Bauer and E.Kuwert in [BK] : they proved that for an arbitrary given closed oriented surface Σ there is an immersion into $\mathbb{R}^m$ that minimizes the Willmore energy among all smooth immersions of that surface. The result of Bauer and Kuwert was using the result of L.Simon whose proof is quite involved. One of the characteristic of this proof is not to work directly with the immersions $\vec{\Phi}$ but mostly instead with its image $\vec{\Phi}(\Sigma)$ [3] ..

In the present paper we first present a new proof of Simon-Bauer-Kuwert's result that will based on the analysis of the immersions themselves . This new proof will be "transposable" to the minimization of the Willmore functional under various constraints, as it arises in several applications, (prescribed effective volume, prescribed conformal class...etc) without having to change the main lines of the proof. Moreover the arguments and tools that we will develop in this work should be sufficiently generic in order to generate new critical points of the Willmore functional under various constraints by applying fundamental principle of the calculus of variation such as the mountain pass lemma...etc as we shall present it in a forthcoming work [Ri3].

The first difficulty encountered while working with immersions $\vec{\Phi}$ instead of working with their image $\vec{\Phi}(\Sigma)$ is the huge invariance group of the functional : the space of diffeomorphisms of Σ, Diff(Σ). Taking for instance a minimizing sequence $\vec{\Phi}_k$ of the Willmore functional (without or with constraints) one can always compose $\vec{\Phi}_k$ with diffeomorphisms that makes the sequence degenerate completely and not reaching an immersion at all ! There is then a "choice of gauge" to be made. By pulling back the standard metric $g_{\mathbb{R}^m}$ of $\mathbb{R}^m$ onto Σ, $\vec{\Phi}$ defines then a metric $g_{\vec{\Phi}}$ on Σ and hence a conformal structure on Σ. There exists then a constant scalar curvature metric h on Σ and a conformal diffeomorphism Ψ from (Σ, h) into $(\Sigma, g_{\vec{\Phi}})$ such that the immersion $\vec{\Phi} \circ \Psi$ is conformal. The space of constant scalar curvature metrics on Σ identifies (modulo dilations) to the space of conformal structures on Σ and hence is finite dimensional see [Jo]. We have then broken the "gauge degeneracy" by replacing $\vec{\Phi}$ by $\vec{\Phi} \circ \Psi$ which satisfies the *Coulomb gauge condition* :

$$
\begin{cases}
div(\vec{e}_1, \nabla \vec{e}_2) = 0 \qquad \text{where} \qquad \vec{e}_j := e^{-\lambda} \, \partial_{x_j}(\vec{\Phi} \circ \Psi) \quad , \\
\\
\text{and} \qquad e^{\lambda} := |\partial_{x_1}(\vec{\Phi} \circ \Psi)| = |\partial_{x_2}(\vec{\Phi} \circ \Psi)|
\end{cases}
$$

and the operator div and ∇ are the standard operators : $divX = \partial_{x_1} X_1 + \partial_{x_2} X_2$ and $\nabla \cdot := (\partial_{x_1} \cdot, \partial_{x_2} \cdot)$ taken in arbitrary complex coordinates $z = x_1 + ix_2$ with respect to the conformal structure given by (Σ, h).

At this stage however the possible perplexity of the reader regarding this choice of gauge is totally justified because what we have gained by fixing the gauge that way is not clear at all at this stage. Indeed, looking again at a minimizing sequence $\vec{\Phi}_k$ of the Willmore functional (without or with constraints) and composing by Ψ_k in order to have a conformal immersion, first one does not have a-priori a control of the conformal class defined by Φ_k : (Σ, h_k) may degenerate to the boundary of the moduli space. A first task in our proof is to exclude this eventuality. More seriously, as k goes to infinity we have a-priori no control at all of the conformal factor e^{λ_k} that could either go to $+\infty$ or 0 at some points and then we would be out of the class of immersions at the limit. This cannot be excluded easily. The problem is that the control of the L^2 norm of the second fundamental form does not provide a global pointwise control of the conformal factor e^{λ} - counter-examples are easy to manufacture. This is just critical : an $L^{2+\varepsilon}$−control of the second fundamental form would have done it. However, below a certain threshold this control exists. This phenomenon has been discovered in a series of works by T.Toro [To1] [To2], S.Müller-V.Sverak [MS] and F.Hélein [Hel]. Precisely one has

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[3]Or more precisely with the rectifiable current $\vec{\Phi}_*[\Sigma]$: the push forward by $\vec{\Phi}$ of the integration current over Σ

Theorem I.1 [Control of local isothermal coordinates] *Let $\vec{\Phi}$ be a conformal immersion of the disc D^2 such that*

$$\int_{D^2} |\nabla \vec{n}_{\vec{\Phi}}|^2 < 8\pi/3 \qquad and \qquad M(\vec{\Phi}_*[D^2]) = \int_{D_2} e^{2\lambda} \, dx_1 \, dx_2 < +\infty \qquad (\text{I.3})$$

where $M(\vec{\Phi}_*[D^2])$ is the mass [4] of the current $\vec{\Phi}_*[D^2]$. Then for any $0 < \rho < 1$ there exists a constant C_ρ independent of $\vec{\Phi}$ such that

$$\sup_{p \in D_\rho^2} e^{\lambda}(p) \le C_\rho \, \left[M(\vec{\Phi}_*[D^2]) \right]^{1/2} \, \exp\left(\int_{D^2} |\nabla \vec{n}_{\vec{\Phi}}|^2 \right) \quad . \qquad (\text{I.4})$$

Moreover, for two given distinct points p_1 and p_2 in the interior of D^2 and again for $0 < \rho < 1$ there exists a constant $C > 0$ independent of $\vec{\Phi}$ such that

$$\begin{aligned}
\|\lambda\|_{L^\infty(D_\rho^2)} \le \quad & C_\rho \, \int_{D^2} |\nabla \vec{n}_{\vec{\Phi}}|^2 + C_\rho \, \log |\vec{\Phi}(p_1) - \vec{\Phi}(p_2)|^{-1} \\
& + C_\rho \, \log M(\vec{\Phi}_*[D^2]) \quad .
\end{aligned} \qquad (\text{I.5})$$

$\square$

Remark I.1 *The existence of two distinct points p_1 and p_2 such that, in the minimization process, $|\vec{\Phi}_k(p_1) - \vec{\Phi}_k(p_2)|$ is not converging to zero - ie the maintenance of the **non-collapsing condition** - will be obtained -see the 3-point normalization lemma A.4 - by the composition with an ad-hoc Möbius transformation of $\mathbb{R}^m$ which does not affect the Willmore energy - see (I.2) - and hence the minimizing nature of the sequence.* $\square$

This Theorem is only implicit in the above mentioned works and therefore we give a proof of it in section III. The main ingredients for proving theorem I.1 are the following. First, under the energy assumption (I.3), one constructs a controlled energy orthonormal frame "lifting" the Gauss map $\vec{n}_{\vec{\Phi}}$. Precisely one has

$$\begin{aligned}
& \forall \vec{n} \in W^{1,2}(D^2, \tilde{G}_2(\mathbb{R}^m)) \quad \text{s.t.} \quad \int_{D^2} |\nabla \vec{n}|^2 < 8\pi/3 \quad , \\
& \exists (\vec{f}_1, \vec{f}_2) \in (S^{m-1})^2 \quad \text{s.t.} \quad \vec{f}_1 \cdot \vec{f}_2 = 0 \qquad , \qquad \vec{n}_{\vec{\Phi}} = \vec{f}_1 \wedge \vec{f}_2 \quad , \\
& and \qquad \int_{D^2} |\nabla \vec{e}_1|^2 + |\nabla \vec{e}_2|^2 \le C \int_{D^2} |\nabla \vec{n}|^2 \quad .
\end{aligned} \qquad (\text{I.6})$$

where $C > 0$ is some universall constant. Next one observes that the logarithm of the conformal factor ,λ, satisfies the following equation[5] :

$$-\Delta \lambda = \partial_{x_1} \vec{f}_1 \cdot \partial_{x_2} \vec{f}_2 - \partial_{x_2} \vec{f}_1 \cdot \partial_{x_1} \vec{f}_2 \quad . \qquad (\text{I.7})$$

The main estimate to exploit this equation and its Jacobian structure in the r.hs. is given by the following theorem of H.Wente which has shown to play a central role in the analysis of 2-dimensional conformally invariant problems (see [Ri1]).

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[4] $M(\vec{\Phi}_*[D^2]) := \sup\{\int_{D^2} \vec{\Phi}^* \omega \; ; \; \|\omega\|_\infty \le 1\}$

[5] In fact equation (I.7) is satisfied by any such a lifting and in particular by $(\vec{e}_1, \vec{e}_2) = e^{-\lambda}(\partial_{x_1} \vec{\Phi}, \partial_{x_2} \vec{\Phi})$. We have indeed $\partial_{x_1} \vec{e}_1 \cdot \partial_{x_2} \vec{e}_2 - \partial_{x_2} \vec{e}_1 \cdot \partial_{x_1} \vec{e}_2 = \partial_{x_1} \vec{f}_1 \cdot \partial_{x_2} \vec{f}_2 - \partial_{x_2} \vec{f}_1 \cdot \partial_{x_1} \vec{f}_2$ however in the present equation the advantage of $(\vec{f}_1, \vec{f}_2)$ over $(\vec{e}_1, \vec{e}_2)$ comes from the fact that we control it's $W^{1,2}-$energy by the L^2 norm of the second fundamental form which is not the case a-priori for $(\vec{e}_1, \vec{e}_2)$.

Theorem I.2 *[We]* [**Regularity by compensation**] *Let a and b be two functions in $W^{1,2}(D^2)$ and φ be the solution to the following equation*

$$\begin{cases} -\Delta\varphi = \partial_{x_1} a \, \partial_{x_2} b - \partial_{x_2} a \, \partial_{x_1} b & in \ D^2 \\[2mm] \varphi = 0 & on \ \partial D^2 \end{cases} \tag{I.8}$$

then the following estimates holds

$$\|\varphi\|_{L^\infty(D^2)} + \|\nabla\varphi\|_{L^2(D^2)} \le C \, \|\nabla a\|_{L^2(D^2)} \, \|\nabla b\|_{L^2(D^2)} \quad . \tag{I.9}$$

where C is some universal constant. $\qquad\square$

λ is then the sum of solutions to equations of the form (I.8) and some harmonic rest. Combining this decomposition, Wente theorem, and Harnack inequalities for the harmonic rest are the main arguments in the proof of theorem I.1. The operation of finding a lifting of the Gauss map $\vec{n}_{\vec{\Phi}}$, $(\vec{f}_1, \vec{f}_2)$, whose energy is controlled by the L^2 norm of the second fundamental form (as in (I.6) is the main limitation for having to restrict to energy below $8\pi/3$. This construction was proved in [Hel] lemma 5.1.4. It is not difficult to construct a counter-example to the statement (I.6) when $8\pi/3$ is replaced by any number strictly larger than 8π. F.Hélein conjectured however that $8\pi/3$ should be replaced by 8π and this would make the statement (I.6) necessarily optimal.

The previous discussion explains how, while minimizing the Willmore functional (without or with constraints), the problem of the indeterminacy due to the huge invariance group $\text{Diff}(D^2)$ is locally solved and, as a consequence of theorem I.1, beside possibly at most isolated points where the second fundamental form is concentrating at least $8\pi/3$ energy, the conformal factor cannot degenerate in the minimization process. However the assumption of having a smooth immersion (beside these isolated points) at the limit could be lost a-priori since locally the L^2−norm of the second fundamental form cannot control more than the L^∞ norm of the conformal factor[6]. It is then necessary, following a classical approach in calculus of variations, to "embed" the problem in a weak class of immersions.

Let g_0 be a reference smooth metric on Σ. One defines the Sobolev spaces $W^{k,p}(\Sigma, \mathbb{R}^m)$ of measurable maps from Σ into $\mathbb{R}^m$ in the following way

$$W^{k,p}(\Sigma, \mathbb{R}^m) = \left\{ f \ \text{meas.} \ \Sigma \to \mathbb{R}^m \ \text{s.t.} \ \sum_{l=0}^{k} \int_\Sigma |\nabla^l f|_{g_0}^p \, dvol_{g_0} < +\infty \right\}$$

Since Σ is assumed to be compact it is not difficult to see that this space is independent of the choice we have made of g_0.

First we need to have a weak first fundamental form that is we need $\vec{\Phi}^* g_{\mathbb{R}^m}$ to define an L^∞ metric with a bounded inverse. The last requirement is satisfied if we assume that $\vec{\Phi}$ is in $W^{1,\infty}(\Sigma)$ and if $d\vec{\Phi}$ has maximal rank 2 at every point with some uniform quantitative control of "how far" $d\vec{\Phi}$ is from being degenerate : there exists $c_0 > 0$ s.t.

$$|d\vec{\Phi} \wedge d\vec{\Phi}|_{g_0} \ge c_0 > 0 \quad . \tag{I.10}$$

where $d\vec{\Phi} \wedge d\vec{\Phi}$ is a 2-form on Σ taking values into 2-vectors from $\mathbb{R}^m$ and given in local coordinates by $2 \, \partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi} \, dx \wedge dy$. The condition (I.10) is again independent of the choice of the metric g_0 . For a Lipschitz immersion satisfying (I.10) we can define the Gauss map as being the following measurable map in $L^\infty(\Sigma)$

$$\vec{n}_{\vec{\Phi}} := \star \frac{\partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi}}{|\partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi}|} \quad .$$

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[6]unless our limit is known to satisfy some special equation of course but we will come to that later.

We then introduce the space $\mathcal{E}_\Sigma$ of Lipschitz immersions of Σ with bounded second fundamental form as follows :

$$\mathcal{E}_\Sigma := \left\{ \begin{array}{c} \vec{\Phi} \in W^{1,\infty}(\Sigma, \mathbb{R}^m) \quad \text{s.t. } \vec{\Phi} \text{ satisfies } (I.10) \text{ for some } c_0 \\[3mm] \text{and} \quad \int_\Sigma |d\vec{n}|_g^2 \; dvol_g < +\infty \end{array} \right\} \quad .$$

Any Lipschitz immersion $\vec{\Phi}$ in $\mathcal{E}_\Sigma$ defines a smooth conformal structure on Σ. This comes again from the works of T.Toro [To1] [To2], S.Müller-V.Sverak [MS] and F.Hélein [Hel] :

Theorem I.3 *([To1],[To2],[MS],[Hel] theorem 5.1.1)* [**Existence of local isothermal coordinates**] *Let $\vec{\Phi} \in \mathcal{E}_{D^2}$ satisfying*

$$\int_{D^2} |d\vec{n}_{\vec{\Phi}}|_g^2 \; dvol_g < \frac{8\pi}{3} \quad , \tag{I.11}$$

then there exists a bilipschitz homeomorphism of the disk $\zeta \in W^{1,\infty}(D^2, D^2)$ such that

$$\left\{ \begin{array}{ll} |\partial_x(\vec{\Phi} \circ \zeta)|^2 - |\partial_y(\vec{\Phi} \circ \zeta)|^2 = 0 & \text{in } D^2 \\[3mm] \partial_x(\vec{\Phi} \circ \zeta) \cdot \partial_y(\vec{\Phi} \circ \zeta) = 0 & . \end{array} \right. \tag{I.12}$$

$\square$

Hence for any lipschitz immersion $\vec{\Phi}$ in $\mathcal{E}_\Sigma$ one takes a finite covering of Σ by disks (U_j) such that $\int_{U_j} |d\vec{n}_{\vec{\Phi}}|_g^2 \; dvol_g < 8\pi/3$, one gets bilipschitz homeomorphisms ζ_j for which $\vec{\Phi} \circ \zeta_j$ satisfies (I.12) and hence the transition functions $\zeta_k^{-1} \circ \zeta_j$ are holomorphic. (U_j, ζ_j) defines then a smooth conformal structure on Σ. Let h be a constant scalar curvature associated to this conformal structure and the smooth diffeomorphism Ψ of Σ such that the maps $\zeta_j^{-1} \circ \Psi$ are conformal from (Σ, h) into D^2, then we have that $\vec{\Phi} \circ \Psi$ is a conformal $W^{1,\infty} \cap W^{2,2}$ immersion of (Σ, h). Using theorem I.1 we can construct local isothermal coordinates for $(\Sigma, g_{\vec{\Phi} \circ \Psi})$ <u>with estimates</u> -i.e. satisfying (I.4) and (I.5) - and work with maps in $\mathcal{E}_\Sigma$ like with smooth embeddings.

The next main difficulty encountered while working with the immersion $\vec{\Phi}$ instead of it's image in the minimization process of Willmore equation comes from the Euler Lagrange equation as it has been written originally in the early 20th century in the works of W.Blaschke[Bla], G.Thomsen [Tho] (in codim 1, i.e. $m = 3$) and J. Weiner [Wei] (arbitrary m). A smooth immersion $\vec{\Phi}$ is a critical point to Willmore functional :

$$\forall \vec{\xi} \in C^\infty(\Sigma, \mathbb{R}^m) \qquad \frac{d}{dt} W(\vec{\Phi} + t\vec{\xi})_{t=0} = 0$$

if and only if $\vec{\Phi}$ satisfy the *Willmore equation*

$$\Delta_\perp \vec{H} - 2|\vec{H}|^2\vec{H} + \tilde{A}(\vec{H}) = 0 \,, \tag{I.13}$$

where $\Delta_\perp$ is the negative covariant Laplacian for the connection[7] D in the normal bundle $N_{\vec{\Phi}}(\Sigma)$ derived from the ambient scalar product in $\mathbb{R}^m$ and where $\tilde{A}_p(\vec{L}) = \sum_{i,j} \mathbb{I}_p(\vec{e}_i, \vec{e}_j)\,\mathbb{I}_p(e_i, e_j) \cdot \vec{L}$ for $\vec{L} \in \mathbb{R}^m$.

In [Ri2] we explained why the Euler Lagrange written in the form (I.13) seems not compatible from the lagrangian it is coming from in the sense that $W(\vec{\Phi})$ only controls the L^2−norm of the mean curvature whereas in order to give a distributional meaning to the non linearities in the equation like $|\vec{H}|^2\vec{H}$ one

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[7]Namely, for every section σ of $N_{\vec{\Phi}}(\Sigma)$, one has $D_X\sigma := \pi_{\vec{n}_{\vec{\Phi}}}(\sigma_* X)$

needs more information on the regularity of $\vec{\Phi}$ ($\vec{H} \in L^3$ for instance for this term). [8] One of the main achievement in [Ri2] was to find a new formulation of the Willmore equation as a conservation law which makes sense for immersions in $\mathcal{E}_\Sigma$.

Theorem I.4 *[Ri2]* [**The conservative Willmore Equation**] *The Willmore equation (I.13) is equivalent to*

$$d\left(*_g d\vec{H} - 3 *_g \pi_{\vec{n}_{\vec{\Phi}}}(d\vec{H}) \right) - d \star \left(d\vec{n}_{\vec{\Phi}} \wedge \vec{H} \right) = 0 \,, \tag{I.14}$$

where $*_g$ is the Hodge operator on Σ associated with the induced metric $g_{\vec{\Phi}}$, and $\star$ is the usual Hodge operator on forms.

In particular, a conformal immersion $\vec{\Phi}$ from the flat unit-disc D^2 into $\mathbb{R}^m$ is Willmore if and only if

$$\Delta \vec{H} - 3 \, div\left(\pi_{\vec{n}_{\vec{\Phi}}}(\nabla \vec{H}) \right) + div \star \left(\nabla^\perp \vec{n}_{\vec{\Phi}} \wedge \vec{H} \right) = 0 \,, \tag{I.15}$$

where the operators ∇, $\nabla^\perp$, Δ, and div are understood with respect to the flat metric on D^2. Namely, $\nabla = (\partial_{x_1}, \partial_{x_2})$, $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$, $\Delta \cdot = \partial_{x_1^2}^2 \cdot + \partial_{x_2^2}^2 \cdot$, and $div \, X = \partial_{x_1} X_1 + \partial_{x_2} X_2$.

This conservative form of the Willmore equation and more conservation laws attached to it permits to pass to the limit in local Palais Smale sequences to the Willmore Lagrangian. The following result is established in [BR]

Theorem I.5 *[BR]* [**Convergence of Willmore Palais Smale sequences.**] *Let $\vec{\Phi}_k$ be a sequence of conformal immersions in $\mathcal{E}_{D^2}$. Assume*

i)
$$\int_{D^2} |\nabla \vec{n}_{\vec{\Phi}_k}|^2 < 8\pi/3 \quad ,$$

ii)
$$\vec{\Phi}_k(D^2) \subset B_1^m(0)$$

iii)
$$\exists \, p_1, \, p_2 \in D^2 \quad s.t. \quad \liminf |\vec{\Phi}_k(p_1) - \vec{\Phi}_k(p_2)| > 0 \quad .$$

Under these assumptions if

$$\Delta \vec{H}_k - 3 \, div\left(\pi_{\vec{n}_{\vec{\Phi}_k}}(\nabla \vec{H}_k) \right) + div \star \left(\nabla^\perp \vec{n}_{\vec{\Phi}_k} \wedge \vec{H}_k \right)$$
$$\longrightarrow 0 \qquad in \; (W^{2,2} \cap W^{1,\infty}(D^2))^* \tag{I.16}$$

then[9] there exists a subsequence $\vec{\Phi}_{k'}$ converges weakly in $W^{2,2}_{loc}$ to an analytic immersion $\vec{\Phi}_\infty$ of the disc satisfying the Conformal Willmore equation

$$\Delta \vec{H}_\infty - 3 \, div\left(\pi_{\vec{n}_{\vec{\Phi}_\infty}}(\nabla \vec{H}_\infty) \right) + div \star \left(\nabla^\perp \vec{n}_{\vec{\Phi}_\infty} \wedge \vec{H}_\infty \right) = \Im \left[f(z) \, \vec{H}_{0,\infty} \right] \tag{I.17}$$

where $f(z)$ is an holomorphic function of the disc D^2 and $\vec{H}_{0,\infty}$ is the Weingarten map of the immersion $\vec{\Phi}_\infty$ given by

$$\vec{H}_{0,\infty} := -2^{-1} \, e^{-2\lambda_\infty} \, \left[\partial_x \vec{n}_{\vec{\Phi}_\infty} \, \llcorner \, \partial_x \vec{\Phi}_\infty - \partial_y \vec{n}_{\vec{\Phi}_\infty} \, \llcorner \, \partial_y \vec{\Phi}_\infty + 2i \, \partial_x \vec{n}_{\vec{\Phi}_\infty} \, \llcorner \, \partial_y \vec{\Phi}_\infty \right]$$

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[8] This is a bit like writting the Euler Lagrange of the Dirichlet energy $\int_{D^2} |\nabla u|^2$ - that is the Laplace equation $\Delta u = 0$ - in the form $\Delta u^2 - |\nabla^2 u|^2 = 0$ which requires u to have at least two derivatives in L^2 though the lagrangian gives only a-priori a $W^{1,2}$−control !

[9] $\vec{H}_k$ denotes the mean-curvature vector of the immersion $\vec{\Phi}_k$.

where $e^{\lambda_\infty} = |\partial_x \vec{\Phi}_\infty| = |\partial_y \vec{\Phi}_\infty|$ and $\llcorner$ is the standard contraction operator in $\mathbb{R}^m$ between a multi-vector and a vector. $\qquad\square$

The *Conformal Willmore equation* also called *Constrained Willmore* is obtained by considering critical points of the Willmore functional among immersions realizing a fixed conformal class and assuming the critical point is not *isothermic* - we shall see this notion a bit below - (see [BPP]). $f(z)$ is just the expression in the conformal chart of an holomorphic quadratic differential $q := f(z) \; dz \otimes dz$ of the Riemann surface whose conformal structure is generated by $\vec{\Phi}$.

The analyticity of the weak limits to local *Palais-Smale sequences* described in the theorem I.5 above was obtained by proving that the *Constrained Willmore equation* (I.17) is equivalent to the the existence of $S \in W^{1,2}(D^2, \mathbb{R})$ and $\vec{R} \in W^{1,2}(D^2, \wedge^2 \mathbb{R}^m)$ satisfying

$$\begin{cases} -\Delta S = -\left(\nabla \star \vec{n}_{\vec{\Phi}}\right) \cdot \nabla^\perp \vec{R} \\[2mm] -\Delta \vec{R} = (-1)^{m-1} \star \left(\nabla \vec{n}_{\vec{\Phi}} \bullet \nabla^\perp \vec{R}\right) + \left(\nabla \star \vec{n}_{\vec{\Phi}}\right) \nabla^\perp S \\[2mm] -\Delta \vec{\Phi} = \nabla \vec{R} \bullet \nabla^\perp \vec{\Phi} + \nabla S \; \nabla^\perp \vec{\Phi}. \end{cases} \qquad (I.18)$$

We call this system the *Conservative Conformal Willmore System*. Observe that the right-hand-sides of this system is only made of linear combinations of jacobians of functions which are at least $W^{1,2}$. Using Wente theorem I.2 one easily bootstrap in the equation and obtain the smoothness of $\vec{\Phi}$. This *conservative form* of the *Conformal Willmore equation* is also the key tool for passing to the limit in *Palais-Smale sequences* of Willmore Lagrangian (see [BR]).

We have then understood how to control isothermal coordinates locally and the local convergence of "almost Willmore surfaces" towards analytic conformal Willmore surfaces Moreover these two operations can be extended to the framework of weak immersions : to the space $\mathcal{E}_\Sigma$ of lipschitz immersions with L^2-bounded second fundamental forms. Our task now is to collect these local procedures in order to be able to deal with the minimization procedure itself. To that aim we introduce a distance d - or more precisely a family of equivalent distances on $\mathcal{E}_\Sigma$ - for which the space $(\mathcal{E}_\Sigma, d)$ will be <u>complete</u>. The details of the construction of this distance are given in section II. As explained in section III, under the assumption that there is a minimizing sequence of Willmore functional with conformal class not diverging in the Moduli space of Σ we can make use of *Ekeland Variational Principle* in order to produce sequences of immersions that will be Palais Smale - satisfying (I.16) - in the *controlled isothermal coordinates* constructed in theorem I.1 and that will converge to an analytic immersion in these charts. A consequence of theorem I.1 is that an extraction of subsequence is possible in order to cover any compact part of the surface minus finitely many fixed points by *controlled isothermal charts*. Hence we obtain at the limit an element in $\mathcal{E}_\Sigma$ which is minimizing W and which is analytic away at most from finitely many points. These points are removable due in one hand to the fact that the total Willmore energy of this immersion has to be less than 8π - for minimality reason - and in the other hand to the fact that the with Li-Yau inequality excludes the possibility of having a branched point below 8π (see lemma A.5 and the argument at the end of section III). Finally we exclude the possibility of the conformal class to degenerate to the "boundary of the moduli space" for energetic reasons (this is explained in proposition III.2) and we have not only given a new proof of the following theorem originally due to L.Simon and M.Bauer-E.Kuwert for the space of smooth immersions (see [Si] and [BK]) but we have been able moreover to extend it to the space $\mathcal{E}_\Sigma$ of Lipschitz immersions with L^2-bounded second fundamental form.

Theorem I.6 [Existence of a minimizer of W in $\mathcal{E}_\Sigma$]. *Let m be an arbitrary dimension larger than 2, let Σ be a smooth compact orientable surface without boundary. Then there exists a smooth Willmore embedding of Σ into $\mathbb{R}^m$ minimizing the Willmore energy among all Lipschitz immersions with L^2-bounded second fundamental form (i.e. elements in $\mathcal{E}_\Sigma$).* $\qquad\square$

In section IV of the paper we explain how to adapt the argument for proving the previous theorem to a minimization problem under constraint. Before to state the result we recall a definition.

Definition I.1 [Local isothermic immersions.] *A C^2 immersion of a surface Σ is called local isothermic if, away from the umbilic points, the curvature lines define conformal coordinates.* □

Local isothermic immersions realizes very particular surfaces that have been studied since the XIXth century. A survey on the classical geometry of *local isothermic immersions* as well as their role in *integrable system theory* can be found in [Bu1], [Bu2] and [Toj]. There are some characterization of *local isothermic immersions* that we recall in the 3 following propositions for which we prove in section V.

