Lectures on Floer homology

Dietmar Salamon
University of Warwick

December 1, 1997

Notes for lectures held at the IAS/Park City Graduate Summer School on Symplectic Geometry and Topology in July 1997.

Contents

Abstract

The purpose of these lectures is to give an introduction to symplectic Floer homology and the proof of the Arnold conjecture. This conjecture gives a lower bound for the number of 1-periodic solutions of a 1-periodic Hamiltonian system in terms of the sum of the Betti numbers.

The first three lectures are introductory, and deal with the basic ideas in Floer's proof of the Arnold conjecture. Topics covered include the Morse-Smale-Witten complex, some basic analysis and Fredholm theory, the spectral flow and the Maslov index, the compactness problem, the construction of Floer homology, the proof that Floer homology is an invariant, and the role of Novikov rings.

The last two lectures deal with more recent developments and lead up to a proof of the Arnold conjecture for general symplectic manifolds and rational coefficients. The fourth lecture gives an introduction to Gromov compactness, stable maps, and the Deligne-Mumford compactification, while the last lecture discusses multi-valued perturbations, branched manifolds, the construction of rational Gromov-Witten invariants, and the proof of the Arnold conjecture for general symplectic manifolds.

Acknowledgement

I would like to thank Joa Weber for drawing the diagrams.

1 Symplectic fixed points and Morse theory

1.1 The Arnold conjecture

Let (M, ω) be a compact symplectic manifold. The form ω determines an isomorphism $I_\omega : T^*M \to TM$ and the image of an exact 1-form $dH : M \to T^*M$ under this isomorphism is called the **Hamiltonian vector field** generated by the **Hamiltonian function** $H : M \to \mathbb{R}$. It is denoted by $X_H : M \to TM$ and is given by $\iota(X_H)\omega = dH$. Let $H_t = H_{t+1} : M \to \mathbb{R}$ be a smooth time dependent 1-periodic family of Hamiltonian functions and consider the Hamiltonian differential equation

$$\dot{x}(t) = X_t(x(t)), \tag{1}$$

where $X_t = X_{H_t}$ for $t \in \mathbb{R}$. The solutions of (1) generate a family of symplectomorphisms $\psi_t : M \to M$ via

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \qquad \psi_0 = \mathrm{id}.$$

The fixed points of the time-1-map $\psi = \psi_1$ are in one-to-one correspondence with the 1-periodic solutions of (1) and we denote the set of such solutions by

$$\mathcal{P}(H) = \{x : \mathbb{R}/\mathbb{Z} \to M \,:\, (1)\}\,.$$

A periodic solution x is called **nondegenerate** if all its Floquet multipliers are not equal to 1, or equivalently,

$$\det(\mathbb{1} - d\psi_1(x(0))) \neq 0. \tag{2}$$

The Arnold conjecture asserts that, in the nondegenerate case, the number of 1-periodic solutions should be bounded below by the sum of the Betti numbers of M.

Conjecture 1.1 (Arnold Conjecture) *Let (M, ω) be a compact symplectic manifold and $H_t = H_{t+1} : M \to \mathbb{R}$ be a smooth time dependent 1-periodic Hamiltonian function. Suppose that the 1-periodic solutions of (1) are all nondegenerate. Then*

$$\#\mathcal{P}(H) \geq \sum_{i=0}^{2n} \dim\, H_i(M, \mathbb{Q})$$

where $H_i(M, \mathbb{Q})$ denotes the singular homology of M with rational coefficients.

In contrast, the Lefschetz fixed point theorem only gives the alternating sum of the Betti numbers as a lower bound. The Lefschetz fixed point theorem is related to the Arnold conjecture in the same way as the Poincaré-Hopf theorem (which asserts that if a vector field has only nondegenerate zeros then the number of zeros is bounded below by the Euler characteristic) is related to Morse theory (which gives the sum of the Betti numbers as a lower bound for number of critical points of a Morse function). In the special case where $H_t = H$ is independent of t, the Arnold conjecture is obvious. In this case all the critical points of H are constant solutions of (1) and, in particular, are 1-periodic. Nondegeneracy of the 1-periodic solutions implies that H is a Morse function, and hence the result follows from Morse theory.

Exercise 1.2 Let x be a critical point of $H \equiv H_t$ and suppose that x is nondegnerate as a 1-periodic solution of (1). Prove that x is nondegenerate as a critical point of H. □

The Arnold conjecture (in the above form) has now been proved in full generality. It was first confirmed by Eliashberg [6] for Riemann surfaces and then by Conley and Zehnder [2] for the $2n$-torus. In [18] Gromov proved the existence of at least one fixed point under the assumption $\pi_2(M) = 0$. The breakthrough came when Floer established the Arnold conjecture for Lagrangian intersections, and hence symplectic fixed points, again under the assumption $\pi_2(M) = 0$. In a series of papers [7, 8, 9, 10] Floer combined the variational approach of Conley and Zehnder with the elliptic techniques of Gromov and the Morse-Smale-Witten complex to develop his infinite dimensional approach to Morse theory which is now called Floer homology. This work culminated in the paper [11], where Floer proved the Arnold conjecture for monotone symplectic manifolds. Floer's proof was extended by Hofer-Salamon [19] and Ono [36] to the weakly monotone case, and recently by Fukaya-Ono [14], Liu-Tian [28], and Hofer-Salamon [20, 21, 22, 23] to the general case. Another proof was announced by Ruan [42].

Remark 1.3 There are many different forms of the Arnold conjecture. For example, it can be formulated with any other coefficient ring (as long as it is a principal ideal domain), or in the form that a lower bound for the number of periodic solutions should (in the nondegenerate case) be the minimal number of critical points of a Morse function. There are many examples of manifolds for which this number is strictly larger than the sum of the Betti numbers (for any coefficient ring).

Another version of the Arnold conjecture gives a lower bound, without the nondegeneracy condition, in terms of the Ljusternik-Schnirelman category, or again in terms of the minimal number of critical points of any function (Morse or not) on the manifold.

Yet another version of the Arnold conjecture refers to intersection points of two Lagrangian submanifolds (which are related by a Hamiltonian isotopy) both in the degenerate and nondegenerate case. There is quite a large literature on this subject, with many partial solutions. Many question are still open, especially concerning lower bounds which go beyond the sum of the Betti numbers, more general coefficient rings, and Ljusternik-Schnirelman estimates for the degenerate case. □

1.2 The monotonicity condition

An almost complex structure J on TM is called **compatible** with ω if the formula

$$\langle \xi, \eta \rangle = \omega(\xi, J\eta) \tag{3}$$

defines a Riemannian metric on M. The space $\mathcal{J}(M, \omega)$ of such almost complex structures is nonempty and contractible. Thus the first Chern class $c_1 = c_1(TM, J) \in H^2(M, \mathbb{Z})$ is independent of the choice of $J \in \mathcal{J}(M, \omega)$. The goal in these lectures is to outline the proof of the Arnold conjecture in the case where the cohomology classes c_1 and $[\omega]$ satisfy the condition

$$\int_{S^2} v^* c_1 = \tau \int_{S^2} v^* \omega \tag{4}$$

for every smooth map $v : S^2 \to M$ and some constant $\tau \in \mathbb{R}$. It is important to distinguish the three cases $\tau > 0$, $\tau = 0$, and $\tau < 0$. Geometrically, this corresponds to the conditions of positive, zero, and negative curvature. The toy models for these cases are the 2-sphere (positive curvature), the 2-torus (zero curvature), and surfaces of higher genus (negative curvature).[1] As these simple examples already indicate, the case $\tau < 0$ is by far the most general. On the other hand the proof of the Arnold conjecture is easier in the case $\tau > 0$ and symplectic manifolds with this property are called **monotone**. This is the case originally treated by Floer in [11] and it led him to the definition of what is now called Floer homology. The case $c_1 = 0$ was treated by Hofer-Salamon in [13, 19]. This is an extension of Floer's work and requires the construction of Floer homology groups with coefficients in a suitable Novikov ring. The case $\tau < 0$, and indeed that of general compact symplectic manifolds was only recently resolved by Fukaya-Ono [14], Liu-Tian [28], and Hofer-Salamon [20, 21, 22, 23].

Remark 1.4 In [35] Ohta and Ono proved that the only symplectic 4-manifolds, in which $c_1(TX, J)$ is a positive multiple of some integral lift of ω, are $S^2 \times S^2$ and $\mathbb{C}P^2$ with up to eight points blown up. Also it is a well known fact in Kähler geometry that the only simply connected Kähler surfaces with $c_1 = 0$ are the K3-surfaces (e.g. hypersurfaces of degree 4 in $\mathbb{C}P^3$) and they are all diffeomorphic. □

Exercise 1.5 Let $X_d \subset \mathbb{C}P^n$ be a hypersurface of degree d. Explicitly, one can think of this as the submanifold cut out by the equation ${z_0}^d + {z_1}^d + \cdots + {z_n}^d = 0$. The Lefschetz hyerplane theorem asserts that this manifold is simply connected. Prove that its first Chern class is given by

$$c_1(X_d) = (n + 1 - d)\iota^* h$$

where $h = \mathrm{PD}([\mathbb{C}P^{n-1}]) \in H^2(\mathbb{C}P^n; \mathbb{Z})$ is the canonical generator, and $\iota : X_d \to \mathbb{C}P^n$ denotes the inclusion. Deduce that X_d satisfies (4) with $\tau > 0$ for $d \leq n$, with $\tau = 0$ for $d = n + 1$, and with $\tau < 0$ for $d \geq n + 2$. **Hint:** The direct sum of the tangent bundle $T\mathbb{C}P^n$ with the trivial line bundle $\mathbb{C}$ is isomorphic to the $(n + 1)$-fold direct sum of the canonical bundle H. The normal bundle of X_d can be identified with the restriction of the dth tensor power of H to X_d. □

Definition 1.6 *Let (M, ω) be a compact symplectic manifold. Then the* **minimal Chern number** *of (M, ω) is the integer*

$$N = \inf \left\{ k > 0 \,\middle|\, \exists\, v : S^2 \to M,\ \int_{S^2} v^* c_1 = k \right\}.$$

If $\int_{S^2} v^* c_1 = 0$ for every $v : S^2 \to M$ we call $N = \infty$ the minimal Chern number. If $N \neq \infty$ then $\langle c_1, \pi_2(M) \rangle = N\mathbb{Z}$.

In the following we shall assume (4) and, in the case $\tau \neq 0$, normalize the symplectic form such that $\int_{S^2} v^* \omega \in \mathbb{Z}$ for every smooth map $v : S^2 \to M$.

1.3 The Morse-Smale-Witten complex

Let M be a compact smooth Riemannian manifold and $f : M \to \mathbb{R}$ be a Morse function. Denote by $\mathrm{Crit}(f) = \{x \in M : df(x) = 0\}$ the set of critical points of f. The Morse condition asserts that the critical points are all nondegenerate. Thus the **Hessian** $d^2 f(x) : T_x M \times T_x M \to \mathbb{R}$ is nondegenerate for every $x \in \mathrm{Crit}(f)$. In local coordinates $d^2 f(x)$ is given by the matrix of second partial derivatives and the nondegeneracy condition asserts that this matrix is nonsingular.

[1] These examples do not quite fit the definition since $\pi_2(\Sigma) = 0$ for Riemann surfaces of genus $g \geq 1$. However, if we strengthen (4) to $c_1 = \tau[\omega]$, then Riemannn surfaces satisfy the condition with $\tau = (\mathrm{Vol}(\Sigma))^{-1}(2 - 2g)$.

Exercise 1.7 Let ∇ denote the Levi-Civita connection of the Riemannian metric. Prove that the linear operator $\nabla^2 f(x) : T_xM \to T_xM$ defined by $\nabla^2 f(x)\xi = \nabla_\xi \nabla f(x)$ for $\xi \in T_xM$ is symmetric with respect to the given Riemannian metric. If $df(x) = 0$ prove that

$$\langle \nabla_\xi \nabla f(x), \eta \rangle = d^2 f(x)(\xi, \eta)$$

for all $\xi, \eta \in T_xM$. □

Consider the (negative) gradient flow

$$\dot{u} = -\nabla f(u) \tag{5}$$

and denote by $\varphi^s : M \to M$ the flow of (5). The Morse condition implies that the critical points of f are hyperbolic fixed points of (5). It follows that the stable and unstable manifolds

$$W^s(x; f) = \left\{ z \in M \, : \, \lim_{s \to \infty} \varphi^s(z) = x \right\},$$

$$W^u(x; f) = \left\{ z \in M \, : \, \lim_{s \to -\infty} \varphi^s(z) = x \right\}$$

are smooth submanifolds of M for every critical point x of f. The Morse index of a critical point is the number of negative eigenvalues of the Hessian (when regarded as a linear operator $\nabla^2 f(x)$) and it agrees with the dimension of the unstable manifold. It is denoted by

$$\mathrm{ind}_f(x) = \nu^-(d^2 f(x)) = \dim W^u(x; f).$$

The gradient flow (5) is called a **Morse-Smale system** if, for any pair of critical points x, y of f, the stable and unstable manifolds intersect transversally. In this case the set

$$\mathcal{M}(y, x; f) = W^s(x; f) \cap W^u(y; f)$$

of points in M whose gradient lines connect y to x (the *space of connecting orbits*) is a smooth submanifold of M whose dimension is given by the difference of the Morse indices:

$$\dim \mathcal{M}(y, x; f) = \mathrm{ind}_f(y) - \mathrm{ind}_f(x).$$

One can think of $\mathcal{M}(y, x; f)$ as the space of gradient flow lines $u : \mathbb{R} \to M$ running from $y = \lim_{s \to -\infty} u(s)$ to $x = \lim_{s \to +\infty} u(s)$. The group $\mathbb{R}$ acts on $\mathcal{M}(y, x; f)$ by translations and the quotient $\widehat{\mathcal{M}}(y, x; f) = \mathcal{M}(y, x; f)/\mathbb{R}$ is a manifold of dimension $\mathrm{ind}_f(y) - \mathrm{ind}_f(x) - 1$ (whenever it is nonempty). Hence the Morse-Smale condition implies that $\mathrm{ind}_f(x) < \mathrm{ind}_f(y)$ whenever there is a connecting orbit from y to x. In short, the index decreases strictly along flow lines.

Exercise 1.8 Prove the dimension formula. □

Exercise 1.9 Prove that for every sequence $u^\nu \in \mathcal{M}(y, x; f)$ there exists a subsequence (still denoted by u^ν), finitely many critical points $x_0 = x, x_1, \ldots, x_m = y$, finitely many gradient flow lines $u_j \in \mathcal{M}(x_j, x_{j-1}; f)$, and sequences $s_j^\nu \in \mathbb{R}$, such that, for every j, $u^\nu(s + s_j^\nu)$ converges to $u_j(s)$, uniformly on compact subsets of $\mathbb{R}$. This limit behaviour is illustrated in Figure 1. (See [45] if you get stuck.) □

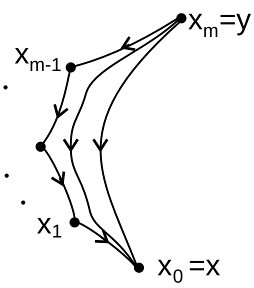

Figure 1: Limit behaviour for connecting orbits

Exercise 1.10 Prove that the quotient space $\widehat{\mathcal{M}}(y,x;f) = \mathcal{M}(y,x;f)/\mathbb{R}$ of gradient flow lines from y to x is a finite set whenever the difference of the Morse indices is equal to 1. $\square$

Exercise 1.11 Fix an orientation o_x of the unstable manifold $W^u(x)$ for every critical point x of f. Show how this gives rise to a natural orientation for each connecting orbit (with index difference 1). **Hint:** For $z \in \mathcal{M}(y,x;f)$ the differential of the gradient flow $d\varphi^t(z)$ determines, for large t, a vector space isomorphism

$$T_zW^u(y) \cap \nabla f(z)^\perp \longrightarrow T_xW^u(x).$$

Define $\varepsilon(z) = \pm 1$, depending on whether this isomorphism is orientation preserving or orientation reversing. This works even if the manifold M is not orientable. $\square$

Let us now assume that the gradient flow of f is a Morse-Smale system and fix an orientation of $W^u(x)$ for every critical point x. Denote by

$$CM_*(f) = \bigoplus_{df(x)=0} \mathbb{Z}\langle x\rangle$$

the free abelian group generated by the critical points of f. This complex is graded by the Morse index and the boundary operator $\partial = \partial^M : CM_k(f) \to CM_{k-1}(f)$ is defined by

$$\partial^M\langle y\rangle = \sum_{\substack{x\in \mathrm{Crit}(f)\\ \mathrm{ind}_f(x)=k-1}} \sum_{[u]\in\widehat{\mathcal{M}}(y,x)} \varepsilon(u)\langle x\rangle$$

for $y \in \mathrm{Crit}(f)$ with $\mathrm{ind}_f(y) = k$. Here the sign $\varepsilon(u)$ is given by Exercise 1.11. This complex $(CM(f), \partial^M)$ is called the **Morse-Smale-Witten complex**. The remarkable observation is that ∂^M is indeed a boundary operator and that the homology of this complex agrees with the homology of M.

Theorem 1.12 (Morse,Smale,Witten) *Suppose that (5) is a Morse-Smale flow. Let $CM(f)$ and ∂^M be defined as above. Then $\partial^M \circ \partial^M = 0$ and there is a natural isomorphism*

$$HM_k(M,f;\mathbb{Z}) = \frac{\ker\,\partial^M}{\mathrm{im}\,\partial^M} \to H_k(M;\mathbb{Z})$$

where $H_k(M;\mathbb{Z})$ denotes the singular homology of M.

Proof: Here is a sketch of the argument which proves $\partial^M \circ \partial^M = 0$. This is equivalent to the formula

$$\sum_{\substack{y \in \mathrm{Crit}(f) \\ \mathrm{ind}_f(y)=k}} \sum_{[v]\in\widehat{\mathcal{M}}(z,y)} \sum_{[u]\in\widehat{\mathcal{M}}(y,x)} \varepsilon(v)\varepsilon(u) = 0 \tag{6}$$

for every pair of critical points $x, z \in \mathrm{Crit}(f)$ with $\mathrm{ind}_f(x) = k-1$ and $\mathrm{ind}_f(z) = k+1$. This is proved by studying the ends of the 1-dimensional moduli space $\widehat{\mathcal{M}}(z,x)$. The endpoints of this moduli space are in one-to-one correspondence with the set of pairs of gradient flow lines (u, v) running from z to x, via some intermediate critical point y, necessarily of index k. This assertion follows from a combination of compactness and gluing arguments. Since every compact 1-manifold has an even number of boundary points, we conclude that *"pairs of connecting orbits come in pairs"* and this proves (6) modulo 2. (See Figure 2.)

Now the manifold $\widehat{\mathcal{M}}(z,x)$ carries a natural orientation inherited from the orientations of $W^u(z)$ and $W^u(x)$. Using this orientation one can show that the indices $\varepsilon(u_0)\varepsilon(v_0)$ and $\varepsilon(u_1)\varepsilon(v_1)$, correponding to the two ends of a component of $\widehat{\mathcal{M}}(z,x)$, cancel out. This proves (6). □

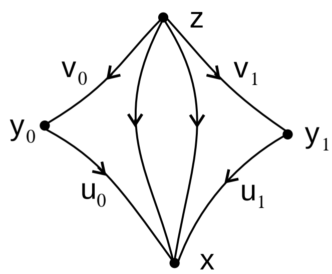

Figure 2: The Morse-Smale-Witten complex

Geometrically, one can think of the formal sum $\sum_i m_i W^u(x_i)$, corresponding to an element $\sum_i m_i x_i \in \ker \partial^M$, as a cycle representing the image in $H_*(M;\mathbb{Z})$ of the Morse-Smale-Witten homology class $[\sum_i m_i x_i] \in HM_*(M,f;\mathbb{Z})$ under the isomorphism of Theorem 1.12. More details of the proof of Theorem 1.12 can be found in [10, 45, 48].

Corollary 1.13 (Morse inequalities) *Let $f : M \to \mathbb{R}$ be a Morse function and denote by c_k the number of critical points of index k, and by $b_k = \mathrm{rank}\, H_k(M,\mathbb{Z})$ the kth Betti number. Then*

$$c_k - c_{k-1} + \cdots + (-1)^k c_0 \geq b_k - b_{k-1} + \cdots + (-1)^k b_0$$

for $0 \leq k \leq n = \dim M$, and equality holds for $k = n$.

Proof: The *weak Morse inequalities* $c_k \geq b_k$ follow from

$$c_k = \mathrm{rank}\, CM_k(f) \geq \mathrm{rank}\, H_k(CM(f), \partial^M) = b_k.$$

The proof of the Morse inequalities in the strong form is left as an exercise. □

Exercise 1.14 Let f be a Morse function with only one critical point on each critical level. For $a \in \mathbb{R}$ denote $M^a = \{x \in M : f(x) \le a\}$. Now let x be a critical point on the level $f(x) = c$. Prove that, for $\varepsilon > 0$ sufficiently small, the relative homology of the pair $(M^{c+\varepsilon}, M^{c-\varepsilon})$ is given by

$$H_k(M^{c+\varepsilon}, M^{c-\varepsilon}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = \mathrm{ind}_f(x), \\ 0, & \text{otherwise.} \end{cases}$$

Use this and the homology exact sequence for triples to deduce the Morse inequalities, without resorting to Theorem 1.12. □

Exercise 1.15 A Morse function $f : M \to \mathbb{R}$ is called **self-indexing** if $f(x) = \mathrm{ind}_f(x)$ for every critical point x. In this case, prove that there is a natural isomorphism $CM_k(f) \to H_k(M^{k+1/2}, M^{k-1/2}; \mathbb{Z})$ and that, under this isomorphism the boundary operator ∂^M corresponds to the boundary operator in the homology exact sequence of a triple. Try to visualize this result geometrically. This can be used to prove Theorem 1.12. □

Exercise 1.16 Identify $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and consider the Morse function $f : \mathbb{T}^2 \to \mathbb{R}$ given by

$$f(x, y) = \cos(2\pi x) + \cos(2\pi y).$$

Find the critical points and the connecting orbits (see Figur 3). Prove that f is a Morse function with a Morse-Smale gradient flow. Compute the Morse-Smale-Witten complex. Give an example of a gradient flow on the 2-torus which is not Morse-Smale. □

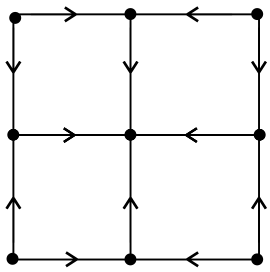

Figure 3: A Morse-Smale gradient flow on the 2-torus

Exercise 1.17 Consider the gradient flow on $\mathbb{R}P^2$ (thought of as the 2-disc with opposite points on the boundary identified) which is depicted in Figure 4. Compute the Morse-Smale-Witten complex of this example. Compute the integral homology (and cohomology) groups of $\mathbb{R}P^2$ from the Morse-Smale-Witten complex. Find an explicit formula of a Morse function with this gradiant flow. □

Exercise 1.18 Consider the function $f : \mathbb{C}P^n \to \mathbb{R}$ given by

$$f([z_0 : z_1 : \cdots : z_n]) = \sum_{j=1}^{n} j|z_j|^2.$$

Find the critical points and compute their Morse indices. Compute the homology groups of $\mathbb{C}P^n$. □

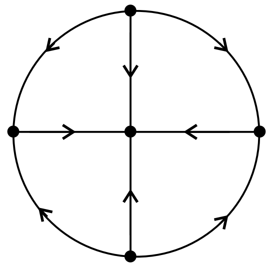

Figure 4: A Morse-Smale gradient flow on the real projective plane

1.4 Symplectic action

Let us now return to Hamiltonian differential equations in the monotone case. In this section we show how the contractible 1-periodic solutions of (1) can be interpreted as the critical points of the (circle valued) symplectic action functional on the space $\mathcal{L}M$ of contractible loops in M. Here is how this works.

Throughout we think of a loop in M as a smooth map $x : \mathbb{R} \to M$ which satisfies $x(t+1) = x(t)$ for $t \in \mathbb{R}$. A tangent vector to $\mathcal{L}M$ at such a loop x is a vector field ξ along x. Explicitly, we think of ξ as a smooth map $\xi : \mathbb{R} \to TM$ which satisfies $\xi(t) \in T_{x(t)}M$ and $\xi(t+1) = \xi(t)$ for $t \in \mathbb{R}$. We denote the space of such vector fields by $C^\infty(\mathbb{R}/\mathbb{Z}, x^*TM) = T_x\mathcal{L}M$. For each 1-periodic Hamiltonian $H_t = H_{t+1}$ as above the loop space $\mathcal{L}M$ carries a natural 1-form $\Psi_H : T\mathcal{L}M \to \mathbb{R}$, defined by

$$\Psi_H(x;\xi) = \int_0^1 \omega(\dot{x}(t) - X_t(x(t)), \xi(t))\, dt$$

for $\xi \in T_x\mathcal{L}M$. The zeros of this 1-form are precisely the 1-periodic solutions of (1).

Exercise 1.19 Prove that the 1-form Ψ_H is closed. **Hint:** Consider a 2-parameter family of loops $\mathbb{R}^2 \to \mathcal{L}M : (s_1, s_2) \to x_{s_1,s_2}$ and denote $\xi_1 = \partial x/\partial s_1$, $\xi_2 = \partial x/\partial s_2$. Then $d\Psi_H(x;\xi_1,\xi_2) = \partial_{s_1}\Psi_H\left(x;\partial_{s_2}x\right) - \partial_{s_2}\Psi_H\left(x;\partial_{s_1}x\right)$. □

The 1-form Ψ_H is not exact. However, it is the differential of a circle valued function $a_H : \mathcal{L}M \to \mathbb{R}/\mathbb{Z}$. This function is defined by

$$a_H(x) = -\int_B u^*\omega - \int_0^1 H_t(x(t))\, dt$$

for $x \in \mathcal{L}M$, where $u : B = \{z \in \mathbb{C} : |z| \le 1\} \to M$ is a smooth map such that $u(e^{2\pi it}) = x(t)$ for $t \in \mathbb{R}$. Such maps u exist whenever x is a contractible loop. The assumption $\int_{S^2} v^*\omega \in \mathbb{Z}$ for every smooth map $v : S^2 \to M$ guarantees that a_H takes values in $\mathbb{R}/\mathbb{Z}$. Sometimes we shall denote by $a_H(x,u)$ the symplectic action of the pair (x,u) which is well defined as a real number.

Exercise 1.20 Prove that the differential of a_H is the 1-form Ψ_H on $\mathcal{L}M$. **Hint:** Consider a path $\mathbb{R} \to \mathcal{L}M : s \mapsto x_s$ where $x_s(t) = x_0$ as $s \le -1$ and define u_s by $u_s(e^{2\pi(r+it)}) = x_{s+r}(t)$ for $r \le 0$ and $t \in \mathbb{R}$. □

Floer's idea is to carry out Morse theory for the symplectic action functional in analogy to the Morse-Smale-Witten complex in finite dimensional Morse theory.

1.5 Connecting orbits

Let us now fix a time dependent Hamiltonian $H_t = H_{t+1} : M \to \mathbb{R}$ such that the 1-periodic solutions $x : \mathbb{R}/\mathbb{Z} \to M$ of (1) are all nondegenerate. We wish to study the gradient flow lines of the action functional $a_H : \mathcal{L}M \to \mathbb{R}/\mathbb{Z}$. For this we must choose a metric on the loop space. Such a metric can be obtained from a 1-periodic family of almost complex structures $J_t = J_{t+1} \in \mathcal{J}(M, \omega)$ with corresponding metrics $\langle \xi, \eta \rangle_t = \omega(\xi, J_t\eta)$. The resulting inner product on the tangent space $T_x\mathcal{L}M = C^\infty(\mathbb{R}/\mathbb{Z}, x^*TM)$ is given by

$$\langle \xi, \eta \rangle = \int_0^1 \langle \xi(t), \eta(t) \rangle_t \, dt.$$

Since the differential of a_H is the 1-form Ψ_H it follows that the gradient of a_H with respect to this metric is given by

$$\operatorname{grad} a_H(x)(t) = J_t(x(t))\dot{x}(t) - \nabla H_t(x(t))$$

where the gradient of H_t is taken with respect to the metric $\langle \cdot, \cdot \rangle_t$ on M. A gradient flow line of a_H is a smooth 1-parameter family of loops $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot)$ which satisfies $\partial u/\partial s + \operatorname{grad} a_H(u(s, \cdot)) = 0$. In view of the above formula for $\operatorname{grad} a_H$ this becomes the partial differential equation

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0 \tag{7}$$

for smooth maps $u : \mathbb{R}^2 \to M$ which satisfy the periodicity condition $u(s, t+1) = u(s, t)$. Note that, in the case where J, H, and u are independent of t, this is the upward gradient flow of $H = H_t$. In the case where $u(s, t) = x(t)$ is independent of s, this reduces to the Hamiltonian equations (1), and in the case $H_t \equiv 0$ and $J_t \equiv J$ this is the equation for J-holomorphic curves.

The construction of the Floer homology groups relies on a careful analysis of the gradient flow lines of the symplectic action, i.e. of the solutions of (7). The **energy** of such a solution is defined by

$$E(u) = \frac{1}{2} \int_0^1 \int_{-\infty}^{\infty} \left(\left| \frac{\partial u}{\partial s} \right|^2 + \left| \frac{\partial u}{\partial t} - X_t(u) \right|^2 \right) ds \, dt.$$

We shall only consider solutions with finite energy. The key observation is that a solution u of (7) has finite energy if and only if it converges to periodic solutions of (1) as $s \to \pm\infty$, provided that all periodic solutions are nondegenerate (see Figure 5).

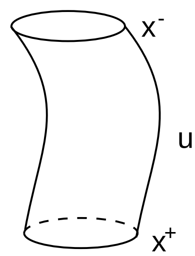

Figure 5: A gradient flow line of the symplectic action

Proposition 1.21 *Let $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ be a solution of (7). Then the following are equivalent.*

(i) $E(u) < \infty$.

(ii) *There exist periodic solutions $x^\pm \in \mathcal{P}(H)$ such that*

$$\lim_{s \to \pm\infty} u(s,t) = x^\pm(t). \tag{8}$$

and $\lim_{s\to\pm\infty} \partial_s u(s,t) = 0$, where both limits are uniform in the t-variable.

(iii) *There exist constants $\delta > 0$ and $c > 0$ such that*

$$|\partial_s u(s,t)| \le ce^{-\delta|s|}$$

for all $s, t \in \mathbb{R}$.

Proof: That (iii) implies (i) is obvious. We prove that (i) implies (ii). The proof relies on the a-priori estimate

$$\int_{B_r(s,t)} |\partial_s u|^2 < \hbar \qquad \Longrightarrow \qquad |\partial_s u(s,t)|^2 \le \frac{8}{\pi r^2}\int_{B_r(s,t)} |\partial_s u|^2 + cr^2 \tag{9}$$

for solutions of (1). Here $\hbar > 0$ and $c > 0$ are constants independent of u. A proof of this estimate can be found in [45].[2] If $E(u) < \infty$ then (9) shows that $\partial_s u$ converges to zero uniformly as $s \to \pm\infty$.[3] Hence $\partial_t u - X_t(u)$ converges to zero, uniformly in t, as $t \to \infty$. Hence it follows from Exercise 1.22 below that, for every $\varepsilon > 0$, there exists a $T > 0$ such that if $|s| > T$ then $u(s,t) \in \bigcup_{x\in\mathcal{P}(H)} B_\varepsilon(x(t))$. This implies (8) for some $x^\pm \in \mathcal{P}(H)$. Thus we have proved that (i) implies (ii). The proof that (ii) implies (iii) will be deferred to Section 2.7. □

Exercise 1.22 Suppose that every 1-periodic solution $x \in \mathcal{P}(H)$ is nondegenerate. Prove that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for every smooth loop $y : \mathbb{R}/\mathbb{Z} \to M$,

$$\int_0^1 |\dot y(t) - X_t(y(t))|^2\, dt < \delta \qquad \Longrightarrow \qquad \sup_{x\in\mathcal{P}(H)} \sup_t d(x(t), y(t)) < \varepsilon.$$

Hint: Argue by contradiction and use the Arzela-Ascoli theorem to show that every sequence $y_\nu : \mathbb{R}/\mathbb{Z} \to M$ with

$$\lim_{\nu\to\infty} \|\dot y_\nu - X_t(y_\nu)\|_{L^2(S^1)} = 0$$

has a subsequence which converges uniformly to a periodic solution of (1). □

---

[2]The proof of (9) is based on an inequality of the form

$$\Delta e \ge -A - Be^2$$

for the *energy density* $e(s,t) = |\partial_s u(s,t)|^2$, where $\Delta = \partial^2/\partial s^2 + \partial^2/\partial t^2$ denotes the standard Laplacian. This inequality holds for all solutions of (7) with constants $A > 0$ and $B > 0$ depending on ω, J, and H. It is worth pointing out that $H \equiv 0 \Longrightarrow A = 0$. Now every function e which satisfies the previous inequality also satisfies the following mean value inequality

$$\int_{B_r(0)} e \le \frac{\pi}{16B} \qquad \Longrightarrow \qquad e(0) \le \frac{8}{\pi r^2}\int_{B_r(0)} e + \frac{Ar^2}{4}.$$

Exercise: Assume the last assertion for $r = 1$, and prove it for general r by rescaling.

[3]Given $\varepsilon > 0$, with $\varepsilon^2 < \hbar$, choose $T > 0$ such that $E(u, [T-1,\infty) \times S^1) < \varepsilon^2$. Then apply (9) with $r = \sqrt{\varepsilon}$ to obtain $|\partial_s u(s,t)|^2 \le (c + 8/\pi)\varepsilon$ for $s \ge T$.

Exercise 1.23 Let $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ be a connecting orbit for the gradient flow of the symplectic action, i.e. a solution of (7) and (8). Assume exponential decay as in Proposition 1.21. Prove that the energy of u is given by

$$E(u) = a_H(x^-, u^-) - a_H(x^+, u^+).$$

where $u^\pm : B \to M$ are smooth functions such that $u^\pm(e^{2\pi it}) = x^\pm(t)$, and u^+ is chosen to agree with the connected sum of u^- and u, i.e. $u^+ = u^- \# u$. □

1.6 Moduli spaces

Denote by

$$\mathcal{M}(x^-, x^+) = \mathcal{M}(x^-, x^+; H, J)$$

the space of all solutions of (7) and (8). For a generic Hamiltonian H_t these spaces are finite dimensional manifolds. However, unless $\tau = 0$ in (4), the dimension of $\mathcal{M}(x^-, x^+)$ depends on the component of the moduli space.

Theorem 1.24 *There exists a subset $\mathcal{H}_{\mathrm{reg}} = \mathcal{H}_{\mathrm{reg}}(J) \subset C^\infty(M \times \mathbb{R}/\mathbb{Z})$ of the second categorie in the sense of Baire (i.e. a countable intersection of open and dense sets) such that the 1-periodic solutions of (1) are all nondegenerate, and the moduli space $\mathcal{M}(x^-, x^+; H, J)$ is a finite dimensional smooth manifold for all $x^\pm \in \mathcal{P}(H)$ and all $H \in \mathcal{H}_{\mathrm{reg}}$.*

Moreover, if (4) holds then there is a function $\eta_H : \mathcal{P}(H) \to \mathbb{R}$ such that for each $u \in \mathcal{M}(x^-, x^+; H, J)$ the dimension of the moduli space is given by

$$\dim_u \mathcal{M}(x^-, x^+; H, J) = \mu(u; H) = \eta_H(x^-) - \eta_H(x^+) + 2\tau E(u) \tag{10}$$

locally near u.

The meaning of this result is not only that, *by accident*, $\mathcal{M}(x^-, x^+)$ is a smooth manifold, but that the natural Fredholm operator, obtained by linearizing (7), is surjective for every connecting orbit. The proof will be outlined in the next lecture.

Remark 1.25 We shall see below that for every nondegenerate 1-periodic solution $x \in \mathcal{P}(H)$ and every smooth map $u : B \to M$ with $u(e^{2\pi it}) = x(t)$ there is a well-defined **Conley-Zehnder index** $\mu_H(x, u)$ which satisfies

$$\mu_H(x, A\#u) = \mu_H(x, u) - 2c_1(A)$$

for every $A \in \pi_2(M)$. Since

$$a_H(x, A\#u) = a_H(x, u) - \omega(A),$$

it follows that the difference

$$\eta_H(x) = \mu_H(x, u) - 2\tau a_H(x, u) \tag{11}$$

is independent of the choice of the function $u : B \to M$ used to define it. That this difference satisfies the requirements of Theorem 1.24 will follow from Exercise 1.23 and the fact that $\mu(u; H) = \mu_H(x^-, u^-) - \mu_H(x^+, u^- \# u)$. Note that, without specifying the map $u : B \to M$, the Conley-Zehnder index of a periodic solution $x \in \mathcal{P}(H)$ is only well defined modulo $2N$. □

2 Fredholm theory

2.1 Fredholm operators

Let X and Y be Banach spaces. A bounded linear operator $D : X \to Y$ is called a **Fredholm operator** if it has a closed range and the kernel and cokernel of D are both finite dimensional. Throughout we think of the cokernel as the quotient $\operatorname{coker} D = Y/\operatorname{im} D$. The **index** of a Fredholm operator D is defined as the difference of the dimensions of kernel and cokernel:

$$\operatorname{index} D = \dim \ker D - \dim \operatorname{coker} D.$$

The Fredholm property and the index are stable under perturbations. In particular, the set of Fredholm operators is open with respect to the norm topology, and the index is constant on each component. Moreover, if D is Fredholm and $K : X \to Y$ is a compact linear operator, then $D + K$ is again a Fredholm operator and it has the same index as D.

Exercise 2.1 Let X, Y, Z be Banach spaces, $D : X \to Y$ be a bounded linear operator, and $K : X \to Z$ be a compact linear operator. Suppose that there exists a constant $c > 0$ such that the following inequality holds for all $x \in X$

$$\|x\|_X \le c\left(\|Dx\|_Y + \|Kx\|_Z\right). \tag{12}$$

Prove that D has a closed range and a finite dimensional kernel. Use this to prove that the Fredholm property of D is invariant under small perturbations. □

A smooth (C^∞) map $f : X \to Y$ is called a **Fredholm map** if its differential $df(x) : X \to Y$ is a (linear) Fredholm operator for every $x \in X$. In this case it follows from the stability of the Fredholm index that the index of $df(x)$ is independent of x and we write $\operatorname{index}(f) = \operatorname{index} df(x)$. A vector $y \in Y$ is called a **regular value** of f if $df(x) : X \to Y$ is onto for every $x \in f^{-1}(y)$. If y is a regular value then, as in the finite dimensional case, the implicit function theorem asserts that

$$\mathcal{M} = f^{-1}(y)$$

is a smooth finite dimensional manifold. Its tangent space at $x \in \mathcal{M}$ is given by

$$T_x\mathcal{M} = \ker df(x)$$

and, since $df(x)$ is onto, the dimension of $\mathcal{M}$ agrees with the index of f.