Proposition I.1 *A C^2 immersion $\vec{\Phi}$ of a surface Σ is local isothermic if and only if there exists an holomorphic quadratic differential q (locally in some complex coordinates $q = f(z)\, dz \otimes dz$) of the Riemann surface $\hat{\Sigma}$, obtained by equiping Σ with the complex structure generated by $\vec{\Phi}$ and by removing the umbilic points of the immersion $\vec{\Phi}$, such that*

$$\langle q, h_0 \rangle_{WP} := \Im\left(f(z)\, \vec{H}_0 \right)\, dz \wedge d\bar{z} \equiv 0 \tag{I.19}$$

where $\vec{H}_0$ is the conjugate of the Weingarten map in local coordinates

$$\vec{H}_0 := -2^{-1}\, e^{-2\lambda}\, \left[\partial_x \vec{n}_{\vec{\Phi}} \llcorner \partial_x \vec{\Phi} - \partial_y \vec{n}_{\vec{\Phi}} \llcorner \partial_y \vec{\Phi} + 2i\, \partial_x \vec{n}_{\vec{\Phi}} \llcorner \partial_y \vec{\Phi} \right]$$

and h_0 is the Weingarten Operator given locally by

$$h_0 := \partial_z \vec{n}_{\vec{\Phi}} \llcorner \partial_z \vec{\Phi}\, dz \otimes dz = 2^{-1}\, e^{2\lambda}\, \overline{\vec{H}_0}\, dz \otimes dz$$

and $\langle \cdot, \cdot \rangle_{WP}$ is the Weil-Peterson pointwise product[10] □

Since an element in $\mathcal{E}_\Sigma$ defines a smooth conformal structure and since it defines an L^2 Weingarten operator one easily extend the notion of isothermic immersions to elements in $\mathcal{E}_\Sigma$. There is a last characterisation of isothermic immersions that we recall and which also permits to define isothermic immersions for an element in $\mathcal{E}_\Sigma$ and which coincide with the previous characterization also for element in $\mathcal{E}_\Sigma$

Proposition I.2 *A Lipschitz immersion $\vec{\Phi}$ in $\mathcal{E}_\Sigma$ is local isothermic if and only if, away from umbilic points, there exists local complex coordinates for the structure defined by $\vec{\Phi}$ and a Lipschitz map $\vec{L}$ on this coordinate domain such that*

$$e^{-2\lambda}\, \partial_z \vec{\Phi} = \partial_{\bar{z}} \vec{L} \quad . \tag{I.20}$$

where $e^{2\lambda} = |\partial_x \vec{\Phi}|^2 = |\partial_y \vec{\Phi}|^2$ and $z = x + iy$. □

This proposition is also proved in section V.

Finally we give a last characterization of local isothermic immersion which we will meet in our proof, this is the following proposition which is also proved in the section V.

Proposition I.3 *A Lipschitz immersion $\vec{\Phi}$ in $\mathcal{E}_\Sigma$ is local isothermic if and only if, away from umbilic points, there exists in local conformal coordinates a Lipschitz map $\vec{L}$ from these coordinates into $\mathbb{R}^m$ such that*

$$\begin{cases} \nabla^\perp \vec{L} \cdot \nabla \vec{\Phi} = 0 \\[2mm] \nabla^\perp \vec{L} \wedge \nabla \vec{\Phi} = 0 \end{cases} \tag{I.21}$$

□

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[10] One verifyies easily that the two form $f(z)\, \vec{H}_0\, dz\, d\bar{z}$ is independent of the local choice of complex coordinates and defines a complex valued 2-form on Σ which is the pointwise Hermitian Weil Peterson product.

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Remark I.2 *The Isothermic equation in the form (I.21) has to be compared with the Conformal Willmore equation written in [BR] : there exists $\vec{L}$ such that*

$$\begin{cases} \nabla^\perp \vec{L} \cdot \nabla \vec{\Phi} = 0 \\[2mm] \nabla^\perp \vec{L} \wedge \nabla \vec{\Phi} = 2\,(-1)^m \nabla^\perp \left(\star(\vec{n}_{\vec{\Phi}} \, \llcorner \, \vec{H}) \right) \llcorner \nabla \vec{\Phi} \quad . \end{cases} \qquad (I.22)$$

There is an apparent strong similarity between the two equations, the isothermic equation (I.21) "corresponds" to the Conformal Willmore equation (I.22) when its right-hand-side of (I.22) is replaced by zero. There is however a major difference between these two equations. Equation (I.22) is <u>*elliptic*</u> *and this can be seen by showing that* $(S, \vec{R})$ *given by*

$$\begin{cases} \nabla S := \nabla \vec{\Phi} \cdot \vec{L} \quad , \\[2mm] \nabla \vec{R} := \nabla \vec{\Phi} \wedge \vec{L} + 2 \nabla^\perp \vec{\Phi} \wedge \vec{H} \end{cases}$$

solves the elliptic system (I.18) (see [Ri2]), whereas (I.21) is <u>*hyperbolic*</u> *(see [Ri1]), which is confirmed by the next remark.* $\qquad \square$

Remark I.3 *It is an interesting question to ask how regular lipschitz local isothermic immersions with L^2- bounded fundamental form can be. They are not necessarily analytic since axially symmetric surfaces are automatically isothermic immersions. Hence an arbitrary axially symmetric surface with L^2-bounded second fundamental form is isothermic but not necessarily C^2 - it is however $C^{1,1/2}$ and it would be interesting either to try to find less regular examples in $\mathcal{E}_\Sigma$ or to prove that isothermic surfaces are necessarily $C^{1,1/2}$.* $\qquad \square$

Finally we need the following more restrictive notion of isothermic immersions that we call *global isothermic*. Because of the lack of regularity of the most elementary examples of isothermic immersions such as rotationally invariant surfaces the definition of *global isothermic immersions* requires a framework that includes non C^2 immersions but which, however, define a <u>smooth</u> complex structure on Σ. The framework of Lipschitz immersions with L^2-bounded second fundamental form $\mathcal{E}_\Sigma$ seems to be the most suitable for that and comes also naturally in the minimization procedure of Willmore surfaces inside a conformal class as we will describe below.

Definition I.2 [Global Isothermic Immersions.] *An immersion $\vec{\Phi}$ in $\mathcal{E}_\Sigma$ is called global isothermic if there exists an holomorphic quadratic differential q of the Riemann surface defined by $\vec{\Phi}$ such that*

$$\langle q, h_0 \rangle_{WP} \equiv 0 \quad .$$

$\qquad \square$

A characteristic of *global isothermic immersions* is to be the degenerate points for the map which to an immersion assigns its conformal class (see lemma V.1 in section V). This is why it is not so surprising to see them appearing as singular points in the minimization process under constrained conformal class - and will appear also as singular points in min-max procedures (see [Ri3]). Adapting the method we used to prove theorem I.6 to the constrained case we establish the following result which is the main result of the present work.

Theorem I.7 [Existence of a minimizer of W in a conformal class]. *Let m be an arbitrary integer larger than 2, let Σ be a smooth compact orientable surface without boundary and c a conformal class for this surface. Then there exists an immersion, away from possibly isolated branched points, minimizing*

the Willmore energy in the sub-space of lipschitz immersions with L^2-bounded second fundamental form, $\mathcal{E}_\Sigma$, realizing the conformal class c. Such a minimal immersion is either a smooth Conformal Willmore immersion of Σ in $\mathbb{R}^m$ satisfying

$$\Delta_\perp \vec{H} - 2|\vec{H}|^2 \vec{H} + \tilde{A}(\vec{H}) = \Im(q, h_0)_{WP}$$

where q is an holomorphic quadratic differential of (Σ, c), h_0 the Weingarten Operator and $(\cdot, \cdot)$ is the pointwise Weil-Peterson product[11] or[12] it is a global isothermic immersion. If the minimal Willmore energy in this conformal class is less than 8π then there the immersion has no branched points and it extends to an embedding of Σ. □

Remark I.4 *The existence of either a **Conformal Willmore surface** or an **Isothermic surfaces** minimizing Willmore energy is already a severe restriction about the nature of the immersion miminizing Willmore energy in it's conformal class. This could however be further explored and it would be interesting to know if there really exists conformal classes with a minimizer which is Isothermic without being Conformal Willmore.* □

Partial existence results of minimizers of the Willmore energy in a given conformal class for the dimensions $m = 3$ and $m = 4$ have been announced in [Scm] and in [Sct]. Moreover, in a recent preprint, [KuLi], the existence of a Minimizer of W within a fixed conformal class in $\mathcal{E}_\Sigma$ is proved. However no information such as the equation that it solves...etc is given about these minimizers in this work. The existence result in [KuLi] is a direct consequence of our normalization lemma A.4 and the lemma III.1. The description of the minimizers we give in section IV and the deduction of the equations they solved and their regularity requires substantial additional work and goes through the use of the new approach of the variations for Willmore functional we aim to present in this work . In a recent preprint [KS] E. Kuwert and R.Schätzle have developed a more thorough analysis than in [KuLi] for the dimensions $m = 3$ and $m = 4$ and under some energy constraint assumption. In this work they were able to rule out the degeneracy case and prove that the conformal Willmore equation is satisfied for any minimizer satisfying this energy constraint.

As a byproduct of our analysis we observe that in a compact subset of the moduli space of the surface Σ the following holds : weak limit of Palais Smale **Willmore** are **Conformal Willmore**, that Weak Limits of Palais Smale sequences of **Conformal Willmore** are either **Conformal Willmore** or **Global Isothermic** and finally we observe also that weakly converging Palais Smale sequences of **Global Isothermic Immersions** are **Global Isothermic**. This notion of global Palais smale will be presented and used in a forthcoming paper [Ri3] to present the Mountain Pass Lemma for Willmore energy in order to produce saddle points for this lagrangian with or without constraints.

Our paper is organized as follows : in section II we define the metric space of Lipschitz conformal immersions with L^2-bounded second fundamental forms. In section III we give a proof of the existence of a minimizer of the Willmore energy for an arbitrary closed surface Σ and an arbitrary codimension (i.e. proof of the theorem I.6). In section IV we show how the proof in section III can be adapted to prove the existence of a minimizer of the Willmore energy in a conformal class (i.e. proof of the theorem I.7). In section V we present isothermic immersion and explain why they are the degenerate points of the conformal class mapping. In the appendix we give the proof of several lemmas and propositions used in the previous sections.

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[11] In local complex coordinates $(q, h_0)_{WP} = e^{-2\lambda} f(z) \vec{H}_0$ where $q = f(z) dz \otimes dz$.
[12] The "or" is not exclusive, there are isothermic immersions which are Conformal Willmore.

II The metric space of lipschitz immersions with L^2-bounded second fundamental form.

II.1 Definitions and notations.

Let $\Phi \in \mathcal{E}_\Sigma$ and Ψ be a Lipschitz homeomorphism from D^2 into Σ. For a.e. $(x,y) \in D^2$ we denote $H(D(\vec{\Phi} \circ \Psi))$ the Hopf differential of $\vec{\Phi} \circ \Psi$:

$$H(\nabla(\vec{\Phi} \circ \Psi)) := \left[|\partial_x(\vec{\Phi} \circ \Psi)|^2 - |\partial_y(\vec{\Phi} \circ \Psi)|^2 \right] - 2\,i\,\partial_x(\vec{\Phi} \circ \Psi) \cdot \partial_y(\vec{\Phi} \circ \Psi)$$

Similarly for a metric $g = \sum_{ij=1}^2 g_{ij}\, dx_i \otimes dx_j$ on the disc D^2 we define

$$H(g) := [g_{11} - g_{22}] - 2\,i\,g_{12} \quad .^{13}$$

Remark that if $H(\nabla(\vec{\Phi} \circ \Psi)) = 0$ then, due to the conformal invariance of the Dirichlet energy one has

$$\int_{\Psi(D^2)} |d\vec{n}_{\vec{\Phi}}|_g^2\, dvol_g = \int_{D^2} |\nabla(\vec{n}_{\vec{\Phi} \circ \Psi})|^2(x,y)\, dx\, dy \quad , \tag{II.23}$$

where ∇ is the usual gradient operator on the disk D^2 for the flat metric : $\nabla := (\partial_x \cdot, \partial_y \cdot)$.

For $\Psi \in W^{1,\infty}(D^2, D^2)$ such that $\log |\nabla \Psi| \in L^\infty(D^2)$ we denote by $Dis(\vec{\Phi} \circ \Psi)(x,y)$ the complex distortion at $(x,y) \in D^2$ given by

$$Dis(\vec{\Phi} \circ \Psi)(x,y) := \frac{H(\nabla(\vec{\Phi} \circ \Psi))}{|\nabla \vec{\Phi} \circ \Psi|^2}(x,y) \quad .$$

Similarly also we define for a metric $g = \sum_{ij=1}^2 g_{ij}\, dx_i \otimes dx_j$ on the disc D^2 the complex distortion of this metric to be

$$Dis(g) := \frac{H(g)}{tr\, g} = \frac{g_{11} - g_{22} - 2\,i\,g_{12}}{g_{11} + g_{22}} \quad .^{14}$$

Observe that

$$|Dis(g)|^2 = 1 - 4\frac{detg}{(tr\, g)^2} < 1 \quad . \tag{II.24}$$

Definition II.3 *An admissible measurable complex structure on Σ is a measurable section[15] J of the endomorphism bundle of Σ satisfying $J^2 = -Id$ and such that*

$$\left\| \ln \frac{|X \wedge JX|_{g_0}}{|X \wedge jX|_{g_0}} \right\|_{L^\infty((T\Sigma)_0)} < +\infty \quad , \tag{II.25}$$

where j is an arbitrary smooth complex structure on Σ, the metric $|\cdot|_{g_0}$ on $T\Sigma \wedge T\Sigma$ is induced from an arbitrary reference metric[16] g_0 on $T\Sigma$ and $(T\Sigma)_0$ is the tangent bundle minus the zero section. □

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[13] Observe that for any immersion $\vec{\Phi}$ of D^2 into $\mathbb{R}^m$ with our notations $H(\nabla \vec{\Phi}) = H(\vec{\Phi}^* g_{\mathbb{R}^m})$.

[14] Once again for any immersion $\vec{\Phi}$ of D^2 into $\mathbb{R}^m$ $Dis(\nabla \vec{\Phi}) = Dis(\vec{\Phi}^* g_{\mathbb{R}^m})$.

[15] i.e. J is a measurable map from Σ into $End(T\Sigma)$ such that for a.e. $p \in \Sigma$ $J(p)$ is an endomorphism of $T_p\Sigma$, the tangent space to Σ at p such that $J(p) \circ J(p)$ is minus the identity map of $T_p\Sigma$ into itself.

[16] Observe that $\ln \frac{|X \wedge JX|_{g_0}}{|X \wedge J_0 X|_{g_0}}$ is independent of the choice of g_0

Remark II.5 *Observe that for $\vec{\Phi} \in \mathcal{E}_\Sigma$ the complex structure induced by $\vec{\Phi}^* g_{\mathbb{R}^m}$ is admissible. Indeed condition(I.10) together with the fact that $\vec{\Phi} \in W^{1,\infty}(\Sigma)$ implies that there exists $C_1 > 0$ such that*

$$\forall p \in \Sigma \quad \forall X \in T_p\Sigma \setminus \{0\} \qquad C_1^{-1} |X|_{g_0} \leq |X|_g \leq C_1 |X|_{g_0} \quad .$$

from which one easily deduce the existence of C_2 such that at every point in Σ

$$C_2^{-1} vol_{g_0} \leq vol_g \leq C_2 vol_{g_0}$$

and we deduce (II.25) by combining the previous equivalences of the metrics and their volume form with the identity

$$X \wedge [g] \cdot X = |d\vec{\Phi} \cdot X|^2 \, vol_g \quad .$$

where $[g]\cdot$ is the action of the complex structure associated to g. $\qquad\square$

Now given a measurable admissible complex structure J on Σ, we define the *complex Distortion with respect to J* of an immersion $\Psi \in W^{1,\infty}(D^2, \Sigma)$ to be the function $Dis^J(\Psi) \in L^\infty(D^2, \mathbb{C})$ given by

$$Dis^J(\Psi) := \frac{|\partial_x \Psi|_g^2 - |\partial_y \Psi|_g^2 - 2\, i \, \langle \partial_x \Psi, \partial_y \Psi \rangle_g}{|\partial_x \Psi|_g^2 + |\partial_y \Psi|_g^2} \quad , \tag{II.26}$$

where g is an arbitrary metric compatible[17] with the complex structure J. It is also not difficult to check that $|Dis^J(\Psi)| < 1$.

II.2 The distance functions d^J.

For any measurable admissible complex structure J and non-negative integer k, we shall consider the following space of quasi-conformal lipschitz parametrization into Σ :

$$\mathcal{Q}_k^J := \left\{ \begin{array}{c} \Psi \in W^{1,\infty}(D^2, \Sigma) \,, \quad \log|\nabla\Psi| \in L^\infty(D^2) \\[2mm] \text{and} \quad \left\| Dis^J(\Psi) \right\|_{L^\infty(D^2)} \leq 1 - 2^{-k} \quad \text{a.e. in } D^2 \end{array} \right\}$$

On $\mathcal{E}_\Sigma \times \mathcal{E}_\Sigma$ we introduce the following non negative function

$$\Delta_k^J(\vec{\Phi}_1, \vec{\Phi}_2) := \sup_{\Psi \in \mathcal{Q}_k^J} \Lambda(\vec{\Phi}_1, \vec{\Phi}_2, \Psi) \quad ,$$

where

$$\Lambda(\vec{\Phi}_1, \vec{\Phi}_2, \Psi) := \left[\iint_{D^2} |\nabla(\vec{n}_1 - \vec{n}_2)|^2 \, dx\, dy \right]^{\frac{1}{2}}$$

$$+ \left\| \log|\nabla(\vec{\Phi}_1 \circ \Psi)| - \log|\nabla(\vec{\Phi}_2 \circ \Psi)| \right\|_{L^\infty(D^2)}$$

$$+ \left\| \nabla\left(\vec{\Phi}_1 \circ \Psi - \vec{\Phi}_2 \circ \Psi \right) \right\|_{L^2(D^2)} \quad ,$$

where we made use of the following notation

$$\vec{n}_i := \vec{n}_{\vec{\Phi}_i \circ \Psi} = \star \frac{\partial_x(\vec{\Phi}_i \circ \Psi) \wedge \partial_y(\vec{\Phi}_i \circ \Psi)}{|\partial_x(\vec{\Phi}_i \circ \Psi) \wedge \partial_y(\vec{\Phi}_i \circ \Psi)|} \quad , \qquad \text{for } i = 1, 2 \quad .$$

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[17]$g(J\cdot, J\cdot) = g(\cdot, \cdot)$

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Proposition II.4 *Let J be an admissible measurable complex structure on Σ, we define d^J to be the following nonnegative function on $\mathcal{E}_\Sigma \times \mathcal{E}_\Sigma$*

$$d^J(\vec{\Phi}_1, \vec{\Phi}_2) := \sum_{k \in \mathbb{N}} 2^{-k} \Delta_k^J(\vec{\Phi}_1, \vec{\Phi}_2) + \| \ln |X|_{g_1} - \ln |X|_{g_2} \|_{L^\infty((T\Sigma)_0)}$$

where $g_i := \vec{\Phi}_i^* g_{\mathbb{R}^m}$ and $(T\Sigma)_0$ is equal to the tangent bundle to Σ minus the zero section. d^J defines a distance-function on $\mathcal{E}_\Sigma$. $\qquad\square$

Proof of proposition II.4. First we have to prove that d^J is a well defined function. There is indeed a sup operation and we have to show that $d^J(\vec{\Phi}_1, \vec{\Phi}_2) < +\infty$ for any pair $(\vec{\Phi}_1, \vec{\Phi}_2) \in \mathcal{E}_\Sigma \times \mathcal{E}_\Sigma$.

Since $\vec{\Phi}_1$ and $\vec{\Phi}_2$ are in $\mathcal{E}_\Sigma$, because of (I.10) - see also remark II.5 - the two metrics $g_1 := \vec{\Phi}_1^* g_{\mathbb{R}^m}$ and $g_2 := \vec{\Phi}_2^* g_{\mathbb{R}^m}$ are equivalent to a reference metric g_0 that we assume to be compatible with g_0 : i.e. there exists C_{g_i, g_0} such that

$$\forall X \in (T\Sigma)_0 \qquad C_{g_i, g_0}^{-1} |X|_{g_i} \le |X|_{g_0} \le C_{g_i, g_0} |X|_{g_i} \quad . \tag{II.27}$$

Hence

$$\| \ln |X|_{g_1} - \ln |X|_{g_2} \|_{L^\infty((T\Sigma)_0)} < +\infty \quad .$$

Let $k \in \mathbb{N}$ and let $\Psi \in \mathcal{Q}_k^J$. Let $\overline{g}_0$, $\overline{g}_1$ and $\overline{g}_2$ be the 3 following metrics on D^2 given by $\overline{g}_0 := \Psi^* g_0$ and $\overline{g}_i := \Psi^* g_i$ where g_0 is a reference metric compatible with J. We have

$$|Dis(\overline{g}_0)| = |Dis^J(\Psi)| < 1 - 2^{-k} \quad .$$

Hence we deduce from lemma A.1 the following inequalities

$$\frac{1}{2} \inf_{X \in (T\Sigma)_0} \frac{|X|_{g_0}^2}{|X|_{g_i}^2} \le \frac{tr(g_0)}{tr(g_i)} \le 2^k \sup_{X \in (T\Sigma)_0} \frac{|X|_{g_0}^2}{|X|_{g_i}^2} \quad . \tag{II.28}$$

From which, together with (II.27), we deduce that

$$2^{-(k+1)} C_{g_2, g_0}^{-1} C_{g_1, g_0}^{-1} \le \frac{tr(\overline{g}_1)}{tr(\overline{g}_2)} = \frac{|\nabla(\vec{\Phi}_1 \circ \Psi)|^2}{|\nabla(\vec{\Phi}_2 \circ \Psi)|^2} \le 2^{k+1} C_{g_2, g_0} C_{g_1, g_0} \quad . \tag{II.29}$$

Let now $\alpha = \alpha_1 \, dx + \alpha_2 \, dy$ be a 1-form on D^2. Denote $G_0 := (\overline{g}_{0,ij})$ and $A := (\alpha_1, \alpha_2)$. we have

$$|\alpha|_{\overline{g}_0}^2 \, dvol_{\overline{g}_0} = A \, G_0^{-1} \, A^T \, \sqrt{det(G_0)} \, dx \, dy \quad .$$

We have also

$$A \, G_0^{-1} \, A^T \, \sqrt{det(G_0)} \ge A \, A^T \, (det(G_0)^{-1})^{-1/2} \inf_{\lambda \in Spec((G_0)^{-1})} \lambda \quad ,$$

where $Spec((G_0)^{-1})$ denotes the spectrum of the inverse of G_0. Hence we have

$$A \, G_0^{-1} \, A^T \, \sqrt{det(G_0)} \ge \mu \, A \, A^T \quad .$$

where

$$\mu := \inf \left\{ \sqrt{\frac{\lambda_1}{\lambda_2}}; \sqrt{\frac{\lambda_2}{\lambda_1}} \right\} \quad ,$$

where λ_1 and λ_2 are the two eigenvalues of G_0^{-1}. Clearly $0 < \mu < 1$. From (II.24) we have

$$\frac{1}{\mu + \frac{1}{\mu}} = \frac{1}{2} \sqrt{1 - (Dis(\overline{g}_0))^2} \quad .$$

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Hence we deduce, since $|Dis(\overline{g}_0)| = |Dis^J(\Psi)| < 1 - 2^{-k} < 1$,

$$\mu > 2^{-1-k/2} \quad .$$

We deduce from the previous identities that for $i = 1, 2$

$$\int_{\Psi(D^2)} |d\vec{n}_{\vec{\Phi}_i}|^2_{g_0} \, dvol_{g_0} \geq 2^{-1-k/2} \int_{D^2} |\nabla \vec{n}_{\vec{\Phi}_i \circ \Psi}|^2 \, dx \, dy \quad .$$

Using now the equivalence of the norms mentioned in remark II.5 we obtain that

$$2^{1+k/2} \int_{\Sigma} |d\vec{n}_{\vec{\Phi}_i}|^2_{g_i} \, dvol_{g_i} \geq \int_{D^2} |\nabla \vec{n}_{\vec{\Phi}_i \circ \Psi}|^2 \, dx \, dy \quad . \tag{II.30}$$

In a similar way we deduce also that

$$2^{1+k/2} \int_{\Sigma} |d\vec{\Phi}_i|^2_{g_i} \, dvol_{g_i} \geq \int_{D^2} |\nabla (\vec{\Phi}_i \circ \Psi)|^2 \, dx \, dy \quad . \tag{II.31}$$

We hence deduce from (II.29) (II.30) and (II.31) that

$$\Delta^J_k(\vec{\Phi}_1, \vec{\Phi}_2) = \sup_{\Psi_1 \in \mathcal{Q}^J_k} \Lambda(\vec{\Phi}_1, \vec{\Phi}_2, \Psi) < C_{\vec{\Phi}_1, \vec{\Phi}_2}(2^{k/4} + k) \quad . \tag{II.32}$$

Combining this fact together with (II.28) we obtain that $d^J(\vec{\Phi}_1, \vec{\Phi}_2) < +\infty$ for any pair $(\vec{\Phi}_1, \vec{\Phi}_2)$ in $\mathcal{E}_{\Sigma} \times \mathcal{E}_{\Sigma}$.

We prove now that d^J is a distance function.

Symmetry. It is clear by definition.

Triangular inequality. Observe first that $\forall \, (\vec{\Phi}_1, \vec{\Phi}_2) \in (\mathcal{E}_{\Sigma})^2 \quad \forall \, \Psi \in \mathcal{Q}^J_k$

$$\forall \, (\vec{\Phi}_1, \vec{\Phi}_2) \in (\mathcal{E}_{\Sigma})^2 \quad \forall \, \Psi \in \mathcal{Q}^J_k \qquad \Lambda(\vec{\Phi}_1, \vec{\Phi}_2, \Psi) = \Lambda(\vec{\Phi}_2, \vec{\Phi}_1, \Psi) \quad . \tag{II.33}$$

and moreover $\forall (\vec{\Phi}_1, \vec{\Phi}_2, \vec{\Phi}_3) \in (\mathcal{E}_{\Sigma})^3$ and $\Psi \in \mathcal{Q}^J_k$

$$\Lambda(\vec{\Phi}_1, \vec{\Phi}_2, \Psi) \leq \Lambda(\vec{\Phi}_1, \vec{\Phi}_3, \Psi) + \Lambda(\vec{\Phi}_3, \vec{\Phi}_2, \Psi) \quad . \tag{II.34}$$

Let $\varepsilon > 0$, there exists $\Psi \in \mathcal{E}_{\Sigma}$ such that $\forall \, \Psi \in \mathcal{Q}^J_k$

$$\Delta^J_k(\vec{\Phi}_1, \vec{\Phi}_2) \leq \Lambda(\vec{\Phi}_1, \vec{\Phi}_2, \Psi) + \varepsilon \quad . \tag{II.35}$$

Combining (II.34) and (II.35) we obtain for any $\varepsilon > 0$

$$\Delta^J_k(\vec{\Phi}_1, \vec{\Phi}_2) \leq \Delta^J_k(\vec{\Phi}_1, \vec{\Phi}_3) + \Delta^J_k(\vec{\Phi}_3, \vec{\Phi}_2) + \varepsilon \quad .$$

which implies the triangular inequality for d^J

discernibility. Assume $d^J(\vec{\Phi}_1, \vec{\Phi}_2) = 0$. Then for any quasi-conformal map $\Psi \, : \, D^2 \to \Sigma$ we have $\vec{\Phi}_1 \circ \Psi = \vec{\Phi}_2 \circ \Psi$. This clearly implies that $\vec{\Phi}_1 = \vec{\Phi}_2$.

This concludes the proof of the fact that d^J is a distance function on $\mathcal{E}_{\Sigma}$ and proposition II.4 is proved. $\square$

Proposition II.5 *Let J and J' be two admissible measurable complex structures on Σ, then d^J and $d^{J'}$ are equivalent distances on $\mathcal{E}_\Sigma$ and there holds*

$$\forall \ (\vec{\Phi}_1, \vec{\Phi}_2) \in (\mathcal{E}_\Sigma)^2$$

$$8^{-1} \ e^{-\delta(J,J')} \ d^J(\vec{\Phi}_1, \vec{\Phi}_2) \le d^{J'}(\vec{\Phi}_1, \vec{\Phi}_2) \le 8 \ e^{\delta(J,J')} \ d^J(\vec{\Phi}_1, \vec{\Phi}_2) \quad , \tag{II.36}$$

where $\delta(J,J')$ is the following distance between the two complex structures :

$$\delta(J,J') := \left\| \ln \frac{|X \wedge JX|_{g_0}}{|X \wedge J'X|_{g_0}} \right\|_{L^\infty((T\Sigma)_0)}$$

(for an arbitrary metric g_0).

$\square$

Proof of proposition II.5. Let Ψ be a quasi-conformal map from the disc D^2 into Σ. (II.26) implies

$$1 - |Dis^J(\Psi)|^2 = 4 \frac{|\partial_x \Psi \wedge \partial_y \Psi|_g^2}{\left[|\partial_x \Psi|_g^2 + |\partial_y \Psi|_g^2\right]^2} \quad .$$

where g is an arbitrary metric compatible with J. We have hence for instance $|\partial_x \Psi|_g^2 = |\partial_x \Psi \wedge J\partial_x \Psi|$ and $|\partial_y \Psi|_g^2 = |\partial_y \Psi \wedge J\partial_y \Psi|$. Let $\mathfrak{J}$ be the complex structure such that $\mathfrak{J}\partial_x \Psi = \partial_y \Psi$. We have

$$\begin{aligned} 1 - |Dis^J(\Psi)|^2 &= 4 \ \left[\frac{|\partial_x \Psi \wedge J\partial_x \Psi|_g}{|\partial_x \Psi \wedge \mathfrak{J}\partial_x \Psi|_g} + \frac{|\partial_y \Psi \wedge J\partial_y \Psi|_g}{|\partial_y \Psi \wedge \mathfrak{J}\partial_y \Psi|_g} \right]^{-1} \\ &= 4 \ \left[\frac{|\partial_x \Psi \wedge J\partial_x \Psi|_{g_0}}{|\partial_x \Psi \wedge \mathfrak{J}\partial_x \Psi|_{g_0}} + \frac{|\partial_y \Psi \wedge J\partial_y \Psi|_{g_0}}{|\partial_y \Psi \wedge \mathfrak{J}\partial_y \Psi|_{g_0}} \right]^{-1} \end{aligned}$$

where g_0 is an arbitrary reference metric on Σ. Using an elementary algebraic inequality[18] we obtain that for any quasi-conformal map Ψ from D^2 into Σ

$$\begin{aligned} \frac{1 - |Dis^J(\Psi)|^2}{1 - |Dis^{J'}(\Psi)|^2} &\le \left[\frac{|\partial_x \Psi \wedge J'\partial_x \Psi|_{g_0}}{|\partial_x \Psi \wedge J\partial_x \Psi|_{g_0}} + \frac{|\partial_y \Psi \wedge J'\partial_y \Psi|_{g_0}}{|\partial_y \Psi \wedge J\partial_y \Psi|_{g_0}} \right] \\ &\le 2e^{\delta(J,J')} \end{aligned} \tag{II.37}$$

Hence we have

$$1 - |Dis^J(\Psi)| \le 4 \ e^{\delta(J,J')} \ [1 - |Dis^{J'}(\Psi)|] \tag{II.38}$$

let $k_0 = [\delta(J,J')/\log 2] + 1$. We have that

$$|Dis^J(\Psi)| < 1 - 2^{-k} \qquad \Longrightarrow \qquad |Dis^{J'}(\Psi)| < 1 - 2^{-k-k_0-2} \quad .$$

Hence

$$\forall k \in \mathbb{N} \qquad 2^{-k} \ \Delta_k^J(\vec{\Phi}_1, \vec{\Phi}_2) \le 2^{k_0+2} \ 2^{-k-k_0-2} \ \Delta_{k+k_0+2}^{J'}(\vec{\Phi}_1, \vec{\Phi}_2) \quad ,$$

from which we deduce

$$d^J(\vec{\Phi}_1, \vec{\Phi}_2) \le 2^{k_0+2} \ d^{J'}(\vec{\Phi}_1, \vec{\Phi}_2) \quad .$$

This last inequality implies proposition II.5.

$\square$

[18]

$$\forall \ a, a', b, b' > 0 \qquad \frac{1}{a+b} \le \left[\frac{a'}{a} + \frac{b'}{b} \right] \ \frac{1}{a'+b'} \quad .$$

II.3 Completeness of the metric spaces $(\mathcal{E}_\Sigma, d^J)$.

In this subsection we prove the following result.

Proposition II.6 *For any admissible measurable complex structure J on Σ, the metric space $(\mathcal{E}_\Sigma, d^J)$ is complete.* $\hfill\square$

Proof of proposition II.6. Because of the equivalence of the d^J we can choose an arbitrary J that we will assume to be smooth. We first choose a finite covering of Σ by open sets U_i such that each U_i is diffeomorphic to a disc and we denote by Ψ_i diffeomorphisms from D^2 into U_i such that $Dis^J(\Psi_i) = 0$. Let now $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ such that $d(\vec{\Phi}_{k-1}, \vec{\Phi}_k) \leq 2^{-k-1}$. Denotes $g_k := \vec{\Phi}_k^* g_{\mathbb{R}^m}$. The assumption implies that $\vec{\Phi}_k \circ \Psi_i$ converges strongly in $W^{1,2}$ to a limit $\vec{\xi}_i : D^2 \to \mathbb{R}^m$. Hence $|\nabla \vec{\Phi}_k \circ \Psi_i|$ converges a.e. to $|\nabla \vec{\xi}_i|$ and hence $\log |\nabla \vec{\Phi}_k \circ \Psi_i|$ converges a.e. to $\log |\nabla \vec{\xi}_i| \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. From the Cauchy sequence assumption for $\vec{\Phi}_k$ w.r.t. d^J we have that $\log |\nabla \vec{\Phi}_k \circ \Psi_i|$ is Cauchy in L^∞ and this limit can only be $\log |\nabla \vec{\xi}_i|$ which is then in $L^\infty(D^2)$. We have moreover

$$\forall\, j,l = 1,2 \qquad \Psi_i^* g_{k,jl} = \partial_{x_j}(\vec{\Phi}_k \circ \Psi_i) \cdot \partial_{x_l}(\vec{\Phi}_k \circ \Psi_i) \longrightarrow \partial_{x_j} \vec{\xi}_i \cdot \partial_{x_l} \vec{\xi}_i \quad \text{a.e.} \quad .$$

Since

$$\forall k \in \mathbb{N} \qquad \| \log |X|_{g_k} - \log |X|_{g_1} \|_{L^\infty((T\Sigma)_0)} \leq 2^{-1} \tag{II.39}$$

we have that

$$\forall\, X \in \mathbb{R}^2 \neq 0 \qquad \log \left[\sum_{j,l=1}^2 \Psi_i^* g_{k,jl} X^j X^l \right] \longrightarrow \log \left[\sum_{j,l=1}^2 \partial_{x_j} \vec{\xi}_i \cdot \partial_{x_l} \vec{\xi}_i \, X^j X^l \right] \qquad \text{a.e.}$$

and hence

$$\forall\, X \in \mathbb{R}^2 \neq 0 \qquad \left| \log \left[\sum_{j,l=1}^2 \partial_{x_j} \vec{\xi}_i \cdot \partial_{x_l} \vec{\xi}_i \, X^j X^l \right] \right| \leq \log |X|_{g_1} + 2^{-1} \quad . \tag{II.40}$$

We deduce from the previous inequality that $\vec{\xi}_i$ is an immersion from D^2 into $\mathbb{R}^m$ and there exists $c_i > 0$ such that

$$|d\vec{\xi}_i \wedge d\vec{\xi}_i| \geq c_i > 0 \qquad \text{on } U_i \quad . \tag{II.41}$$

Hence the $\vec{\xi}_i$ are lipschitz immersions. We have, for any pair i,j, $\vec{\xi}_i^{-1} \circ \vec{\xi}_j = \Psi_i^{-1} \circ \Psi_j$, hence there exists $\vec{\xi}$ a Lipschitz immersion from Σ into $\mathbb{R}^m$ such that $\vec{\xi}_i = \vec{\xi} \circ \Psi_i$ for all i. Let

$$\vec{n}_{i,k} := \vec{n}_{\vec{\Phi}_k \circ \Psi_i} = \star \frac{\partial_x(\vec{\Phi}_k \circ \Psi_i) \wedge \partial_y(\vec{\Phi}_k \circ \Psi_i)}{|\partial_x(\vec{\Phi}_k \circ \Psi_i) \wedge \partial_y(\vec{\Phi}_k \circ \Psi_i)|} \qquad .$$

From the Cauchy sequence assumption for $\vec{\Phi}_k$ we deduce that $\vec{n}_{i,k}$ converges strongly in $\dot{W}^{1,2}(D^2)$ to a limit that we denote $\vec{n}_i$, moreover, from the above we have that

$$\star \frac{\partial_x(\vec{\Phi}_k \circ \Psi_i) \wedge \partial_y(\vec{\Phi}_k \circ \Psi_i)}{|\partial_x(\vec{\Phi}_k \circ \Psi_i) \wedge \partial_y(\vec{\Phi}_k \circ \Psi_i)|} \longrightarrow \star \frac{\partial_x \vec{\xi}_i \wedge \partial_y \vec{\xi}_i}{|\partial_x \vec{\xi}_i \wedge \partial_y \vec{\xi}_i|} \qquad \text{a.e.} \quad .$$

Hence we have that

$$\star \frac{\partial_x \vec{\xi}_i \wedge \partial_y \vec{\xi}_i}{|\partial_x \vec{\xi}_i \wedge \partial_y \vec{\xi}_i|} = \vec{n}_i \quad \in \quad W^{1,2}(D^2) \quad . \tag{II.42}$$

Hence $\vec{\xi}$ is a Lipschitz immersion inducing a metric comparable to the smooth reference metric g_0 whose Gauss map is in $W^{1,2}$ with respect to this metric. This implies that $\vec{\xi} \in \mathcal{E}_\Sigma$ and it can be proved with moderate efforts that $d^J(\vec{\Phi}_k, \vec{\xi}) \longrightarrow 0$. This concludes the proof of proposition II.6. $\hfill\square$

II.4 Control of $d^g(\vec{\Phi}, \vec{\Phi}+t\vec{w})$ for conformal $\vec{\Phi}$ and for $W^{2,2} \cap W^{1,\infty}$ perturbations $\vec{w}$.

Lemma II.1 *There exists $\varepsilon_0 > 0$ such that for any conformal immersion $\vec{\Phi}$ of the disc D^2 into $\mathbb{R}^m$ in $W^{2,2} \cap W^{1,\infty}$ satisfying*

$$\int_{D^2} |\nabla \vec{n}_{\vec{\Phi}}|^2 \ dx \ dy < \frac{4\pi}{3} \quad , \tag{II.43}$$

Let $\vec{w} \in W^{1,\infty} \cap W^{2,2}(D^2, \mathbb{R}^m)$ compactly supported in $D^2_{1/2}$ such that

$$\|\nabla \vec{w}\|_{L^\infty(D^2)} + \|\nabla^2 \vec{w}\|_{L^2(D^2)} \leq 1 \tag{II.44}$$

Denote $\vec{\Phi}_t := \vec{\Phi} + t\vec{w}$. Then there exists $C > 0$ independent of $\vec{\Phi}$ and $\vec{w}$ such that, for $|t| < [\inf_{D^2_{1/2}} |\nabla \vec{\Phi}|]/4$,

$$d^g(\vec{\Phi}_t, \vec{\Phi}) \leq \frac{C}{\inf_{D^2_{1/2}} |\nabla \vec{\Phi}|} \ |t| \ \|\nabla^2 w\|_2$$

$$+ \frac{C}{\inf_{D^2_{1/2}} |\nabla \vec{\Phi}|} \left[1 + \frac{\|\nabla \vec{\Phi}\|_\infty \ \|\nabla \vec{n}\|_2 + \|\nabla \vec{\Phi}\|_2}{\inf_{D^2_{1/2}} |\nabla \vec{\Phi}|} \right] \ |t| \ \|\nabla w\|_\infty \quad . \tag{II.45}$$

where $g := \vec{\Phi}^* g_{\mathbb{R}^m}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of lemma II.1.

We denote the conformal factor as usual as follows : $e^\lambda := |\partial_x \vec{\Phi}| = |\partial_y \vec{\Phi}|$. Denote $e^{\overline{\lambda}} = \|\nabla \vec{\Phi}\|_\infty$ and $e^{\underline{\lambda}} := \inf_{D^2_{1/2}} e^\lambda$. Consider t such that $4\,|t| < e^{\underline{\lambda}}$. Since w is supported in $D^2_{1/2}$ and since $\|\nabla w\|_\infty \leq 1$, we have

$$|\partial_x \vec{\Phi}_t \wedge \partial_y \vec{\Phi}_t| \quad \geq e^{2\lambda} - |t| \ |\partial_x \vec{w} \wedge \partial_y \vec{\Phi}| - |t| \ |\partial_x \vec{\Phi} \wedge \partial_y \vec{w}|$$

$$-t^2 |\partial_x \vec{w} \wedge \partial_y \vec{w}|$$

$$\geq e^{2\lambda} - 2\,|t|\,e^\lambda - t^2 \geq \frac{7}{16} \ e^{2\lambda} = \frac{7}{16} \ |\partial_x \vec{\Phi} \wedge \partial_y \vec{\Phi}|$$

A straightforward but a bit lengthy computation shows that

$$|\nabla(\vec{n}_{\vec{\Phi}} - \vec{n}_{\vec{\Phi}_t})| \leq C \ |t| \ e^{-\lambda} \ \left[|\nabla \vec{w}| \ e^{-\lambda} \ |\nabla^2 \vec{\Phi}| + |\nabla^2 \vec{w}| \right] \quad , \tag{II.46}$$

where C is independent of all the datas $\vec{\Phi}$, $\vec{w}$ and t. Since

$$\Delta \vec{\Phi} = 2e^{2\lambda} \ \vec{H}$$

where $\vec{H}$ is the mean curvature vector of the immersion of D^2 which is pointwisely controled by $|\nabla \vec{n}|$, standard elliptic estimates imply

$$\int_{D^2_{1/2}} |\nabla^2 \vec{\Phi}|^2 \leq C \ e^{2\overline{\lambda}} \int_{D^2} |\nabla \vec{n}|^2 + C \ \int_{D^2} |\nabla \vec{\Phi}|^2 \quad .$$

Integrating hence (II.46) on D^2, since $|\nabla(\vec{n}_{\vec{\Phi}} - \vec{n}_{\vec{\Phi}_t})|$ is supported on $D^2_{1/2}$, we obtain

$$\|\nabla(\vec{n}_{\vec{\Phi}} - \vec{n}_{\vec{\Phi}_t})\|_2 \leq \frac{C}{\inf_{D^2_{1/2}} |\nabla \vec{\Phi}|} \left[|t| \ \|\nabla^2 \vec{w}\|_2 \right.$$

$$\left. + \frac{\|\nabla \vec{\Phi}\|_\infty \ \|\nabla \vec{n}\|_2 + \|\nabla \vec{\Phi}\|_2}{\inf_{D^2_{1/2}} |\nabla \vec{\Phi}|} \ |t| \ \|\nabla \vec{w}\|_\infty \right] \quad . \tag{II.47}$$

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We have the pointwise identity

$$|\nabla\vec{\Phi}_t - \nabla\vec{\Phi}| \le |t|\,|\nabla\vec{w}|$$

and hence

$$\int_{D^2} |\nabla\vec{\Phi}_t - \nabla\vec{\Phi}|^2 \le C\,|t|^2\,\|\nabla\vec{w}\|_\infty^2 \quad . \tag{II.48}$$

We have also

$$\left|\log\frac{|\nabla\vec{\Phi}_t|^2}{|\nabla\vec{\Phi}|^2}\right| = \left|\log\left[1 + 2e^{-2\lambda}\,t\,\nabla\vec{w}\cdot\nabla\vec{\Phi} + e^{-2\lambda}\,t^2\,|\nabla\vec{w}|^2\right]\right| \quad .$$

Hence for $4\,|t| < e^{\lambda}$ we deduce

$$\left\|\log|\nabla\vec{\Phi}_t| - \log|\nabla\vec{\Phi}|\right\|_{L^\infty(D^2)} \le \frac{C}{\inf_{D^2_{1/2}}|\nabla\vec{\Phi}|}\,|t|\,\|\nabla\vec{w}\|_\infty \quad . \tag{II.49}$$

Combining (II.47), (II.48) and (II.49), we obtain

$$\Lambda(\vec{\Phi},\vec{\Phi}_t, id_{D^2}) \le \frac{C}{\inf_{D^2_{1/2}}|\nabla\vec{\Phi}|}\,|t|\,\|\nabla^2\vec{w}\|_2$$

$$+ \frac{C}{\inf_{D^2_{1/2}}|\nabla\vec{\Phi}|}\left[1 + \frac{\|\nabla\vec{\Phi}\|_\infty\,\|\nabla\vec{n}\|_2 + \|\nabla\vec{\Phi}\|_2}{\inf_{D^2_{1/2}}|\nabla\vec{\Phi}|}\right]|t|\,\|\nabla w\|_\infty \;. \tag{II.50}$$

Let now Ψ be an arbitrary map in $\mathcal{Q}_k^g(\vec{\Phi})$. Since $\vec{\Phi}$ is conformal Ψ is a quasiconformal map satisfying

$$|Dis^g(\Psi)| = |Dis(\vec{\Phi}\circ\Psi)| \le 1 - 2^{-k} \quad .$$

This implies

$$(2^{k+1} - 1)^{-1}|\partial_y\Psi|^2 < |\partial_x\Psi|^2 \le (2^{k+1} - 1)\,|\partial_y\Psi|^2 \quad , \tag{II.51}$$

and

$$\left|\frac{\partial_x\Psi}{|\partial_x\Psi|}\cdot\frac{\partial_y\Psi}{|\partial_y\Psi|}\right|^2 < 1 - 2^{-k-1} \quad . \tag{II.52}$$

After some short computation, we deduce from the previous line

$$2^{-5/2}\,2^{-3k/2}\,|\nabla\Psi|^2 \le \det\nabla\Psi = \partial_x\Psi \times \partial_y\Psi \le |\nabla\Psi|^2/2 \quad . \tag{II.53}$$

Hence we have

$$\int_{D^2} |\nabla(\vec{n}_{\vec{\Phi}\circ\Psi} - \vec{n}_{\vec{\Phi}_t\circ\Psi})|^2 \; dx\,dy$$

$$\le \int_{D^2} |\nabla(\vec{n}_{\vec{\Phi}} - \vec{n}_{\vec{\Phi}_t})|^2 \circ \Psi\,|\nabla\Psi|^2 \; dx\,dy$$

$$\le 2^{5/2}\,2^{3k/2}\int_{D^2} |\nabla(\vec{n}_{\vec{\Phi}} - \vec{n}_{\vec{\Phi}_t})|^2 \circ \Psi\;\det\nabla\Psi\;dx\,dy \tag{II.54}$$

$$\le 2^{5/2}\,2^{3k/2}\int_{D^2} |\nabla(\vec{n}_{\vec{\Phi}} - \vec{n}_{\vec{\Phi}_t})|^2\;dx\,dy$$

$$\le 2^{5/2}\,2^{3k/2}\,\Lambda^2(\vec{\Phi},\vec{\Phi}_t, id_{D^2}) \quad .$$

Similarly we have

$$\int_{D^2} |\nabla(\vec{\Phi} \circ \Psi - \vec{\Phi}_t \circ \Psi)|^2 \, dx \, dy$$

$$\leq 2^{5/2} \, 2^{3k/2} \int_{D^2} |\nabla(\vec{\Phi} - \vec{\Phi}_t)|^2 \circ \Psi \, \det \nabla \Psi \, dx \, dy \qquad \text{(II.55)}$$

$$\leq 2^{5/2} \, 2^{3k/2} \, \Lambda^2(\vec{\Phi}, \vec{\Phi}_t, id_{D^2}) \quad .$$

We have moreover, since $|\nabla(\vec{\Phi} \circ \Psi)| = e^\lambda \, |\nabla\Psi|$,

$$\left| \log \frac{|\nabla(\vec{\Phi}_t \circ \Psi)|^2}{|\nabla(\vec{\Phi} \circ \Psi)|^2} \right| = \left| \log \left[1 + 2e^{-2\lambda} \, |\nabla\Psi|^{-2} t \, \nabla(\vec{w} \circ \Psi) \cdot \nabla(\vec{\Phi} \circ \Psi) \right. \right.$$

$$\left. \left. + e^{-2\lambda} \, t^2 \, |\nabla\Psi|^{-2} \, |\nabla\vec{w} \circ \Psi|^2 \right] \right| \quad .$$

Using the fact that $|\nabla(\vec{w} \circ \Psi)| \leq |\nabla\vec{w}| \, |\nabla\Psi|$, we then have for $|t| < [\inf_{D^2_{1/2}} |\nabla\vec{\Phi}|]/4$

$$\left\| \log |\nabla(\vec{\Phi}_t \circ \Psi)| - \log |\nabla(\vec{\Phi} \circ \Psi)| \right\|_{L^\infty(D^2)} \leq \frac{C}{\inf_{D^2_{1/2}} |\nabla\vec{\Phi}|} \, |t| \, \|\nabla\vec{w}\|_\infty \quad . \qquad \text{(II.56)}$$

Hence combining (II.54), (II.55) and (II.56) we have obtained the existence of $C > 0$ independent of $\vec{\Phi}$ and $\vec{w}$ such that for $|t| < [\inf_{D^2_{1/2}} |\nabla\vec{\Phi}|]/4$, for any $\Psi \in \mathcal{Q}_k^{[g]}$

$$\Lambda(\vec{\Phi}, \vec{\Phi}_t, \Psi) \leq \frac{2^{3k/4} \, C}{\inf_{D^2_{1/2}} |\nabla\vec{\Phi}|} \, |t| \, \|\nabla^2\vec{w}\|_2$$

$$+ \frac{2^{3k/4} \, C}{\inf_{D^2_{1/2}} |\nabla\vec{\Phi}|} \left[1 + \frac{\|\nabla\vec{\Phi}\|_\infty \, \|\nabla\vec{n}\|_2 + \|\nabla\vec{\Phi}\|_2}{\inf_{D^2_{1/2}} |\nabla\vec{\Phi}|} \right] \, |t| \, \|\nabla\vec{w}\|_\infty \quad . \qquad \text{(II.57)}$$

where C is independent of Ψ. Hence we deduce

$$\sum_{k \in \mathbb{N}} 2^{-k} \Delta_k^{[g]}(\vec{\Phi}_t, \vec{\Phi}) \leq \frac{C}{\inf_{D^2_{1/2}} |\nabla\vec{\Phi}|} \, |t| \, \|\nabla^2\vec{w}\|_2$$

$$+ \frac{C}{\inf_{D^2_{1/2}} |\nabla\vec{\Phi}|} \left[1 + \frac{\|\nabla\vec{\Phi}\|_\infty \, \|\nabla\vec{n}\|_2 + \|\nabla\vec{\Phi}\|_2}{\inf_{D^2_{1/2}} |\nabla\vec{\Phi}|} \right] \, |t| \, \|\nabla\vec{w}\|_\infty \quad . \qquad \text{(II.58)}$$

at a point $p \in D^2$ we have for $X = X_1 \, \partial_{x_1} + X_2 \, \partial_{x_2}$, denoting $|X|_0^2 = X_1^2 + X_2^2$,

$$\frac{|X|_{g_{\vec{\Phi}_t}}}{|X|_{g_{\vec{\Phi}}}} = 1 + 2 \, e^{-2\lambda} \, t \, \nabla\vec{w} \frac{X}{|X|_0} \cdot \nabla\vec{\Phi} \frac{X}{|X|_0} + e^{-2\lambda} \, t^2 \, |\nabla\vec{w} \frac{X}{|X|_0}|^2$$

Hence for $|t| < e^\lambda$ we have that

$$\| \log |X|_{g_{\vec{\Phi}_t}} - \log |X|_{g_{\vec{\Phi}}} \|_{L^\infty((T\Sigma)_0)} \leq \frac{C}{\inf_{D^2_{1/2}} |\nabla\vec{\Phi}|} \, |t| \, \|\nabla\vec{w}\|_\infty \qquad \text{(II.59)}$$

where we are using the fact that, for such t and $\vec{w}$, $e^{-\lambda} \, |t| \, |\nabla\vec{w}| < 1$. Inequality (II.58) together with inequality (II.59) imply (II.45) and Lemma II.1 is proved. $\qquad \square$

III   Existence of Minimizers of the Willmore Energy.

In this section we prove the following result

Theorem III.1 *Let Σ be an abstract closed two dimensional smooth manifolds. Assume that there exists a minimizing sequence $\vec{\Phi}_k \in \mathcal{E}_\Sigma$ of the Willmore energy such that the conformal class induced by $\vec{\Phi}_k^* g_{\mathbb{R}^m}$ stays in the compact subset of the Riemann Moduli Space of Σ and assume that*

$$\limsup_{k \to +\infty} W(\vec{\Phi}_k) = \int_\Sigma |\vec{H}_k|^2_{g_k} \; dvol_{g_k} < 8\pi \quad .$$

where $g_k := \vec{\Phi}_k^* g_{\mathbb{R}^m}$ and $\vec{H}_k$ is the mean-curvature vector of the immersion $\vec{\Phi}_k$, then

$$\inf_{\vec{\Phi} \in \mathcal{E}_\Sigma} W(\vec{\Phi})$$

is achieved by a smooth embedding. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Before to prove theorem III.1 we first state the following proposition which is a direct application of Ekeland's Variational Principle (see theorem 5.1 in [St]).

Proposition III.1 *Let J be an arbitrary smooth complex structure on Σ and let $\vec{\Phi}_k$ be a minimizing sequence such that*

$$W(\vec{\Phi}_k) \le \inf_{\vec{\Phi} \in \mathcal{E}_\Sigma} W(\vec{\Phi}) + 2^{-k} \quad ,$$

then there exists $\vec{\xi}_k \in \mathcal{E}_\Sigma$ such that

i) $\vec{\xi}_k$ minimizes in $\mathcal{E}_\Sigma$ the following functional

$$W(\vec{\xi}_k) = \inf_{\vec{\Phi} \in \mathcal{E}_\Sigma} W(\vec{\Phi}) + 2^{-k/2} \; d^J(\vec{\Phi}, \vec{\xi}_k) \quad , \tag{III.1}$$

ii)

$$W(\vec{\xi}_k) \le W(\vec{\Phi}_k) \quad , \tag{III.2}$$

iii)

$$d^J(\vec{\xi}_k, \vec{\Phi}_k) \le 2^{-k/2} \quad . \tag{III.3}$$

$\square$

Proof of theorem III.1. We can assume that Σ is not S^2 since a classical result implies that

$$\inf_{\vec{\Phi} \in \mathcal{E}_{S^2}} W(\vec{\Phi}) = 4\pi$$

and is achieved by the unit sphere of $\mathbb{R}^3 \subset \mathbb{R}^m$ (see for instance [Wi]). Let $\vec{\Phi}_k$ be a minimizing sequence of the Willmore energy W in the space $\mathcal{E}_\Sigma$ satisfying

$$W(\vec{\Phi}_k) \le \inf_{\vec{\Phi} \in \mathcal{E}_\Sigma} W(\vec{\Phi}) + 2^{-k} \quad . \tag{III.4}$$

A straightforward molification argument allows to work under the assumption that $\vec{\Phi}_k \in C^\infty(\Sigma, \mathbb{R}^m)$. The assumption of the theorem is telling us that the conformal class of the induced metric $\vec{\Phi}_k^* g_{\mathbb{R}^m}$ is

contained in a compact subset of the Moduli space of Σ. Therefore, modulo extraction of a subsequence, we can find a sequence of complex structure J_k and diffeomorphisms f_k of Σ such that

$$\vec{\Phi}_k \circ f_k \ : \ (\Sigma, J_k) \longrightarrow \mathbb{R}^m \quad \text{is conformal} \quad ,$$

and, if h_k denotes

$$J_k \longrightarrow J_\infty \quad \text{w.r.t. } \delta \quad . \tag{III.5}$$

$\vec{\Phi}_k \circ f_k$ satisfies of course still (III.4). Denote by h_k, resp. h_∞, the smooth constant scalar curvature metric compatible with J_k resp. J_∞ having a fixed volume 1 on Σ, we may also ensure that

$$\| \log |X|_{h_k} - \log |X|_{h_\infty} \|_{L^\infty((T\Sigma)_0)} \longrightarrow 0 \quad . \tag{III.6}$$

We are now using lemma A.4 in order to "normalize" the embeddings $\vec{\Phi}_k \circ f_k$.

For the minimizing sequence $\vec{\Phi}_k \circ f_k$, modulo extraction of a subsequence, we can find a sequence of geodesic balls $B_{2r_k}(p_k)$ such that $r_k \to r_\infty > 0$, $p_k \to p_\infty \in \Sigma$ and

$$\int_{B_{2r_k}(p_k)} |d\vec{n}_{\vec{\Phi}_k}|_{g_k}^2 \ dvol_{g_k} < \delta$$

For each k we consider the "Normalization Moebius Transformations" Ξ_k given by lemma A.4 for the ball $B_{2r_k}(p_k)$, and we replace our minimizing sequence $\vec{\Phi}_k \circ f_k$ by $\Xi_k \circ \vec{\Phi}_k \circ f_k$.