2.2 The linearized operator

We wish to prove that the moduli spaces $\mathcal{M}(x^-, x^+; H, J)$ are smooth finite dimensional manifolds. Hence we must express these spaces as zero sets of functions between suitable Banach spaces. It is useful to abbreviate the left hand side of (7) as

$$\bar\partial_{H,J}(u) = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - \nabla H_t(u).$$

This is a vector field along u. Let us fix an element $u \in \mathcal{M}(x^-, x^+)$ and consider a vector space

$$\mathcal{X}_u \subset C^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, u^*TM)$$

of all vector fields ξ along u which satisfy a suitable exponential decay condition as $s \to \pm\infty$. Explicitly, think of ξ as a smooth function $\mathbb{R} \times \mathbb{R}/\mathbb{Z} \to TM$ such that $\xi(s,t) \in T_{u(s,t)}M$. A function near u which also satisfies the limit condition (8) can be expressed uniquely in the form $u' = \exp_u(\xi)$ for some $\xi \in \mathcal{X}_u$. Hence the set of solutions of (7) and (8) can be expressed as the zero set of a function

$$\mathcal{F}_u : \mathcal{X}_u \to \mathcal{X}_u.$$

Explicitly, $\mathcal{F}_u$ is defined by

$$\mathcal{F}_u(\xi) = \Phi_u(\xi)^{-1}\bar{\partial}_{H,J}(\exp_u(\xi))$$

for $\xi \in \mathcal{X}_u$, where $\Phi_u(\xi) : T_uM \to T_{\exp_u(\xi)}M$ denotes parallel transport along the geodesic $\tau \mapsto \exp_u(\tau\xi)$. The differential of $\mathcal{F}_u$ at 0 is the linear first order differential operator $D_u = d\mathcal{F}_u(0)$ given by

$$D_u\xi = \nabla_s\xi + J(u)\nabla_t\xi + \nabla_\xi J(u)\partial_t u - \nabla_\xi\nabla H_t(u). \tag{13}$$

It turns out that D_u is a Fredholm operator between suitable Sobolev completions of $\mathcal{X}_u$. We introduce the Sobolev norms

$$\|\xi\|_{L^p} = \left(\int_{-\infty}^{\infty}\int_0^1 |\xi|^p\right)^{1/p}, \qquad \|\xi\|_{W^{1,p}} = \left(\int_{-\infty}^{\infty}\int_0^1 |\xi|^p + |\nabla_s\xi|^p + |\nabla_t\xi|^p\right)^{1/p},$$

for $1 < p < \infty$. The corresponding completions of $\mathcal{X}_u$ will be denoted by

$$L^p = L^p(\mathbb{R} \times S^1, u^*TM), \qquad W^{1,p} = W^{1,p}(\mathbb{R} \times S^1, u^*TM).$$

We shall prove that $D_u : W^{1,p} \to L^p$ is a Fredholm operator and express its index in terms of a suitable Maslov index.

It is useful to simplify the formula for D_u by choosing a unitary trivialization of the vector bundle $u^*TM \to \mathbb{R} \times S^1$. Such a trivialization takes the form of a smooth family of vector space isomorphisms $\Phi(s,t) : \mathbb{R}^{2n} \to T_{u(s,t)}M$ which identify the standard symplectic and complex structures ω_0 and J_0 on $\mathbb{R}^{2n}$ with the corresponding structures ω and J on TM. In such a frame the operator D_u has the form

$$D\xi = \partial_s\xi + J_0\partial_t\xi + S\xi \tag{14}$$

for $\xi : \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$. Here the matrices $S(s,t) \in \mathbb{R}^{2n\times 2n}$ are defined by

$$S = \Phi^{-1}D_u\Phi = \Phi^{-1}\left(\nabla_s\Phi + J(u)\nabla_t\Phi + \nabla_\Phi J(u)\partial_t u - \nabla_\Phi\nabla H_t(u)\right).$$

The limit matrices

$$S^\pm(t) = \lim_{s\to\pm\infty} S(s,t) = \Phi^{-1}J\left(\nabla_t\Phi - \nabla_\Phi X_t(u)\right)$$

are symmetric and hence, modulo some compact perturbation, we may as well assume that S is symmetric for all s and t. Associated to a symmetric matrix valued function $S : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^{2n\times 2n}$ is a symplectic matrix valued function $\Psi : \mathbb{R} \times \mathbb{R} \to \mathrm{Sp}(2n)$ given by[4]

$$J_0\partial_t\Psi + S\Psi = 0, \qquad \Psi(s,0) = \mathbb{1}. \tag{15}$$

Denote $\Psi^\pm(t) = \lim_{s\to\pm\infty}\Psi(s,t)$.

[4]Recall that the group of symplectic matrices is given by

$$\mathrm{Sp}(2n) = \left\{\Psi \in \mathbb{R}^{2n\times 2n} \,:\, \Psi^T J_0 \Psi = J_0\right\}.$$

Here $J_0 \in \mathbb{R}^{2n\times 2n}$ denotes the standard complex structure given by $(x,y) \mapsto (-y,x)$ for $x, y \in \mathbb{R}^n$.

Theorem 2.2 *Suppose that* $\det(\mathbb{1} - \Psi^{\pm}(1)) \neq 0$. *Then the operator*

$$D : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$

given by (14) is Fredholm for $1 < p < \infty$. *Its Fredholm index is given by the difference of the Conley-Zehnder indices:*

$$\operatorname{index} D = \mu_{\mathrm{CZ}}(\Psi^+) - \mu_{\mathrm{CZ}}(\Psi^-). \tag{16}$$

The index formula in terms of the Conley-Zehnder index is due to Salamon-Zehnder [47], and an alternative proof was given by Robbin-Salamon [41]. The proof of Theorem 2.2 and the relevant definitions occupy the next three sections.

2.3 L^p-estimates

For $p = 2$ the proof of the Fredholm property is fairly straight forward and details have been carried out by several authors (cf [9, 40, 47, 49]). For $p \neq 2$ the Fredholm property was proved in [29]. The case $p = 2$ is the Sobolev borderline case, and the nonlinear Fredholm theory requires the case $p > 2$. Roughly speaking, the reason is that, in dimension 2, the Sobolev space $W^{1,p}$ embeds into the space of continuous functions, while there are $W^{1,2}$-functions on $\mathbb{R}^2$ which are discontinuous. For $p > 2$ the proof of the Fredholm property relies on the following two lemmata. We follow the line of argument in [3].

Lemma 2.3 *There exists a constant* $c > 0$ *such that*

$$\|\xi\|_{W^{1,p}} \leq c\,(\|D\xi\|_{L^p} + \|\xi\|_{L^p}) \tag{17}$$

Proof: This is essentially the Calderon-Zygmund inequality which asserts that there exists a constant $c_0 > 0$ such that every compactly supported function $u : \mathbb{R}^m \to \mathbb{R}$ satisfies

$$\sum_{i,j=1}^{m} \|\partial_i \partial_j u\|_{L^p(\mathbb{R}^m)} \leq c\,\|\Delta u\|_{L^p(\mathbb{R}^m)}$$

where

$$\Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial {x_j}^2}$$

denotes the Laplace operator. Once this is established, the proof of Lemma 2.3 is an easy exercise. Details are left to the reader. **Hint:** Use the formula $(\partial_s - J_0\partial_t)(\partial_s + J_0\partial_t) = \Delta$. Consult Appendix B in [31] if you get stuck. □

Lemma 2.4 *Suppose that* $S(s,t) = S(t)$ *is independent of* s *and that*

$$\det(\mathbb{1} - \Psi(1)) \neq 0$$

where $\Psi : [0,1] \to \mathrm{Sp}(2n)$ *is defined by* $\dot{\Psi}(t) = J_0 S(t)\Psi(t)$ *and* $\Psi(0) = \mathbb{1}$. *Then the operator*

$$D = \partial_s + J_0\partial_t + S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

is bijective for $1 < p < \infty$.

Proof: We shall only consider the case $p \geq 2$. The proof consists of four steps.

Step 1: *The result holds for* $p = 2$.

Consider the operator

$$A = J_0\partial_t + S : W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$$

This is an unbounded self-adjoint operator on the Hilbert space $H = L^2(S^1, \mathbb{R}^{2n})$ with domain $W = W^{1,2}(S^1, \mathbb{R}^{2n})$. The assumption $\det(\mathbb{1} - \Psi(1)) \neq 0$ guarantees that A is invertible, i.e. 0 is not an eigenvalue. Hence there is a splitting

$$H = E^+ \oplus E^-$$

into the positive and negative eigenspaces of A. Denote $A^\pm = A|_{E^\pm}$ and denote by $P^\pm : H \to E^\pm$ the orthogonal projections. The operator $-A^+$ generates a strongly continuous semigroup of operators on E^+ and A^- generates a strongly continuous semigroup of operators on E^-. Denote these semigroups by $s \mapsto e^{-A^+s}$ and $s \mapsto e^{A^-s}$, respectively, where both are defined for $s \geq 0$. Now define $K : \mathbb{R} \to \mathcal{L}(H)$ by

$$K(s) = \begin{cases} e^{-A^+s}P^+, & \text{for } s \geq 0, \\ -e^{-A^-s}P^-, & \text{for } s < 0. \end{cases}$$

This function is discontinuous at $s = 0$, strongly continuous for $s \neq 0$, and satisfies

$$\|K(s)\|_{\mathcal{L}(H)} \leq e^{-\delta|s|} \tag{18}$$

for some constant $\delta > 0$. Consider the operator $Q : L^2(\mathbb{R}, H) \to W^{1,2}(\mathbb{R}, H) \cap L^2(\mathbb{R}, W)$ defined by

$$Q\eta(s) = \int_{-\infty}^{\infty} K(s - \tau)\eta(\tau)\, d\tau$$

for $\eta \in L^2(R, H)$. We claim that this is the inverse of D. To see this note that $\xi = Q\eta = \xi^+ + \xi^-$ where

$$\xi^+(s) = \int_{-\infty}^{s} e^{-A^+(s-\tau)}\eta^+(\tau)d\tau, \qquad \xi^-(s) = -\int_{s}^{\infty} e^{-A^-(s-\tau)}\eta^-(\tau)d\tau.$$

A simple calculation now shows that $\dot{\xi}^\pm + A^\pm\xi^\pm = \eta^\pm$ and hence $\dot{\xi} + A\xi = \eta$. Finally, note that the space $W^{1,2}(\mathbb{R}, H) \cap L^2(\mathbb{R}, W)$ agrees with $W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.

Step 2: *There exists a constant* $c_1 > 0$ *such that*

$$\|\xi\|_{W^{1,p}([0,1]\times S^1)} \leq c_1 \left(\|D\xi\|_{L^p([-1,2]\times S^1)} + \|\xi\|_{L^2([-1,2]\times S^1)}\right)$$

for $\xi \in W^{1,p}([-1,2] \times S^1)$. *Moreover, if* $\xi \in W^{1,2}$ *and* $D\xi \in W^{k,p}_{\mathrm{loc}}$, *then* $\xi \in W^{k+1,p}_{\mathrm{loc}}$.

The inequality is proved in three stages. The first is the same inequality with the L^2-norm on the right replaced by the L^p-norm. This follows directly from the Calderon-Zygmund inequality. The second stage uses the Sobolev embedding $W^{1,2} \hookrightarrow L^p$ with corresponding estimate $\|\xi\|_{L^p} \leq c\,\|\xi\|_{W^{1,2}}$. The third stage is the elliptic estimate for $W^{1,2}$. In both the first and third stage the domain has to be increased. Details are easy and are left to the reader. Finally, that $D\xi \in L^p_{\mathrm{loc}}$ implies $\xi \in W^{1,p}_{\mathrm{loc}}$ is the standard elliptic regularity result.

Step 3: *Consider the norm*

$$\|\xi\|_{2,p} = \left(\int_{-\infty}^{\infty} \|\xi(s,\cdot)\|_{L^2(S^1)}^p \, ds \right)^{1/p}.$$

There exist constants $c_2, c_3 > 0$ such that, if $\xi \in W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ and $D\xi \in L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$, then $\xi \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ and

$$\|\xi\|_{2,p} \le c_2 \|D\xi\|_{L^p}, \qquad \|\xi\|_{W^{1,p}} \le c_3 \left(\|D\xi\|_{L^p} + \|\xi\|_{2,p} \right).$$

It follows from Step 2 that $\xi \in W^{1,p}_{\mathrm{loc}}$. Hence, to establish the first assertion, it only remains to prove that $\|\xi\|_{W^{1,p}} < \infty$ and this will follow from the two estimates. The first of these is just Young's convolution inequality. Namely, by assumption, we have $\eta = D\xi \in L^2(\mathbb{R}, H) \cap L^p(\mathbb{R}, H)$. Step 1 shows that $\xi = Q\eta = K * \eta$. Hence, by Young's inequality,

$$\|Q\eta\|_{2,p} = \|K * \eta\|_{L^p(\mathbb{R},H)} \le \|K\|_{L^1(\mathbb{R},\mathcal{L}(H))} \|\eta\|_{L^p(\mathbb{R},H)} \le \frac{2}{\delta} \|\eta\|_{L^p}.$$

The last inequality uses (18) and $\|\eta(s)\|_{L^2(S^1)} \le \|\eta(s)\|_{L^p(S^1)}$ for $p \ge 2$.

To prove the second inequality in Step 3, we use Step 2 and $(a+b)^p \le 2^p(a^p + b^p)$:

$$\begin{aligned}
\|\xi\|_{W^{1,p}([k,k+1]\times S^1)}^p &\le 2^p {c_1}^p \left(\int_{k-1}^{k+2} \|D\xi\|_{L^p(S^1)}^p \, ds + \left(\int_{k-1}^{k+2} \|\xi\|_{L^2(S^1)}^2 \, ds \right)^{p/2} \right) \\
&\le 2^p {c_1}^p \left(\int_{k-1}^{k+2} \|D\xi\|_{L^p(S^1)}^p \, ds + 3^{p/2-1} \int_{k-1}^{k+2} \|\xi\|_{L^2(S^1)}^p \, ds \right) \\
&\le 3^{p/2-1} 2^p {c_1}^p \int_{k-1}^{k+2} \left(\|D\xi\|_{L^p(S^1)}^p + \|\xi\|_{L^2(S^1)}^p \right) ds
\end{aligned}$$

Take the sum over all k to obtain

$$\|\xi\|_{W^{1,p}}^p \le 3^{p/2} 2^p {c_1}^p \int_{-\infty}^{\infty} \left(\|D\xi\|_{L^p(S^1)}^p + \|\xi\|_{L^2(S^1)}^p \right) ds,$$

and hence

$$\|\xi\|_{W^{1,p}} \le c_3 \left(\|D\xi\|_{L^p} + \|\xi\|_{2,p} \right).$$

This proves the inequalities and Step 3.

Step 4: *We prove the lemma.*

Putting the two estimates of Step 3 together we obtain

$$\|\xi\|_{W^{1,p}} \le c_3(1 + c_2) \|D\xi\|_{L^p}$$

for every $\xi \in C_0^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Since C_0^∞ is dense in $W^{1,p}$, this estimate continues to hold for all $\xi \in W^{1,p}$. It follows that $D : W^{1,p} \to L^p$ is injective and has a closed range. Thus it suffices to prove that the range is dense. Let

$$\eta \in L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \cap L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

be given. Then, by Step 1, there exists a $\xi \in W^{1,2}$ such that $D\xi = \eta$. By Step 3, $\xi \in W^{1,p}$ and this proves that η is in the range of $D : W^{1,p} \to L^p$. Hence D has a dense range, and hence it is onto. This proves the lemma in the case $p \ge 2$. □

Exercise 2.5 Prove that Lemma 2.4 continues to hold with p replaced by $q \leq 2$. **Hint:** Choose $p \geq 2$ such that $1/p + 1/q = 1$. Define $W^{-1,q}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ as the dual space of $W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Prove that the operator D satisfies an estimate of the form

$$\|\xi\|_{L^q} \leq c \|D\xi\|_{W^{-1,q}}$$

To see this interprete $D : L^q \to W^{-1,q}$ as the functional analytic adjoint of the operator

$$D^* = -\partial_s + J_0\partial_t + S : W^{1,p} \to L^p.$$

Now prove that $\|\partial_s \xi\|_{W^{-1,q}} \leq \|\xi\|_{L^q}$ and use this to establish an estimate of the form

$$\|\xi\|_{W^{1,q}} \leq c \|D\xi\|_{L^q}.$$

Finally, prove that $D : W^{1,q} \to L^q$ has a dense range. □

Proof of Theorem 2.2 (the Fredholm property): If $\det(\mathbb{1} - \Psi^{\pm}(1)) \neq 0$, then it follows from Lemma 2.4 that there exist constants $T > 0$ and $c > 0$ such that, for every $\xi \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^n)$,

$$\xi(s,t) = 0 \quad \text{for} \quad |s| \leq T - 1 \qquad \Longrightarrow \qquad \|\xi\|_{W^{1,p}} \leq c \|D\xi\|_{L^p}. \tag{19}$$

Now choose a smooth cutoff function $\beta : \mathbb{R} \to [0,1]$ such that $\beta(t) = 0$ for $|t| \geq T$ and $\beta(t) = 1$ for $|t| \leq T - 1$. Using the estimate (17) for $\beta\xi$ and (19) for $(1-\beta)\xi$ we obtain

$$\begin{aligned}
\|\xi\|_{W^{1,p}} &\leq \|\beta\xi\|_{W^{1,p}} + \|(1-\beta)\xi\|_{W^{1,p}} \\
&\leq c_1 \left(\|\beta\xi\|_{L^p} + \|D(\beta\xi)\|_{L^p} + \|D((1-\beta)\xi)\|_{L^p}\right) \\
&\leq c_2 \left(\|\xi\|_{L^p[-T,T]} + \|D\xi\|_{L^p}\right).
\end{aligned}$$

Thus we have established an estimate of the form

$$\|\xi\|_{W^{1,p}} \leq c \left(\|\xi\|_{L^p[-T,T]} + \|D\xi\|_{L^p}\right) \tag{20}$$

for $\xi \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Since the restriction operator

$$W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^n) \to L^p([-T,T] \times S^1, \mathbb{R}^n)$$

is compact it follows from Exercise 2.1 that D has a finite dimensional kernel and a closed range. That D also has a finite dimensional cokernel, follows from elliptic regularity. Namely, suppose that $\eta \in L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ (with $1/p + 1/q = 1$) annihilates the image of D. Then it follows from local elliptic regularity that $\eta \in W^{1,q}_{\mathrm{loc}}$ and there is a constant $c > 0$ such that

$$\begin{aligned}
\|\eta\|_{W^{1,q}([k,k+1]\times S^1)} &\leq c \left(\|D^*\eta\|_{L^q([k-1,k+2]\times S^1)} + \|\eta\|_{L^q([k-1,k+2]\times S^1)}\right) \\
&= c \|\eta\|_{L^q([k-1,k+2]\times S^1)}.
\end{aligned}$$

Here $D^* = -\partial_s + J_0\partial_t + S$ denotes the formal adjoint operator and the last equality holds since $D^*\eta = 0$. Taking the q-th power of this inequality and summing over k we find that $\eta \in W^{1,q}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ and $D^*\eta = 0$. Thus the cokernel of $D : W^{1,p} \to L^p$ agrees with the kernel of $D^* : W^{1,q} \to L^q$ and therefore is also finite dimensional. □

2.4 The Conley-Zehnder index

Denote by $\mathrm{Sp}(2n)$ the group of symplectic $2n \times 2n$-matrices. In [2] Conley and Zehnder introduced a Maslov type index for paths of symplectic matrices. Their index assigns an integer $\mu_{\mathrm{CZ}}(\Psi)$ to every path $\Psi : [0,1] \to \mathrm{Sp}(2n)$ such that $\Psi(0) = \mathbb{1}$ and $\det(\mathbb{1} - \Psi(1)) \neq 0$. Other expositions are given in Salamon-Zehnder [47] and Robbin-Salamon [40].