In order to simplify the notations we then write $\vec{\Phi}_k$ instead of $\Xi_k \circ \vec{\Phi}_k \circ f_k$.

To each $x \in \Sigma$ we assign $\rho_x > 0$ such that

$$\int_{B_{\rho_x}(x)} |d\vec{n}|_{h_k}^2 \ dvol_{h_k} = \int_{B_{\rho_x}(x)} |d\vec{n}|_{g_k}^2 \ dvol_{g_k} = 8\pi/3 \quad ,$$

where $B_{\rho_x}(x)$ is the geodesic ball in (Σ, h_k) of center x and radius ρ_x and $g_k := f_k^* \vec{\Phi}_k^* g_{\mathbb{R}^m}$. We extract a finite Besicovitch covering : each point in Σ is covered by at most N of such balls where N only depends on (Σ, g_∞). Let $(B_{\rho_k^i}(x_k^i))_{i \in I}$ be this finite covering. We can extract a subsequence such that I is independent of k, such that each x_k^i converges to a limit x_∞^i and each ρ_k^i converges to a limit ρ_∞^i. Let

$$I_0 := \{i \in I \quad \text{s. t.} \quad \rho_\infty^i = 0\} \quad .$$

Let $I_1 := I \setminus I_0$. It is clear that the union of the closures of the balls $\cup_{i \in I_1} \overline{B}_{\rho_\infty^i}(x_\infty^i)$ covers Σ. Because of the strict convexity of the balls with respect either to the euclidian distance ($\Sigma = T^2$) or the hyperbolic distance (genus(Σ) > 1) the points in Σ which are not contained in the union of the <u>open</u> balls $\cup_{i \in I} B_{\rho_\infty^i}(x_\infty^i)$ cannot accumulate and therefore are isolated and hence finite. Denote

$$\{a_1 \cdots a_N\} := \Sigma \setminus \cup_{i \in I_1} B_{\rho_\infty^i}(x_\infty^i) \quad . \tag{III.7}$$

Hence from now on we have a sequence of complex structures J_k on Σ such that

$$J_k \longrightarrow J_\infty \quad \text{w.r.t. } \delta \quad .$$

with associated constant scalar curvature metrics h_k of volume 1 and satisfying

$$\| \log |X|_{h_k} - \log |X|_{h_\infty} \|_{L^\infty((T\Sigma)_0)} \longrightarrow 0 \quad .$$

and we have a sequence of smooth immersions $\vec{\Phi}_k$ of Σ into $\mathbb{R}^m$ satisfying (III.4) and the following five conditions

i)
$$\vec{\Phi}_k \qquad \text{is conformal from } (\Sigma, J_k) \quad \text{into } \mathbb{R}^m \quad . \qquad \text{(III.8)}$$

ii) There exists finitely many points $a_1 \cdots a_N$ in Σ and a fixed finite covering $(B_{\rho_\infty^i}(x_\infty^i))_{i \in I_1}$ of $\Sigma \setminus \{a_1 \cdots a_N\}$ such that for any $i \in I_1$, $0 < \rho < \rho_\infty$ and k large enough

$$\int_{B_\rho(x_k^i)} |d\vec{n}_{\vec{\Phi}_k}|_{g_k}^2 \, dvol_{g_k} < 8\pi/3 \quad . \qquad \text{(III.9)}$$

where $g_k := \vec{\Phi}_k^* g_{\mathbb{R}^m}$.

iii) There exists a positive real $R > 0$ such that

$$\vec{\Phi}_k(\Sigma) \subset B_R(0) \quad . \qquad \text{(III.10)}$$

iv) There exists a constant $C > 0$ such that

$$\mathcal{H}^2(\vec{\Phi}_k(\Sigma)) \leq C \quad . \qquad \text{(III.11)}$$

v) There exist an index $i \in I_1$ and positive real number $1 > c > 0$, independent of k such that

$$c \, (\rho_\infty^i)^2 \, h \leq g_{\vec{\Phi}_k} \leq c^{-1} \, (\rho_\infty^i)^2 \, h \qquad . \qquad \text{(III.12)}$$

Using now proposition III.1 we construct $\vec{\xi}_k$ satisfying (III.1), (III.2) and (III.3).

We claim now the following

Lemma III.1 *For any compact $K \subset \Sigma \setminus \{a_1 \cdots a_N\}$ there exists $C_K > 0$ and $k_K \in \mathbb{N}$ such that*

$$\sup_{k \geq k_K} \| \log |d\vec{\Phi}_k|_{h_k} \|_{L^\infty(K)} \leq C_K < +\infty \quad . \qquad \text{(III.13)}$$

$\square$

Proof of the lemma III.1. For any compact subset K of $\Sigma \setminus \{a_1 \cdots a_N\}$ there exists $\delta > 0$ such that $K \subset \Sigma \setminus \cup_{i=1}^N \overline{B}_\delta(a_i)$. Since $\Sigma \setminus \cup_{i=1}^N B_\delta(a_i) \subset \cup_{i \in I_1} B_{\rho_\infty^i}(x_\infty^i)$, there exist $\rho_\infty^i > r_i > 0$ such that

$$\Sigma \setminus \cup_{i=1}^N B_\delta(a_i) \subset \cup_{i \in I_1} B_{r^i}(x_\infty^i) \qquad \text{(III.14)}$$

and for k large enough one has for any $i \in I_1$ $B_{r^i}(x_\infty^i) \subset B_{\rho_k^i}(x_k^i)$. Let $s^i = (r^i + \rho_\infty^i)/2$. We consider k large enough in such a way that $B_{s^i}(x_\infty^i) \subset B_{\rho_k^i}(x_k^i)$ for any $i \in I_1$. On the ball $B_{s^i}(x_\infty^i)$ for the index i_0 such that (III.12) holds, one has

$$\sup_{k \geq k_K} \| \log |d\vec{\Phi}_k|_{h_k} \|_{L^\infty(B_{s^{i_0}}(x_\infty^{i_0}))} \leq C < +\infty$$

Considering now any other ball $B_{r^i}(x_\infty^i)$ which intersection with $B_{r^{i_0}}(x_\infty^{i_0})$ is non empty. Since (III.9) holds, there exists a moving frame $(\vec{e}_1, \vec{e}_2)$ satisfying (A.45)...(A.48). Using the same arguments as in the beginning of the proof of lemma A.4 we have that for any radius $r^i < r < s^i$

$$\| \lambda_k - \overline{\lambda}_k \|_{L^\infty(B_r(x_\infty^i))} \leq C_r$$

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where $g_{\vec{\Phi}_k} = e^{2\lambda_k} h_k$ and $\overline{\lambda}_k$ is the average of λ_k over $B_r(x^i_\infty)$. Since $B_r(x^i)$ intersects $B_{s^{i_0}}(x^{i_0}_\infty)$ for some r we deduce that, for such a r

$$\sup_{k \geq k_K} \| \log |d\vec{\Phi}_k|_{h_k}\|_{L^\infty(B_r(x^i_\infty))} \leq C < +\infty \quad .$$

We iterate this procedure until having reached every ball $B_{r^i}(x^i_\infty)$ for $i \in I_1$ since Σ is assumed to be connected. Hence the claim (III.13) is proved and this finishes the proof of lemma III.1. $\qquad\square$

Proof of theorem III.1 continued . Since , from (III.3), $d^J(\vec{\Phi}_k, \vec{\xi}_k) \leq 2^{-k/2}$, by taking on each $B_{r^i}(x^i_\infty)$, Ψ^i to be the canonical coordinate map (x, y) of the Poincaré half plane $\mathbb{H}$ once $B_{r^i}(x^i_\infty)$ has been identified with a connected part of a fundamental domain associated to (Σ, J_k) we have

$$\| \log |d\vec{\Phi}_k|_{h_k} - \log |d\vec{\xi}_k|_{h_k}\|_{L^\infty(B_{r^i}(x^i_\infty))}$$

$$\leq \| \log |\nabla(\vec{\Phi}_k \circ \Psi^i)| - \log |\nabla(\vec{\xi}_k \circ \Psi^i)|\|_{L^\infty(D^2)} \tag{III.15}$$

$$\leq d^{J_k}(\vec{\Phi}_k, \vec{\xi}_k) \leq C\, 2^{-k/2}$$

Where we have used proposition II.5 and the fact that $\delta(J_k, J)$ is uniformly bounded since J_k converges to a limit J_∞. Hence we deduce that for any compact $K \subset \Sigma \setminus \{a_1 \cdots a_N\}$ there exists $C_K > 0$ and $k_K \in \mathbb{N}$ such that

$$\sup_{k \geq k_K} \| \log |d\vec{\xi}_k|_{h_k}\|_{L^\infty(K)} \leq C_K < +\infty \quad . \tag{III.16}$$

Moreover, since for the same Ψ^i on $B_{r^i}(x^i_\infty)$, $\vec{\Phi}_k \circ \Psi^i$ is conformal from D^2 into $\mathbb{R}^m$, combining the fact that

$$\| \log |X|_{g_{\vec{\xi}_k}} - \log |X|_{g_{\vec{\Phi}_k}}\|_{L^\infty((T\Sigma)_0)} \leq d^{J_k}(\vec{\xi}_k, \vec{\Phi}_k) \leq C\, 2^{-k/2}$$

together with inequality (III.15) and identitiy (A.2), we obtain

$$\left\| \log_{\mathbb{C}} \left[1 + Dis(\vec{\xi}_k \circ \Psi^i) \right] \right\|_{L^\infty(D^2)} \leq 2^{-k/2} \quad . \tag{III.17}$$

This implies that for any compact $K \subset \Sigma \setminus \{a_1 \cdots a_N\}$ there exists $C_K > 0$ for any $k \in \mathbb{N}$

$$\left\| \log \frac{|d\vec{\xi}_k \cdot X \wedge d\vec{\xi}_k \cdot J_k X|}{|d\vec{\xi}_k \cdot X|^2} \right\|_{L^\infty(TK)} \leq C_K\, 2^{-k/2} \quad . \tag{III.18}$$

We have proved that

$$\sup_{k \geq k_K} \| \log |\nabla \vec{\Phi}_k \circ \Psi^i|\|_{L^\infty(B_{r^i}(x^i_\infty))} \leq C^i < +\infty \quad . \tag{III.19}$$

Moreover

$$4^{-1} \sup_{k \geq k_K} \int_{D^2} |\Delta(\vec{\Phi}_k \circ \Psi^i)|^2 e^{-2\lambda_k}\, dx\, dy \;=\; \sup_{k \geq k_K} \int_{B_{r^i}(x^i_\infty)} |\vec{H}|^2_{g_k}\, dvol_{g_k} \tag{III.20}$$

$$\leq 8\pi/3 \quad .$$

Hence, combining (III.19) and (III.20) we deduce that

$$\sup_{k \geq k_K} \int_{\cup_{i \in I_1} B_{r^i}(x^i_\infty)} |\Delta_{h_k} \vec{\Phi}_k|^2\, dvol_{h_k} < +\infty \quad . \tag{III.21}$$

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Combining this fact and the fact that

$$\sup_{k \geq k_K} \int_{\cup_{i \in I_1} B_{r^i}(x^i_\infty)} |d\vec{\Phi}_k|^2_{h_k} \; dvol_{h_k} = \sup_{k \geq k_K} \int_{\cup_{i \in I_1} B_{r^i}(x^i_\infty)} |d\vec{\Phi}_k|^2_{g_k} \; dvol_{g_k} \tag{III.22}$$

$$\leq \sup_{k \geq k_K} \mathcal{H}^2(\vec{\Phi}_k(\Sigma)) < +\infty \quad ,$$

we have that, modulo extraction of a subsequence, $d\vec{\Phi}_k$ converges strongly in $L^p(K)$ ($\forall p < +\infty$) w.r.t. h_k (which itself converges to h_∞ in every norm). This implies that $\nabla(\vec{\Phi}_k \circ \Psi^i)$ converges strongly in $L^p(K \cap B_{r^i}(x^i_\infty))$ for all $p < +\infty$. From (III.15) and (III.17) we deduce that $d\vec{\xi}_k$ converges also strongly in $L^p(K)$ to a limit $d\vec{\xi}_\infty$ where $\vec{\xi}_\infty$ is a Lipschitz conformal immersion of (K, J_∞) into $\mathbb{R}^m$. We have then, using also (III.16), for all $p < +\infty$

$$\vec{n}_{\vec{\xi}_k} = *_{h_k} \frac{d\vec{\xi}_k \wedge d\vec{\xi}_k}{|d\vec{\xi}_k \wedge d\vec{\xi}_k|_{h_k}} \longrightarrow *_{h_\infty} \frac{d\vec{\xi}_\infty \wedge d\vec{\xi}_\infty}{|d\vec{\xi}_\infty \wedge d\vec{\xi}_\infty|_{h_\infty}} = \vec{n}_{\vec{\xi}_\infty} \quad \text{in } L^p(K) \tag{III.23}$$

From the definition of d^J have for any $i \in I_1$

$$\int_{D^2} |\nabla(\vec{n}_{\vec{\Psi}_k} \circ \Psi^i - \vec{n}_{\vec{\xi}_k} \circ \Psi^i)|^2 \; dx \, dy \quad \leq d^{J_k}(\vec{\xi}_k, \vec{\Phi}_k) \tag{III.24}$$

$$\leq C \; d^J(\vec{\xi}_k, \vec{\Phi}_k) \leq C \; 2^{-k/2}$$

Hence we have

$$\limsup_{k \to +\infty} \int_K |d\vec{n}_{\vec{\xi}_k}|^2_{h_k} \; dvol_{h_k} \leq \limsup_{k \to +\infty} \int_K |d\vec{n}_{\vec{\Phi}_k}|^2_{h_k} \; dvol_{h_k} \tag{III.25}$$

$$= \limsup_{k \to +\infty} \int_K |d\vec{n}_{\vec{\Phi}_k}|^2_{g_k} \; dvol_{g_k} < +\infty$$

Combining (III.23) and (III.25) we deduce that

$$\int_K |d\vec{n}_{\vec{\xi}_\infty}|^2_{h_\infty} \; dvol_{h_\infty} = \int_K |d\vec{n}_{\vec{\xi}_\infty}|^2_{g_\infty} \; dvol_{g_\infty} \tag{III.26}$$

$$\leq \limsup_{k \to +\infty} \int_K |d\vec{n}_{\vec{\Phi}_k}|^2_{g_k} \; dvol_{g_k} \leq \inf_{\vec{\Phi} \in \mathcal{E}_\Sigma} 4W(\vec{\Phi}) - 4\pi\chi(\Sigma) \quad ,$$

where $g_\infty := \vec{\xi}_\infty^* g_{\mathbb{R}^m}$. Hence, by iterating the previous facts for a sequence of compacts K_l of $\Sigma \backslash \{a_1 \cdots a_N\}$ $\vec{\xi}_\infty$ such that $\cup_l K_l = \Sigma \backslash \{a_1 \cdots a_N\}$ one obtains that $\vec{\xi}_\infty$ realizes a conformal, locally lipschitz, immersion of $\Sigma \backslash \{a_1 \cdots a_N\}$ such that

$$\int_{\Sigma \backslash \{a_1 \cdots a_N\}} |d\vec{n}_{\vec{\xi}_\infty}|^2_{g_\infty} \; dvol_{g_\infty} \leq \inf_{\vec{\Phi} \in \mathcal{E}_\Sigma} 4W(\vec{\Phi}) - 4\pi\chi(\Sigma) \quad . \tag{III.27}$$

We claim now that

Lemma III.2

$$\vec{\xi}_\infty \;\; \text{is a Conformal Willmore immersion of } \Sigma \backslash \{a_1 \cdots a_N\} \tag{III.28}$$

and hence is analytic on $\Sigma \backslash \{a_1 \cdots a_N\}$ *(see [BR]).* □

Proof of the lemma III.2. Denote γ_k the metric $\gamma_k = \vec{\xi}_k^* g_{\mathbb{R}^m}$. Because of (III.17) We have that

$$\|Dis^{J_k}(\vec{\xi}_k)\|_{L^\infty(B_{r^i}(x_\infty^i))} \leq C \, 2^{-k/2} \quad \longrightarrow \quad 0$$

We can then apply lemma A.3 in order to obtain the existence of a diffeomorphism ζ_k from D^2 into D^2 such that $\vec{\Xi}_k^i := \vec{\xi}_k \circ \Psi^i \circ \zeta_k$ is conformal and satisfy

$$\limsup_{k \to +\infty} \|\zeta_k\|_{C^{0,\alpha}(D^2)} + \|\zeta_k^{-1}\|_{C^{0,\alpha}(D^2)} < +\infty \tag{III.29}$$

and for any $\rho < 1$, (A.13) implies

$$\limsup_{k \to +\infty} \|\log|\nabla\vec{\Xi}_k|\|_{L^\infty(D_\rho^2)} + \|\vec{\Xi}_k^j\|_{W^{2,2}(D_\rho^2)} < +\infty \quad . \tag{III.30}$$

Combining this with (III.15) and (III.17) we also obtain

$$\limsup_{k \to +\infty} \|\log|\nabla\vec{\zeta}_k|\|_{L^\infty(D_\rho^2)} < +\infty \quad . \tag{III.31}$$

Let $\rho < 1$ and $\vec{w} \in W^{1,\infty} \cap W^{2,2}(D^2, \mathbb{R}^m)$ such that $\vec{w} \in C_0^\infty(D_\rho^2, \mathbb{R}^m)$ and

$$\|\nabla\vec{w}\|_{L^\infty(D^2)} + \|\nabla^2\vec{w}\|_{L^2(D^2)} \leq 1 \quad .$$

Denote γ_k the metric $\gamma_k = \vec{\xi}_k^* g_{\mathbb{R}^m}$. Because of (III.16) and (III.17) we have, because of proposition II.5,

$$d^{\gamma_k} \simeq d^{g_k} \simeq d^J \qquad \text{indep. of } k \quad . \tag{III.32}$$

Since $\vec{\xi}_k$ minimizes $W(\cdot) + 2^{-k/2} \, d^J(\cdot, \vec{\xi}_k)$, we have that for any such $\vec{w}$ and for $|t|$ small enough, independent of k, say $|t| < t_0$, denoting

$$\vec{\xi}_k^t := \vec{\xi}_k + t \, \chi_{(B_{r^i}(x_\infty^i))} \, \vec{w} \circ \zeta_k^{-1} \circ (\Psi^i)^{-1} \in \mathcal{E}_\Sigma$$

where $\chi_{(B_{r^i}(x_\infty^i))}$ is the characteristic function of the ball $B_{r^i}(x_\infty^i)$,

$$\begin{aligned} W(\vec{\xi}_k) &\leq W(\vec{\xi}_k^t) + 2^{-k/2} \, d^J(\vec{\xi}_k^t, \vec{\xi}_k) \\[2mm] &\leq W(\vec{\xi}_k^t) + C \, 2^{-k/2} \, d^{\gamma_k}(\vec{\xi}_k^t, \vec{\xi}_k) \end{aligned} \tag{III.33}$$

Using now lemma II.1, we deduce the existence of a constant $C > 0$ independent of k, $\vec{w}$ and t such that

$$W(\vec{\xi}_k) \leq W(\vec{\xi}_k^t) + C \, 2^{-k/2} \, |t| \, \left[\|\nabla\vec{w}\|_\infty + \|\nabla^2\vec{w}\|_2\right] \tag{III.34}$$

We have

$$W(\vec{\xi}_k) - W(\vec{\xi}_k^t) = 4^{-1} \int_{B_{r^i}(x_\infty^i)} |d\vec{n}_{\vec{\xi}_k}|^2_{\gamma_k} \, dvol_{\gamma_k} - |d\vec{n}_{\vec{\xi}_k^t}|^2_{\gamma_k^t} \, dvol_{\gamma_k^t}$$

where $\gamma_k^t := (\vec{\xi}_k^t)^* g_{\mathbb{R}^m}$. A straightforward but a bit lengthy argument shows that

$$|d\vec{n}_{\vec{\xi}_k^t}|^2_{\gamma_k^t} \, dvol_{\gamma_k^t} = |d\vec{n}_{\vec{\xi}_k}|^2_{\gamma_k} \, dvol_{\gamma_k} + t \, \vec{w} \circ \zeta_k^{-1} \circ (\Psi^i)^{-1} \cdot \vec{F}_{\vec{\xi}_k} \, dvol_{\vec{\xi}_k}$$

$$+ t^2 G(\vec{\xi}_k, \vec{w}, t) \, dvol_{\vec{\xi}_k} \tag{III.35}$$

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where

$$\limsup_{k \to +\infty} \sup_{|t| < t_0} \int_{B_{r^i}(x^i_\infty)} |G(\vec{\xi}_k, \vec{w}, t)| \, dvol_{\vec{\xi}_k} < +\infty \quad . \tag{III.36}$$

Now a classical computation from Blashke [Bla] for $m = 3$ and [Wei] for arbitrary m gives for a regular immersion $\vec{\xi}$ from $B_{r^i}(x^i_\infty)$ into $\mathbb{R}^m$ that

$$4^{-1} \vec{F}_{\vec{\xi}} = \Delta_\perp \vec{H}_{\vec{\xi}} + \tilde{A}(\vec{H}_{\vec{\xi}}) - 2|\vec{H}_{\vec{\xi}}|^2 \vec{H}_{\vec{\xi}} \tag{III.37}$$

where $\Delta_\perp$ is the negative covariant laplacian on the normal bundle of the immersion $\vec{\xi}$, moreover for any $\vec{L} \in \mathbb{R}^m$, $\tilde{A}(\vec{L}) := \sum_{i,j=1}^2 \vec{B}(\vec{e}_i, \vec{e}_j) \, \vec{B}(\vec{e}_i, \vec{e}_j) \cdot \vec{L}$ where $\vec{B}$ is the second fundamental form of the immersion $\vec{\xi}$. At this stage it is very important to observe that we are computing $\vec{F}_{\vec{\xi}}$ for a smooth immersion ξ. It does not make sense for an immersion in $\mathcal{E}_\Sigma$ such as $\vec{\xi}_k$. One of the main computation in [Ri2] establishes that in conformal coordinates Ψ from D^2 into $B_{r^i}(x^i_\infty)$ one has

$$div \left[\nabla \vec{H}_{\vec{\xi}} - 3\pi_{\vec{n}_{\vec{\xi}}}(\nabla \vec{H}_{\vec{\xi}}) + \star(\nabla^\perp \vec{n}_{\vec{\xi}} \wedge \vec{H}_{\vec{\xi}}) \right]$$

$$= -2 \, e^{2\lambda} \left[\Delta_\perp \vec{H}_{\vec{\xi}} + \tilde{A}(\vec{H}_{\vec{\xi}}) - 2|\vec{H}_{\vec{\xi}}|^2 \vec{H}_{\vec{\xi}} \right] \quad . \tag{III.38}$$

where e^λ is the conformal factor of the immersion in conformal coordinates w.r.t. these coordinates (x, y) and $\pi_{\vec{n}_{\vec{\xi}}}$ is the orthogonal projection onto the normal space to the immersion $\vec{\xi}$. One observe that $e^{2\lambda} [dx^2 + dy^2] = dvol_{\vec{\xi}}$. Hence (III.38) implies that for any function $\vec{f}$ in $C_0^\infty(D^2, \mathbb{R}^m)$, for any smooth immersion $\vec{\xi}$ from $B_{r^i}(x^i_\infty)$ into $\mathbb{R}^m$ and for any conformal coordinates Ψ one has

$$\int_{D^2} \nabla \vec{f} \cdot \left[\nabla \vec{H}_{\vec{\xi}} - 3\pi_{\vec{n}_{\vec{\xi}}}(\nabla \vec{H}_{\vec{\xi}}) + \star(\nabla^\perp \vec{n}_{\vec{\xi}} \wedge \vec{H}_{\vec{\xi}}) \right] \, dx \, dy$$

$$= 2 \int_{B_{r^i}(x^i_\infty)} \vec{f} \circ \Psi^{-1} \cdot \left[\Delta_\perp \vec{H}_{\vec{\xi}} + \tilde{A}(\vec{H}_{\vec{\xi}}) - 2|\vec{H}_{\vec{\xi}}|^2 \vec{H}_{\vec{\xi}} \right] \, dvol_\gamma \tag{III.39}$$

where $\gamma = \xi^* g_{\mathbb{R}^m}$. As observed in [Ri2], The projection $\pi_{\vec{n}_{\vec{\xi}}}$ can be expressed using the Gauss $m - 2$-vector $\vec{n}_{\vec{\xi}}$ and the interior multiplication $\llcorner$ between multivectors

$$\pi_{\vec{n}_{\vec{\xi}}}(\vec{v}) := \vec{n}_{\vec{\xi}} \llcorner (\vec{n}_{\vec{\xi}} \llcorner \vec{v}) \quad .$$

Hence we have in particular

$$\pi_{\vec{n}_{\vec{\xi}}}(\nabla \vec{H}_{\vec{\xi}}) = (\nabla \vec{n}_{\vec{\xi}}) \llcorner (\vec{n}_{\vec{\xi}} \llcorner \vec{H}_{\vec{\xi}}) + \vec{n}_{\vec{\xi}} \llcorner ((\nabla \vec{n}_{\vec{\xi}}) \llcorner \vec{H}_{\vec{\xi}}) + \vec{n}_{\vec{\xi}} \llcorner (\vec{n}_{\vec{\xi}} \llcorner \nabla \vec{H}_{\vec{\xi}}) \tag{III.40}$$

Taking now $\varphi \in C_0^\infty(D^2)$ such that $\int_{D^2} \varphi = 1$ and denote $\varphi_\varepsilon(x) := \varepsilon^{-2} \varphi(\varepsilon^{-1} x)$. We also denote $\vec{\xi}_{\varepsilon,k} := \varphi_\varepsilon \star \vec{\xi}_k$ and

$$\vec{\xi}_{\varepsilon,k}^t := \varphi_\varepsilon \star \vec{\xi}_k + t \, \chi_{(B_{r^i}(x^i_\infty))} \, \vec{w} \circ \zeta_k^{-1} \circ (\Psi^i)^{-1}$$

We have that $\vec{\xi}_{\varepsilon,k} \longrightarrow \vec{\xi}$ strongly in $W^{2,2}(B_{r^i}(x^i_\infty), \mathbb{R}^m)$ and for ε small enough $\| \log |\nabla \vec{\xi}_{\varepsilon,k}| \|_\infty$ remains uniformly bounded. Hence we deduce that, as ε goes to zero

$$\vec{n}_{\vec{\xi}_{\varepsilon,k}} \longrightarrow \vec{n}_{\vec{\xi}_k} \quad \text{strongly in } W^{1,2}(B_{r^i}(x^i_\infty), \mathbb{R}^m) \quad , \tag{III.41}$$

and

$$\vec{H}_{\vec{\xi}_{\varepsilon,k}} \longrightarrow \vec{H}_{\vec{\xi}_k} \quad \text{strongly in } L^2(B_{r^i}(x^i_\infty), \mathbb{R}^m) \quad . \tag{III.42}$$

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Hence, combining (III.40), (III.41) and (III.42) we obtain that, for any $\vec{f} \in W^{1,\infty} \cap W^{2,2}(D^2, \mathbb{R}^m)$, as ε goes to zero

$$\int_{D^2} \nabla \vec{f} \cdot \left[\nabla \vec{H}_{\vec{\xi}_{\varepsilon,k}} - 3\pi_{\vec{n}_{\vec{\xi}_{\varepsilon,k}}}(\nabla \vec{H}_{\vec{\xi}_{\varepsilon,k}}) + \star(\nabla^{\perp} \vec{n}_{\vec{\xi}_{\varepsilon,k}} \wedge \vec{H}_{\vec{\xi}_{\varepsilon,k}}) \right] \, dx\, dy$$

$$\downarrow \qquad\qquad\qquad\qquad (III.43)$$

$$\int_{D^2} \nabla \vec{f} \cdot \left[\nabla \vec{H}_{\vec{\xi}_k} - 3\pi_{\vec{n}_{\vec{\xi}_k}}(\nabla \vec{H}_{\vec{\xi}_k}) + \star(\nabla^{\perp} \vec{n}_{\vec{\xi}_k} \wedge \vec{H}_{\vec{\xi}_k}) \right] \, dx\, dy \quad .$$

One verifies easily moreover that $G(\vec{\xi}_{\varepsilon,k}, \vec{w}, t) \longrightarrow G(\vec{\xi}_k, \vec{w}, t)$ in L^1. Hence applying (III.35) and (III.38) to $\vec{\xi} := \vec{\xi}_{\varepsilon,k}$ and passing to the limit as ε goes to zero using again (III.41)

$$W(\vec{\xi}_k) - W(\vec{\xi}_k^t)$$

$$= -2^{-1} t \int_{D^2} \nabla \vec{w} \cdot \left[\nabla \vec{H}_{\vec{\xi}_k} - 3\pi_{\vec{n}_{\vec{\xi}_k}}(\nabla \vec{H}_{\vec{\xi}_k}) + \star(\nabla^{\perp} \vec{n}_{\vec{\xi}_k} \wedge \vec{H}_{\vec{\xi}_k}) \right] \, dx\, dy \qquad (III.44)$$

$$-8^{-1} t^2 \int_{B_{r^i}(x_{\infty}^i)} G(\vec{\xi}_k, \vec{w}, t)$$

Thus, combining (III.34) and (III.44), dividing by $|t|$ taking respectively the limit as $t \to 0^+$ and $t \to 0^-$ one obtains for any $\vec{w}$ in $W^{1,\infty} \cap W^{2,2}(D^2, \mathbb{R}^m)$ supported in a strict open subset to D^2

$$\left| \int_{D^2} \nabla \vec{w} \cdot \left[\nabla \vec{H}_{\vec{\xi}_k} - 3\pi_{\vec{n}_{\vec{\xi}_k}}(\nabla \vec{H}_{\vec{\xi}_k}) + \star(\nabla^{\perp} \vec{n}_{\vec{\xi}_k} \wedge \vec{H}_{\vec{\xi}_k}) \right] \, dx\, dy \right|$$

$$\leq C \, 2^{-k/2} \left[\|\nabla \vec{w}\|_{\infty} + \|\nabla^2 \vec{w}\|_2 \right] \qquad (III.45)$$

This implies that

$$div \left[\nabla \vec{H}_{\vec{\xi}_k} - 3\pi_{\vec{n}_{\vec{\xi}_k}}(\nabla \vec{H}_{\vec{\xi}_k}) + \star(\nabla^{\perp} \vec{n}_{\vec{\xi}_k} \wedge \vec{H}_{\vec{\xi}_k}) \right]$$

$$\longrightarrow 0 \quad \text{in} \quad (W^{1,\infty} \cap W^{2,2})^* \quad . \qquad (III.46)$$

Using theorem II.1 and theorem II.2 of [BR] we deduce that $\vec{\xi}_{\infty}$ is conformal willmore on $B_{r^i}(x_{\infty}^i)$ but since $\vec{\xi}_{\infty}$ is conformal from $(B_{r^i}(x_{\infty}^i), J_{\infty})$ into $\mathbb{R}^m$, using again theorem II.2 of [BR] we obtain that $\vec{\xi}_{\infty}$ is analytic on $B_{r^i}(x_{\infty}^i)$. This holds for any i in I_1 and hence we have proved lemma III.2. $\qquad\square$

Proof of theorem III.1 continued.