Denote by $\mathrm{Sp}^*(2n)$ the open and dense set of all symplectic matrices which do not have 1 as an eigenvalue. This set has two components, distinguished by the sign of $\det(\mathbb{1} - \Psi)$. Its complement is called the **Maslov cycle**. It is an algebraic variety of codimension 1 and admits a natural coorientation. The intersection number of a loop $\Phi : S^1 \to \mathrm{Sp}(2n)$ with the Maslov cycle is always even and the Maslov index $\mu(\Phi)$ is half this intersection number. Alternatively, the Maslov index can be defined as the degree

$$\mu(\Phi) = \deg(\rho \circ \Phi)$$

where $\rho : \mathrm{Sp}(2n) \to S^1$ is a continuous extension of the determinant map $\det : \mathrm{U}(n) = \mathrm{Sp}(2n) \cap \mathrm{O}(2n) \to S^1$. The map ρ is not a homomorphism but can be chosen to be multiplicative with respect to direct sums, invariant under similarity, and taking the value ± 1 for symplectic matrices with no eigenvalues on the unit circle. These properties determine ρ uniquely (cf. [47]).

Now denote by $\mathrm{SP}(n)$ the space of paths $\Psi : [0,1] \to \mathrm{Sp}(2n)$ with $\Psi(0) = \mathbb{1}$ and $\Psi(1) \in \mathrm{Sp}^*(2n)$. Any such path admits an extension $\Psi : [0,2] \to \mathrm{Sp}(2n)$, unique up to homotopy, such that $\Psi(s) \in \mathrm{Sp}^*(2n)$ for $s \geq 1$ and $\Psi(2)$ is one of the matrices $W^+ = -\mathbb{1}$ and $W^- = \mathrm{diag}(2, -1, \ldots, -1, 1/2, -1, \ldots, -1)$. Since $\rho(W^\pm) = \pm 1$ it follows that $\rho^2 \circ \Psi : [0,2] \to S^1$ is a loop and the **Conley-Zehnder index** of Ψ is defined as its degree

$$\mu_{\mathrm{CZ}}(\Psi) = \deg(\rho^2 \circ \Psi).$$

The Conley-Zehnder index has the following properties. It is uniquely determined by the homotopy, loop, and signature properties [47].

(Naturality) For any path $\Phi : [0,1] \to \mathrm{Sp}(2n)$, $\mu_{\mathrm{CZ}}(\Phi\Psi\Phi^{-1}) = \mu_{\mathrm{CZ}}(\Psi)$.

(Homotopy) The Conley-Zehnder index is constant on the components of $\mathrm{SP}(n)$

(Zero) If $\Psi(s)$ has no eigenvalue on the unit circle for $s > 0$ then $\mu_{\mathrm{CZ}}(\Psi) = 0$.

(Product) If $n' + n'' = n$ identify $\mathrm{Sp}(2n') \oplus \mathrm{Sp}(2n'')$ with a subgroup of $\mathrm{Sp}(2n)$ in the obvious way. Then $\mu_{\mathrm{CZ}}(\Psi' \oplus \Psi'') = \mu_{\mathrm{CZ}}(\Psi') + \mu_{\mathrm{CZ}}(\Psi'')$.

(Loop) If $\Phi : [0,1] \to \mathrm{Sp}(2n, \mathbb{R})$ is a loop with $\Phi(0) = \Phi(1) = \mathbb{1}$ then

$$\mu_{\mathrm{CZ}}(\Phi\Psi) = \mu_{\mathrm{CZ}}(\Psi) + 2\mu(\Phi).$$

(Signature) If $S = S^T \in \mathbb{R}^{2n \times 2n}$ is a symmetric matrix with $\|S\| < 2\pi$ and $\Psi(t) = \exp(J_0 S t)$ then

$$\mu_{\mathrm{CZ}}(\Psi) = \frac{1}{2}\mathrm{sign}(S)$$

where $\mathrm{sign}\, S$ is the signature (the number of positive minus the number of negative eigenvalues).

(Determinant) $(-1)^{n - \mu_{\mathrm{CZ}}(\Psi)} = \mathrm{sign}\det(\mathbb{1} - \Psi(1))$.

(Inverse) $\mu_{\mathrm{CZ}}(\Psi^{-1}) = \mu_{\mathrm{CZ}}(\Psi^T) = -\mu_{\mathrm{CZ}}(\Psi)$.

Here is an alternative definition of the Conley-Zehnder index in terms of crossing numbers, as in [40]. Any path $\Psi \in \mathrm{SP}(n)$ can be expressed as a solution of an ordinary differential equation

$$\dot{\Psi}(t) = J_0 S(t)\Psi(t), \qquad \Psi(0) = \mathbb{1},$$

where $t \mapsto S(t) = S(t)^T$ is a smooth path of symmetric matrices. A number $t \in [0,1]$ is called a **crossing** if $\det(\mathbb{1} - \Psi(t)) = 0$. If t is a crossing then the **crossing form** is the quadratic form $\Gamma(\Psi, t) : \ker(\mathbb{1} - \Psi(t)) \to \mathbb{R}$ defined by

$$\Gamma(\Psi, t)\xi_0 = \omega_0(\xi_0, \dot{\Psi}(t)\xi_0) = \langle \xi_0, S(t)\xi_0 \rangle$$

for $\xi_0 \in \ker(\mathbb{1} - \Psi(t))$. A crossing t is called **regular** if the crossing form is non-degenerate. Regular crossings are isolated. For a path $\Psi \in \mathrm{SP}(n)$ with only regular crossings the Conley-Zehnder index is given by the formula

$$\mu_{\mathrm{CZ}}(\Psi) = \frac{1}{2}\mathrm{sign}(S(0)) + \sum_{t>0} \mathrm{sign}\,\Gamma(\Psi, t)$$

where the sum runs over all crossings $t > 0$. That both definitions of the Conley-Zehnder index agree is proved in Robbin-Salamon [40].

2.5 The spectral flow

The operator $D = \partial_s + J_0\partial_t + S$ can be written in the form

$$D = \frac{\partial}{\partial s} + A(s)$$

where $A(s) : W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ is given by

$$A(s) = J_0 \frac{\partial}{\partial t} + S(s, \cdot). \tag{21}$$

This is a smooth family of unbounded self-adjoint operators on the Hilbert space $H = L^2(S^1, \mathbb{R}^{2n})$. It turns out that the limit operators

$$A^{\pm} = \lim_{s \to \pm\infty} A(s)$$

are both invertible. In this case the Fredholm index of D is given by the spectral flow of A (see Atiyah-Patodi-Singer [1]). In our discussion we follow the exposition in [41].

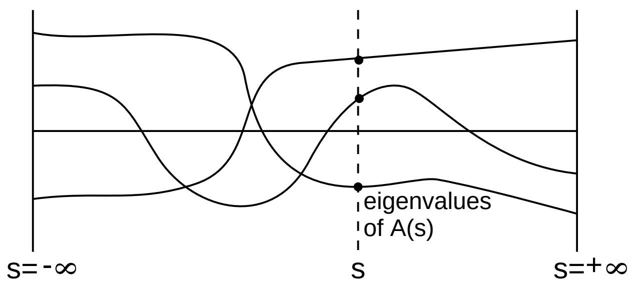

Figure 6: The spectral flow

Intuitively, the spectral flow is the number of eigenvalues of $A(s)$ crossing zero from negative to positive as s moves from $-\infty$ to $+\infty$ (see Figure 6). More formally, the spectral flow can be defined as follows. A number $s \in \mathbb{R}$ is called a **crossing** if $\ker A(s) \neq \{0\}$. If s is a crossing then the **crossing form** is the quadratic form $\Gamma(A,s) : \ker A(s) \to \mathbb{R}$ defined by

$$\Gamma(A,s)\xi = \langle \xi, \dot{A}(s)\xi \rangle_H$$

for $\xi \in \ker A(s)$. A crossing s is called **regular** if the crossing form is nondegenerate. Regular crossings are isolated. For a smooth family $s \mapsto A(s)$ with only regular crossings the **spectral flow** is defined by

$$\mu^{\mathrm{spec}}(A) = \sum_s \operatorname{sign} \Gamma(A,s)$$

where the sum runs over all crossings (cf. Robbin-Salamon [41]).

Lemma 2.6 *Let $A(s)$ be given by (21) and $\Psi(s,t)$ by (15). Then the operator path $s \mapsto A(s)$ has the same crossings as the symplectic path $s \mapsto \Psi(s,1)$ and the crossing forms are isomorphic.*

Proof: A function $\xi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^{2n}$ is in the kernel of $A(s)$ if and only if

$$\xi(t) = \Psi(s,t)\xi_0, \qquad \xi_0 \in \ker(\mathbb{1} - \Psi(s,1)).$$

This shows that the crossings are the same. Next we claim that the crossing forms agree under the natural identification of $\ker A(s)$ with $\ker(\mathbb{1} - \Psi(s,1))$. This means that

$$\begin{aligned} \Gamma(A,s)\xi &= \int_0^1 \langle \Psi(s,t)\xi_0, \partial_s S(s,t)\Psi(s,t)\xi_0 \rangle \, dt \\ &= \langle \xi_0, \hat{S}(s,1)\xi_0 \rangle \\ &= \Gamma(\Psi(\cdot,1),s)\xi_0 \end{aligned} \tag{22}$$

for $\xi_0 \in \ker(\mathbb{1} - \Psi(s,1))$, where $\hat{S}(s,t)$ is defined by

$$\partial_s \Psi = J_0 \hat{S} \Psi.$$

To prove (22) note that

$$\begin{aligned} \partial_t(\Psi^T \hat{S} \Psi) &= (\partial_t \Psi)^T \hat{S} \Psi + \Psi^T \partial_t(\hat{S}\Psi) \\ &= -\Psi^T S J_0 \hat{S} \Psi - \Psi^T J_0 \partial_t \partial_s \Psi \\ &= -\Psi^T S (\partial_s \Psi) - \Psi^T J_0 \partial_s (J_0 S \Psi) \\ &= \Psi^T (\partial_s S) \Psi. \end{aligned}$$

Integrating over t from 0 to 1 we obtain

$$\Psi(s,1)^T \hat{S}(s,1) \Psi(s,1) = \int_0^1 \Psi(s,t)^T \partial_s S(s,t) \Psi(s,t) \, dt$$

and this implies (22). □

Proof of Theorem 2.2 (the index formula): Suppose, without loss of generality, that $\Psi(s,t) = \Psi^-(t)$ for $s \le -T$ and $\Psi(s,t) = \Psi^+(t)$ for $s \ge T$, and that the path $s \mapsto \Psi(s,1)$ has only regular crossings. The symplectic loop around the boundary of the square $[-T,T] \times [0,1]$ is obviously contractible. Hence the difference of the Conley-Zehnder indices can be expressed in the form

$$\mu_{\mathrm{CZ}}(\Psi^+) - \mu_{\mathrm{CZ}}(\Psi^-) = \mu(\{\Psi(s,1)\}_{-T\le s\le T}) = \sum_s \operatorname{sign}\Gamma(\Psi(\cdot,1),s).$$

By Lemma 2.6, the term on the right agrees with the spectral flow of the operator family $s \mapsto A(s)$, and hence we have

$$\mu_{\mathrm{CZ}}(\Psi^+) - \mu_{\mathrm{CZ}}(\Psi^-) = \mu^{\mathrm{spec}}(A) = \operatorname{index} D.$$

The last identity is proved in [41]. This proves the theorem. □

2.6 Transversality

Let us now return to the manifold situation of Theorem 1.24. In order to apply Theorem 2.2 we must assign a Conley-Zehnder index to every nondegenerate periodic solution $x \in \mathcal{P}(H)$. This can be done as follows. Linearizing the differential equation (1) along $x(t)$ we obtain linear symplectomorphisms $d\psi_t(x(0)) : T_{x(0)}M \to T_{x(t)}M$. In order to obtain a symplectic path $\Psi_x \in \mathrm{SP}(n)$ we must trivialize the tangent bundle x^*TM over x:

$$\begin{array}{ccc} T_{x(0)}M & \stackrel{d\psi_t(x(0))}{\longrightarrow} & T_{x(t)}M \\ \uparrow & & \uparrow \\ \mathbb{R}^{2n} & \stackrel{\Psi_x(t)}{\longrightarrow} & \mathbb{R}^{2n} \end{array}.$$

Such a trivialization can be obtained by specifying a disc $u : B \to M$ such that $u(e^{2\pi i t}) = x(t)$ and then trivializing u^*TM. This gives rise to a Conley-Zehnder index

$$\mu_H(x,u) = n - \mu_{\mathrm{CZ}}(\Psi_x).$$

This index satisfies the following.

Corollary 2.7 *Let $x^\pm \in \mathcal{P}(H)$ be two nondegenerate periodic solutions of (1). Moreover, let $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ be a smooth map which satisfies the limit condition (8). Let $u^\pm : B \to M$ satisfy $u^\pm(e^{2\pi i t}) = x^\pm(t)$ and $u^+ = u^- \# u$. Then*

$$D_u : W^{1,p}(u^*TM) \to L^p(u^*TM)$$

is a Fredholm operator and its Fredholm index is given by

$$\operatorname{index} D_u = \mu(u,H) = \mu_H(x^-,u^-) - \mu_H(x^+,u^+). \tag{23}$$

Proof: Theorem 2.2. □

Exercise 2.8 Suppose that $H_1 \equiv H$ is a Morse function with sufficiently small second derivatives. Let $x(t) \equiv x$ be a critical point of H and define $u : B \to M$ as the constant disc $u(z) = x$ for $z \in B$. Prove that

$$\mu_H(x,u) = \mathrm{ind}_{-H}(x),$$

i.e. the Maslov index of the pair (x,u) is equal to the Morse index of x as a critical point of $-H$. □

Exercise 2.9 Let $x \in \mathcal{P}(H)$ and $u : B \to M$ be as above. Prove that

$$\mu_H(x, A\#u) = \mu_H(x, u) - 2c_1(A)$$

for every $A \in \pi_2(M)$, where $c_1(A) \in \mathbb{Z}$ denotes the integral of the first Chern class $c_1 \in H^2(M, \mathbb{Z})$ of the tangent bundle over A and $A\#u$ denotes the disc obtained by taking the connected sum of a representative of A with u. □

Exercise 2.10 Given a Hamiltonian H with only nondegenerate 1-periodic solutions and $x^\pm \in \mathcal{P}(H)$, consider the space $\mathcal{Z}(x^-, x^+)$ of all smooth maps $u : \mathbb{R} \times S^1 \to M$ which satisfy (8). Abbreviate

$$\mathcal{Z}(H) = \bigcup_{x^\pm \in \mathcal{P}(H)} \mathcal{Z}(x^-, x^+).$$

If $u_{01} \in \mathcal{Z}(x_0, x_1)$ and $u_{12} \in \mathcal{Z}(x_1, x_2)$ such that $u_{01}(s,t) = x_1(t)$ for $s \geq 0$ and $u_{12}(s,t) = x_1(t)$ for $s \leq 0$ define the **catenation** $u_{01}\#u_{12} \in \mathcal{Z}(x_0, x_2)$ by

$$u_{01}\#u_{12}(s,t) = \begin{cases} u_{01}(s,t), & s \leq 0, \\ u_{12}(s,t), & s \geq 0. \end{cases}$$

Think of the Maslov index as a function $\mathcal{Z}(H) \to \mathbb{Z} : u \mapsto \mu(u, H)$ defined by (23). Prove that this function has the following properties.

(Homotopy) The Maslov index is constant on the components of $\mathcal{Z}(H)$.

(Zero) If $x^- = x^+ = x$ and $u(s,t) = x(t)$ then

$$\mu(u, H) = 0.$$

(Catenation)

$$\mu(u_{01}\#u_{12}, H) = \mu(u_{01}, H) + \mu(u_{12}, H).$$

(Chern class) If $v : S^2 \to M$ then

$$\mu(u\#v, H) = \mu(u, H) + 2\int_{S^2} v^* c_1.$$

(Morse index) Assume $H_t = H : M \to \mathbb{R}$ is a Morse function with sufficiently small second derivatives. Then the 1-periodic solutions $x \in \mathcal{P}(H)$ are the critical points of H and, for every $u \in \mathcal{Z}(x^-, x^+)$ with $u(s,t) \equiv u(s)$,

$$\mu(u, H) = \mathrm{ind}_H(x^+) - \mathrm{ind}_H(x^-).$$

(Fixed point index) For $u \in \mathcal{Z}(x^-, x^+)$,

$$(-1)^{\mu(u,H)} = \mathrm{sign}\det(\mathbb{1} - d\varphi_H(x^-(0)))\det(\mathbb{1} - d\varphi_H(x^+(0)))$$

where $\varphi_H = \psi_1 : M \to M$ is the time-1-map of (1). □

Exercise 2.9 shows that, in the case where all 1-periodic solutions $x \in \mathcal{P}(H)$ are nondegenerate and ω satisfies (4), there exists a well defined function $\eta_H : \mathcal{P}(H) \to \mathbb{R}$ which satisfies

$$\eta_H(x) = \mu_H(x, u) - 2\tau a_H(x, u)$$

for every $u : B \to M$ with $u(e^{2\pi it}) = x(t)$. By Exercise 1.23,

$$\mu(u, H) = \eta_H(x^-) - \eta_H(x^+) + 2\tau E(u)$$

for every $u \in \mathcal{M}(x^-, x^+; H, J)$. This proves the index formula in Theorem 1.24. Let us now define

$$\mathcal{H}_{\mathrm{reg}} \subset C^\infty(M \times \mathbb{R}/\mathbb{Z})$$

as the subset of all those Hamiltonians for which all the periodic solutions $x \in \mathcal{P}(H)$ are nondegenerate and for which the operator D_u is surjective for every pair $x^\pm \in \mathcal{P}(H)$ and every connecting orbit $u \in \mathcal{M}(x^-, x^+; H, J)$. Then the implicit function theorem shows that the moduli spaces $\mathcal{M}(x^-, x^+; H, J)$ of connecting orbits are finite dimensional smooth manifolds of the right dimension whenever $H \in \mathcal{H}_{\mathrm{reg}}$. Hence, to complete the proof of Theorem 1.24, it remains to show that the set $\mathcal{H}_{\mathrm{reg}}$ is indeed of the second category in the sense of Baire, i.e. can be expressed as a countable intersection of open and dense sets (and hence, in particular, is nonempty). This follows, via standard transversality arguments, from an infinite dimensional version of Sard's theorem which is due to Smale [52]. The details are carefully carried out in [13] and [47] and will not be reproduced here.

2.7 Exponential convergence

The purpose of this final section of the Fredholm chapter is to prove that all finite energy solutions of (7) converge exponentially to periodic orbits, and thus to complete the proof of Proposition 1.21.

Lemma 2.11 *Suppose that the operator $D = \partial_s + J_0\partial_t + S$ satisfies the requirements of Theorem 2.2 and, in addition,*

$$\lim_{s\to\pm\infty} \sup_{0\le t\le 1} \|\partial_s S(s,t)\| = 0, \qquad \sup_{s,t} \|\partial_t S(s,t)\| < \infty.$$

Then there exists a constant $\delta > 0$ such that the following holds. For every C^2-function $\xi : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^{2n}$ which satisfies $D\xi = 0$ and does not diverge to ∞ as $s \to \pm\infty$ there exists a constant $c > 0$ such that, for all $s, t \in \mathbb{R}$,

$$|\xi(s,t)| \le ce^{-\delta|s|}.$$

Proof: Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(s) = \frac{1}{2}\int_0^1 |\xi(s,t)|^2\, dt.$$

By assumption, this function is twice continuously differentiable and its second derivative is given by

$$\begin{aligned}
f''(s) &= \int_0^1 \Big(|\partial_s\xi(s,t)|^2 + \langle \xi(s,t), \partial_s\partial_s\xi(s,t)\rangle \Big)\, dt \\
&= 2\int_0^1 |\partial_s\xi(s,t)|^2\, dt - \int_0^1 \langle \xi(s,t), \partial_s S(s,t)\xi(s,t)\rangle\, dt \\
&\ge 2\int_0^1 |J_0\partial_t\xi(s,t) + S(s,t)\xi(s,t)|^2\, dt - \varepsilon \int_0^1 |\xi(s,t)|^2\, dt \\
&\ge \delta^2 \int_0^1 |\xi(s,t)|^2\, dt \\
&= \delta^2 f(s).
\end{aligned}$$

Here the penultimate inequality only holds for sufficiently large s and it uses the fact that the operator $A(s) = J_0\partial_t + S$ is invertible for large s. Now the inequality

$$f'' \geq \delta^2 f$$

inplies that $e^{-\delta s}(f'(s) + \delta f(s))$ is nondecreasing. Hence $f'(s) + \delta f(s) \leq 0$ for $s \geq s_0$ since otherwise $f(s)$ would diverge exponentially as $s \to \infty$. Hence the function $e^{\delta s} f(s)$ is nonincreasing and this implies $f(s) \leq e^{-\delta(s-s_0)} f(s_0)$ for $s \geq s_0$. The argument for $s \to -\infty$ is similar, and this proves an estimate of the form

$$\int_0^1 |\xi(s,t)|^2 \, dt \leq ce^{-\delta|s|}$$

for all s. To get the pointwise inequality, apply the operator $\partial_s - J_0\partial_t$ to the equation $D\xi = 0$ to obtain

$$\Delta\xi = J_0\partial_t(S\xi) - \partial_s(S\xi),$$

where $\Delta = \partial^2/\partial s^2 + \partial^2/\partial t^2$. This implies that there is a constant $c > 0$ such that

$$\Delta|\xi|^2 \geq -c|\xi|^2$$

for all $\xi \in \ker D$. This inequality in turn can be used to derive a mean value inequality of the form

$$|\xi(s,t)|^2 \leq \frac{c}{r^2} \int_{B_r(s,t)} |\xi|^2$$

for $r > 0$ and $s, t \in \mathbb{R}$. With $r = 1$, say, we obtain the required pointwise exponential decay. □

Proof of Proposition 1.21: (ii) $\Longrightarrow$ (iii): First it follows from standard elliptic estimates that every finite energy solution $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ of (7) satisfies

$$\lim_{s\to\pm\infty} \sup_{0\leq t\leq 1} \Big(|\nabla_s\partial_s u(s,t)| + |\nabla_t\partial_s u(s,t)| \Big) = 0, \qquad \sup_{s,t} |\nabla_t\partial_t u(s,t)| < \infty.$$

It then follows by inspection that the matrix function $S(s,t)$ in (14) satisfies the requirements of Lemma 2.11. Since $D_u\partial_s u = 0$, it follows from Lemma 2.11 that $\partial_s u$ converges to zero exponentially as $s \to \pm\infty$. This proves the proposition. □

3 Floer homology

The goal of this lecture is to explain the definition of the Floer homology groups of a Hamiltonian $H \in C^\infty(M \times \mathbb{R}/\mathbb{Z})$ for monotone symplectic manifolds. The first section is devoted to the fundamental compactness result which asserts that, for a generic Hamiltonian, there are only finitely many connecting orbits of index 1. Section 3.2 explains the definition of the Floer homology groups and their main properties. The following three sections provide some details of the proofs. Section 3.3 explains Floer's gluing construction, Section 3.4 outlines the proof that Floer homology is an invariant, and Section 3.5 explains the isomorphism between Floer homology and Morse homology. Sections 3.6 and 3.7 discuss the extension of Floer homology to symplectic manifolds with vanishing first Chern class.

3.1 Compactness

Denote by $\mathcal{M}^1(x^-, x^+; H, J) = \{u \in \mathcal{M}(x^-, x^+; H, J) : \mu(u; H) = 1\}$ the one dimensional part of the moduli space of connecting orbits. The goal of this section is to prove the following.

Proposition 3.1 *If (M, ω) is monotone and $H \in \mathcal{H}_{\mathrm{reg}}$ then the quotient space*

$$\widehat{\mathcal{M}}^1(x^-, x^+; H, J) = \mathcal{M}^1(x^-, x^+; H, J)/\mathbb{R}$$

is a finite set for every pair $x^\pm \in \mathcal{P}(H)$.

For $v : S^2 \to M$ and $J \in \mathcal{J}(M, \omega)$ define

$$\bar{\partial}_J(v) = \frac{1}{2}(dv + J \circ dv \circ i) \in \Omega_J^{0,1}(S^2, v^*TM),$$

where i denotes the standard complex structure on S^2. The function v is called a J-holomorphic sphere if $\bar{\partial}_J(v) = 0$. The **energy** of v is the integral

$$E(v) = \int_{S^2} v^*\omega.$$

This is equal to half the L^2-norm of dv and hence is positive whenever v is nonconstant.

Lemma 3.2 *For every $J \in \mathcal{J}(M, \omega)$ there exists a constant $\hbar = \hbar(M, \omega, J) > 0$ such that $E(v) \geq \hbar$ for every nonconstant J-holomorphic sphere $v : S^2 \to M$.*

Proof: Choose $\hbar > 0$ such that that the following holds for every J-holomorphic curve $v : \{z \in \mathbb{C} : |z - z_0| < r\} \to M$

$$\int_{|z-z_0|<r} |dv(z)|^2 < \hbar \qquad \Longrightarrow \qquad |dv(z_0)|^2 \leq \frac{8}{\pi r^2} \int_{|z-z_0|<r} |dv(z)|^2. \tag{24}$$

The proof, that such a constant $\hbar > 0$ exists, relies on a partial differential inequality of the form $\Delta e \geq -Be^2$ for the *energy density* $e = |dv|^2$. (see footnote on page 12). Now suppose that $v : S^2 = \mathbb{C} \cup \{\infty\} \to M$ is a J-holomorphic curve with energy $E(v) < \hbar$. Then the a-priori estimate (24) holds for all $r > 0$, hence $dv \equiv 0$, and hence v is constant. □

If $t \mapsto J_t = J_{t+1}$ is a smooth family of almost complex structures compatible with ω, then $\hbar = \min_t \hbar(M, \omega, J_t) > 0$. Moreover, if (M, ω) is monotone with minimal Chern number N, then $\hbar = N/\tau$ satisfies the requirements of Lemma 3.2 for every $J \in \mathcal{J}(M, \omega)$.

Proposition 3.3 (Bubbling) *Let $u^\nu \in \mathcal{M}(x^-, x^+; H, J)$ be a sequence such that*

$$\sup_\nu E(u^\nu) < \infty. \tag{25}$$

Then there exist finitely many points $z_j = s_j + it_j \in \mathbb{R} \times S^1$, $j = 1, \ldots, \ell$, and a solution u of (7) such that a subsequence of u^ν converges to u, uniformly with all derivatives on compact subsets of $\mathbb{R} \times S^1 - \{z_1, \ldots, z_\ell\}$. Moreover, the limit solution satisfies

$$E(u) \le \liminf_{\nu\to\infty} E(u^\nu) - \ell\hbar$$

where $\hbar = \min_t \hbar(M, \omega, J_t)$.

Proof: We sketch the main ideas. First, it follows from basic elliptic bootstrapping techniques that every sequence $u^\nu \in \mathcal{M}(x^-, x^+; H, J)$ with uniformly bounded first derivatives, i.e.

$$\sup_\nu \|\partial_s u^\nu\|_{L^\infty} < \infty, \tag{26}$$

has a subsequence which converges, uniformly with all derivatives on compact subsets of $\mathbb{R}\times S^1$, to a solution u of (7). Secondly, if (26) is not satisfied then, for every sequence $\zeta^\nu \to z$ with $|\partial_s u^\nu(\zeta^\nu)| \to \infty$, one can prove that there exists another sequence $z^\nu \to z = s + it$ and a sequence $\varepsilon^\nu \to 0$ such that the rescaled sequence $v^\nu(z) = u^\nu(z^\nu + \varepsilon^\nu z)$ has a subsequence which converges to a nonconstant J_t-holomorphic map $v : \mathbb{C} \to M$.[5] The removable singularity theorem asserts that v extends to a nonconstant J_t-holomorphic sphere and hence, by Lemma 3.2, $E(v) \ge \hbar$. Now the subsequences (still denoted by u^ν and v^ν) satisfy the following, for every (small) $\varepsilon > 0$ and every (large) $R > 0$,

$$\begin{aligned}
\liminf_{\nu\to\infty} E(u^\nu, B_\varepsilon(z)) &\ge \liminf_{\nu\to\infty} E(u^\nu, B_{R\varepsilon^\nu}(z^\nu)) \\
&= \liminf_{\nu\to\infty} E(v^\nu, B_R(0)) \\
&= E(v, B_R(0)).
\end{aligned}$$

Taking the limit $R \to \infty$ we find

$$\liminf_{\nu\to\infty} E(u^\nu, B_\varepsilon(z)) \ge \hbar$$

for every $\varepsilon > 0$. Since the energy of u^ν is uniformly bounded above, this can only happen at finitely many points $z_1, \ldots, z_\ell$. We can then choose a further subsequence which converges, uniformly with all derivatives on compact subsets of $\mathbb{R} \times S^1 - \{z_1, \ldots, z_\ell\}$,

[5]The sequences z^ν and ε^ν can be found by using Hofer's lemma: Let (X, d) be a complete metric space, $f : X \to \mathbb{R}$ be a nonnegative continuous function, $x \in X$, and $\delta > 0$. Then there exists a $z \in X$ and a positive number $\varepsilon \le \delta$ such that

$$d(x, z) < 2\delta, \qquad \sup_{B_\varepsilon(z)} f \le 2f(z), \qquad \varepsilon f(z) \ge \delta f(x).$$

Apply this lemma to the function $f = |\partial_s u^\nu|$, the point $x = \zeta^\nu$, and the constant $\delta = |\partial_s u^\nu(z^\nu)|^{-1/2}$ to obtain sequences $z^\nu \to z$ and $\varepsilon^\nu > 0$ such that

$$\varepsilon^\nu \to 0, \qquad \varepsilon^\nu |\partial_s u^\nu(z^\nu)| \to \infty, \qquad \sup_{B_{\varepsilon^\nu}(z^\nu)} |\partial_s u^\nu| \le 2|\partial_s u^\nu(z^\nu)|.$$

These sequences satisfy the assertion.

to a function $u : \mathbb{R} \times S^1 - \{z_1, \dots, z_\ell\} \to M$. The limit necessarily satisfies (7) and

$$
\begin{aligned}
E\left(u, \mathbb{R} \times S^1 - \bigcup_i B_\varepsilon(z_i)\right) &= \lim_{\nu \to \infty} E\left(u^\nu, \mathbb{R} \times S^1 - \bigcup_i B_\varepsilon(z_i)\right) \\
&\leq \limsup_{\nu \to \infty} E(u^\nu) - \sum_i \liminf_{\nu \to \infty} E(u^\nu, B_\varepsilon(z_i)) \\
&\leq \limsup_{\nu \to \infty} E(u^\nu) - \ell\hbar
\end{aligned}
$$

for every $\varepsilon > 0$. Taking the limit $\varepsilon \to 0$ we find

$$E(u) \leq \limsup_{\nu \to \infty} E(u^\nu) - \ell\hbar.$$

Finally, the removable singularity theorem for solutions of (7) shows that u extends to a smooth function on all of $\mathbb{R} \times S^1$. This proves the proposition. □

The convergence in the complement of a finite set as in the assertion of Proposition 3.3 will be called **convergence modulo bubbling**. The limit solution u has finite energy and hence, by Proposition 1.21, is again a connecting orbit. One can now argue as in the finite dimensional case (Exercise 1.9) to prove the following corollary (see Figure 7).

Corollary 3.4 *Suppose that the periodic solutions $x \in \mathcal{P}(H)$ are all nondegenerate and let $u^\nu \in \mathcal{M}(x^-, x^+; H, J)$ be a sequence which satisfies (25). Then there exist a subsequence (still denoted by u^ν), finitely many periodic orbits $x_0, \dots, x_m \in \mathcal{P}(H)$ with $x_0 = x^+$ and $x_m = x^-$, finitely many connecting orbits*

$$u_j \in \mathcal{M}(x_j, x_{j-1}; H, J), \qquad j = 1, \dots, m,$$

and finitely many sequences s_j^ν, such that $u^\nu(s + s_j^\nu, t)$ converges modulo bubbling to $u_j(s,t)$. Moreover, the limit solutions satisfy

$$\sum_{j=0}^{m} E(u_j) \leq \limsup_{\nu \to \infty} E(u^\nu) - \ell\hbar. \tag{27}$$

where ℓ is the total number of bubbles.

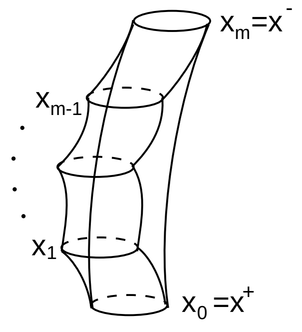

Figure 7: Limit behaviour for Floer's connecting orbits with bounded derivatives

Proof of Proposition 3.1: The formula (10) shows that

$$\eta_H(x^-) - \eta_H(x^+) + 2\tau E(u) = \mu(u;H) = 1$$

for $u \in \mathcal{M}^1(x^-, x^+; H, J)$ and hence the energy inequality (25) is automatically satisfied for sequences $u^\nu \in \mathcal{M}^1(x^-, x^+; H, J)$. Moreover, the limit solutions $u_j \in \mathcal{M}(x_j, x_{j-1}; H, J)$ in Corollary 3.4 satisfy

$$\begin{aligned}
\sum_{j=1}^m \mu(u_j;H) &= \sum_{j=1}^m \Big(\eta_H(x_j) - \eta_H(x_{j-1}) + 2\tau E(u_j)\Big) \\
&= \eta_H(x^-) - \eta_H(x^+) + 2\tau \sum_{j=1}^m E(u_j) \\
&\le \eta_H(x^-) - \eta_H(x^+) + 2\tau E(u^\nu) - 2\tau\ell\hbar \\
&= 1 - 2\ell N.
\end{aligned}$$

Here ℓ is the number of bubbles, the first equation follows from (10), the third inequality from (27), and the last equation from (10) and the fact that $\tau\hbar = N$. If $H \in \mathcal{H}_{\mathrm{reg}}$ then Theorem 1.24 guarantees that $\mu(u_j;H) \ge 1$ for every j. This implies that $m = 1$ and $\ell = 0$, i.e. there is no bubbling and there is a single limit solution $u = u_1 \in \mathcal{M}(x^-, x^+; H, J)$. In this case one can show that $u^\nu(s + s_1^\nu, t)$ converges to $u(s,t)$ in the $W^{1,p}$-norm on the noncompact domain $\mathbb{R} \times S^1$. This shows that, in the monotone case, the moduli space $\widehat{\mathcal{M}}^1(x^-, x^+; H, J)$ is compact for every pair $x^\pm \in \mathcal{P}(H)$. Since this space is also a zero dimensional manifold it must be a finite set. □

3.2 Floer homology

Continue with the monotone case and, for $H \in \mathcal{H}_{\mathrm{reg}}$, consider the chain complex

$$CF_k(H) = \bigoplus_{\substack{x\in\mathcal{P}(H) \\ \mu_{\mathrm{CZ}}(x;H)=k (\mathrm{mod}\ 2N)}} \mathbb{F}\langle x\rangle.$$

Here $\mathbb{F}$ is a principal ideal domain (which we shall choose to be either of $\mathbb{Z}_2$, $\mathbb{Z}$, or $\mathbb{Q}$). In the case $\mathbb{F} = \mathbb{Z}_2$ the boundary operator is defined by counting the connecting orbits u running from x^- to x^+ which satisfy $\mu(u;H) = 1$. For other coefficient rings one has to take account of orientations. The latter is discussed in detail in Floer-Hofer [12]. The rough idea is to prove first that the moduli spaces $\mathcal{M}(x^-, x^+)$ are all orientable, and then to choose a system of *coherent orientations* under which Floer's gluing maps (discussed below) are orientation preserving. That such orientations exist is proved in [12]. These orientations are not unique. As in the finite dimensional case they involve one choice of orientation for each critical point $x \in \mathcal{P}(H)$. With these orientations in place one can define a number $\varepsilon(u) \in \{\pm 1\}$ for $u \in \mathcal{M}^1(x^-, x^+; H, J)$ by comparing this coherent orientation of $\mathcal{M}^1$ with the obvious flow orientation. The Floer boundary operator is now defined by

$$\partial^F\langle y\rangle = \sum_{\substack{x\in\mathcal{P}(H) \\ \mu_{\mathrm{CZ}}(x;H)=k-1(\mathrm{mod}\ 2N)}} \sum_{[u]\in\widehat{\mathcal{M}}^1(y,x;H,J)} \varepsilon(u)\langle x\rangle \tag{28}$$

for a periodic orbit $y \in \mathcal{P}(H)$ with $\mu_{\mathrm{CZ}}(y;H) = k (\mathrm{mod}\ 2N)$.

Theorem 3.5 (Floer) *If (M,ω) is monotone and $H \in \mathcal{H}_{\mathrm{reg}}$ then $\partial^F \circ \partial^F = 0$.*

Proof: In the case $N \geq 2$ the proof is as in the finite dimensional case. The key is to understand the ends of the 2-dimensional moduli space $\mathcal{M}^2(z,x;H,J)$. If $N \geq 2$ one proves as before that no bubbling can occur and the ends correspond to pairs of connecting orbits $v \in \mathcal{M}^1(z,y)$ and $u \in \mathcal{M}^1(y,x)$. In other words, we can think of $\widehat{\mathcal{M}}^2(z,x) = \mathcal{M}^2(z,x)/\mathbb{R}$ as a compact 1-manifold whose boundary is given by

$$\partial\widehat{\mathcal{M}}^2(z,x) = \bigcup_{y \in \mathcal{P}(H)} \widehat{\mathcal{M}}^1(z,y) \times \widehat{\mathcal{M}}^1(y,x). \tag{29}$$

This follows from Floer's gluing theorem (see Section 3.3 below). Hence one obtains

$$\sum_{\substack{y \in \mathcal{P}(H) \\ \mu_{\mathrm{CZ}}(y;H)=k (\mathrm{mod}\ 2N)}} \sum_{[v] \in \widehat{\mathcal{M}}^1(z,y)} \sum_{[u] \in \widehat{\mathcal{M}}^1(y,x)} \varepsilon(v)\varepsilon(u) = 0 \tag{30}$$

for every pair $x,z \in \mathcal{P}(H)$ with $\mu_{\mathrm{CZ}}(z;H) = k+1 (\mathrm{mod}\ 2N)$ and $\mu_{\mathrm{CZ}}(x;H) = k-1(\mathrm{mod}\ 2N)$.