The goal now is to extend $\vec{\xi}_{\infty}$ as a smooth embedding through the points a_j.

Let a_j be such a point. Let Ψ_j be a positive conformal diffeomorphism from D^2, equipped with the canonical complex structure, into a neighborhood U_j of a_j in (Σ, J_{∞}) and such that $\Psi_j(0) = a_j$ We keep denoting $\vec{\xi}_{\infty}$ the composition $\vec{\xi}_{\infty} \circ \Psi_j$.

$\vec{\xi}_{\infty}$ is conformal from $D^2 \setminus \{0\}$ into $\mathbb{R}^m$ and we have that $\mathcal{H}^2(\vec{\xi}_{\infty}(D^2 \setminus \{0\})) < +\infty$ moreover $\vec{\xi}_{\infty}(D^2 \setminus \{0\}) \subset B_R(0)$. Hence $\vec{\xi}_{\infty}$ is in $L^{\infty} \cap W^{1,2}(D^2 \setminus \{0\}, \mathbb{R}^m)$. Since the 2-capacity of a point in 2 dimension is zero we deduce that

$$\vec{\xi}_{\infty} \in L^{\infty} \cap W^{1,2}(D^2, \mathbb{R}^m) \quad .$$

Similarly, $\vec{n}_{\vec{\xi}_{\infty}}$ realizes a map in $W^{1,2}(D^2 \setminus \{0\}, Gr_{m-2}(\mathbb{R}^m))$. For the same reason as before, $\vec{n}_{\vec{\xi}_{\infty}}$ extends to a map in $W^{1,2}(D^2, Gr_{m-2}(\mathbb{R}^m))$. We now use the lemma A.5 which is already implicitly present in [Hub], [MS] and [Hel] but for which we thought that it could have been useful for the reader to have

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the details of a proof of it presented in the appendix. We can deduce from this lemma that $\vec{\xi}_\infty$ extends to a Lipshitz map through 0 and that there exists an integer n such that

$$(C - o(1)) \, |z|^{n-1} \leq \left| \frac{\partial \vec{\xi}_\infty}{\partial z} \right| \leq (C + o(1)) \, |z|^{n-1} \quad . \tag{III.47}$$

We claim that $n = 1$. Because of this estimate, for any $\delta > 0$ there exists $r_\delta > 0$ such that, for any $r < r_\delta$, $\vec{\xi}_\infty(B_r(0)) \subset B_\rho(\vec{\xi}_\infty(0))$ and $|\partial_x \vec{\xi}_\infty| = |\partial_x \vec{\xi}_\infty| = e_\infty^\lambda \geq C(1-\delta)/\sqrt{2} \, |z|^{n-1}$. where $\rho = C(\sqrt{2})^{-1} \, n^{-1} \, (1+\delta) \, r^m$. We have then that the mass of $\vec{\xi}_\infty(\Sigma)$ present in $B_\rho(\vec{\xi}_\infty(0))$ can be estimated from below as follows

$$\begin{aligned} M\left(\vec{\xi}_\infty(\Sigma) \llcorner B_\rho(\vec{\xi}_\infty(0)) \right) \; &\geq C^2 \, \frac{(1-\delta)^2}{2} \int_{B_r(0)} |z|^{2n-2} \\ &\geq \frac{\pi}{n} \, C^2 \, \frac{(1-\delta)^2}{2} \, r^{2n} \\ &\geq n \, \left(\frac{1-\delta}{1+\delta} \right)^2 \, \pi \, \rho^2 \end{aligned} \tag{III.48}$$

This implies that the lower 2-density $\theta_*^2((\vec{\xi}_\infty)_*[\Sigma], \vec{\xi}_\infty(0))$ of $(\vec{\xi}_\infty)_*[\Sigma]$ at $\vec{\xi}_\infty(0)$ is larger or equal to n. The Li-Yau inequality (see [LY]) which also holds for varifolds with weak L^2−bounded mean curvature which are smooth outside one point (as proved in [KS2]) implies

$$n \leq \theta_*^2((\vec{\xi}_\infty)_*[\Sigma], \vec{\xi}_\infty(0)) \leq \frac{W(\vec{\xi}_\infty(\Sigma))}{4\pi} \quad . \tag{III.49}$$

Because of the lower semi-continuity of W and the assumption that $W(\vec{\xi}_k(\Sigma)) < 8\pi - \delta$ for some $\delta > 0$, we deduce that $n = 1$.

We have then proved that $\vec{\xi}_\infty$ is a $W^{2,2} - lipschitz$ immersion, that is an element from $\mathcal{E}_\Sigma$, which is smooth on $\Sigma \setminus \{a_1 \cdots a_N\}$ and satisfying

$$W(\vec{\xi}_\infty) \leq \inf_{\vec{\Phi} \in \mathcal{E}_\Sigma} W(\vec{\Phi}) \quad .$$

Because of the minimality of $\vec{\xi}$, for any i and any $\vec{w} \in C_0^\infty(B_r(a_i))$ for some r, we have that, for t small enough

$$W(\vec{\xi}_\infty) \leq W(\vec{\xi}_\infty + t\vec{w}) \quad . \tag{III.50}$$

Arguing like above we have the asymptotic expansion

$$\begin{aligned} &W(\vec{\xi}_\infty) - W(\vec{\xi}_\infty + t\vec{w}) \\ &= -2^{-1} \, t \int_{D^2} \nabla \vec{w} \cdot \left[\nabla \vec{H}_{\vec{\xi}_\infty} - 3\pi_{\vec{n}_{\vec{\xi}_\infty}}(\nabla \vec{H}_{\vec{\xi}_\infty}) + \star(\nabla^\perp \vec{n}_{\vec{\xi}_\infty} \wedge \vec{H}_{\vec{\xi}_\infty}) \right] \, dx \, dy \\ &\quad - 8^{-1} \, t^2 \int_{B_r(a_i)} G(\vec{\xi}_\infty, \vec{w}, t) \end{aligned} \tag{III.51}$$

where we are using some holomorphic chart on $B_r(a_i)$ which identifies $B_r(a_i)$ with D^2 and where $\int_{B_r(a_i)} G(\vec{\xi}_\infty, \vec{w}, t)$ is uniformly bounded w.r.t. t as before. Combining (III.50) and (III.51) we deduce that $\vec{\xi}_\infty$ realizes a weak Willmore $W^{2,2} \cap W^{1,\infty}$ immersion and from [Ri2] we deduce that $\vec{\xi}_\infty$ is an

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analytic immersion. Since $W(\vec{\xi}_\infty) < 8\pi$ we deduce that $\vec{\xi}_\infty$ realizes an embedding which concludes the proof of theorem III.1. □

In order to complete the proof of theorem I.6 we have to exclude the possibility for the conformal class to degenerate while considering a minimizing sequence. This is a consequence of the following result obtained in [Ri4] (see also [Ri1]) which has also been proved in [KuLi].

Theorem III.2 *Let (Σ, c_k) be a sequence of closed riemann surface of fixed topology but with degenerating conformal class c_k diverging to the boundary of the Moduli Space of Σ. Let $\vec{\Phi}_k$ be a sequence of conformal immersions in $\mathcal{E}_\Sigma$ then*

$$\liminf_{k \to +\infty} \int_\Sigma |\vec{H}_{\vec{\Phi}_k}|^2 \; dvol_{\vec{\Phi}_k^* g_{\mathbb{R}^m}} > \inf_{\vec{\Phi} \in \mathcal{E}_\Sigma} W(\vec{\Phi}) \quad . \tag{III.52}$$

□

IV   Existence of Minimizers of the Willmore Energy in a Conformal Class.

IV.1   The completeness of the metric space of $W^{2,2}$ lipschitz immersions of Σ in a given conformal class.

We assume in this section that Σ is a connected closed smooth two dimensional manifold of genus larger or equal to 1. Let c be a conformal class Σ which is represented by a smooth complex structure J on Σ. Denote g be an arbitrary smooth metric on g that we can choose to be compatible with J. We introduce the subspace space of $\mathcal{E}_\Sigma$ of lipschitz immersions realizing a complex structure equivalent to J :

$$\mathcal{E}_\Sigma^c := \left\{ \begin{array}{c} \vec{\Phi} \in \mathcal{E}_\Sigma \quad \text{s.t. } \exists \; \Psi \in W^{2,2}(\Sigma, \Sigma) \quad \text{s.t.} \\[2mm] \Psi \text{ is a bilipschitz diffeomorphism} \\[2mm] \vec{\Phi} \circ \Psi \; : \; (\Sigma, J) \longrightarrow \mathbb{R}^m \quad \text{is conformal} \end{array} \right\}$$

We are now proving the following proposition

Proposition IV.1 *The metric space $(\mathcal{E}_\Sigma^c, d^J)$ is complete.* □

Proof of proposition IV.1. Let $\vec{\Phi}_k$ be a Cauchy sequence for d^J. From proposition II.6 there exists a limit $\vec{\Phi}_\infty$ in $\mathcal{E}_\Sigma$. Denote by Ψ_k a lipschitz diffeomorphism such that $\vec{\Phi}_k \circ \Psi_k$ is conformal. Denote by $g_k := \vec{\Phi}_k^* g_{\mathbb{R}^m}$ and J_k the associated complex structure. Ψ_k realizes then a bilipschitz conformal diffeomorphism between (Σ, J) and (Σ, J_k). Because of the d^J convergence J_k converges in $L^\infty \cap W^{1,2}$ norm to J_∞ the complex structure associated to $g_\infty := \vec{\Phi}_\infty^* g_{\mathbb{R}^m}$. Denote by $(U^i)_{i \in I}$ a finite covering by balls of Σ chosen in such a way that $\int_{U_i} |d\vec{n}_{\vec{\Phi}_\infty}|^2 \; dvol_{g_\infty} < 4\pi/3$. Denote by $\zeta_\infty^i \; D^2 \; : \to (U^i, J_\infty)$ the conformal parametrization given by lemma 5.1.4 of [Hel] combined with the moving frame technic of the proof of theorem 5.4.3 that we exposed also in the proofs of lemma A.3 and lemma A.5 below. For each i in I we use lemma A.3 in order to construct $\varphi_k : D^2 \to D^2$ such that $\zeta_k^i := \zeta_\infty^i \circ \varphi_k \; D^2 \; : \to \; (U_i, J_k)$ is conformal and ζ_k^i is uniformly bounded in $W^{2,2}$ and $\log |\nabla \zeta_k^i|$ is also uniformly bounded in L^∞ (by taking possibly U^i a bit smaller but still realizing a covering of Σ). Denote by $f_k^i(z) := (\Psi_k^i)^{-1} \circ \zeta_k^i$ the maps from D^2 into (Σ, J). These sequences realize sequences of conformal maps which are harmonic if one equips (Σ, J) with a corresponding constant scalar curvature metric h and denote the corresponding

volume form. Observe that since Ψ_k is a conformal diffeomorphism one has

$$\int_\Sigma \omega = \int_\Sigma (\Psi_k^{-1})^* \omega = \frac{1}{2} \int_\Sigma |d\Psi_k^{-1}|_{h,g_k}^2 \ dvol_{g_k}$$

$$\geq \frac{1}{card\,I} \sum_{i\in I} \int_{U_i} |\nabla f_k^i|_h^2 \ dx_1\,dx_2 \quad .$$

Hence Ψ_k^{-1} is a uniformly bounded sequence in $W^{1,2}(\Sigma,\Sigma)$ (the metric g_k remains comparable to an arbitrary smooth fixed metric on Σ because of the d^J convergence) and the f_k^i are uniformly bounded sequences of conformal maps in $W^{1,2}(D^2,\Sigma)$. Hence the f_k^i are uniformly bounded energy harmonic maps. Since the constant scalar curvature of the metric h is non-positive (genus(Σ)¿0) the sequences converge strongly in C^l-norm in the interiors of U_i (see for instance [Jo]). Since the ζ_k^i are uniformly bounded in $W^{2,2}$, since $\log|\nabla\zeta_k^i|$ is also uniformly bounded in L^∞ and since $\|Dis^{J_\infty}(\zeta_k^i)\|_\infty \longrightarrow 0$, we deduce that Ψ_k^{-1} converges to a bilipschitz diffeomorphism Ψ_∞^{-1} which is conformal between (Σ,J_∞) and (Σ,J). This implies that the d^J limit $\vec{\Phi}_\infty$ is in $\mathcal{E}_\Sigma^c$ and this concludes the proof of proposition IV.1. $\quad\square$

IV.2 Minimizing Willmore energy in a conformal class.

In this section we prove the following theorem

Theorem IV.1 *Let Σ be a closed surface let c be a conformal class on Σ and m an integer larger or equal to 3. Assume that*

$$\inf_{\vec{\Phi}\in\mathcal{E}_\Sigma^c} W(\vec{\Phi}) \leq 8\pi$$

Then the infimum is achieved by either

   i) a C^∞ Conformally Willmore embedding of Σ into $\mathbb{R}^m$

   ii) or a global isothermic embedding of (Σ,c).

If $\inf_{\vec{\Phi}\in\mathcal{E}_\Sigma^c} W(\vec{\Phi}) > 8\pi$ the results is the same modulo the possible existence of isolated branched points. $\quad\square$

Proof of theorem IV.1. Let $\vec{\Phi}_k$ be a minimizing sequence of W in $\mathcal{E}_\Sigma^c$. Applying the *3-points renormalization lemma* and arguing exactly like in the beginning of the proof of theorem III.1 we can assume that

   i)

$$\vec{\Phi}_k \qquad \text{is conformal from } (\Sigma,J) \quad \text{into } \mathbb{R}^m \quad . \tag{IV.1}$$

   ii) There exists finitely many points $a_1\cdots a_N$ in Σ and a fixed finite covering $(B_{\rho_\infty^i}(x_\infty^i))_{i\in I_1}$ of $\Sigma \setminus \{a_1\cdots a_N\}$ such that for any $i\in I_1$, $0 < \rho < \rho_\infty$ and k large enough

$$\int_{B_\rho(x_\infty^i)} |d\vec{n}_{\vec{\Phi}_k}|_{g_k}^2 \ dvol_{g_k} < 8\pi/3 \quad . \tag{IV.2}$$

     where $g_k := \vec{\Phi}_k^* g_{\mathbb{R}^m}$.

   iii) There exists a positive real $R > 0$ such that

$$\vec{\Phi}_k(\Sigma) \subset B_R(0) \quad . \tag{IV.3}$$

iv) There exists a constant $C > 0$ such that

$$\mathcal{H}^2(\vec{\Phi}_k(\Sigma)) \leq C \quad . \tag{IV.4}$$

v) There exist a positive real number $r > 0$, independent of k and three distinct points P_1, P_2 and P_3, independent of k too, in the interior of one ball $B_{\rho_\infty^i}(x_\infty^i)$ such that

$$\forall\, i \neq j \qquad |\vec{\Phi}_k(P_i) - \vec{\Phi}_k(P_j)| \geq r > 0 \qquad . \tag{IV.5}$$

The following proposition is a direct application of Ekeland's Variational Principle since $(\mathcal{E}_\Sigma^c, d^J)$ is a complete metric space as we showed in the previous subsection.

Proposition IV.2 *Let J be an arbitrary smooth complex structure on Σ and c be the conformal class of (Σ, J). Let $\vec{\Phi}_k$ be a minimizing sequence for W in $\mathcal{E}_\Sigma^c$ such that*

$$W(\vec{\Phi}_k) \leq \inf_{\vec{\Phi} \in \mathcal{E}_\Sigma^c} W(\vec{\Phi}) + 2^{-k} \quad ,$$

then there exists $\vec{\xi}_k \in \mathcal{E}_\Sigma^c$ such that

i) $\vec{\xi}_k$ minimizes in $\mathcal{E}_\Sigma^c$ the following functional

$$W(\vec{\xi}_k) = \inf_{\vec{\Phi} \in \mathcal{E}_\Sigma^c} W(\vec{\Phi}) + 2^{-k/2}\, d^J(\vec{\Phi}, \vec{\xi}_k) \quad , \tag{IV.6}$$

ii)

$$W(\vec{\xi}_k) \leq W(\vec{\Phi}_k) \quad , \tag{IV.7}$$

iii)

$$d^J(\vec{\xi}_k, \vec{\Phi}_k) \leq 2^{-k/2} \quad . \tag{IV.8}$$

$\square$

As in the previous section we prove that $d\vec{\xi}_k$ converges strongly in $L^p_{loc}(\Sigma \setminus \{a_1 \cdots a_N\})$ to a limiting $W^{2,2}$ immersion $\vec{\xi}_\infty$ of $\Sigma \setminus \{a_1 \cdots a_N\}$ and moreover we have that

$$\int_{\Sigma \setminus \{a_1 \cdots a_N\}} |d\vec{n}_{\vec{\xi}_\infty}|^2_{g_\infty}\, dvol_{g_\infty} \leq \inf_{\vec{\Phi} \in \mathcal{E}_\Sigma^c} 4W(\vec{\Phi}) - 4\pi\chi(\Sigma) \quad . \tag{IV.9}$$

We claim now that

Lemma IV.1 *Under the previous notations we have that either*

$$\vec{\xi}_\infty \text{ is a Conformal Willmore immersion of } \Sigma \setminus \{a_1 \cdots a_N\} \tag{IV.10}$$

and hence is analytic on $\Sigma \setminus \{a_1 \cdots a_N\}$ (see [BR]) or

$$\vec{\xi}_\infty \text{ is an isothermic immersion of } (\Sigma, J) \quad .$$

$\square$

Proof of the lemma IV.1.
First case : there exists a subsequence $\vec{\xi}_k$ which is not made of isothermic surfaces.

Let Ψ^i be a local conformal chart on $B_{r^i}(x^i_\infty)$ for the complex structure J. We have that $\vec{\xi}_k \circ \Psi^i$ is conformal on D^2. From lemma V.1 we know that a perturbation $\vec{w} \in W^{1,\infty} \cap W^{2,2}$ of $\vec{\xi}_k$ keeps infinitesimally the conformal class c if and only if for any holomorphic quadratic differential q of (Σ, J) which is an holomorphic section of $K \otimes K$, where K is the canonical bundle $T^{(0,1)}\Sigma$ of (1-0)-forms over (Σ, J), one has

$$(\partial_z \vec{w} \cdot \partial_z \vec{\xi}_k \ dz \otimes dz, q)_{WP} = 0 \tag{IV.11}$$

where $(\cdot, \cdot)_{WP}$ is the Weil-Petersson Hermitian product given locally (assuming $\vec{w}$ is supported in a ball $B_{r^i}(x^i_\infty)$ on which we have holomorphic chart given by Ψ^i that we simply denote by z), writing $q = f(z) \ dz \otimes dz$,

$$(\partial_z \vec{w} \cdot \partial_z \vec{\xi}_k \ dz \otimes dz, q)_{WP} := \frac{i}{2} \int_{D^2} e^{-2\lambda_k} \ \partial_z \vec{w} \cdot \partial_z \vec{\xi}_k \ \overline{f(z)} \ dz \wedge d\overline{z} \quad .$$

where $e^{2\lambda_k} = |\partial_x \vec{\xi}_k|^2 = |\partial_y \vec{\xi}_k|^2$.

Consider hence $\vec{w} \in W^{1,\infty} \cap W^{2,2}(D^2, \mathbb{R}^m)$ supported in the interior of D^2, satisfying (IV.11) and such that

$$\|\nabla \vec{w}\|_{L^\infty(D^2)} + \|\nabla^2 \vec{w}\|_{L^2(D^2)} \le 1 \quad .$$

Using Lemma V.1 and the implicit function theorem there exists a family $\vec{\xi}_k^t$ in $\mathcal{E}_\Sigma^c$ such that

$$\vec{\xi}_k^t := \vec{\xi}_k + t \ \chi_{(B_{r^i}(x^i_\infty))} \ \vec{w} \circ (\Psi^i)^{-1} + o(t) \quad .$$

From now on we shall omit to write explicitly the composition with $(\Psi^i)^{-1}$ and write simply $\vec{w}$ instead of $\vec{w} \circ (\Psi^i)^{-1}$.

Arguing exactly like in the previous section this implies that there exists a constant $C > 0$ such that, for all $\vec{w}$ supported in a strict open subset of D^2 and satisfying (IV.11) one has

$$\left| \int_{D^2} \nabla \vec{w} \cdot \left[\nabla \vec{H}_k - 3\pi_{\vec{n}_k}(\nabla \vec{H}_k) + \star(\nabla^\perp \vec{n}_k \wedge \vec{H}_k) \right] \ dx \, dy \right|$$
$$\le C \ 2^{-k/2} \ \left[\|\nabla \vec{w}\|_\infty + \|\nabla^2 \vec{w}\|_2 \right] \tag{IV.12}$$

where $\vec{H}_k := \vec{H}_{\vec{\xi}_k}$ and $\vec{n}_k := \vec{n}_{\vec{\xi}_k}$. Using the notations and computations in appendix V.1 the constraints (IV.11) on $\vec{w}$ becomes

$$\forall j = 1 \cdots Q \qquad \int_{D^2} f^j(z) \ \vec{H}_{0,k} \cdot \vec{w} \ \frac{i}{2} \ dz \wedge d\overline{z} = 0 \quad . \tag{IV.13}$$

where we recall that $f^j(z) \ dz \otimes dz$ is the expression in the Ψ^i conformal chart of the different element q^j of a <u>fixed</u> basis of the $Q-$dimensional complex space of holomorphic quadratic differentials $Q(J)$ of (Σ, J) and where $\vec{H}_{0,k}$ is the Weingarten operator associated to the immersion $\vec{\xi}_k$. . Combining (IV.12) and (IV.13) we then obtain the existence of a sequence $\mu_k = (\mu_k^j)_{j=1\cdots N} \in \mathbb{C}^Q$ such that

$$div \left[\nabla \vec{H}_k - 3\pi_{\vec{n}_k}(\nabla \vec{H}_k) + \star(\nabla^\perp \vec{n}_k \wedge \vec{H}_k) \right] + \Im \left[f_k(z) \ \vec{H}_{0,k} \right]$$
$$\longrightarrow 0 \quad \text{in} \quad (W^{1,\infty} \cap W^{2,2})^* \quad . \tag{IV.14}$$

where

$$f_k(z) := \sum_{j=1}^{Q} \mu_k^j \ f^j(z) \quad .$$

Using the computations in [BR] section III.2.2 we have

$$e^{2\lambda_k} \left[\Delta_\perp \vec{H}_k + 2 \Re\big((\overline{\vec{H}_{0,k} \cdot \vec{H}_k}) \vec{H}_{0,k}\big) \right] - \Im\big(f_k(z) \vec{H}_{k,0}\big) \ . \tag{IV.15}$$

$$\longrightarrow 0 \quad \text{in} \quad (W^{1,\infty} \cap W^{2,2})^* \quad .$$

The covariant laplacian in conformal coordinates is given by

$$e^{2\lambda_k} \Delta_\perp \vec{H}_k = \pi_{\vec{n}_k}\Big(div \ \pi_{\vec{n}_k}\big(\nabla \vec{H}_k\big)\Big) \equiv 4 \Im\Big[i \ \pi_{\vec{n}_k} \ \partial_{\overline{z}} \pi_{\vec{n}_k} \ \partial_z \vec{H}_k\Big] \quad . \tag{IV.16}$$

Combining (IV.15) and (IV.16) we obtain that

$$\Im\Big[4i \ \pi_{\vec{n}_k} \ \partial_{\overline{z}} \pi_{\vec{n}_k} \ \partial_z \vec{H}_k + 2i \ e^{2\lambda_k} \ (\overline{\vec{H}_{0,k} \cdot \vec{H}_k}) \vec{H}_{0,k} - f_k(z) \vec{H}_{k,0}\Big] \tag{IV.17}$$

$$\longrightarrow 0 \quad \text{in} \quad (W^{1,\infty} \cap W^{2,2})^* \quad .$$

It is convenient to introduce $A_k \in \mathbb{C}$ given by

$$A_k = e^{-\lambda_k} f_k(z) - 2 \, i \, e^{\lambda_k} \ \overline{\vec{H}_{0,k} \cdot \vec{H}_k} \quad . \tag{IV.18}$$

With this notation we have in particular

$$\partial_{\overline{z}}(e^{\lambda_k} \ A_k) = - 2 \, i \, \partial_{\overline{z}}\Big(e^{2\lambda_k} \ \overline{\vec{H}_{0,k} \cdot \vec{H}_k}\Big) \quad . \tag{IV.19}$$

Using the general equation $\partial_{\overline{z}}(e^{-\lambda_k} e_{\overline{z}}) = 2^{-1} \vec{H}_0$ (see again [BR] section III.2.2), we have

$$\partial_{\overline{z}}(A_k \ e_{\overline{z}}) = - 2 \, i \, e^{-\lambda_k} \ \partial_{\overline{z}}\Big(e^{2\lambda_k} \ \overline{\vec{H}_{0,k} \cdot \vec{H}_k}\Big) e_{\overline{z}} + \frac{e^{\lambda_k}}{2} \ A_k \ \vec{H}_{0,k} \quad . \tag{IV.20}$$

We recall at this stage the Codazzi-Mainardi equation [19]

$$e^{-2\lambda_k} \ \partial_{\overline{z}}\big(e^{2\lambda_k} \ \overline{\vec{H}_{0,k} \cdot \vec{H}_k}\big) = \vec{H}_k \cdot \partial_z \vec{H}_k + \overline{\vec{H}_{0,k}} \cdot \partial_{\overline{z}} \vec{H}_k \ . \tag{IV.21}$$

Combining (IV.18), (IV.20) and (IV.21) we obtain that

$$\partial_{\overline{z}}(A_k \ \vec{e}_{\overline{z}}) = - 2 \, i \, e^{\lambda_k} \ \Big[\vec{H}_k \cdot \partial_z \vec{H}_k + \overline{\vec{H}_{0,k}} \cdot \partial_{\overline{z}} \vec{H}_k\Big] \vec{e}_{\overline{z}} + \frac{e^{\lambda_k}}{2} \ A_k \ \vec{H}_{0,k}$$

$$= - 2 \, i \, e^{\lambda_k} \ \Big[\vec{H}_k \cdot \partial_z \vec{H}_k + \overline{\vec{H}_{0,k}} \cdot \partial_{\overline{z}} \vec{H}_k\Big] \vec{e}_{\overline{z}} + \frac{1}{2} \ f_k(z) \ \vec{H}_{0,k} \tag{IV.22}$$

$$- i \, e^{2\lambda_k} \ \Big(\overline{\vec{H}_{0,k} \cdot \vec{H}_k}\Big) \ \vec{H}_{0,k}$$

Another computation in section III.2.2 of [BR] gives

$$-2i\partial_{\overline{z}}\pi_{\vec{n}_k}\partial_z \vec{H}_k = -2i \ \pi_{\vec{n}_k} \partial_{\overline{z}} \pi_{\vec{n}_k} \partial_z \vec{H}_k$$

$$+2i \ e^{\lambda_k} \ \Big[(\vec{H}_k \cdot \partial_z \vec{H}_k) \ \vec{e}_{\overline{z}} + (\vec{H}_{0,k} \cdot \partial_z \vec{H}_k) \ \vec{e}_z\Big] \tag{IV.23}$$

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[19] See [BR] lemma A.3 for a proof.

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Combining (IV.22) and (IV.23) we obtain

$$\partial_{\overline{z}}(A_k \, \vec{e}_{\overline{z}}) - 2i\partial_{\overline{z}}\pi_{\vec{n}_k}\partial_z\vec{H}_k = -2i \, \pi_{\vec{n}_k}\partial_{\overline{z}}\pi_{\vec{n}_k}\partial_z\vec{H}_k$$

$$-2i \, e^{\lambda_k} \left[(\overline{\vec{H}_{0,k} \cdot \partial_{\overline{z}}\vec{H}_k}) \, \vec{e}_{\overline{z}} - (\vec{H}_{0,k} \cdot \partial_z\vec{H}_k) \, \vec{e}_z \right] \tag{IV.24}$$

$$+\frac{1}{2} f_k(z) \, \vec{H}_{0,k} - i \, e^{2\lambda_k} \, \left(\overline{\vec{H}_{0,k} \cdot \vec{H}_k} \right) \, \vec{H}_{0,k}$$

Observe that

$$\Im \left(-2i \, e^{\lambda_k} \left[(\overline{\vec{H}_{0,k} \cdot \partial_{\overline{z}}\vec{H}_k}) \, \vec{e}_{\overline{z}} - (\vec{H}_{0,k} \cdot \partial_z\vec{H}_k) \, \vec{e}_z \right] \right) = 0 \tag{IV.25}$$

Combining (IV.17), (IV.24) and (IV.25) we obtain

$$\Im \left(\partial_{\overline{z}} \left[A_k \, \vec{e}_{\overline{z}} - 2i \, \pi_{\vec{n}_k}\partial_z\vec{H}_k \right] \right) \longrightarrow 0 \quad \text{in} \quad (W^{1,\infty} \cap W^{2,2})^* \quad . \tag{IV.26}$$

or in other words

$$\Im \left(\partial_{\overline{z}} \left[e^{-\lambda_k} \, f_k(z) \, \vec{e}_{\overline{z}} - 2 \, i \, e^{\lambda_k} \, \vec{H}_{0,k} \cdot \vec{H}_k \, \vec{e}_z - 2 \, i \, \pi_{\vec{n}_k}\partial_z\vec{H}_k \right] \right)$$

$$\longrightarrow 0 \quad \text{in} \quad (W^{1,\infty} \cap W^{2,2})^* \quad . \tag{IV.27}$$

Let $\vec{F}_k = \vec{F}_k^{\Re} + i\vec{F}_k^{\Im} \in L^{2,\infty}(D^2, \mathbb{R}^m \otimes \mathbb{C})$ be the unique solution of

$$\begin{cases} \partial_z\vec{F}_k = e^{-\lambda_k} \, f_k(z) \, \vec{e}_{\overline{z}} - 2 \, i \, e^{\lambda_k} \, \vec{H}_{0,k} \cdot \vec{H}_k \, \vec{e}_z - 2 \, i \, \pi_{\vec{n}_k}\partial_z\vec{H}_k & \text{in } D^2 \\[2mm] \partial_\nu\vec{F}_k^{\Im} = 0 & \text{on } \partial D^2 \end{cases} \tag{IV.28}$$

Hence combining (IV.27) and (IV.28) we have

$$\begin{cases} \Delta\vec{F}_k^{\Im} \longrightarrow 0 \quad \text{in} \quad (W^{1,\infty} \cap W^{2,2})^* \\[2mm] \partial_\nu\vec{F}_k^{\Im} = 0 & \text{on } \partial D^2 \end{cases} \tag{IV.29}$$

This implies in particular that

$$\nabla\vec{F}_k^{\Im} \longrightarrow 0 \qquad \text{strongly in } (W^{1,q}(D^2))^* \qquad \forall \, q > 2 \quad . \tag{IV.30}$$

Let $\vec{Q}_k^{\mathbb{C}} := \vec{Q}_k^{\Re} + i\vec{Q}_k^{\Im} = -4 \, e^{\lambda_k} \, \vec{H}_{0,k} \cdot \vec{H}_k \, \vec{e}_z - 4 \, \pi_{\vec{n}_k}\partial_z\vec{H}_k \in \mathbb{R}^m \otimes \mathbb{C}$. It is proved in [Ri2] that for any conformal immersion $\vec{\xi}_k$, $\vec{Q} := (\vec{Q}^{\Re}, \vec{Q}^{\Im})$ satisfies

$$\begin{cases} \vec{Q}_k \cdot \nabla\vec{\xi}_k := \vec{Q}_k^{\Re} \cdot \partial_x\vec{\xi}_k + \vec{Q}_k^{\Im} \cdot \partial_y\vec{\xi}_k = 0 \\[2mm] \vec{Q}_k \wedge \nabla\vec{\xi}_k := \vec{Q}_k^{\Re} \wedge \partial_x\vec{\xi}_k + \vec{Q}_k^{\Im} \wedge \partial_y\vec{\xi}_k = 2 \, (-1)^m \, \nabla^\perp \left(\star(\vec{n}_k \llcorner \vec{H}_k) \right) \llcorner \nabla\vec{\xi}_k \end{cases} \tag{IV.31}$$

We rewrite (IV.28) in the form

$$\nabla^\perp\vec{F}_k^{\Re} + \nabla\vec{F}_k^{\Im} = \vec{Q}_k + e^{2\lambda_k} \begin{pmatrix} f_k^{\Im}(z) \, \partial_x\vec{\xi}_k - f_k^{\Re}(z) \, \partial_y\vec{\xi}_k \\[4mm] -f_k^{\Re}(z) \, \partial_x\vec{\xi}_k - f_k^{\Im}(z) \, \partial_y\vec{\xi}_k \end{pmatrix} \tag{IV.32}$$

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Combining (IV.31 and (IV.32) gives finally

$$
\begin{cases}
\left(\nabla^{\perp}\vec{F}_{k}^{\Re}+\nabla\vec{F}_{k}^{\Im}\right)\cdot\nabla\vec{\xi}_{k}=0 \\[3mm]
\left(\nabla^{\perp}\vec{F}_{k}^{\Re}+\nabla\vec{F}_{k}^{\Im}\right)\wedge\nabla\vec{\xi}_{k}:=2\,(-1)^{m}\,\nabla^{\perp}\left(\star(\vec{n}_{k}\llcorner\vec{H}_{k})\right)\llcorner\nabla\vec{\xi}_{k}\quad.
\end{cases}
\tag{IV.33}
$$

As in the proof of theorem III.1 we can extract a subsequence to $\vec{\xi}_{k}$ that weakly converges to a limiting conformal immersion $\vec{\xi}_{\infty}$ in $W^{1,\infty}\cap W^{2,2}$ and $\vec{n}_{k}$ weakly converges in $W^{1,2}$ to the Gauss map $\vec{n}_{\infty}$ of ξ_{∞}. Because of (IV.29) and (IV.30) we have

$$
\nabla\vec{F}_{k}^{\Im}\cdot\nabla\vec{\xi}_{k}=div(\nabla\vec{F}_{k}^{\Im}\cdot\vec{\xi}_{k})-\Delta\vec{F}_{k}^{\Im}\cdot\vec{\xi}_{k}\longrightarrow 0 \qquad \text{in } \mathcal{D}'(D^{2}),
\tag{IV.34}
$$

and

$$
\nabla\vec{F}_{k}^{\Im}\wedge\nabla\vec{\xi}_{k}=div(\nabla\vec{F}_{k}^{\Im}\wedge\vec{\xi}_{k})-\Delta\vec{F}_{k}^{\Im}\wedge\vec{\xi}_{k}\longrightarrow 0 \qquad \text{in } \mathcal{D}'(D^{2}),.
\tag{IV.35}
$$

Assume first there exists a subsequence - that we still denote $\vec{\xi}_{k}$ - such that $|\mu_{k}|$ is uniformly bounded and hence a subsequence such that $\mu_{k}\longrightarrow\mu_{\infty}=(\mu_{\infty}^{j})_{j=1\ldots Q}$. This implies that

$$
f_{k}(z)\longrightarrow f_{\infty}(z):=\sum_{j=1}^{Q}\mu_{\infty}^{j}\,f^{j}(z) \qquad \text{in } C^{l}(D^{2})\quad\forall l\in\mathbb{N}\quad.
\tag{IV.36}
$$

Standard elliptic estimates applied to the system (IV.28) imply that, modulo extraction of a subsequence $\vec{F}_{k}$ converges weakly in L^{p} for any $p<2$ [20]to a map $\vec{F}_{\infty}$ which is <u>real</u> because of (IV.30). By Rellich Kondrachov compact embedding $\nabla\vec{\xi}_{k}$ strongly converges to $\nabla\vec{\xi}_{\infty}$ in L^{q} for any $q<+\infty$. Hence using the Jacobian structures we have

$$
\begin{aligned}
&\nabla^{\perp}\vec{F}_{k}^{\Re}\cdot\nabla\vec{\xi}_{k}=-div\left[\vec{F}_{k}^{\Re}\cdot\nabla^{\perp}\vec{\xi}_{k}\right] \\[2mm]
&\longrightarrow -div\left[\vec{F}_{\infty}\cdot\nabla^{\perp}\vec{\xi}_{\infty}\right]=\nabla^{\perp}\vec{F}_{\infty}^{\Re}\cdot\nabla\vec{\xi}_{\infty} \\[3mm]
&\nabla^{\perp}\vec{F}_{k}^{\Re}\wedge\nabla\vec{\xi}_{k}=-div\left[\vec{F}_{k}^{\Re}\wedge\nabla^{\perp}\vec{\xi}_{k}\right] \\[2mm]
&\longrightarrow -div\left[\vec{F}_{\infty}\wedge\nabla^{\perp}\vec{\xi}_{\infty}\right]=\nabla^{\perp}\vec{F}_{\infty}\wedge\nabla\vec{\xi}_{\infty} \\[3mm]
&\nabla^{\perp}\left(\star(\vec{n}_{k}\llcorner\vec{H}_{k})\right)\llcorner\nabla\vec{\xi}_{k}=-div\left[\left(\star(\vec{n}_{k}\llcorner\vec{H}_{k})\right)\llcorner\nabla^{\perp}\vec{\xi}_{k}\right] \\[2mm]
&\longrightarrow -div\left[\left(\star(\vec{n}_{\infty}\llcorner\vec{H}_{\infty})\right)\llcorner\nabla^{\perp}\vec{\xi}_{\infty}\right]=\nabla^{\perp}\left(\star(\vec{n}_{\infty}\llcorner\vec{H}_{\infty})\right)\llcorner\nabla\vec{\xi}_{\infty}\quad.
\end{aligned}
\tag{IV.37}
$$

Hence we have proved that $\vec{\xi}_{\infty}$ satisfies the following system : $\exists\vec{F}_{\infty}\in L^{2,\infty}(D^{2},\mathbb{R}^{m})$ such that

$$
\begin{cases}
\nabla^{\perp}\vec{F}_{\infty}\cdot\nabla\vec{\xi}_{\infty}=0 \\[3mm]
\nabla^{\perp}\vec{F}_{\infty}\wedge\nabla\vec{\xi}_{\infty}=2\,(-1)^{m}\nabla^{\perp}\left(\star(\vec{n}_{\infty}\llcorner\vec{H}_{\infty})\right)\llcorner\nabla\vec{\xi}_{\infty}\quad.
\end{cases}
$$

This is equivalent to the fact that $\vec{\xi}_{\infty}$ satisfies the *Conformal Willmore* equation and $\vec{\xi}_{\infty}$ is analytic.

---

[20]Also weakly* in $L^{2,\infty}$.

Assume now that

$$\lim_{k \to +\infty} |\mu_k| = +\infty \quad .$$

Then we consider $\vec{F}_k / |\mu_k|$ and dividing by $|\mu_k|$ equation (IV.28) and passing to the limit in the Jacobian expressions as above we get the existence of a map $\vec{L}_\infty$ and a non zero holomorphic function $g_\infty(z)$ contained in the span of f^j such that

$$\partial_z \vec{L}_\infty = e^{-2\lambda_\infty} \, g_\infty(z) \, \partial_{\overline{z}} \vec{\xi}_\infty \quad . \tag{IV.38}$$

This could have been done on all the balls $B_{r^i}(x^i_\infty)$ simultaneously [21] and hence, like in the proof of lemma V.1, since $\partial_{\overline{z}}(e^{-2\lambda_\infty} \, \partial_{\overline{z}} \vec{\xi}_\infty) = 2^{-1} \vec{H}_{0,\infty}$, (IV.37) implies the existence of a non trivial holomorphic form q of $Q(J)$ such that

$$\Im(< q, h_{0,\infty} >_{WP}) \equiv 0 \quad ,$$

where locally in holomophic coordinates $h_0 := \overline{\vec{H}_0} \, dz \otimes dz$. This is equivalent to the fact that $\vec{\xi}_\infty$ is isothermic. We have then proved lemma IV.1 in the first case : when there exists a subsequence $\vec{\xi}_k$ which is not made of isothermic surfaces.

Second case : all the $\vec{\xi}_k$ are isothermic conformal immersion of (Σ, J).

This would mean that there exists a sequence of holomorphic quadratic differentials $q_k \neq 0$ such that

$$\Im(< q_k, h_{0,k} >_{WP}) \equiv 0 \quad , \tag{IV.39}$$

We can normalize q_k in such a way that $< q_k, q_k >_{WP} = 1$ and since $Q(J)$ is finite dimensional we can extract a subsequence such that q_k converges strongly in any C^l norm (for any $l \in \mathbb{N}$) to a non zero limiting holomorphic quadratic differential q_∞. We have seen that $d\vec{\xi}_k$ converges strongly in $L^p_{loc}(\Sigma \setminus \{a_1 \cdots a_N\})$ to a limiting $W^{2,2}$ immersion $\vec{\xi}_\infty$ of $\Sigma \setminus \{a_1 \cdots a_N\}$ and since the second fundamental form of $\vec{\xi}_k$ is uniformly bounded in L^2, $h_{0,k}$ converges weakly in L^2 to the Weingarten Operator $h_{0,\infty}$ of $\vec{\xi}_\infty$. We can then pass in the limit in the identity (IV.39). This implies that

$$\Im(< q_\infty, h_{0,\infty} >_{WP}) \equiv 0 \quad , \tag{IV.40}$$

from which we deduce that $\vec{\xi}_\infty$ is an isothermic immersion of $\Sigma \setminus \{a_1, \cdots a_N\}$ into $\mathbb{R}^m$. This concludes the proof of lemma IV.1 in all cases. $\qquad\square$

The proof theorem IV.1 can be finished exactly like in the proof of theorem III.1 in order to exploit the assumption $\inf_{\vec{\Phi} \in \mathcal{E}^c_\Sigma} W(\vec{\Phi}) \leq 8\pi$ and "remove" the singularity points a_i. $\qquad\square$

V   The conformal constraint and isothermic immersions.

We first describe the immersions which are the singular points for the map which assigns to an immersion it's conformal class as we will prove in lemma V.1 : the isothermic immersions.

Proof of proposition I.1.

Isothermic immersions have been defined in definition I.1. First let consider an isothermic immersion. There are locally, away from umbilic points, conformal coordinates in which the second fundamental form is diagonal. Hence this means that the Weingarten map is real in such charts. Take two such complex

---

[21]Either $q_k := \sum_{j=1}^N \mu_k^j f^j(z) \, dz \otimes dz$ is bounded in the space $Q(J)$ of holomorphic quadratic forms of (Σ, J) or goes to infinity in norm (for the Weil-Peterson hermitian product).

charts $z = x_1 + ix_2$ and $\xi = \xi_1 + i\xi_2$ overlaping on some open set. The Weingarten operator is independent of the complex chart and we have

$$h_0 = \partial_z \vec{n}_{\vec{\Phi}} \llcorner \partial_z \vec{\Phi} \, dz \otimes dz = \partial_\xi \vec{n}_{\vec{\Phi}} \llcorner \partial_\xi \vec{\Phi} \, d\xi \otimes d\xi \quad .$$

Our assumption reads $\Im(\partial_z \vec{n}_{\vec{\Phi}} \llcorner \partial_z \vec{\Phi}) = \Im(\partial_\xi \vec{n}_{\vec{\Phi}} \llcorner \partial_\xi \vec{\Phi}) = 0$, moreover, since we are away from umbilic points $h_0 \neq 0$ that is then $\Re(\partial_z \vec{n}_{\vec{\Phi}} \llcorner \partial_z \vec{\Phi}) \neq 0$. Thus we have that

$$\frac{\Re(\partial_\xi \vec{n}_{\vec{\Phi}} \llcorner \partial_\xi \vec{\Phi})}{\Re(\partial_z \vec{n}_{\vec{\Phi}} \llcorner \partial_z \vec{\Phi})} = (z'(\xi))^2$$

This implies that the imaginary part of the holomorphic function $(z'(\xi))^2$ is zero which implies that $z'(\xi)$ is constant and $(z'(\xi))^2$ is a real constant. Thus $dz \otimes dz$ and $d\xi \otimes d\xi$ are proportional to eachother by a real non zero constant and this implies that the form $dz \otimes dz$ extends to an holomorpfhic quadratic differential q of the riemann surface $\tilde{\Sigma}$ obtained by withdrawing to Σ the umbilic points of the immersion $\vec{\Phi}$ and we have by construction $< q, h_0 >_{WP} = 0$.

We are now proving the reciproque. Let q be an holomorphic quadratic differential of $\tilde{\Sigma}$. Away from the isolated zeros of q we can choose complex coordinate z such that $q(z) = dz \otimes dz$ (indeed in arbitrary complex coordinates $q(\xi) = f(\xi) \, d\xi \otimes d\xi$ where f is holomorphic and just choose $z(\xi) = \sqrt{f(\xi)}$. In these coordinates the condition (I.19) implies $\Im(\vec{H}_0) = 0$ which means that the second fundamental form is diagonal in these complex coordinates and hence $\vec{\Phi}$ is local isothermic. $\qquad\square$

Proof of proposition I.2.

Let $\vec{\Phi}$ be an immersion. From computations in section III.2.2 of [BR] we have in complex coordinales

$$\partial_{\bar{z}} \left(e^{-2\lambda} \, \partial_{\bar{z}} \vec{\Phi} \right) = 2^{-1} \, \vec{H}_0 \quad . \tag{V.41}$$

Assume $\vec{\Phi}$ is local isothermic, because of the previous proposition there exists complex coordinates in which $\Im[\vec{H}_0] = 0$. Hence in these coordinates

$$\Im \left[\partial_{\bar{z}} \left(e^{-2\lambda} \, \partial_{\bar{z}} \vec{\Phi} \right) \right] = 0 \qquad \text{in } D^2 . \tag{V.42}$$

Let $\vec{L} = \vec{L}_\Re + i\vec{L}_\Im \in \mathbb{R}^m \otimes \mathbb{C}$ be the unique solution to the following elliptic system

$$\begin{cases} \partial_z \vec{L} = e^{-2\lambda} \, \partial_{\bar{z}} \vec{\Phi} & \text{in } D^2 \\[2mm] \vec{L}_\Im = 0 & \text{on } \partial D^2 \end{cases}$$

Then, because of (V.42), $\vec{L}_\Im$ solves

$$\begin{cases} \Delta \vec{L}_\Im = 0 & \text{in } D^2 \\[2mm] \vec{L}_\Im = 0 & \text{on } \partial D^2 \quad . \end{cases}$$

This implies that $\vec{L} = \vec{L}_\Re \in \mathbb{R}^m$. Hence we have proved (I.20). Assuming now that (I.20) holds, we obtain the existence of complex coordinates such that (V.42) is satisfied which implies from (V.41) that $\Im(\vec{H}_0) = 0$ and from which we deduce that $\vec{\Phi}$ is isothermic. This finishes the proof of proposition I.2. $\square$

Proof of proposition I.3.

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An elementary computation gives for any pair $\vec{\Phi}$ and $\vec{L}$ from D^2 into $\mathbb{R}^m$

$$4 \, \Im \left(\partial_{\bar{z}} \vec{L} \cdot \partial_z \vec{\Phi} \right) = \nabla \vec{L} \cdot \nabla^\perp \vec{\Phi} \qquad \text{and} \qquad 4 \, \Im \left(\partial_{\bar{z}} \vec{L} \wedge \partial_z \vec{\Phi} \right) = \nabla \vec{L} \wedge \nabla^\perp \vec{\Phi}$$

Hence (I.20) clearly implies (I.21).

Assuming now that (I.21) holds, we have then the existence of $\vec{L} \in \mathbb{R}^m$ such that

$$\begin{cases} \vec{e}_1 \wedge \partial_y \vec{L} = \vec{e}_2 \wedge \partial_x \vec{L} \\[2mm] \vec{e}_1 \cdot \partial_y \vec{L} = \vec{e}_2 \cdot \partial_x \vec{L} \quad . \end{cases}$$

A short computation shows that this implies the existence of $(a, b) \in \mathbb{R}^2$ such that

$$\nabla^\perp \vec{L} = \begin{pmatrix} b & a \\ a & -b \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}$$

or in other words, introducing $f := e^\lambda (a + ib)$, one has

$$\partial_z \vec{L} = f \, e^{-2\lambda} \partial_{\bar{z}} \vec{\Phi} \quad . \tag{V.43}$$

Since the components of $\vec{L}$ are real, and by consequence the components of $\Delta \vec{L}$ are real as well, we have that

$$\Im \left[\partial_{\bar{z}} \left(f \, e^{-2\lambda} \partial_{\bar{z}} \vec{\Phi} \right) \right] = 0 \quad .$$

Using (V.41) this gives

$$\partial_{\bar{z}} f \, \vec{e}_{\bar{z}} - \partial_z \bar{f} \, \vec{e}_z = -\frac{e^\lambda}{2} \left[f \, \vec{H}_0 - \bar{f} \, \overline{\vec{H}_0} \right] \quad .$$

Since $\vec{H}_0$ is orthogonal to the tangent plane of the immersion and since $\vec{e}_z$ and $\vec{e}_{\bar{z}}$ are in the complexified space to the tangent space and are linearly independent we deduce

$$\partial_{\bar{z}} f = 0 \quad .$$

Take now $w = \sqrt{f}$ equation (V.43) becomes

$$\partial_w \vec{L} = e^{-2\lambda} \, |f| \, \partial_{\bar{w}} \vec{\Phi}$$

and one observes that $e^\lambda \, |f|^{-1/2}$ is the new conformal factor of $\vec{\Phi}$ in the coordinate w, which means that $\vec{\Phi}$ satisfies (I.20) in these coordinates and hence, from the previous proposition, $\vec{\Phi}$ is an isothermic immersion. $\qquad \square$

Finally we prove that the *global isothermic immersions* are the degenerate points for the conformal class mapping. Precisely we prove the following result.

Lemma V.1 *Let $\vec{\Phi}$ be a conformal $W^{2,2} \cap W^{1,\infty}$ immersion of a closed Riemann surface (Σ, J) of genus larger or equal to one. Consider in a neighborhood of 0 the map*

$$\mathcal{C} \; : \; \vec{w} \in W^{2,2} \cap W^{1,\infty} \; \longrightarrow \; \mathcal{C}(\vec{w}) \in \mathcal{T}_\Sigma \quad ,$$

where $\mathcal{T}_\Sigma$ is the Teichmüller Space associated to the surface Σ and $\mathcal{C}(\vec{w})$ is the Teichmüller class issued from the immersion $\vec{\Phi} + \vec{w}$ with fixed generators of the π_1 on Σ. The map $\mathcal{C}$ is C^1 in a neighborhood of

0. Identifying $\mathcal{T}_\Sigma$ with the space $Q(J)$ of holomorphic quadratic differentials[22] on (Σ, J), its differential at 0 is given by

$$dC(0) \cdot \vec{\nu} = 8 \sum_{j=1}^{Q} q_j \left\langle q_j, \partial_z \vec{\nu} \cdot \partial_z \vec{\Phi} \, dz \otimes dz \right\rangle_{WP} \quad , \qquad (V.44)$$

where $< \cdot, \cdot >_{WP}$ is the Weil Peterson Hermitian product and $(q_j)_{j=1\ldots Q}$ is an orthonormal basis of $Q(J)$ for this product. Moreover, if $\vec{\Phi}$ is not an isothermic surface, $dC(0)$ is a submersion onto the space of holomorphic quadratic differentials of (Σ, J). □

Proof of lemma V.1. Let $\vec{\Phi}$ be a conformal $W^{2,2} \cap W^{1,\infty}$ immersion of (Σ, J) and $\vec{w}$ be a map in $W^{2,2} \cap W^{1,\infty}(\Sigma, \mathbb{R}^m)$, small enough in this space, in such a way that $\vec{\Phi} + \vec{w}$ still defines an immersion. Denote by $J^{\vec{w}}$ the $W^{1,2}$ complex structure defined by $(\vec{\Phi} + \vec{w})^* g_{\mathbb{R}^m}$. Using lemma A.3 there exists a covering of Σ by disks $(U_i)_{i \in I}$ and $W^{1,\infty} \cap W^{2,2}$ diffeomorphisms $\psi_i^{\vec{w}}$ from D^2 into U_i such that $(\vec{\Phi} + \vec{w}) \circ \psi_i$ is conformal. Considering now Σ together with the covering U_i and the holomorphic transition maps

$$h_{ij}^{\vec{w}}(z) := (\psi_j^{\vec{w}})^{-1} \circ \psi_i^{\vec{w}} \quad ,$$

which satisfy of course the cocycle condition $h_{ij}^{\vec{w}} \circ h_{jk}^{\vec{w}} \circ h_{ki}^{\vec{w}}(z) = z$, we have defined a new smooth complex structure on Σ, $\tilde{J}^{\vec{w}}$ which is equivalent to $(\Sigma, J^{\vec{w}})$: there exist smooth conformal diffeomorphisms, $\varphi^{\vec{w}}$ from D^2 into $(U_i, \tilde{J}^{\vec{w}})$ and an homeomorphism[23] $\Psi^{\vec{w}}$ of Σ, isotopic to the identity which is conformal from $(\Sigma, J^{\vec{w}})$ into $(\Sigma, \tilde{J}^{\vec{w}})$ and such that $\Psi^0 = id_\Sigma$ and $\tilde{J}^0 = J$. Hence $\Psi^{\vec{w}}$ is bilipschitz and both $\Psi^{\vec{w}}$ and $(\Psi^{\vec{w}})^{-1}$ are $W^{2,2}$. By replacing now $\vec{\Phi}$ by $(\vec{\Phi} + \vec{w}) \circ (\Psi^{\vec{w}})^{-1}$, if one shows that that $\mathcal{C}$ is C^1 at 0 one has shown that $\mathcal{C}$ is C^1 in a neighborhood of the origin.

In order to show that $\mathcal{C}$ is C^1 at 0 it suffices to show that the mappings wich to $\vec{w}$ assigns the family of holomorphic transition functions $h_{ij}^{\vec{w}}$ is C^1 from $W^{1,\infty} \cap W^{2,2}$ into C^0 (which implies that it is C^1 from $W^{1,\infty} \cap W^{2,2}$ into C^l for an arbitrary l on a slightly small covering). In order to show that it suffices to show that the mappings which to $\vec{w}$ assigns $\psi_i^{\vec{w}}$ and $(\psi_i^{\vec{w}})^{-1}$ are C^1 from $W^{1,\infty} \cap W^{2,2}$ into $W^{2,p}$ for some $p > 1$. This can be done following carefully the construction of conformal coordinates in lemma A.3. We leave the details to the reader.