In the case of minimal Chern number $N = 1$ there is an additional subtlety arising from the presence of J-holomorphic spheres with Chern number

$$c_1(v) = \int_{S^2} v^* c_1 = 1.$$

The bubbling of such spheres cannot be excluded apriori. However, if such bubbling does occur, then the remaining limit solution u must have zero energy and hence be of the form $u(s,t) = x(t) = z(t)$. Now there are two facts which prevent such bubbles. The first is a more subtle version of the compactness theorem (discussed in Chapter 4 below) which asserts that the image of the bubbling sphere $v : S^2 \to M$ must intersect the remaining limit curve $u(s,t) = x(t)$. But now, for a generic J, the J-holomorphic curves with Chern number 1 form moduli space $\mathcal{M}(1;J) \subset \mathrm{Map}(S^2,M)$ of dimension $2n+2$. Dividing by the 6-dimensional conformal group $\mathrm{G} = \mathrm{PSL}(2,\mathbb{C})$ we obtain a moduli space $\mathcal{M}(1;J)/\mathrm{G}$ of dimension $2n-4$. Taking account of the fact that each sphere is 2-dimensional we obtain finally a space $\mathcal{M}(1;J) \times S^2/\mathrm{G}$ of dimension $2n-2$. Thus the points on J-holomorphic spheres of Chern number 1 form, for a generic J, a codimension-2 subset of M, and for a generic H these spheres therefore do not intersect the 1-dimensional periodic orbits of H. Hence no such bubbling can occur, and we can proceed with the same argument as above. □

The Floer homology groups of a regular pair (H,J) are defined as the homology of the chain complex $(CF_*(H), \partial^F)$:

$$HF_*(M,\omega,H,J;\mathbb{F}) = \frac{\ker \partial^F}{\mathrm{im}\, \partial^F}.$$

The key observation is that these homology groups are an invariant, i.e. they do not depend on the almost complex structure or the Hamiltonian. Moreover, they are naturally isomorphic to the ordinary homology groups of M. This is the content of the following two theorems.

Theorem 3.6 (Floer) *Let (M,ω) be a monotone symplectic manifold. For two pairs (H^α, J^α) and (H^β, J^β), which satisfy the regularity requirements for the definition of Floer cohomology, there exists a natural isomorphism*

$$\Phi^{\beta\alpha} : HF_*(M,\omega,H^\alpha,J^\alpha) \to HF_*(M,\omega,H^\beta,J^\beta).$$

If (H^γ, J^γ) *is another such pair then*

$$\Phi^{\gamma\alpha} = \Phi^{\gamma\beta} \circ \Phi^{\beta\alpha}, \qquad \Phi^{\alpha\alpha} = \mathrm{id}.$$

Theorem 3.7 (Floer) *Let* (M, ω) *be a monotone symplectic manifold. Then, for every regular pair* (H^α, J^α)*, there exists an isomorphism*

$$\Phi^\alpha : HF_k(M, \omega, H^\alpha, J^\alpha; \mathbb{F}) \to QH_k(M; \mathbb{F}) = \bigoplus_{j \equiv k (\mathrm{mod}\ 2N)} H_j(M; \mathbb{F}).$$

These maps are natural in the sense that

$$\Phi^\beta \circ \Phi^{\beta\alpha} = \Phi^\alpha.$$

The Arnold conjecture for monotone symplectic manifolds (and general coefficient rings) follows immediately from Theorem 3.7. The proofs will be explained in the next three sections.

Exercise 3.8 Prove the naturality of Floer's gradient flow equation (7). More precisely, suppose that $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ is a solution of (7) and $\varphi_t = \varphi_{t+1}$ is a loop of Hamiltonian symplectomorphisms, generated by the Hamiltonian functions $K_t = K_{t+1} : M \to \mathbb{R}$ via

$$\frac{d}{dt}\varphi_t = X_{K_t} \circ \varphi_t, \qquad \varphi_0 = \mathrm{id}.$$

Prove that the function $\tilde{u}(s,t) = {\varphi_t}^{-1}(u(s,t))$ satisfies

$$\partial_s \tilde{u} + \tilde{J}_t(\tilde{u})\partial_t \tilde{u} - \nabla \tilde{H}_t(\tilde{u}) = 0,$$

where

$$\tilde{J}_t = {\varphi_t}^* J_t, \qquad \tilde{H}_t = (H_t - K_t) \circ \varphi_t,$$

and the gradient is computed with respect to the metric induced by $\tilde{J}_t$. Deduce that the Hamiltonian loop φ_t induces an isomorphism of Floer homologies.

$$HF_*(M, \omega, H_t, J_t) \to HF_*(M, \omega, (H_t - K_t) \circ \varphi_t, {\varphi_t}^* J_t).$$

This isomorphism will not, in general, agree with the one of Theorem 3.6. One can think of this as an action of the fundamental group of the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian symplectomorphisms on Floer homology. This action has recently been used by Seidel and Lalonde-McDuff-Polterovich to derive nontrivial information about the fundamental group of $\mathrm{Ham}(M, \omega)$. □

3.3 Floer's gluing theorem

The goal of this section is to provide more details for the proof of Theorem 3.5 and to prepare the background for the proofs of Theorems 3.6 and 3.7. The basic construction is very simple. Given two connecting orbits

$$v \in \mathcal{M}(z, y; H, J), \qquad u \in \mathcal{M}(y, x; H, J),$$

with surjective Fredholm operators D_v and D_u, one constructs a one parameter family of approximate solution

$$\tilde{w}_R = v \#_R u$$

of (7) running directly from z to x. One then proves that the linearized operator $D_{\tilde{w}_R}$ is surjective for large R and has a right inverse which satisfies a uniform bound, independent of R. It then follows from the infinite dimensional implicit function theorem that near $\tilde{w}_R$ there is a true solution $w_R \in \mathcal{M}(z,x;H,J)$ of (7). This construction gives rise to a **gluing map**

$$\iota_{z,y,x} : \widehat{\mathcal{M}}(z,y;H,J) \times (R_0,\infty) \times \widehat{\mathcal{M}}(y,x;H,J) \to \widehat{\mathcal{M}}(z,x;H,J). \tag{31}$$

This is not quite accurate, unless the index differences are 1 and so the moduli spaces $\widehat{\mathcal{M}}(z,y;H,J)$ and $\widehat{\mathcal{M}}(y,x;H,J)$ are compact. In general, this map is only defined on any given compact subset of $\widehat{\mathcal{M}}(z,y;H,J) \times \widehat{\mathcal{M}}(y,x;H,J)$ and the constants in the estimtes will depend on this subset. With this understood, the gluing map (31) is a diffeomorphism onto an open subset of $\widehat{\mathcal{M}}(z,x;H,J)$.

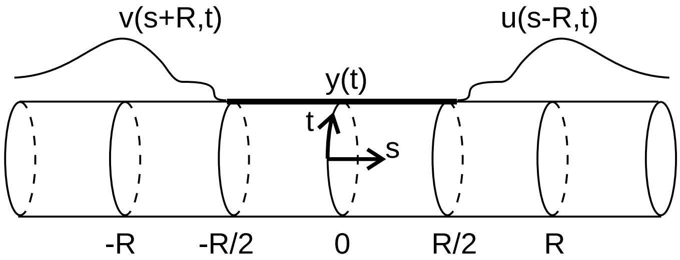

Figure 8: Floer's gluing construction

Here are some more details of Floer's gluing construction. The approximate solution is illustrated in Figure 8. Explicitly, it can be defined by

$$v\#_R u(s,t) = \begin{cases} v(s+R,t), & s \le -R/2-1, \\ \exp_{y(t)}(\beta(-s-R/2)\eta(s+R,t)), & -R/2-1 \le s \le -R/2, \\ y(t), & -R/2 \le s \le R/2, \\ \exp_{y(t)}(\beta(s-R/2)\xi(s-R,t)), & R/2 \le s \le R/2+1, \\ u(s-R,t), & s \ge R/2+1, \end{cases}$$

where $\xi(s,t), \eta(s,t) \in T_{y(t)}M$ are chosen such that $u(s,t) = \exp_{y(t)}(\xi(s,t))$ for all t and large negative s and $v(s,t) = \exp_{y(t)}(\eta(s,t))$ for all t and large positive s. Here $\beta : \mathbb{R} \to [0,1]$ is a cutoff function equal to 1 for $s \ge 1$ and equal to 0 for $s \le 0$. Let us fix the two solutions u and v and assume that D_u and D_v are surjective. The next proposition shows that there is a uniformly bounded family of right inverses for the operators $D_R = D_{v\#_R u}$ for R sufficiently large.

Proposition 3.9 *Suppose that $H \in \mathcal{H}_{\mathrm{reg}}$, $x,y,z \in \mathcal{P}(H)$, $u \in \mathcal{M}(y,x;H,J)$, and $v \in \mathcal{M}(z,y;H,J)$. Then there exist constants $c > 0$ and $R_0 > 0$ such that, for every $R > R_0$ and every $\eta \in W^{2,p}(\mathbb{R}\times S^1, (v\#_R u)^*TM)$,*

$$\left\|{D_R}^*\eta\right\|_{W^{1,p}} \le c \left\|D_R {D_R}^*\eta\right\|_{L^p}. \tag{32}$$

Proof: Let us denote

$$v_R(s,t) = \begin{cases} v\#_R u(s,t), & s \le 0, \\ y(t), & s \ge 0, \end{cases} \qquad u_R(s,t) = \begin{cases} y(t), & s \le 0, \\ v\#_R u(s,t), & s \ge 0. \end{cases}$$

Note that $v_R(s,t) = v(s+R,t)$ for $s \le -R/2 - 1$ and $u_R(s,t) = u(s-R,t)$ for $s \ge R/2+1$. By Proposition 1.21, the difference between $v_R(s,t)$ and $v(s+R,t)$ (in the C^ℓ-norm for any ℓ) is exponentially small as $R \to \infty$, and so is the difference between $u_R(s,t)$ and $u(s-R,t)$. Hence there exist constants $R_0 > 0$, $c_0 > 0$, and $c_1 > 0$ such that, for every $R \ge R_0$ and every $\eta_u \in W^{1,p}(\mathbb{R} \times S^1, u_R{}^*TM)$,

$$\|\eta_u\|_{W^{1,p}} \le c_0 \left\|D_{u_R}{}^*\eta_u\right\|_{L^p}, \qquad \left\|D_{u_R}{}^*\eta_u\right\|_{W^{1,p}} \le c_1 \left\|D_{u_R} D_{u_R}{}^*\eta_u\right\|_{L^p}.$$

Similar inequalities hold with D_{u_R} replaced by D_{v_R}.

Now, for $R > 2$, we have

$$v\#_R u(s,t) = \begin{cases} v_R(s,t), & \text{if } s \le 0, \\ u_R(s,t), & \text{if } s \ge 0. \end{cases}$$

Note that $v\#_R u(s,t) = y(t)$ for $-R/2 \le s \le R/2$. In order to establish the required estimate for $D_R = D_{v\#_R u}$ we fix a vector field $\eta \in W^{1,p}(\mathbb{R} \times S^1, (v\#_R u)^*TM)$ and define

$$\eta_u(s,t) = \beta_R(s)\eta(s,t) \in T_{u_R(s,t)}M,$$
$$\eta_v(s,t) = (1-\beta_R(s))\eta(s,t) \in T_{v_R(s,t)}M,$$

where $\beta_R(s) = \beta(s/R + 1/2)$ is a smooth cutoff function such that

$$\beta_R(s) = \begin{cases} 0, & \text{if } s \le -R/2, \\ 1, & \text{if } s \ge R/2, \end{cases} \qquad 0 \le \beta_R{}'(s) \le 2R^{-1}, \quad -R^{-1} \le \beta_R{}''(s) \le R^{-1},$$

for $R \ge R_0$. Note that $D_R{}^*\eta_u = D_{u_R}{}^*\eta_u$ and $D_R{}^*\eta_v = D_{v_R}{}^*\eta_v$. Hence we obtain the following inequality.

$$\begin{aligned} \|\eta\|_{W^{1,p}} &\le \|\eta_u\|_{W^{1,p}} + \|\eta_v\|_{W^{1,p}} \\ &\le c_0 \left(\left\|D_{u_R}{}^*\eta_u\right\|_{L^p} + \left\|D_{v_R}{}^*\eta_v\right\|_{L^p}\right) \\ &= c_0 \left(\left\|D_R{}^*(\beta_R\eta)\right\|_{L^p} + \left\|D_R{}^*((1-\beta_R)\eta)\right\|_{L^p}\right) \\ &\le 2c_0 \left\|D_R{}^*\eta\right\|_{L^p} + \frac{4c_0}{R}\|\eta\|_{L^p}. \end{aligned}$$

The last inequality follows from the fact that $D_R{}^*(\beta_R\eta) = \beta_R D_R{}^*\eta - \beta_R{}'\eta$ and $|\beta_R{}'(s)| \le 2/R$ for all s. With $4c_0/R \le 1/2$ we obtain an inequality

$$\|\eta\|_{W^{1,p}} \le 4c_0 \left\|D_R{}^*\eta\right\|_{L^p} \tag{33}$$

for all $\eta \in W^{1,p}(\mathbb{R} \times S^1, (v\#_R u)^*TM)$. Now observe that

$$D_{u_R}{}^*\eta_u = \beta_R D_R{}^*\eta - \beta_R{}'\eta, \qquad D_{v_R}{}^*\eta_v = (1-\beta_R) D_R{}^*\eta + \beta_R{}'\eta.$$

In particular, $D_{u_R}{}^*\eta_u + D_{v_R}{}^*\eta_v = D_R{}^*\eta$ and hence

$$\begin{aligned} \left\|D_R{}^*\eta\right\|_{W^{1,p}} &\le \left\|D_{u_R}{}^*\eta_u\right\|_{L^p} + \left\|D_{v_R}{}^*\eta_v\right\|_{L^p} \\ &\le c_1 \left(\left\|D_{u_R} D_{u_R}{}^*\eta_u\right\|_{L^p} + \left\|D_{v_R} D_{v_R}{}^*\eta_v\right\|_{L^p}\right) \\ &\le c_1 \Big(\left\|D_R(\beta_R D_R{}^*\eta - \beta_R{}'\eta)\right\|_{L^p} \\ &\qquad + \left\|D_R((1-\beta_R) D_R{}^*\eta + \beta_R{}'\eta)\right\|_{L^p}\Big) \\ &\le 2c_1 \left\|D_R D_R{}^*\eta\right\|_{L^p} + \frac{2c_1}{R}\left\|D_R{}^*\eta - D_R\eta\right\|_{L^p} + \frac{4c_1}{R}\|\eta\|_{L^p} \\ &\le 2c_1 \left\|D_R D_R{}^*\eta\right\|_{L^p} + \frac{2c_1}{R}\left\|D_R{}^*\eta\right\|_{L^p} + \frac{c_2}{R}\|\eta\|_{W^{1,p}} \\ &\le 2c_1 \left\|D_R D_R{}^*\eta\right\|_{L^p} + \frac{2c_1 + 4c_0c_2}{R}\left\|D_R{}^*\eta\right\|_{L^p}. \end{aligned}$$

In the last step we have used (33). With $(2c_1 + 4c_0c_2)/R \leq 1/2$ we obtain

$$\|D_R{}^*\eta\|_{W^{1,p}} \leq 4c_1 \|D_R D_R{}^*\eta\|_{L^p}$$

as claimed. This proves the proposition. $\square$

With the uniform estimates for the inverse established, it follows easily from the implicit function theorem that, for $R > 0$ sufficiently large, there exists a unique solution $w_R \in \mathcal{M}(z, x; H, J)$ near $v\#_R u$ of the form

$$w_R = \exp_{v\#_R u}(D_R{}^*\eta)$$

for some $\eta \in W^{2,p}(\mathbb{R} \times S^1, (v\#_R u)^*TM)$. The details are standard and will not be carried out here. If the original moduli spaces $\widehat{\mathcal{M}}(z, y; H, J)$ and $\widehat{\mathcal{M}}(y, x; H, J)$ are zero dimensional, then the images of the gluing maps describe precisely the ends of the one dimensional moduli space $\widehat{\mathcal{M}}(z, x; H, J)$. In other words, the complement of these images is a compact 1-manifold. This justifies the formula (29) in the proof of Theorem 3.5.

3.4 Invariance of Floer homology

The proof of Theorem 3.6 is based on the following construction. We assume throughout that J is fixed and discuss the dependence of the Floer homology groups on the Hamiltonian function. Variations of the almost complex structure can easily be incorporated by an analogout construction. Given two regular Hamiltonians $H^\alpha, H^\beta \in \mathcal{H}_{\mathrm{reg}}$ choose a homotopy $H^{\alpha\beta} = \{H^{\alpha\beta}_{s,t}\}$ from H^α to H^β such that

$$H^{\alpha\beta}_{s,t} = \begin{cases} H^\alpha_t, & \text{for } s \leq -1, \\ H^\beta_t, & \text{for } s \geq 1. \end{cases}$$

Now consider the time-dependent version of equation (7) with H_t replaced by $H_{s,t}$. This equation has the form

$$\partial_s u + J(u)\partial_t u - \nabla H^{\alpha\beta}_{s,t}(u) = 0. \tag{34}$$

We consider solutions of (34) which satisfy the limit conditions

$$\lim_{s\to-\infty} u(s,t) = x^\alpha(t), \qquad \lim_{s\to\infty} u(s,t) = x^\beta(t) \tag{35}$$

for some periodic solutions $x^\alpha \in \mathcal{P}(H^\alpha)$ and $x^\beta \in \mathcal{P}(H^\beta)$. The solutions of (34) and (35) form a moduli space

$$\mathcal{M}(x^\alpha, x^\beta) = \mathcal{M}(x^\alpha, x^\beta; H^{\alpha\beta}) = \{u : \mathbb{R} \times S^1 \to M \,:\, (34),\, (35)\}$$

which, for a generic homotopy $H^{\alpha\beta}$, is a smooth manifold of dimension

$$\dim \mathcal{M}(x^\alpha, x^\beta; H^{\alpha\beta}) = \mu_{H^\alpha}(x^\alpha) - \mu_{H^\beta}(x^\beta) (\mathrm{mod}\ 2N).$$

As before, the local dimension near u will be denoted by $\mu(u, H^{\alpha\beta})$ and we write

$$\mathcal{M}^0(x^\alpha, x^\beta; H^{\alpha\beta}) = \left\{u \in \mathcal{M}(x^\alpha, x^\beta; H^{\alpha\beta}) \,:\, \mu(u, H^{\alpha\beta}) = 0\right\}.$$

This is the zero dimensional part of the moduli space. As in Proposition 3.1 one proves that this is a finite set for a generic homotopy $H^{\alpha\beta}$. Counting the elements of these sets with appropriate signs gives rise to a homomorphism

$$\Phi^{\beta\alpha} : CF_*(H^\alpha) \to CF_*(H^\beta)$$

defined by

$$\Phi^{\beta\alpha}\langle x^\alpha\rangle = \sum_{x^\beta \in \mathcal{P}(H^\beta)} \sum_{u \in \mathcal{M}^0(x^\alpha, x^\beta)} \varepsilon(u)\langle x^\beta\rangle.$$

The next lemma shows that this is a chain homomorphism and hence descends to a homomorphism of Floer homologies. The subsequent lemma shows that the chain maps $\Phi^{\beta\alpha}$ satisfy the obvious composition rule (for catenation of homotopies) and the third lemma shows that the induced map on Floer homology is independent of the choice of the homotopy. The main technical ingredients in the proofs of all three lemmata are Floer's gluing construction and Gromov compactness.