We compute now the differential of $\mathcal{C}$ at the origin. As above $h^{\vec{w}}$ denotes the metric of constant scalar curvature compatible with $(\Sigma, \tilde{J}^{\vec{w}})$ and we denote simply by h the constant scalar curvature compatible with (Σ, J). Let $u^{\vec{w}}$ be the harmonic map from (Σ, J) into $(\Sigma, \tilde{J}^{\vec{w}})$ isotopic to the identity given by corollary 3.10.1 in [Jo]. The map $C(\vec{w})$ is given explicitly by

$$C(\vec{w}) = \sum_{j=1}^{Q} A_j^{\vec{w}} \, q_j \quad ,$$

where

$$A_j^{\vec{w}} := \left\langle q_j, \left[\left\langle \partial_x u^{\vec{w}}, \partial_x u^{\vec{w}} \right\rangle_{h^{\vec{w}}} - \left\langle \partial_y u^{\vec{w}}, \partial_y u^{\vec{w}} \right\rangle_{h^{\vec{w}}} - 2i \left\langle \partial_x u^{\vec{w}}, \partial_y u^{\vec{w}} \right\rangle_{h^{\vec{w}}} \right] (dz)^2 \right\rangle_{WP}$$

Denote $v^{\vec{w}} := (\Psi^{\vec{w}})^{-1} \circ u^{\vec{w}}$ and $e^{\mu^{\vec{w}}} g^{\vec{w}} = (\Psi^{\vec{w}})^* h^{\vec{w}}$. Hence we have in particular

$$A_j^{\vec{w}} := \left\langle q_j, e^{2\mu^{\vec{w}}} \left[\left\langle \partial_x v^{\vec{w}}, \partial_x v^{\vec{w}} \right\rangle_{g^{\vec{w}}} - \left\langle \partial_y v^{\vec{w}}, \partial_y v^{\vec{w}} \right\rangle_{g^{\vec{w}}} - 2i \left\langle \partial_x v^{\vec{w}}, \partial_y v^{\vec{w}} \right\rangle_{g^{\vec{w}}} \right] (dz)^2 \right\rangle_{WP}$$

---

[22]See for instance theorem 4.2.2 in [Jo].

[23]Σ together with the charts (U_i, ψ_i) defines a smooth complex manifold since $\psi_j^{-1} \circ \psi_i$, are holomorphic, the smooth complex structure being given by the multiplication by i in the charts. It admits then a constant scalar curvature metric $h^{\vec{w}}$ and $\Psi^{\vec{w}}$ is the harmonic diffeomorphism from $(\Sigma, g^{\vec{w}})$ into $(\Sigma, h^{\vec{w}})$ isotopic to the identity, see [Jo].

Let $X^{\vec{\nu}} := dv^{\vec{w}}(0) \cdot \nu$. Since $v^0 = id_\Sigma$, we have, writing locally $q_j = q_j^z(z)\, dz \otimes dz$ in complex coordinates [24] satisfying in particular $|\partial_x \vec{\Phi}| = |\partial_y \vec{\Phi}| = e^\lambda$ and then in which $h = e^{2\mu(0)}\, g = e^{2\mu(0)+2\lambda}\, [dx^2 + dy^2]$

$$
dA_j^{\vec{w}}(0) \cdot \nu = \int_\Sigma e^{-2\lambda - 2\mu(0)}\, q_j^z(z)\, e^{2\mu(0)} \left[2 \left\langle \partial_x X^{\vec{\nu}}, \partial_x id_\Sigma \right\rangle_g - 2 \left\langle \partial_y X^{\vec{\nu}}, \partial_y id_\Sigma \right\rangle_g \right.
$$

$$
+ 2i \left\langle \partial_x X^{\vec{\nu}}, \partial_y id_\Sigma \right\rangle_g + 2i \left\langle \partial_y X^{\vec{\nu}}, \partial_x id_\Sigma \right\rangle_g
$$

$$
\left. + dg_{11}^{\vec{w}}(0) \cdot \nu - dg_{22}^{\vec{w}}(0) \cdot \nu + 2i\, dg_{12}^{\vec{w}}(0) \cdot \nu \right]\ \frac{i}{2}\, dz \wedge d\overline{z}
$$

Decomposing the vector-field $X^{\vec{\nu}}$ as follows : $X^{\vec{\nu}} = X_z^{\vec{\nu}}\, \partial_z + X_{\overline{z}}^{\vec{\nu}}\, \partial_{\overline{z}}$ (where $\partial_z := 2^{-1}(\partial_x - i\partial_y)$) and observing that $dg^{\vec{w}}(0) \cdot \nu = (\partial_{x_j} \vec{\Phi} \cdot \partial_{x_i} \vec{\nu} + \partial_{x_i} \vec{\Phi} \cdot \partial_{x_j} \vec{\nu})_{ij}$ gives

$$
dA_j^{\vec{w}}(0) \cdot \nu = 2i \int_\Sigma q_j^z(z)\, \partial_{\overline{z}} X_z^{\vec{\nu}}\, dz \wedge d\overline{z}
$$

$$
+ 4i \int_\Sigma e^{-2\lambda}\, q_j^z(z)\, \partial_{\overline{z}} \vec{\Phi} \cdot \partial_{\overline{z}} \vec{\nu}\, dz \wedge d\overline{z}
$$

(V.45)

Observe that

$$
q_j^z(z)\, \partial_{\overline{z}} X_z^{\vec{\nu}}\, dz \wedge d\overline{z} = q_j^z(z)\, dz \wedge d(X_z^{\vec{\nu}}) = d\left[X_z^{\vec{\nu}}\, q_j^z(z)\, dz \right]\quad .
$$

Let ξ be another complex coordinates. We have that $q_j = q_j^z(z)\, dz \otimes dz = (\xi')^{-2}\, q_j^z(z)\, d\xi \otimes d\xi$. Then $q_j^\xi(\xi) = (\xi')^{-2}\, q_j^z(z)$. We also have $X = X_z^{\vec{\nu}}\, \partial_z + X_{\overline{z}}^{\vec{\nu}}\, \partial_{\overline{z}} = \xi'\, X_z^{\vec{\nu}}\, \partial_\xi + \overline{\xi}'\, X_{\overline{z}}^{\vec{\nu}}\, \partial_{\overline{\xi}}$. Hence $\xi'\, X_z^{\vec{\nu}} = X_\xi^{\vec{\nu}}$ and

$$
\alpha = X_z^{\vec{\nu}}\, q_j^z(z)\, dz = X_\xi^{\vec{\nu}}(\xi')^{-1}\, q_j^\xi(\xi)\, (\xi')^2\, \xi'\, d\xi = X_\xi^{\vec{\nu}}\, q_j^\xi(\xi)\, d\xi
$$

is an intrinsic 1-form globally defined on Σ. Thus

$$
\int_\Sigma q_j^z(z)\, \partial_{\overline{z}} X_z^{\vec{\nu}}\, dz \wedge d\overline{z} = \int_\Sigma d\alpha = 0
$$

and (V.45) implies (V.44). It remains to prove that $d\mathcal{C}(0)$ is a submersion onto the space of holomorphic quadratic differentials of (Σ, J).

In local conformal coordinates for $\vec{\Phi}$ we denote $\vec{e}_i = e^{-\lambda}\partial_{x_i} \vec{\Phi}$ where $e^\lambda = |\partial_{x_i} \vec{\Phi}|$. Let $(\vec{n}_\alpha)_{\alpha=1\cdots m-2}$ be a local orthonormal frame of the normal bundle to $\vec{\Phi}(\Sigma)$. We denote $h_{ij}^\alpha := -e^{-\lambda}\, \vec{e}_i \cdot \partial_{x_j} \vec{n}_\alpha$. The *Weingarten map* is given by

$$
\vec{H}_0 = \sum_{\alpha=1}^{m-2} H_0^\alpha\, \vec{n}_\alpha := \frac{1}{2} \sum_{\alpha=1}^{m-2} (h_{11}^\alpha - h_{22}^\alpha + 2i\, h_{12}^\alpha)\, \vec{n}_\alpha\quad .
$$

Denote $\vec{e}_z := e^{-\lambda}\partial_z \vec{\Phi} = 2^{-1}(\vec{e}_1 - i\,\vec{e}_2)$ and $\vec{e}_{\overline{z}} := e^{-\lambda}\partial_{\overline{z}} \vec{\Phi} = 2^{-1}(\vec{e}_1 + i\,\vec{e}_2)$. Some elementary computations give (see for instance [BR] section III.2.2)

$$
\partial_{\overline{z}}(e^{-\lambda} \vec{e}_{\overline{z}}) = 2^{-1}\, \vec{H}_0\quad .
$$

(V.46)

Let $\vec{\nu}$ be a map supported in the domain of definition for the local conformal charts that we identify with D^2. We assume $\vec{\nu}$ to be in $W_0^{2,2} \cap W^{1,\infty}(D^2)$. Denote by $f_j(z)\, dz \otimes dz$ the expression of the basis q_j in

this conformal charts in such a way that $f_j(z)$ are holomorphic functions on D^2. The expression (V.44) of $d\mathcal{C}(0) \cdot \nu$ gives

$$
\begin{aligned}
d\mathcal{C}(0) \cdot \vec{\nu} \;\; &= 8 \sum_{j=1}^{Q} q_j \; \int_{D^2} f_j(z) \; e^{-\lambda} \vec{e_{\overline{z}}} \cdot \partial_{\overline{z}} \vec{\nu} \; \frac{i}{2} dz \wedge d\overline{z} \\
&= -8 \sum_{j=1}^{Q} q_j \; \int_{D^2} f_j(z) \; \partial_{\overline{z}}(e^{-\lambda}\vec{e_{\overline{z}}}) \cdot \vec{\nu} \; \frac{i}{2} dz \wedge d\overline{z} \qquad\qquad (V.47) \\
&= -8 \sum_{j=1}^{Q} q_j \; \int_{D^2} f_j(z) \; \vec{H}_0 \cdot \vec{\nu} \; \frac{i}{2} dz \wedge d\overline{z}
\end{aligned}
$$

If

$$
\nu \longrightarrow \left(\int_{D^2} f_j(z) \; \vec{H}_0 \cdot \vec{\nu} \; \frac{i}{2} dz \wedge d\overline{z} \right)_{j=1 \cdots Q}
$$

does not have a complex N dimensional Range then it would mean that the real $2N$ linear forms on $W^{2,2} \cap W^{1,\infty}$ given by

$$
\nu \longrightarrow \left(\Re \left[\int_{D^2} f_j(z) \; \vec{H}_0 \cdot \vec{\nu} \; \frac{i}{2} dz \wedge d\overline{z} \right], \Im \left[\int_{D^2} f_j(z) \; \vec{H}_0 \cdot \vec{\nu} \; \frac{i}{2} dz \wedge d\overline{z} \right] \right)_{j=1 \cdots Q}
$$

are linearly dependent. This is equivalent to the existence of a non trivial family of real numbers $(\mu_j, \delta_j)_{j=1 \cdots Q} \in \mathbb{R}^{2Q}$ such that
$\forall \vec{\nu} \in W^{2,2} \cap W^{1,\infty}$

$$
\sum_{j=1}^{Q} \mu_j \; \Re \left[\int_{D^2} f_j(z) \; \vec{H}_0 \cdot \vec{\nu} \; \frac{i}{2} dz \wedge d\overline{z} \right] + \delta_j \; \Im \left[\int_{D^2} f_j(z) \; \vec{H}_0 \cdot \vec{\nu} \; \frac{i}{2} dz \wedge d\overline{z} \right] = 0
$$

or in other words there would exist $(\delta_j + i\mu_j) \in \mathbb{C}^Q$, being not all equal to zero, such that

$$
\Im \left[\sum_{j=1}^{N} (\delta_j + i\mu_j) \; f_j(z) \; \vec{H}_0 \right] \equiv 0
$$

In other words again, this would mean that there exist a non zero holomorphic quadratic form q in Q_J such that

$$
\Im(< q, h_0 >_{WP}) \equiv 0 \quad,
$$

where locally $h_0 := e^{2\lambda} \; \overline{\vec{H}_0} \; dz \otimes dz$. This is equivalent to the fact that $\vec{\Phi}$ is isothermic. Hence if we make the assumption that $\vec{\Phi}$ is not isothermic the dimension of the range of $d\mathcal{C}(0)$ is $Q = dim(Q(J))$, which concludes the proof of lemma V.1. $\qquad\square$

A Appendix

Lemma A.1 *Let g and h be two metrics at a point $p \in D^2$ such that*

$$
|Dis(g)| < 1 - 2^{-k} \quad.
$$

where $k \in \mathbb{N}$. Then the following inequality holds

$$
\frac{1}{2} \; \inf_{X \in \mathbb{R}^2 \setminus \{0\}} \frac{|X|_g^2}{|X|_h^2} \leq \frac{tr(g)}{tr(h)} \leq 2^k \; \sup_{X \in \mathbb{R}^2 \setminus \{0\}} \frac{|X|_g^2}{|X|_h^2} \quad. \qquad\qquad (A.1)
$$

$\square$

Proof of Lemma A.1. Observe that

$$|X|_g^2 = \left[\frac{tr(g) + \Re[H(g)]}{2}\right] X_1^2 + \left[\frac{tr(g) - \Re[H(g)]}{2}\right] X_2^2$$

$$-\Im[H(g)] \, X_1 \, X_2 \quad .$$

Hence, denoting also $X = X_1 + iX_2$

$$|X|_g^2 = \Re\left[\left(\frac{H(g)}{2} X + \frac{tr(g)}{2} \overline{X}\right) X\right] \quad .$$

We deduce that

$$\frac{|X|_g^2}{|X|_h^2} = \frac{tr(g)}{tr(h)} \frac{1 + \Re\left[Dis(g) \, e^{2i\,\theta}\right]}{1 + \Re\left[Dis(h) \, e^{2i\,\theta}\right]} \quad , \tag{A.2}$$

where $X = |X| \, e^{2i\,\theta}$.

Since $|Dis(g)| < 1 - 2^{-k}$, using (A.2) we have that

$$\frac{tr(g)}{tr(h)} \frac{2^{-k}}{\inf_\theta |1 + \Re\left[Dis(h) \, e^{2i\,\theta}\right]|} \leq \sup_{X \in \mathbb{R}^2 \setminus \{0\}} \frac{|X|_g^2}{|X|_h^2} \quad .$$

Hence in particular $\inf_\theta \left|1 + \Re\left[Dis(h) \, e^{2i\,\theta}\right]\right| > 0$ and by taking $e^{2i\theta} := Dis(h)/|Dis(h)|$ (in the case when $Dis(h) \neq 0$) we see that there is θ such that $1 + \Re\left[Dis(h) \, e^{2i\,\theta}\right] > 0$, by continuity this implies that

$$\forall \theta \in \mathbb{R} \qquad 1 + \Re\left[Dis(h) \, e^{2i\,\theta}\right] > 0$$

This implies that $\inf_\theta \left|1 + \Re\left[Dis(h) \, e^{2i\,\theta}\right]\right| = 1 - |Dis(h)| < 1$ from which we deduce the upper bound in (A.1). Take now again $X = |X| \, e^{i\theta}$ in such a way that $e^{2i\theta} := Dis(h)/|Dis(h)|$ (still in the cas when $Dis(h) \neq 0$). For this X we have

$$\frac{tr(h)}{tr(g)} = \frac{|X|_h^2}{|X|_g^2} \frac{1 + \Re\left[Dis(g) \, e^{2i\,\theta}\right]}{1 + |Dis(h)|} \leq 2 \sup_{X \neq 0} \frac{|X|_h^2}{|X|_g^2}$$

which gives the lower bound in (A.1). $\qquad\qquad\square$

Lemma A.2 *There exists $\varepsilon_0 > 0$ such that for any $\vec{\xi} \in W^{2,2} \cap W^{1,\infty}(D^2, \mathbb{R}^m)$ satisfying*

$$|Dis(\vec{\xi})| < \varepsilon_0 \tag{A.3}$$

and

$$\int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|_g^2 \, dvol_g < \frac{4\pi}{3} \quad ,$$

where $g := \vec{\xi}^* g_{\mathbb{R}^m}$, there exists $\vec{e}_1$ and $\vec{e}_2$ in $W^{1,2}(D^2, S^{m-1})$ such that

$$\vec{e}_1 \cdot \vec{e}_2 = 0 \qquad , \qquad n_{\vec{\xi}} = \vec{e}_1 \wedge \vec{e}_2 \quad , \tag{A.4}$$

$$\int_{D^2} \left[|\nabla \vec{e}_1|_g^2 + |\nabla \vec{e}_2|_g^2\right] \, dvol_g \leq 2 \int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|_g^2 \, dvol_g \tag{A.5}$$

and

$$\begin{cases} d(*_g(\vec{e}_1, d\vec{e}_2)) = 0 \quad , \\[2mm] \iota_{\partial D^2}^* *_g (\vec{e}_1, d\vec{e}_2) = 0 \quad . \end{cases} \tag{A.6}$$

where $\iota_{\partial D^2}$ is the canonical inclusion of ∂D^2 in $\mathbb{R}^2$. $\qquad\qquad\square$

Proof of lemma A.2. This lemma is proved in [Hel] lemma 5.1.4 for $\vec{\xi}$ conformal - which implies that $g = \vec{\xi}^* g_{\mathbb{R}^m} = e^\lambda \, dx^2 + dy^2$ and hence, in that case, for any function f

$$\int_{D^2} |df|_g^2 \, dvol_g = \int_{D^2} |\nabla f|^2 \, dx \, dy \quad .$$

We now explain how the strategy in [Hel] adapts to the case when $Dis(\vec{\xi}) = 0$ is replaced by (A.3). The assumption (A.3) implies that

$$\forall (x,y) \in D^2 \quad \forall X \in T_{(x,y)} D^2$$

$$(1 - \varepsilon_0^2) \, |X|_{g_0}^2 \leq (detg)^{-1} \, |X|_g^2 \leq (1 + \varepsilon_0^2) \, |X|_{g_0}^2 \quad , \tag{A.7}$$

where g_0 is the flat metric $dx^2 + dy^2$. We can first assume that $n_{\vec{\xi}}$ is a smooth map from D^2 into the Grassman Space of oriented 2-planes in $Gr_2(\mathbb{R}^m)$ - which are dense in $W^{1,2}(D^2, Gr_2(\mathbb{R}^m))$ see step 6 of the proof of lemma 5.1.4 of [Hel]. Let $\tilde{e} := (\tilde{e}_1, \tilde{e}_2)$ be a smooth orthonormal 2-frame[25] in $\mathbb{R}^m$ realizing (A.4). For each $r \in (0,1]$ we minimize

$$F_r(\theta) = \int_{D_r^2} |(e_1, de_2)|_g^2 \, dvol_g \quad ,$$

among $\theta \in W^{1,2}(D_r^2, \mathbb{R})$ and $e_1 + ie_2 = e^{i\theta}(\tilde{e}_1 + i\tilde{e}_2)$. Since $(e_1^r, de_2^r) = d\theta + (\tilde{e}_1, d\tilde{e}_2)$ F_r is convex and the minimum is achieved by a unique e^r satisfying

$$\begin{cases} d(*_g(e_1^r, de_2^r)) = 0 & \text{in } D_r^2 \\ \\ \iota_{\partial D_r^2}^* *_g (e_1^r, de_2^r) = 0 \end{cases} \tag{A.8}$$

where $\iota_{\partial D_r^2}$ is the canonical embedding of ∂D_r^2 in $\mathbb{R}^2$. Hence there exists a unique function $f^r \in W_0^{1,2}(D_r^2, \mathbb{R})$ such that $*_g(e_1^r, de_2^r) = df^r$ and f^r satisfies

$$\begin{cases} \Delta_{g_0} f^r = \partial_x e_1^r \cdot \partial_y e_2^r - \partial_x e_2^r \cdot \partial_y e_1^r + *_{g_0} d((*_{g_0} - *_g) \, df^r) & \text{in } D_r^2 \\ \\ f^r = 0 & \text{on } \partial D_r^2 \end{cases} \tag{A.9}$$

where Δ_{g_0} is the Laplace operator for the flat metric $\Delta_{g_0} = -[\partial_x^2 + \partial_y^2]$. Because of (A.7) we have that $\forall \alpha$ one-form on D^2 $|(*_g - *_{g_0}) \alpha| \leq \varepsilon_0 \, |\alpha|$. Hence Wente estimates together with more standard elliptic estimates gives the bound

$$\int_{D_r^2} |\nabla f^r|^2 \, dx \, dy \quad \leq \frac{3}{16\pi} \int_{D_r^2} |\nabla e_1^r|^2 \, dx \, dy \, \int_{D_r^2} |\nabla e_2^r|^2 \, dx \, dy$$

$$+ C\varepsilon_0 \int_{D_r^2} |\nabla f^r|^2 \, dx \, dy \tag{A.10}$$

for some universal $C > 0$. Thus for ε_0 chosen small enough we obtain the existence of $C > 0$ independent of r and the data of the lemma such that

$$\int_{D_r^2} |\nabla f^r|^2 \, dx \, dy \leq \frac{3}{16\pi}(1 + 2C\varepsilon_0) \int_{D_r^2} |\nabla e_1^r|^2 \, dx \, dy \, \int_{D_r^2} |\nabla e_2^r|^2 \, dx \, dy \tag{A.11}$$

Once this estimate is established, the rest of the arguments of F. Hélein carries over and we obtain lemma A.2. $\qquad\square$

---

[25] this trivialization exists since we are now working with a smooth $\vec{n}_\xi$ and the pull-back over D^2 by $\vec{n}_\xi$ of the tautological bundle $SO(m)/SO(m-2)$ over $Gr_2(\mathbb{R}^m)$ is trivial since D^2 is contractible

Lemma A.3 *There exists $\varepsilon_0 > 0$ and $0 < \alpha < 1$ such that for any $\vec{\xi} \in W^{2,2} \cap W^{1,\infty}(D^2, \mathbb{R}^m)$ satisfying*

$$|Dis(\vec{\xi})| < \varepsilon_0$$

and

$$\int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|^2_g \; dvol_g < \frac{4\pi}{3} \quad,$$

where $g := \vec{\xi}^* g_{\mathbb{R}^m}$ there exists $\zeta \in W^{1,\infty}_{loc} \cap W^{2,2}_{loc}(D^2, D^2)$ such that $\vec{\xi} \circ \zeta$ is conformal,

$$\|\zeta\|_{C^{0,\alpha}(D^2)} + \|\zeta^{-1}\|_{C^{0,\alpha}(D^2)} \leq C \left[\exp(C \,\| \log |\nabla \vec{\xi}| \|_{L^\infty(D^2)}) \right] \quad, \tag{A.12}$$

where $C > 0$ is independent of $\vec{\xi}$ and

$$\forall \; 1 > r > 0 \qquad \exists \; C_r > 0 \quad s. \; t.$$

$$\tag{A.13}$$

$$\| \log |\nabla(\vec{\xi} \circ \zeta)| \|_{L^\infty(D^2_r)} + \|\vec{\xi} \circ \zeta\|_{W^{2,2}(D^2_r)} \leq C_r \left[1 + \exp(\| \log |\nabla \vec{\xi}| \|_\infty) \right]$$

where $C_r > 0$ only depends on r and not on $\vec{\xi}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of lemma A.3. Let $(\vec{e}_1, \vec{e}_2)$ be the orthonormal 2-framing given by lemma A.2 such that $\vec{n}_{\vec{\xi}} = \vec{e}_1 \wedge \vec{e}_2$ and let f such that $f \equiv 0$ on ∂D^2 and $df = *_g(\vec{e}_1, d\vec{e}_2)$. Hence it solves

$$\begin{cases} \Delta_g f = (\nabla^\perp \vec{e}_1, \nabla \vec{e}_2) & \text{on } D^2 \\[2mm] f = 0 & \text{on } \partial D^2 \end{cases} \quad.$$

writing as in the proof of lemma A.2

$$\begin{cases} \Delta_{g_0} f = \partial_x \vec{e}_1 \cdot \partial_y \vec{e}_2 - \partial_x \vec{e}_2 \cdot \partial_y \vec{e}_1 + *_{g_0} d((*_{g_0} - *_g) \, df^r) & \text{in } D^2 \\[2mm] f = 0 & \text{on} \quad \partial D^2 \end{cases} \tag{A.14}$$

Using integrability compensation result Theorem 3.4.1 we get the a-priori estimate

$$\|\nabla f\|_{L^{2,1}(D^2)} \leq C \,\|\nabla \vec{e}_1\|_{L^2(D^2)} \,\|\nabla \vec{e}_2\|_{L^2(D^2)} + C \,\varepsilon_0 \,\|\nabla f\|_{L^{2,1}(D^2)} \tag{A.15}$$

where we have used the fact that $\forall \alpha$ one-form on D^2 $|(*_g - *_{g_0}) \alpha| \leq \varepsilon_0 \,|\alpha|$. Hence by density of smooth maps, for ε_0 small enough, we convert the a-priori estimate (A.15) into an estimate and by Lorentz-Sobolev embedding one has

$$\|f\|_{L^\infty(D^2)} \leq C \,\|\nabla f\|_{L^{2,1}(D^2)} \leq C \int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|^2 \; dvol_g \tag{A.16}$$

Let $\varepsilon_i := d\vec{\xi}^{-1} \vec{e}_i$ and let ε_i^* be the dual unit frame for the $g = \vec{\xi}^* g_{\mathbb{R}^m}$ metric of $(\varepsilon_1, \varepsilon_2)$. We have that

$$\begin{cases} df \wedge \varepsilon_1^* = (*_g df) \wedge (*_g \varepsilon_1^*) = -(\vec{e}_1, d\vec{e}_2) \wedge \varepsilon_2^* \quad, \\[2mm] df \wedge \varepsilon_2^* = (*_g df) \wedge (*_g \varepsilon_2^*) = (\vec{e}_1, d\vec{e}_2) \wedge \varepsilon_1^* \quad, \end{cases} \tag{A.17}$$

Moreover Cartan formula gives

$$\begin{aligned} d\varepsilon_i^*(\vec{e}_1, \vec{e}_2) &= d(\varepsilon_i^*(\vec{e}_2)) \cdot \vec{e}_2 - d(\varepsilon_i^*(\vec{e}_1)) \cdot \vec{e}_2 - \varepsilon_i^*([\vec{e}_1, \vec{e}_2]) \\[2mm] &= -\varepsilon_i^*([\vec{e}_1, \vec{e}_2]) = -\varepsilon_i^*(D_{\vec{e}_1} \vec{e}_2 - D_{\vec{e}_2} \vec{e}_1) \end{aligned} \tag{A.18}$$

where D is the Levi-Civita connection associated to the induced metric g. Using the immersion $\vec{\xi}$ we have that $D_X \vec{e}_i = P_{\vec{\xi}}(d\vec{e}_i \cdot X)$ where $P_{\vec{\xi}}$ is the orthonormal projection in $\mathbb{R}^m$ onto $T\vec{\xi}(D^2)$. Hence $D_X \vec{e}_i = (\vec{e}_1, d\vec{e}_i \cdot X)\, \vec{e}_1 + (\vec{e}_2, d\vec{e}_i \cdot X)\, \vec{e}_2$. Combining this later fact with (A.18) gives

$$
\begin{cases}
d\varepsilon_1^* = -(\vec{e}_1, d\vec{e}_2) \wedge \varepsilon_2^* \quad, \\[2mm]
d\varepsilon_2^* = -\varepsilon_1^* \wedge (\vec{e}_1, d\vec{e}_2) \quad.
\end{cases}
\tag{A.19}
$$

Combining (A.17) and (A.19) gives,

$$
d(e^{-f}\varepsilon_1^*) = 0 \qquad \text{and} \qquad d(e^{-f}\varepsilon_2^*) = 0 \quad.
\tag{A.20}
$$

Hence there exists $\sigma := (\sigma_1, \sigma_2) \in W^{2,2} \cap W^{1,\infty}(D^2, \mathbb{R}^2)$ such that

$$
d\sigma_i = e^{-f}\, \varepsilon_i^* \qquad \text{and} \qquad \sigma_i(0) = 0 \quad.
\tag{A.21}
$$

$\nabla \sigma$ has maximal rank equal to 2 at every point, therefore it realizes a lipschitz diffeomorphism from D^2 into $\Omega := \sigma(D^2)$. Let $\frac{\partial}{\partial \sigma_i}$ be the dual basis to $d\sigma_i$. Since $d\vec{\xi} = \sum_{i=1}^{2} d\vec{\xi} \cdot \varepsilon_i\, \varepsilon_i^*$, one has $d\vec{\xi} \cdot \frac{\partial}{\partial \sigma_i} = e^f\, \vec{e}_i$ Hence $\vec{\xi} \circ \sigma^{-1}$ is a conformal immersion. For every one-form α on D^2 one has

$$
(1 - \varepsilon_0^2)\, |\alpha|_{g_0}^2 \le (det g)\, |\alpha|_g^2 \le (1 + \varepsilon_0^2)\, |\alpha|_{g_0}^2 \quad,
\tag{A.22}
$$

Hence, since $|d\sigma|_g^2 = e^{2f}$, since $det g = |\partial_x \vec{\xi} \times \partial_y \vec{\xi}|^2$ and since $1/2 - Dis(\vec{\xi})^2/2 \le |\nabla \vec{\xi}|^{-2}\, |\partial_x \vec{\xi} \times \partial_y \vec{\xi}|^2 \le 1/2 + Dis(\vec{\xi})^2/2$, we deduce from (A.22) and (A.16) the following estimate

$$
\| \log |\nabla \sigma|_{g_0} \|_{L^\infty(D^2)} \le C\, \| \log |\nabla \vec{\xi}| \|_{L^\infty(D^2)} + C \int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|^2\, dvol_g \quad.
\tag{A.23}
$$

Moreover from the above remark[26] we have that $|Dis(\sigma)| < 2\,\varepsilon_0$. Hence we deduce

$$
\begin{cases}
|Dis(\sigma^{-1})| < 2\varepsilon_0 \qquad \text{and} \\[3mm]
\| \log |\nabla \sigma^{-1}|_{g_0} \|_{L^\infty(\Omega)} \le C\, \| \log |\nabla \vec{\xi}| \|_{L^\infty(D^2)} + C \int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|^2\, dvol_g \quad.
\end{cases}
\tag{A.24}
$$

Let h be the solution to the Riemann Mapping Theorem for Ω :

$$
h \;:\; \Omega \longrightarrow D^2 \quad \text{is holomorphic}
$$

$$
h(0) = 0 \qquad \text{and} \quad h'(0) \in \mathbb{R} \quad \text{with} \quad h'(0) > 0 \quad.
$$

The Riemann Mapping Theorem asserts that h is bi-holomorphic and we will denote k it's holomorphic inverse from D^2 into Ω. Finally let $\zeta := \sigma^{-1} \circ k$. (We shall often see ζ as a $\mathbb{C}-$valued map). ζ satisfies

$$
\partial_{\bar{z}} \zeta = \nu(z)\, \overline{\partial_z \zeta} \quad,
\tag{A.25}
$$

where $\nu \circ h := \partial_{\bar{z}} \sigma^{-1} / \overline{\partial_z \sigma^{-1}} = H(\nabla \sigma^{-1}) / [|\nabla \sigma^{-1}|^2 + 2\, det \nabla \sigma^{-1}]$. Since $det \nabla \sigma^{-1} \ge 0$, we obtain

$$
\|\nu\|_{L^\infty(D^2)} \le 2\varepsilon_0 \quad.
\tag{A.26}
$$

---

[26] We use that $|d\sigma_1 \wedge d\sigma_2|_g = |d\sigma|_g^2 / 2$ and (A.22).