Lemma 3.10 *The above homomorphisms $\Phi^{\beta\alpha} : CF_*(H^\alpha) \to CF_*(H^\beta)$ satisfies*

$$\partial^\beta \circ \Phi^{\beta\alpha} = \Phi^{\beta\alpha} \circ \partial^\alpha.$$

Proof: Examining the 1-dimensional moduli space $\mathcal{M}^1(y^\alpha, x^\beta)$ one finds that

$$\begin{aligned} \partial\mathcal{M}^1(y^\alpha, x^\beta; H^{\alpha\beta}) &= \bigcup_{x^\alpha} \widehat{\mathcal{M}}^1(y^\alpha, x^\alpha; H^\alpha) \times \mathcal{M}^0(x^\alpha, x^\beta; H^{\alpha\beta}) \\ &\quad \cup \bigcup_{y^\beta} \mathcal{M}^0(y^\alpha, y^\beta; H^{\alpha\beta}) \times \widehat{\mathcal{M}}^1(y^\beta, x^\beta; H^\beta). \end{aligned}$$

This equality is to be understood with appropriate orientations and hence is equivalent to the assertion of the lemma. (See Figure 9.) □

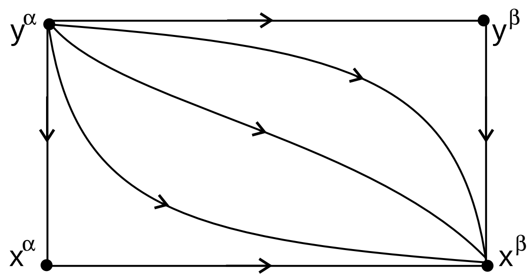

Figure 9: Floer's chain homomorphism

Lemma 3.11 *Let $H^{\alpha\beta}$ be a regular homotopy from H^α to H^β and $H^{\beta\gamma}$ a regular homotopy from H^β to H^γ. Define*

$$H^{\alpha\gamma}_{R,s,t} = \begin{cases} H^{\alpha\beta}_{s+R,t}, & \text{for } s \le 0, \\ H^{\beta\gamma}_{s-R,t}, & \text{for } s \ge 0, \end{cases}$$

for $R > 0$ sufficiently large. Then there exists an $R_0 > 0$ such that, for every $R > R_0$, $H^{\alpha\gamma}$ is a regular homotopy from H^α to H^γ and the morphism $\Phi^{\gamma\alpha}_R : CF_*(H^\alpha) \to CF_*(H^\gamma)$ induced by this homotopy is given by

$$\Phi^{\gamma\alpha}_R = \Phi^{\gamma\beta} \circ \Phi^{\beta\alpha}.$$

Proof: This is again a gluing theorem which asserts that for R sufficiently large one can glue together solutions in $\mathcal{M}(x^\alpha, x^\beta; H^{\alpha\beta})$ with those in $\mathcal{M}(x^\beta, x^\gamma; H^{\beta\gamma})$ to obtain direct connecting orbits from x^α to x^γ corresponding to the homotopy $H_R^{\alpha\gamma}$. A compactness argument then shows that there are no other solutions for large R. Hence there is a bijection

$$\bigcup_{x^\beta} \mathcal{M}^0(x^\alpha, x^\beta; H^{\alpha\beta}) \times \mathcal{M}^0(x^\beta, x^\gamma; H^{\beta\gamma}) \longrightarrow \mathcal{M}^0(x^\alpha, x^\gamma; H_R^{\alpha\gamma})$$

for R sufficiently large. One can show that this bijection is orientation preserving and this proves the lemma. □

Lemma 3.12 *If $H_0^{\alpha\beta}$ and $H_1^{\alpha\beta}$ are two regular homotopies from H^α to H^β, then the two corresponding chain homomorphisms $\Phi_0^{\beta\alpha}$ and $\Phi_1^{\beta\alpha}$ are chain homotopy equivalent. In other words, there exists a homomorphism $T : CF_*(H^\alpha) \to CF_*(H^\beta)$ such that*

$$\Phi_1^{\beta\alpha} - \Phi_0^{\beta\alpha} = \partial^\beta T + T\partial^\alpha.$$

Proof: The idea is to choose a regular homotopy of homotopies $H_{\lambda,s,t}^{\alpha\beta}$ from H^α to H^β which agrees with $H_{0,s,t}^{\alpha\beta}$ for $\lambda = 0$ and $H_{1,s,t}^{\alpha\beta}$ for $\lambda = 1$. Then one considers the parametrized moduli space

$$\mathcal{M}^{-1}(x^\alpha, y^\beta; \{H_\lambda^{\alpha\beta}\}) = \left\{(\lambda, u) \,:\, 0 \le \lambda \le 1,\, u \in \mathcal{M}(x^\alpha, x^\beta; H_\lambda^{\alpha\beta}), \mu(u; H_\lambda^{\alpha\beta}) = -1\right\}.$$

This moduli space is a zero dimensional manifold (for a generic homotopy of homotopies) and the chain homotopy $T : CF_*(H^\alpha) \to CF_*(H^\beta)$ is defined by

$$T\langle x^\alpha\rangle = \sum_{y^\beta} \#\mathcal{M}^{-1}(x^\alpha, y^\beta)\langle y^\beta\rangle$$

where $\#\mathcal{M}^{-1}(x^\alpha, y^\beta)$ is to be understood as counting with appropriate signs. Note that this zero dimensional moduli space consists of finitely many pairs (λ_j, u_j) with $0 < \lambda_j < 1$. Since the homotopies $H_0^{\alpha\beta}$ and $H_1^{\alpha\beta}$ are regular there cannot be any connecting orbits with index -1 for $\lambda = 0$ and $\lambda = 1$. However, in a generic 1-parameter family such orbits do occur for isolated parameter values. Counting these gives rise to the chain homotopy equivalence T. That T satisfies the required equation follows again by examining the boundaries of the 1-dimensional moduli spaces (see Figure 10). One finds

$$\begin{aligned}\partial\mathcal{M}^0(y^\alpha, y^\beta; \{H_\lambda^{\alpha\beta}\}) &= \mathcal{M}^0(y^\alpha, y^\beta; H_0^{\alpha\beta}) \cup \mathcal{M}^0(y^\alpha, y^\beta; H_1^{\alpha\beta}) \\ &\quad \cup \bigcup_{z^\beta} \mathcal{M}^{-1}(y^\alpha, z^\beta; \{H_\lambda^{\alpha\beta}\}) \times \widehat{\mathcal{M}}^1(z^\beta, y^\beta; H^\beta) \\ &\quad \cup \bigcup_{x^\alpha} \widehat{\mathcal{M}}^1(y^\alpha, x^\alpha; H^\alpha) \times \mathcal{M}^{-1}(x^\alpha, y^\beta; \{H_\lambda^{\alpha\beta}\}).\end{aligned}$$

The proof involves again Floer's gluing argument and Gromov compactness. The identity is to be understood with orientations, and this proves the lemma. □

Proof of Theorem 3.6: Let $H^{\alpha\beta}$ be any regular homotopy from H^α to H^β and denote by $H^{\beta\alpha}$ the inverse homotopy. By Lemma 3.10, these homotopies induce chain

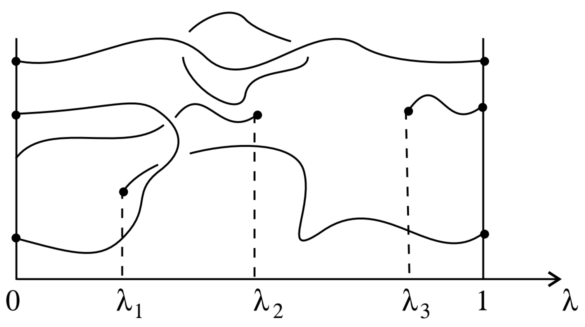

Figure 10: The parametrized moduli space $\mathcal{M}^0(y^\alpha, y^\beta; \{H_\lambda^{\alpha\beta}\})$

maps $\Phi^{\alpha\beta} : CF_*(H^\alpha) \to CF_*(H^\beta)$ and $\Phi^{\beta\alpha} : CF_*(H^\beta) \to CF_*(H^\alpha)$. By Lemma 3.11, the composition $\Phi^{\alpha\beta} \circ \Phi^{\beta\alpha}$ is equal to the map induced by some homotopy from H^α to itself. Now the constant homotopy induces the identity map on $CF_*(H^\alpha)$. Hence it follows from Lemma 3.12 that $\Phi^{\alpha\beta} \circ \Phi^{\beta\alpha}$ is chain homotopy equivalent to the identity. Hence $\Phi^{\alpha\beta}$ is a chain homotopy equivalence with chain homotopy inverse $\Phi^{\beta\alpha}$. Hence $\Phi^{\alpha\beta}$ induces an isomorphism on Floer homology. □

3.5 A natural isomorphism

There are essentially two ways to prove that the Floer homology groups of a pair (H, J) agree with the ordinary homology of M. The first is to use the independence of the Floer homology groups of the Hamiltonian, and then to prove that, if $H_t \equiv H$ is a smooth time independent Morse function with sufficiently small second derivatives, then the 1-dimensional moduli spaces of Floer's connecting orbits consist entirely of gradient flow lines of H. Once this is established, computing the Floer chain complex reduces to the computation of the Morse complex of H, and hence the resulting Floer homology groups agree with Morse homology, and hence, by Theorem 1.12, with the singular homology of M. This method was used by Floer [11] and also in [47] and [19].

An entirely different approach was found by Piunikhin-Salamon-Schwarz [38]. The idea is to consider perturbed J-holomorphic planes $u : \mathbb{C} \to M$ which satisfy the following conditions.

- $z \mapsto u(z)$ is a J-holomorphic curve for $|z| < 1$.
- $u(e^{2\pi(s+it)})$ satisfies (34) for $s > 0$ where $H_{s,t} = 0$ for $s \leq 0$ and $H_{s,t} = H_t$ is independent of s for $s > 1$.
- $u(e^{2\pi(s+it)})$ converges to a periodic solution $x(t) = x(t+1)$ of (1) as $s \to \infty$.
- $u(0) \in W^u(y; f)$ for some Morse function $f : M \to \mathbb{R}$ and a critical point y.

One can think of these as J-holomorphic **spiked disks**, where the spike is the gradient flow line from y to $u(0)$ (see Figure 11). Without the condition $u(0) \in W^u(y; f)$ these perturbed J-holomorphic planes form a manifold of local dimension $2n - \mu_H(x, u)$ near u. Hence the condition $u(0) \in W^u(y; f)$ gives rise to a zero dimensional moduli space whenever the index difference is zero, i.e. $\mathrm{ind}_f(y) = \mu_H(x, u)$. In the zero dimensional case counting the solutions with suitable orientations gives rise to a chain homomorphism $CM_*(f) \to CF_*(H)$. In [38] it is shown that this map induces an isomorphism on homology $HM_*(f) \to HF_*(M, \omega; H, J)$. This construction can also

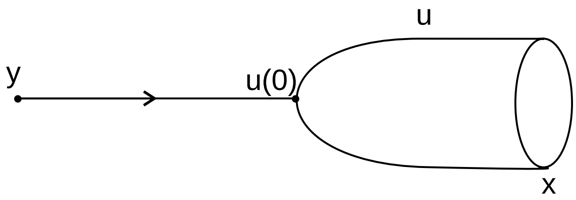

Figure 11: The isomorphism between Morse homology and Floer homology

be used to prove that the corresponding isomorphism on cohomology identifies the quantum cohomology structure on $H^*(M)$ with the pair-of-pants product on Floer cohomology. For details see [38].

3.6 Calabi-Yau manifolds

Let (M, ω) be a compact symplectic manifold whose first Chern class vanishes over $\pi_2(M)$, i.e.

$$\int_{S^2} v^* c_1 = 0 \tag{36}$$

for every smooth map $v : S^2 \to M$. It is an immediate consequence of this condition that the Conley-Zehnder index of a (nondegenerate) periodic solution $x \in \mathcal{P}(H)$ gives rise to a well defined integer

$$\mu_H(x) = n - \mu_{\mathrm{CZ}}(\Psi_x)$$

(see page 23), and hence Floer homology will be graded over the integers. However, care must be taken with bubbling of J-holomorphic spheres of Chern number zero. Such bubbling no longer leads to connecting orbits of strictly lower index. But the good news is that spheres with Chern number zero form subsets of M of codimension 4, and hence will generically avoid the spaces of connecting orbits with index difference 1 or 2, since these form geometrically at most 3-dimensional sets.

More precisely, if c_1 vanishes on $\pi_2(M)$ then, for a generic $J \in \mathcal{J}(M, \omega)$, the moduli space

$$\mathcal{M}^s(J) = \{v : S^2 \to M \; : \; \bar{\partial}_J(v) = 0, \, v \text{ is simple}\}$$

of J-holomorphic spheres which are not multiply covered is a smooth manifold of dimension $2n$. Dividing by the action of the reparametrization group $\mathrm{G} = \mathrm{PSL}(2, \mathbb{C})$ gives a space $\mathcal{M}^s(J)/\mathrm{G}$ of dimension $2n - 6$. Since each sphere is 2-dimenional we obtain a space $\mathcal{W}^s(J) = \mathcal{M}^s(J) \times S^2/\mathrm{G}$ of dimension $2n-4$ with the obvious evaluation map $\mathrm{ev} : \mathcal{W}^s(J) \to M$. The image of this map is the compact codimension-4 subset of all points in M which lie on some J-holomorphic sphere (with Chern number zero). For a generic Hamiltonian $H \in \mathcal{H}_{\mathrm{reg}}$ this set will not intersect the moduli spaces $\mathcal{M}^1(x^-, x^+; H, J)$ and $\mathcal{M}^2(x^-, x^+; H, J)$ of connecting orbits with index 1 or 2. This shows that no bubbling can occur for sequences of such connecting orbits.

With this understood, there is an additional difficulty arising from the presence of possibly infinitely many connecting orbits with index difference 1, with energy diverging to infinity. In other words, the moduli space $\widehat{M}^1(x, y; H, J)$ may not be a finite set but there are finitely many connecting orbits in each homology class. Counting the connecting orbits in their homology classes leads naturally to Floer homology over Novikov rings [19].

3.7 Novikov rings

Continue to assume (36) and let $\Gamma \subset H_2(M) = H_2(M,\mathbb{Z})/\text{torsion}$ be the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$. Associated to the homomorphism $\omega : \Gamma \to \mathbb{R}$ is the Novikov ring $\Lambda = \Lambda_\omega$. This is a kind of completion of the group ring of Γ, reminiscent of the ring of Laurent series. The elements of the Novikov ring Λ are formal sums of the form

$$\lambda = \sum_{A\in\Gamma} \lambda_A e^{2\pi i A}$$

with $\lambda_A \in \mathbb{Z}$, which satisfy the finiteness condition

$$\#\{A \in \Gamma : \lambda_A \neq 0,\, \omega(A) \leq c\} < \infty$$

for every $c > 0$. In other words, for each $c > 0$, there are only finitely many nonzero coefficients λ_A with energy $\omega(A) \leq c$. The ring structure is given by

$$\lambda * \mu = \sum_{A,B} \lambda_A \mu_B e^{2\pi i(A+B)}.$$

Thus $(\lambda * \mu)_A = \sum_B \lambda_{A-B}\mu_B$. It is a simple matter to check that the finiteness condition is preserved under this multiplication.

Remark 3.13 **(i)** The Novikov ring can also be defined if the first Chern class does not vanish over $\pi_2(M)$. In that case the Novikov ring carries a natural grading given by the first Chern class via

$$\deg(e^{2\pi i A}) = 2c_1(A).$$

(ii) If $c_1 \neq 0$ we denote by $\Lambda_k \subset \Lambda$ the subset of all elements of degree k. Then Λ_0 is a ring, but in general multiplication changes the degree via the formula

$$\deg(\lambda * \mu) = \deg(\lambda) + \deg(\mu).$$

In other words, Λ_k is a module over Λ_0. Moreover, multiplication by any element of degree k provides a bijection $\Lambda_0 \to \Lambda_k$. Note that $\Lambda_k \neq \emptyset$ if and only if k is an integer multiple of $2N$ where N is the minimal Chern number of M (defined by $\langle c_1, \pi_2(M)\rangle = N\mathbb{Z}$).

(iii) If $\Gamma = \mathbb{Z}$ then Λ is the ring of Laurent series with integer coefficients. This is a principal ideal domain and if the coefficients are taken in a field then Λ is a field. These observations remain valid when the homomorphism $\omega : \Gamma \to \mathbb{R}$ is injective. (See for example [19].) In the case $\pi_2(M) = \mathbb{Z}$ it is interesting to note the difference between $c_1 = 0$ and $c_1 = [\omega]$. In both cases Λ_ω is the ring of Laurent series but if $c_1 = 0$ then this ring is not graded and in general we cannot exclude the possibility of infinitely many nonzero coefficients.

(iv) Novikov first introduced a ring of the form Λ_ω in the context of his Morse theory for closed 1-forms (cf. [34]). In that case Γ is replaced by the fundamental group and the homomorphism $\pi_1(M) \to \mathbb{R}$ is induced by the closed 1-form. □

Floer homology revisited

It is useful to introduce a covering of the space $\mathcal{L}M$ of contractible loops in M with covering group Γ. For every contractible loop $x : \mathbb{R}/\mathbb{Z} \to M$ choose a smooth map $u : B \to M$ defined on the unit disc $B = \{z \in \mathbb{C} : |z| \le 1\}$ which satisfies $u(e^{2\pi it}) = x(t)$. Two such maps u_1 and u_2 are called **equivalent** if their sum $u_1\#(-u_2)$ represents a torsion homology class. We use the notation $[x, u_1] \sim [x, u_2]$ for equivalent pairs and denote by $\widetilde{\mathcal{L}M}$ the space of equivalence classes. The elements of $\widetilde{\mathcal{L}M}$ will also be denoted by $\tilde{x}$. The space $\widetilde{\mathcal{L}M}$ is the unique covering space of $\mathcal{L}M$ whose group of deck transformations is the image $\Gamma \subset H_2(M)$ of the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$. We denote by

$$\Gamma \times \widetilde{\mathcal{L}M} \to \widetilde{\mathcal{L}M} : (A, \tilde{x}) \mapsto A\#\tilde{x}$$

the obvious action of Γ on $\widetilde{\mathcal{L}M}$. The **symplectic action functional** $a_H : \widetilde{\mathcal{L}M} \to \mathbb{R}$ is defined by

$$a_H([x, u]) = -\int_B u^*\omega - \int_0^1 H_t(x(t))\,dt$$

and satisfies

$$a_H(A\#\tilde{x}) = a_H(\tilde{x}) - \omega(A)$$

for $A \in \Gamma$. Let us denote by $\widetilde{\mathcal{P}}(H) \subset \widetilde{\mathcal{L}M}$ the covering of the set $\mathcal{P}(H)$ of contractible periodic solutions of (1).

The Floer chain complex can now be introduced as the set $CF_*(H)$ of formal sums of the form

$$\xi = \sum_{\tilde{x} \in \widetilde{\mathcal{P}}(H)} \xi_{\tilde{x}} \langle \tilde{x} \rangle$$

which satisfy the finiteness condition

$$\#\left\{\tilde{x} \in \widetilde{\mathcal{P}}(H) \,:\, \xi_{\tilde{x}} \ne 0,\, a_H(\tilde{x}) \ge c\right\} < \infty$$

for every constant $c \in \mathbb{R}$. The Novikov ring Λ_ω obviously acts on $CF_*(H)$ by

$$\lambda * \xi = \sum_{A \in \Gamma} \sum_{\tilde{x} \in \widetilde{\mathcal{P}}(H)} \lambda_A \xi_{\tilde{x}} \langle A\#\tilde{x} \rangle.$$

Thus $(\lambda * \xi)_{\tilde{x}} = \sum_A \lambda_A \xi_{(-A)\#\tilde{x}}$ and the reader may check that these elements still satisfy the required finiteness condition. Now one can proceed as before and define the Floer boundary map

$$\partial^F : CF_*(H) \to CF_*(H)$$

by counting the connecting orbits in $\mathcal{M}(\tilde{y}, \tilde{x}; H, J)$ in the case of index difference 1. The pair $(\tilde{y}, \tilde{x})$ encodes the homology class of the connecting orbits, and in the case of index difference 1 this moduli space is a finite set. The resulting Floer boundary map is then a module homomorphism over the Novikov ring Λ_ω and hence the Floer homology groups are modules over Λ_ω. The analogue of Theorem 3.7 for Calabi-Yau manifolds is that the Floer homology groups of H and J are naturally isomorphic to the singular homology of M with coefficients in the Novikov ring:

$$HF_*(M, \omega, H, J) \cong H_*(M; \Lambda_\omega).$$

The Arnold conjecture for Calabi-Yau manifolds follows immediately from this assertion. For details see Hofer-Salamon [19].