Let $\delta := dist(0, \partial\Omega)$. Integrating $d\sigma^{-1}$ on a segment S connecting 0 to one of it's nearest point P on $\partial\Omega$ gives $1 = |\int_S d\sigma^{-1}| \leq \delta \|\nabla\sigma^{-1}\|_\infty$ and integrating now $d\sigma$ on a ray R issued from zero and connecting $\sigma^{-1}(P)$ gives $\delta = |P - 0| = |\int_R d\sigma| \leq \|\nabla\sigma\|_\infty$. Hence we have

$$\frac{1}{\|\nabla\sigma^{-1}\|_\infty} \leq \delta \leq \|\nabla\sigma\|_\infty \quad . \tag{A.27}$$

Combining (A.23), (A.24) and (A.27) gives

$$|\log\delta| \leq C \; \| \log|\nabla\vec{\xi}| \|_{L^\infty(D^2)} + C \int_{D^2} |\nabla\vec{n}_{\vec{\xi}}|^2 \; dvol_g \tag{A.28}$$

Since h is holomorphic and $h(\Omega) = D^2$ we have

$$\|h'\|_{L^\infty(B_{\delta/2}(0))} \leq \frac{C_1}{\delta} \; \|h\|_\infty \leq \frac{C_1}{\delta} \quad . \tag{A.29}$$

This implies that $h(B_r^2(0)) \subset B_{C_1 r/\delta}^2(0)$ and hence for instance $k(\partial B_{1/4}(0)) \subset \Omega \setminus B_{\delta/4C_1}(0)$ Hence

$$\| \log|\xi| \|_{L^\infty(\partial B_{1/4}(0))} \quad \leq |\log[4C_1\|\nabla\sigma\|_\infty]| + |\log\delta|$$

$$\leq C \; \left[1 + \| \log|\nabla\vec{\xi}| \|_{L^\infty(D^2)} \right] + C \int_{D^2} |\nabla\vec{n}_{\vec{\xi}}|^2 \; dvol_g \tag{A.30}$$

We have

$$\|k'\|_{L^\infty(B_{1/2}(0))} \leq 2C_1 \; \|k\|_\infty = 2 \; C_1 \|\nabla\sigma\|_\infty \quad , \tag{A.31}$$

This implies that

$$\|\nabla\zeta\|_{L^\infty(B_{1/4}(0))} \quad \leq 2 \; C_1 \; \|\nabla\sigma^{-1}\|_\infty \; \|\nabla\sigma\|_\infty$$

$$\leq 2 \; C_1 \; \exp\left[C \; \| \log|\nabla\vec{\xi}| \|_{L^\infty(D^2)} + C \int_{D^2} |\nabla\vec{n}_{\vec{\xi}}|^2 \right] \quad , \tag{A.32}$$

In $B_1(0) \setminus B_{1/4}(0)$ we write $\zeta i = e^{\lambda + i\mu}$ where λ is a real-valued function and where μ takes value into $\mathbb{R}/2\pi\mathbb{Z}$. Using this notation (A.25) becomes

$$\frac{[1 - \nu \; e^{-2i\mu}]}{[1 + \nu \; e^{-2i\mu}]} \; \partial_{\bar{z}}\lambda = -i \; \partial_{\bar{z}}\mu \quad , $$

which implies

$$\Re\left[\frac{\partial}{\partial z} \left[(1 + \beta)\frac{\partial\lambda}{\partial\bar{z}} \right] \right] = 0 \quad , \tag{A.33}$$

where $\beta := \beta_1 + i\beta_2 = -2\nu \; e^{-2i\mu} \; [1 + \nu \; e^{2i\mu}]^{-1}$. Hence λ satisfies

$$\begin{cases} \displaystyle\sum_{i,j=1}^{2} \partial_{x_i}[a_{ij} \; \partial_{x_j}\lambda] = 0 \qquad \text{in } B_1^2(0) \setminus B_{1/4}^2(0) \\[2mm] \lambda = 0 \qquad \text{on } \partial B_1^2(0) \\[2mm] \lambda = \log|\zeta| \qquad \text{on } \partial B_{1/4}^2(0) \end{cases} \tag{A.34}$$

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where $a_{11} = a_{22} = 1 + \beta_1$ and $a_{12} = -a_{21} = -\beta_2$. From (A.26) we have that $|\beta| < 4\varepsilon_0$. Hence for $\varepsilon_0 > 0$ small enough De Giorgi-Nash result (see for instance [TA3]) gives the existence of $0 < \alpha < 1$ such that

$$\|\lambda\|_{C^{0,\alpha}(B_1^2 \setminus B_{1/4}^2)} \leq C \, \| \log|\zeta| \|_{W^{1,\infty}(\partial B_{1/4}(0))} \quad .$$

Using (A.32) we then obtain

$$\|\zeta\|_{C^{0,\alpha}(B_1^2 \setminus B_{1/4}^2)} \leq 2 \, C_1 \, \exp\left[C \, \| \log|\nabla\vec{\xi}| \|_{L^\infty(D^2)} + C \int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|^2 \right] \quad . \tag{A.35}$$

Since

$$\Re\left[\frac{\partial}{\partial z} \left[(1+\beta)^{-1} \frac{\partial \mu}{\partial \overline{z}} \right] \right] = 0 \quad , \tag{A.36}$$

we get a similar control to (A.35) for $arg\Psi$ on $B_1^2 \setminus B_{1/4}^2$. Hence combining these last estimates together with (A.32) again, we finally obtain

$$\|\zeta\|_{C^{0,\alpha}(D^2)} \leq 2 \, C_1 \, \exp\left[C \, \| \log|\nabla\vec{\xi}| \|_{L^\infty(D^2)} + C \int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|^2 \right] \quad , \tag{A.37}$$

which finishes the proof of lemma A.3. $\qquad\square$

Lemma A.4 [Normalization Lemma] *Let (Σ, h) be a riemman surface where h denotes the hyperbolic metric of volume 1 associated to the conformal class c. Assume c is contained in a compact class $\mathcal{K}$ of the Moduli space of Σ. For any $\Lambda > 0$ there exists $R > 0$, $1 > r > 0$ and $\delta > 0$ such that, for any conformal embedding $\vec{\Phi}$ of Σ into $\mathbb{R}^m$ satisfying*

$$\int_\Sigma |d\vec{n}_{\vec{\Phi}}|_g^2 \, dvol_g < \Lambda \quad , \tag{A.38}$$

where $g := \vec{\Phi}^* g_{\mathbb{R}^m}$, and for any choice of a geodesic disc $B_{2r}(p)$ for the metric h satisfying

$$\int_{B_{2r}(p)} |d\vec{n}_{\vec{\Phi}}|_g^2 \, dvol_g < \delta$$

then there exists a Moebius transformation Ξ of $\mathbb{R}^m$ such that

   i)
$$\Xi \circ \vec{\Phi}(\Sigma) \subset B_R^m(0) \quad , \tag{A.39}$$

   ii)
$$\mathcal{H}^2(\Xi \circ \vec{\Phi}(\Sigma)) \leq 3 \, R^2 \, \Lambda \quad , \tag{A.40}$$

   iii)
$$c \, r^{-2} \, h \leq g_{\Xi \circ \vec{\Phi}} \leq c^{-1} \, r^{-2} \, h \qquad \text{on } B_r(p) \quad , \tag{A.41}$$

   iv)
$$\int_{B_r(p)} |d\vec{n}_{\Xi \circ \vec{\Phi}}|_g^2 \, dvol_g < 8\pi/3 \tag{A.42}$$

$\qquad\square$

Proof of lemma A.4. Let λ be the function given by $e^{2\lambda} h = \vec{\Phi}^* g_{\mathbb{R}^m}$. λ satisfies

$$-\Delta_h \lambda = K_g \ e^{2\lambda} + K_h$$

where $K_h = 4\pi(1 - genus(\Sigma))$ is the scalar curvature of the metric h and K_g the scalar curvature of the metric $g := \vec{\Phi}^* g_{\mathbb{R}^m}$. Since the L^1 norm of $K_g \ e^{2\lambda}$ is bounded by Λ, standard elliptic estimates gives the existence of a constant $C(\Sigma, h)$ such that

$$\|d\lambda\|_{L^{2,\infty}(\Sigma)} \le C(\Sigma, h) \ (\Lambda^2 + 1) \tag{A.43}$$

where the $L^{2,\infty}$ norm is taken w.r.t h. Since h is assumed to be contained in a compact subpart of the moduli space of Σ we have that $C(\Sigma, h)$ is uniformly bounded. On the disc $B_{2r}(p)$ we take conformal coordinates in such a way that the flat metric in these coordinates is comparable to h (i.e. $h = e^\nu[dx_1^2 + dx_2^2]$ and e^ν is comparable to Cr^2 where C is universal) and we identify $B_{2r}(p)$ with the flat unit disc D^2. Let f such that $e^{2f}[dx_1^2 + dx_2^2] = g$ (i.e. $f = \lambda + \nu$). We have

$$\|\nabla f\|_{L^{2,\infty}} \le C \ [\Lambda^2 + 1] \quad . \tag{A.44}$$

We assume $\delta < 8\pi/3$ and then using lemma 5.1.4 of [Hel] there exists a $W^{1,2}$ orthonormal frame $(\vec{e}_1, \vec{e}_2)$ on $B_{2r}(p)$ such that

$$\vec{n}_{\vec{\Phi}} = \vec{e}_1 \wedge \vec{e}_2 \tag{A.45}$$

which is in $W^{1,2}(D^2, S^{m-1} \times S^{m-1})$ such that

$$\vec{e}_1 \cdot \vec{e}_2 = 0 \qquad , \qquad n_{\vec{\xi}} = \vec{e}_1 \wedge \vec{e}_2 \quad , \tag{A.46}$$

$$\int_{D^2} \left[|\nabla \vec{e}_1|^2 + |\nabla \vec{e}_2|^2\right] \ dx \, dy \le 2 \int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|^2 \ dx \, dy \tag{A.47}$$

and

$$\begin{cases} div(\vec{e}_1, \nabla \vec{e}_2)) = 0 \qquad \text{in } D^2 \quad , \\[2mm] \left(\vec{e}_1, \dfrac{\partial \vec{e}_2}{\partial \nu}\right) = 0 \qquad \text{on } \partial D^2 \quad . \end{cases} \tag{A.48}$$

With this frame, as in [Hel], we can express Δf and we have

$$\Delta f = (\nabla^\perp \vec{e}_1, \nabla \vec{e}_2) \quad . \tag{A.49}$$

Let μ be the solution of

$$\begin{cases} \Delta \mu = (\nabla^\perp \vec{e}_1, \nabla \vec{e}_2) \qquad \text{on } D^2 \\[2mm] \mu = 0 \qquad \text{on } \partial D^2 \end{cases} \tag{A.50}$$

Wente inequality gives

$$\|\mu\|_{L^\infty(D^2)} + \|\nabla \mu\|_{L^2(D^2)} \le C \sum_{i=1}^{2} \int_{D^2} |\nabla \vec{e}_i|^2 \le C\delta \tag{A.51}$$

Combining (A.44) and (A.51) we obtain, since $f - \mu$ is harmonic that

$$\|f - \overline{f}\|_{L^\infty(D^2_{1/2})} \le C\delta + C \ [\Lambda^2 + 1] \quad . \tag{A.52}$$

where $\overline{f}$ is the average of f on the 2 disc of radius $1/2$, $D^2_{1/2}$.

We translate the surface $\vec{\Phi}(\Sigma)$ in such a way that $\vec{\Phi}(p) = 0$. Denote by $\Lambda_t(x) = e^t x$ the dilation of rate e^t. Observe that

$$\vec{n}_{\vec{\Phi}} = \vec{n}_{\Lambda_t \circ \vec{\Phi}} \quad .$$

We consider now the new immersion $\Lambda_t \circ \vec{\Phi}$ that we denote $\vec{\Phi}_t$ and $f_t := f + t$. We are going to fix t such that $\overline{f} + t$ will be comparable to 1 but will be relatively small.

From (A.6) in [KS2] for any $x_0 \in \mathbb{R}^m$ and $0 < \sigma << +\infty$

$$\sigma^{-2}\, \mathcal{H}^2(\vec{\Phi}_t(\Sigma) \cap B_\sigma(x_0)) \leq 3 \left[r^{-2}\, \mathcal{H}^2(\vec{\Phi}_t(\Sigma) \cap B_r(x_0)) + \int_{\vec{\Phi}_t^{-1}(B_r(x_0))} |\vec{H}|^2\, dvol_g \right] \quad . \tag{A.53}$$

We claim that there exists ρ_0 depending only on $\Lambda > W(\vec{\Phi})$ and $x_1 \in B_1(0)$ such that $\vec{\Phi}_t(\Sigma) \cap B_{\rho_0}(x_1) = \emptyset$. For $y \in \vec{\Phi}(\Sigma)$ one has

$$\lim_{\sigma \to 0} \sigma^{-2}\, \mathcal{H}^2(\vec{\Phi}(\Sigma) \cap B_\sigma(y)) = \pi \quad . \tag{A.54}$$

For $0 < \rho < 1/2$ we consider a regular covering of $B_1(0)$ by balls $B_\rho(z_l)$ in such a way that any point in $B_1(0)$ is contained in at most $C(m)$ balls of the form $B_{2\rho}(z_l)$. The number of l such that $\int_{\vec{\Phi}^{-1}(B_{2\rho}(z_l))} |\vec{H}|^2\, dvol_g > C^{-1}\, \pi/2$ is bounded by $2\,\Lambda\,C\,C(m)$. For an l such that $\int_{\vec{\Phi}^{-1}(B_{2\rho}(z_l))} |\vec{H}|^2\, dvol_g < C^{-1}\,\pi/2$ and such that there exists $y \in B_\rho(z_l) \cap \Sigma \neq \emptyset$, combining (A.53) and (A.54) one obtains that

$$(2\rho)^{-2}\, \mathcal{H}^2(\vec{\Phi}(\Sigma) \cap B_{2\rho}(z_l)) > C^{-1}\,\pi/2 \quad .$$

the number of such l is then bounded by ρ^{-2} times a number depending only on m and Λ - where we are using again (A.53) but for $x_0 = 0$, $\sigma = 1$ and $r \to +\infty$. The total number of ball $B_\rho(z_l)$ is proportional to ρ^{-m}. Since $m > 2$, for $\rho = \rho_0$ chosen small enough, depending only on m and Λ we deduce the claim.

Let x_1 and ρ_0 given by the claim. Let $I(x) := x - x_1/|x - x_1|^2$ to be the inversion with respect to x_1. We will choose $\Xi_t = I \circ \Lambda_t$ for some suitable t. We have then

$$\Xi_t(\vec{\Phi}(\Sigma)) \subset B_{1/\rho_0}(0) \quad , \tag{A.55}$$

Because of (A.52)

$$\Lambda_t \circ \vec{\Phi}(D_{1/2}^2) \subset B_{\alpha\, e^{t+\overline{f}}}(0) \tag{A.56}$$

for α being the exponential of the r.h.s. of (A.52) $\alpha := e^{C\delta + C\,[\Lambda^2+1]}$. Moreover, still because of (A.52), we have

$$\pi \alpha^{-2} e^{2(t+\overline{f})} \leq \mathcal{H}^2(\Lambda_t \circ \vec{\Phi}(D_{1/2}^2)) = \int_{D_{1/2}^2} e^{2(t+f)} \leq \pi \alpha^2 e^{2(t+\overline{f})} \tag{A.57}$$

Choosing already t such that $\alpha\, e^{t+\overline{f}} < 1$, (A.57) implies that

$$\frac{1}{4\alpha^2}\, e^{2(\overline{f}+t)} \leq g_{\Xi_t \circ \vec{\Phi}} \leq \frac{\alpha^2}{\rho_0^2}\, e^{2(\overline{f}+t)} \tag{A.58}$$

from which we deduce

$$4^{-1}\alpha\, e^{t+\overline{f}} \leq \mathcal{H}^2(\Xi_t \circ \vec{\Phi}(D_{1/2}^2)) \leq \frac{\pi}{\rho_0^2}\alpha^2 e^{2(t+\overline{f})} \quad . \tag{A.59}$$

An exact computation gives

$$\vec{n}_{\Xi_t \circ \vec{\Phi}} = \vec{n}_{\vec{\Phi}} - 2(\vec{n}_{\vec{\Phi}} \llcorner (\vec{\Phi}_t - x_1)) \wedge \frac{\vec{\Phi}_t - x_1}{|\vec{\Phi}_t - x_1|^2}$$

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where we see $\vec{n}$ as a 2-vector. Hence we obtain that

$$|\nabla \vec{n}_{\Xi_t \circ \vec{\Phi}}| \leq 4\,|\nabla \vec{n}_{\vec{\Phi}}| + 4\,\rho_0^{-1}\,|\nabla \vec{\Phi}_t|$$

This implies, using (A.59),

$$\int_{D_{1/2}^2} |\nabla \vec{n}_{\Xi_t \circ \vec{\Phi}}| \leq 4\delta + 4\,\frac{\pi}{\rho_0^4}\alpha^2 e^{2(t+\overline{f})} \tag{A.60}$$

We choose now t such that $4\delta + 4\frac{\pi}{\rho_0^4}\alpha^2 e^{2(t+\overline{f})} = 8\pi/3$ and collecting (A.55), (A.58) and (A.60) gives the result. Indeed it remains only to check (A.40). We first observe that inequality (A.53) (which holds also for $\vec{\Phi}$ replaced by $\Xi \circ \vec{\Phi}$) implies that , for any $r \geq \rho_0^{-1}$

$$\mathcal{H}^2(\Xi \circ \vec{\Phi}(\Sigma)) \leq C\rho_0^{-2}r^{-2}\,\mathcal{H}^2(\Xi \circ \vec{\Phi}(\Sigma)) + \rho_0^{-2}\,\Lambda \quad .$$

Letting ρ converge to $+\infty$ yields the desired estimate (A.40). Hence lemma A.4 is proved. $\qquad\square$

Lemma A.5 *Let $\vec{\xi}$ be a conformal immersion of $D^2 \setminus \{0\}$ into $\mathbb{R}^m$ in $W_{loc}^{2,2}(D^2 \setminus \{0\}, \mathbb{R}^m)$ and such that $\log |\nabla \vec{\xi}| \in L_{loc}^\infty(D^2 \setminus \{0\})$. Assume $\vec{\xi}$ extends to a map in $W^{1,2}(D^2)$ and that the corresponding Gauss map $\vec{n}_{\vec{\xi}}$ also extends to a map in $W^{1,2}(D^2, Gr_{m-2}(\mathbb{R}^m))$. Then $\vec{\xi}$ realizes a lipshitz conformal immersion of the whole disc D^2 and there exits a positive integer n and a constant C such that*

$$(C - o(1))\,|z|^{n-1} \leq \left|\frac{\partial \vec{\xi}}{\partial z}\right| \leq (C + o(1))\,|z|^{n-1} \quad . \tag{A.61}$$

$$\square$$

Proof of Lemma A.5. We can always localize in order to ensure that

$$\int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|^2 \; dx\,dy < \frac{8\pi}{3} \quad .$$

Using lemma 5.1.4 of [Hel] we deduce the existence of a framing $\vec{e} := (\vec{e}_1, \vec{e}_2)$ which is in $W^{1,2}(D^2, S^{m-1} \times S^{m-1})$ such that

$$\vec{e}_1 \cdot \vec{e}_2 = 0 \qquad , \qquad n_{\vec{\xi}} = \vec{e}_1 \wedge \vec{e}_2 \quad , \tag{A.62}$$

$$\int_{D^2} \left[|\nabla \vec{e}_1|^2 + |\nabla \vec{e}_2|^2\right]\;dx\,dy \leq 2\int_{D^2} |\nabla \vec{n}_{\vec{\xi}}|^2\;dx\,dy \tag{A.63}$$

and

$$\begin{cases} div(\vec{e}_1, \nabla \vec{e}_2)) = 0 & \text{in } D^2 \quad , \\[2mm] \left(\vec{e}_1, \dfrac{\partial \vec{e}_2}{\partial \nu}\right) = 0 & \text{on } \partial D^2 \quad . \end{cases} \tag{A.64}$$

Similarly as in the proof of lemma A.3, we introduce $\varepsilon_i := d\vec{\xi}^{-1}\vec{e}_i$ and ε_i^* to be the dual framing. Denoting $|\partial_x \vec{\xi}|^2 = |\partial_y \vec{\xi}|^2 = e^{2\lambda}$ we have that the metric $g_\infty := \vec{\xi}^* g_{\mathbb{R}^m}$ is given by $g = e^{2\lambda}\left[dx^2 + dy^2\right]$. Hence with respect to the flat metric $g_0 := \left[dx^2 + dy^2\right]$ one has

$$|\varepsilon_i|_{g_0}^2 = g_0(\varepsilon_i, \varepsilon_i) = e^{-2\lambda}\,g_\infty(\varepsilon_i, \varepsilon_i) = e^{-2\lambda} \quad .$$

and since $< \varepsilon_i, \varepsilon_j^* > = \delta_{ij}$ we have that $|\varepsilon_i^*|_{g_0}^2 = e^{2\lambda}$. Thus we deduce that the 1-forms ε_i^* are in $L^2(D^2)$. Since $\vec{\xi}$ is in $W^{1,\infty} \cap W_{loc}^{2,2}(D^2 \setminus \{0\}, \mathbb{R}^m)$ and $\log |\nabla \vec{\xi}| \in L_{loc}^\infty(D^2 \setminus \{0\})$ we have that the framing given by

$\vec{f_i} := e^{-\lambda} \, \partial_{x_i} \vec{\xi}$ is in $L^\infty_{loc} \cap W^{1,2}_{loc}(D^2 \setminus \{0\}, \mathbb{R}^m)$. Since $\vec{\xi}$ is conformal the unit framing $(\vec{f_1}, \vec{f_2})$ is Coulomb[27] :

$$ div(\vec{f_1}, \nabla \vec{f_2}) = 0 \qquad \text{in } D^2 \setminus \{0\} \quad . $$

Denoting $e^{i\theta}$ the rotation which passes[28] from $(\vec{f_1}, \vec{f_2})$ to $(\vec{e_1}, \vec{e_2})$. The Coulomb condition satisfied by the two framings implies that $d\theta := (ie^{i\theta}, d(e^{i\theta}))$ is an harmonic 1-form on $D^2 \setminus \{0\}$ and hence analytic on this domain. This implies that

$$ \varepsilon_i^* \in L^\infty_{loc} \cap W^{1,2}_{loc}(D^2 \setminus \{0\}) \quad . $$

Like again in lemma A.3 we introduce[29] $f \in C^0 \cap W^{1,2}(D^2)$ to be the solution of

$$ \begin{cases} \Delta f = (\nabla^\perp \vec{e_1}, \nabla \vec{e_2}) & \text{on } D^2 \\[2mm] f = 0 & \text{on } \partial D^2 \quad . \end{cases} $$

As in lemma A.3, the computations give in $D^2 \setminus \{0\}$

$$ \forall i = 1,2 \qquad d[e^{-f} \varepsilon_i^*] = 0 \qquad \text{a.e. in } D^2 \setminus \{0\} \quad . $$

By schwartz lemma the distribution $d[e^{-f}\varepsilon_i^*]$ is a finite linear combination of successive derivatives of the Dirac Mass at the origin but since $e^{-f}\varepsilon_i^* \in L^2(D^2)$, this linear combination can only be 0. Hence we have

$$ \forall i = 1,2 \qquad d[e^{-f}\varepsilon_i^*] = 0 \qquad \text{in } \mathcal{D}'(D^2) \quad . $$

Hence, by Poincaré Lemma, there exists $(\sigma_1, \sigma_2) \in W^{1,2}(D^2, \mathbb{R}^2)$ such that $d\sigma_i = e^{-f}\varepsilon_i^*$. The dual basis $(\partial/\partial\sigma_1, \partial/\partial\sigma_2) = e^f(\varepsilon_1, \varepsilon_2)$ is positive, orthogonal on $D^2 \setminus \{0\}$ and integrable by nature. hence $\sigma = \sigma_1 + i\sigma_2$ is an holomorphic function on $D^2 \setminus \{0\}$ which extends to a $W^{1,2}-$map on D^2. The classical point removability theorem for holomorphic map implies that σ extends to an holomorphic function on D^2. We can choose it in such a way that $\sigma(0) = 0$. The holomorphicity of σ implies in particular that $|d\sigma|_{g_0} = e^{\lambda - f}$ is uniformly bounded and, since $f \in L^\infty(D^2)$, we deduce that λ is bounded from above on D^2. This later fact implies that $\vec{\xi}$ extends to a Lipshitz map on D^2. Though $|d\sigma|_{g_0} = e^{\lambda - f}$ has no zero on $D^2 \setminus \{0\}$, σ' might have a zero at the origin : there exists an holomorphic function $h(z)$ on D^2 satisfying $h(0) = 0$, a complex number c_0 and an integer n such that

$$ \sigma(z) = c_0 \, z^n \, (1 + h(z)) \qquad . \tag{A.65} $$

We have that locally

$$ \frac{\partial\vec{\xi}}{\partial\sigma} = \partial_{\sigma_1}\vec{\xi} - i\partial_{\sigma_2}\vec{\xi} = d\vec{\xi} \, e^f \varepsilon_1^* - id\vec{\xi} \, e^f \varepsilon_2^* = e^f[\vec{e_1} - i\vec{e_2}] $$

Hence, since f is continuous, we have that

$$ \left| \frac{\partial\vec{\xi}}{\partial\sigma} \right| = \sqrt{2} \, e^{f(0)} \, (1 + o(1)) \quad . \tag{A.66} $$

Combining (A.65) and (A.66) gives

$$ \left| \frac{\partial\vec{\xi}}{\partial z} \right| = \left| \frac{\partial\vec{\xi}}{\partial\sigma} \right| \, \left| \frac{\partial\sigma}{\partial z} \right| = c_0 \, n \, \sqrt{2} \, e^{f(0)} \, |z|^{n-1} \, (1 + o(1)) \quad . \tag{A.67} $$

This last identity implies the lemma A.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

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[27] This follows from a straightforward computation presented in [Hel] chapter 5.

[28] $e^{i\theta} \, (\vec{f_1} + i\vec{f_2}) = (\vec{e_1} + i\vec{e_2})$

[29] By virtue of Wente's theorem (see theorem 3.1.2 of [Hel]).

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