4 Gromov compactness and stable maps

The purpose of this lecture is to discuss how Gromov compactness for J-holomorphic curves leads to Kontsevich's notion of stable maps and to describe a natural topology on the space of stable maps. The last section deals with the Deligne-Mumford compactification of the moduli space of Riemann surfaces of genus zero with marked points.

4.1 Bubbling

Let (M, ω) be a compact symplectic manifold of dimension $2n$ and $J \in \mathcal{J}(M, \omega)$ be a compatible almost complex structure. Suppose that $u : \mathbb{C} \to M$ is a J-holomorphic curve. With coordinates $z = s + it$ on $\mathbb{C}$ this means that u satisfies the PDE

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0.$$

The **energy** of u is defined as the integral

$$E(u) = \int_{\mathbb{C}} |\partial_s u|^2 = \int_{\mathbb{C}} u^*\omega.$$

Throughout we shall only consider J-holomorphic curves with finite energy. In the first section we shall discuss the limit behaviour of sequences with uniformly bounded energy. The following three facts play a central role.

Remark 4.1 **(i)** The **removable singularity theorem** asserts that every J-holomorphic curve $u : \mathbb{C} \to M$ with finite energy extends to $S^2 = \mathbb{C} \cup \{\infty\}$. This means that the function $\mathbb{C} - \{0\} \to M : z \mapsto u(1/z)$ extends to a smooth map on $\mathbb{C}$. A proof can be found in [31].

(ii) By Lemma 3.2, there exists, for every $J \in \mathcal{J}(M, \omega)$, a constant $\hbar = \hbar(M, \omega, J) > 0$ such that

$$E(u) \geq \hbar$$

for every J-holomorphic sphere $u : S^2 \to M$.

(iii) If $u^\nu : \mathbb{C} \to M$ is a sequence of J-holomorphic curves such that

$$\sup_\nu \|du^\nu\|_{L^\infty} < \infty$$

then u^ν has a convergent subsequence. More precisely, the subsequence can be chosen to converge uniformly with all derivatives on compact sets. The limit curve is again a J-holomorphic curve and, if $E(u^\nu) \leq c$ for all ν, then the energy of the limit curve is also bounded by c. This result extends to the case where u^ν is a sequence of J^ν-holomorphic curves and J^ν converges in the C^∞-topology to $J \in \mathcal{J}(M, \omega)$. The proof follows from standard elliptic bootstrapping techniques. For details see [31]. □

Let us now examine the case of a sequence of J-holomorphic curves $u^\nu : \mathbb{C} \to M$ with uniformly bounded energy but unbounded derivatives:

$$\sup_\nu E(u^\nu) < \infty, \qquad \lim_{\nu \to \infty} \|du^\nu\|_{L^\infty} = \infty.$$

Such sequences do exist. Note, in particular, that the condition $\sup_\nu E(u^\nu) < \infty$ is equivalent to a uniform L^2-bound on the first derivatives of u^ν. In contrast, a uniform L^p-bound on the first derivatives for some $p > 2$ would imply a uniform L^∞-bound and hence, by Remark 4.1 (iii), the existence of a convergent subsequence. That such bounds cannot be obtained in the case $p = 2$ has an analytical and a geometric reason. The analytical reason is the fact that $p = 2$ is a borderline case for the Sobolev estimates (see Section 2.3). The geometric reason is the conformal invariance of the energy and the resulting bubbling phenomenon, which was first observed by Sacks and Uhlenbeck in the context of harmonic maps [44]. Here is how this works.

Suppose that $z^\nu \in \mathbb{C}$ is a sequence such that

$$\lim_{\nu\to\infty} |du^\nu(z^\nu)| = \infty, \qquad \lim_{\nu\to\infty} z^\nu = z.$$

Modifying the sequence slightly, without changing its limit, we may assume that there exists a sequence $\delta^\nu > 0$ such that

$$\lim_{\nu\to\infty} \delta^\nu = 0, \qquad \lim_{\nu\to\infty} \delta^\nu |du^\nu(z^\nu)| = \infty, \qquad \sup_{B_{\delta^\nu}(z^\nu)} |du^\nu| \le 2|du^\nu(z^\nu)|.$$

(See the footnote on page 28.) Now consider the rescaled sequence

$$v^\nu(z) = u^\nu(z^\nu + \varepsilon^\nu z), \qquad \varepsilon^\nu = \frac{1}{|du^\nu(z^\nu)|}.$$

This sequence has uniformly bounded derivatives on any compact subset of $\mathbb{C}$. Namely, for any $R > 0$ there exists a ν_R such that $R\varepsilon^\nu \le \delta^\nu$ for $\nu \ge \nu_R$. Hence $|dv^\nu(z)| \le 2$ for $|z| \le R$ and $\nu \ge \nu_R$. By Remark 4.1 (iii), this implies that there exists a subsequence which converges, uniformly with all derivatives on compact subsets of $\mathbb{C}$, to a J-holomorphic curve $v : \mathbb{C} \to M$. This curve has finite energy and so, by Remark 4.1 (i), extends to a J-holomorphic sphere. Since $|dv(0)| = \lim_{\nu\to\infty} |dv^\nu(0)| = 1$, the map v is nonconstant. Hence, by Remark 4.1 (ii), it has energy

$$E(v) \ge \hbar.$$

Now the energy of v^ν in $B_R(0)$ is equal to the energy of u^ν in $B_{R\varepsilon^\nu}(z^\nu)$ which in turn is bounded above by the energy of u^ν in an arbitrarily small ball $B_\varepsilon(z)$ for ν sufficiently large. This means that in the large ν limit the sequence u^ν has an energy of at least $\hbar$ concentrated in an arbitrarily small ball about z. Hence, as in the proof of Proposition 3.3, we have

$$\lim_{\varepsilon\to 0} \liminf_{\nu\to\infty} E(u^\nu, B_\varepsilon(z)) \ge \hbar.$$

This implies that there can be only finitely many points $z_1, \dots, z_\ell$ near which the derivative of u^ν tends to infinity. After passing to a suitable subsequence, we may assume, by Remark 4.1 (iii), that u^ν converges uniformly with all derivatives on compact subset of $\mathbb{C} - \{z_1, \dots, z_\ell\}$ to a J-holomorphic curve u. Invoking the removable singularity theorem, we find that u extends to a J-holomorphic curve on $S^2 = \mathbb{C} \cup \{\infty\}$.

These arguments exhibit the convergence behaviour of a sequence u^ν of J-holomorphic spheres with uniformly bounded energy. They do not, however, give a complete picture of the bubble tree. At each point, near which the derivatives of u^ν blow up, several J-holomorphic spheres may bubble off and, moreover, the collection of all these spheres forms a connected set. To see this it is necessary to refine the above rescaling argument.

Soft rescaling

Let us assume that $u^\nu : \mathbb{C} \to M$ is a sequence of J-holomorphic curves with uniformly bounded energy $E(u^\nu) \le c$ and that a subsequenc has been chosen (still denoted by u^ν) which converges, uniformly with all derivatives on compact subsets of $\mathbb{C} - \{z_1, \dots, z_\ell\}$, to a J-holomorphic curve $u : \mathbb{C} \to M$. Suppose further that the function $z \mapsto |du^\nu(z)|$ attains its maximum in the (small) ball $B_r(z_j)$ at the point $z_j^\nu \to z_j$ and that this maximum tends to ∞. Then the above discussion shows that

$$m(z_j) = \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E(u^\nu, B_\varepsilon(z_j)) \ge \hbar$$

for every j. Here we assume that a subsequence has been chosen such that the relevant limits exist. We shall see that this number $m(z_j)$ is equal to the total energy of all the bubbles splitting off at the point z_j. To capture the *"first"* J-holomorphic sphere bubbling off at z_j^ν we choose $\varepsilon_j^\nu > 0$ such that

$$E(u^\nu, B_{\varepsilon_j^\nu}(z_j^\nu)) = m(z_j) - \frac{\hbar}{2}. \tag{37}$$

By definition of $m(z_j)$, the sequence $\varepsilon_j^\nu > 0$ converges to zero. Now consider the sequence

$$v_j^\nu(z) = u^\nu(z_j^\nu + \varepsilon_j^\nu z).$$

A suitable subsequence, still denoted by v_j^ν, has the following properties (see Figure 12).

(a) The function $z \mapsto |dv_j^\nu(z)|$ takes on its supremum (over a large ball) at 0:

$$|dv_j^\nu(0)| = \sup_{B_{r/\varepsilon_j^\nu}} |dv_j^\nu|.$$

Here we abbreviate $B_\rho = B_\rho(0)$.

(b) The energy of v_j^ν outside the ball of radius 1 is bounded by $\hbar/2$. More precisely, for every $R > 0$ there exists a $\nu_R \in \mathbb{N}$ such that $E(v_j^\nu, B_R - B_1) \le \hbar/2$ for $\nu \ge \nu_R$. This implies that no bubbling can occur outside the unit ball and hence the derivative of v_j^ν on the annulus $B_R - B_{1+\varepsilon}$ is uniformly bounded for every $R > 0$ and every $\varepsilon > 0$.

(c) The sequence v_j^ν converges, modulo further bubbling, to a J-holomorphic sphere $v_j : \mathbb{C} \cup \{\infty\} \to M$. Moreover, the image of the limit sphere v_j is connected to image of the original limit curve u. Namely, $v_j(\infty) = u(z_j)$. A detailed proof can be found in [19, 31, 37].

(d) For every j,

$$m(z_j) = \lim_{R \to \infty} \lim_{\nu \to \infty} E(v_j^\nu, B_R). \tag{38}$$

(e) If the limit curve v_j is constant then, for every $r > 1$,

$$\lim_{\nu \to \infty} E(v_j^\nu, B_r - B_1) = \frac{\hbar}{2}. \tag{39}$$

This means that the energy $\hbar/2$ of the approximating curves v_j^ν in the domain $B_r - B_1$ is concentrated in an arbitrarily small neighbourhood of the unit circle.

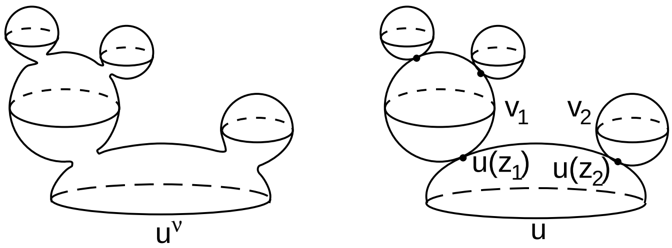

Figure 12: The bubbling phenomenon

To prove (d) one can choose a sequence $\rho_j^\nu \to 0$ such that $E(u^\nu, B_{\rho_j^\nu}(z_j^\nu)) \to m(z_j)$ and examine the quotient $\rho_j^\nu/\varepsilon_j^\nu$. If (38) does not hold then this quotient tends to infinity and one can prove that the energy of u^ν in the annulus $B_{\rho_j^\nu}(z_j^\nu) - B_{\varepsilon_j^\nu}(z_j^\nu)$, which converges to $\hbar/2$, is concentrated in the subset $B_{R\varepsilon_j^\nu}(z_j^\nu) - B_{\varepsilon_j^\nu}(z_j^\nu)$. More precisely, there exists a constant $c > 0$ such that

$$\lim_{\nu\to\infty} E(u^\nu, B_{R\varepsilon_j^\nu}(z_j^\nu) - B_{\varepsilon_j^\nu}(z_j^\nu)) \geq \left(1 - \frac{c}{\log R}\right)\frac{\hbar}{2}.$$

By (37), this implies

$$\lim_{\nu\to\infty} E(v_j^\nu, B_R) = \lim_{\nu\to\infty} E(u^\nu, B_{R\varepsilon_j^\nu}(z_j^\nu)) \geq m(z_j) - \frac{c}{\log R}\frac{\hbar}{2},$$

in contradiction to the assumption that (38) does not hold. This proves (d). Full details are given in [19, 31]. If v is constant then, by (b), the limit on the right hand side of (38) is independent of $R > 1$. Hence (39) follows from (d). In turn it follows from (39) that, for the sequence v_j^ν, bubbling occurs on the unit circle. By (a), this means that there is also a bubble at the origin. In summary, if the limit curve v_j is constant then the sequence v_j^ν exhibits bubbling at two or more points inside the unit ball. This has two crucial consequences. Firstly, since the total mass of all the bubbles of v_j^ν is equal to the mass $m(z_j)$ of the original sequence u^ν, it follows that, if this mass is divided among two bubbles, each part is at most $m(z_j) - \hbar$. This enables us to carry out an induction argument, replacing u^ν by v_j^ν, which must terminate after finitely many steps. Secondly, we observe that, if the limit curve v_j is constant, it is connected at three or more points to other J-holomorphic curves in the bubble tree. Namely, at $z = \infty$ it is connected to u, and at $z = 0$ and some point on the unit circle it is connected to the next bubbles in the induction argument. This observation leads naturally to the notion of a stable map as a tree of J-holomorphic spheres, where each constant sphere is connected to at least three other curves in the tree (at distinct points). This condition ensures that there are only finitely many automorphisms which preserve the curve.

4.2 Stable maps

The concept of a stable map was introduced by Kontsevich [25]. We begin with the definition of a tree as a connected graph without cycles. Think of a tree as a finite set (of vertices) equipped with a relation E such that two vertices are related by E iff they are connected by an edge.

Definition 4.2 *A* **tree** *is a finite set T with a relation $E \subset T \times T$ satisfying the following axioms.*

(symmetric) *If $\alpha E \beta$ then $\beta E \alpha$.*

(anti-reflexive) *If $\alpha E \beta$ then $\alpha \neq \beta$.*

(connected) *For all $\alpha, \beta \in T$ with $\alpha \neq \beta$ there exist $\gamma_0, \ldots, \gamma_m \in T$ with $\gamma_0 = \alpha$ and $\gamma_m = \beta$ such that $\gamma_i E \gamma_{i+1}$ for all i.*

(no cycles) *If $\gamma_0, \ldots, \gamma_m \in T$ with $\gamma_i E \gamma_{i+1}$ and $\gamma_i \neq \gamma_{i+2}$ for all i then $\gamma_0 \neq \gamma_m$.*

A map $f : (T, E) \to (\tilde{T}, \tilde{E})$ is called a **tree homomorphism** *if $f^{-1}(\tilde{\alpha})$ is a tree for all $\tilde{\alpha} \in \tilde{T}$ and, for all $\alpha, \beta \in T$ with $\alpha E \beta$ and $f(\alpha) \neq f(\beta)$, we have $f(\alpha) \tilde{E} f(\beta)$. It is called a* **tree isomorphism** *if it is bijective and both f and f^{-1} are tree homomorphisms.*

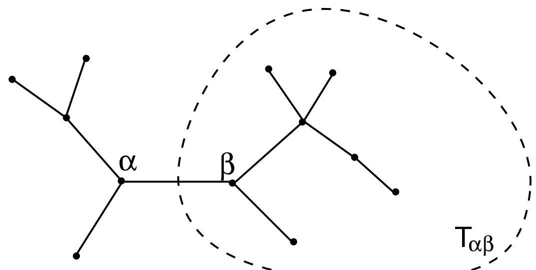

Figure 13: Trees

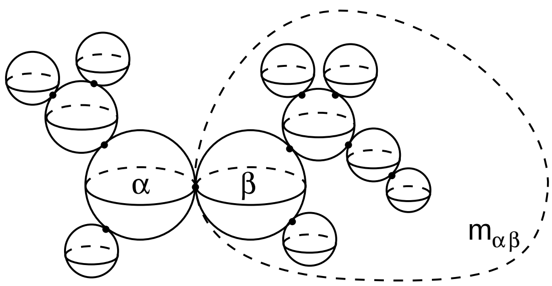

Figure 14: Stable maps

For future reference it is useful to introduce some notation. For every pair $\alpha, \beta \in T$ with $\alpha \neq \beta$ there exists a unique ordered set of vertices $\gamma_0, \ldots, \gamma_m \in T$ such that $\gamma_i E \gamma_{i+1}$, $\gamma_i \neq \gamma_{i+2}$, $\gamma_0 = \alpha$, and $\gamma_m = \beta$. We call this the **chain (of edges) running from** α **to** β and denote the set of vertices belonging to this chain by

$$[\alpha, \beta] = [\beta, \alpha] = \{\gamma_i : i = 0, \ldots, m\}.$$

Cutting any edge $\alpha E \beta$ decomposes the tree T into two components. The component containing β will be denoted by $T_{\alpha\beta}$ and is given by $T_{\alpha\beta} = \{\gamma \in T : \beta \in [\alpha, \gamma]\}$. This set is called a **branch** of the tree T (see Figure 13). Note that T is the disjoint union of $\{\alpha\}$ and the branches $T_{\alpha\beta}$ over all $\beta \in T$ with $\alpha E \beta$. Moreover, $T = T_{\alpha\beta} \cup T_{\beta\alpha}$ whenever $\alpha E \beta$.

Definition 4.3 (Stable maps) *Let* (M, ω) *be a compact symplectic manifold and* $J \in \mathcal{J}(M, \omega)$. *A* **stable J-holomorphic curve of genus zero in M with k marked points**, **modelled over a tree** (T, E), *is a tuple*

$$(\mathbf{u}, \mathbf{z}) = (\{u_\alpha\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le k})$$

with the following properties. For each $\alpha \in T$, $u_\alpha : S^2 \to M$ *is a* J*-holomorphic sphere, for all* $\alpha, \beta \in T$ *with* $\alpha E \beta$, $z_{\alpha\beta} \in S^2$, *and* $(\alpha_1, z_1), \ldots, (\alpha_k, z_k)$ *are finitely many points in* $T \times S^2$, *satisfying the following conditions (see Figure 14).*

(i) *If* $\alpha, \beta \in T$ *with* $\alpha E \beta$ *then* $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$.

(ii) *If* $\alpha E \beta$, $\alpha E \gamma$, *and* $\beta \neq \gamma$, *then* $z_{\alpha\beta} \neq z_{\alpha\gamma}$. *If* $\alpha_i = \alpha_j$ *with* $i \neq j$ *then* $z_i \neq z_j$. *If* $\alpha_i = \alpha$ *and* $\alpha E \beta$ *then* $z_i \neq z_{\alpha\beta}$.

(iii) *If* u_α *is a constant function then the set*

$$Z_\alpha = Z_\alpha(\mathbf{u}, \mathbf{z}) = \{z_{\alpha\beta} : \beta \in T, \alpha E \beta\} \cup \{z_i : 1 \le i \le k, \alpha_i = \alpha\}$$

consists of at least three elements.

If $(\mathbf{u}, \mathbf{z})$ is a stable map then the tree T carries natural weights

$$m_\alpha(\mathbf{u}) = E(u_\alpha) = \int_{S^2} {u_\alpha}^* \omega.$$

The weight m_α can only be zero if the α-sphere carries at least three special points. It is useful to introduce the notation

$$E_\alpha(\mathbf{u}, \Omega) = \int_\Omega {u_\alpha}^* \omega + \sum_{\substack{\alpha E \beta \\ z_{\alpha\beta} \in \Omega}} m_{\alpha\beta}(\mathbf{u}), \qquad m_{\alpha\beta}(\mathbf{u}) = \sum_{\gamma \in T_{\alpha\beta}} E(u_\gamma),$$

for $\alpha, \beta \in T$ with $\alpha E \beta$ and any open set $\Omega \subset S^2$. Then the total energy

$$E(\mathbf{u}) = \sum_{\alpha \in T} E(u_\alpha)$$

of the stable map $(\mathbf{u}, \mathbf{z})$ is equal to $E_\alpha(\mathbf{u}, S^2)$ for any $\alpha \in T$.

There is a natural equivalence relation on the set of stable maps.[6] The equivalence relation is essentially given by complex diffeomorphisms of the domains of the curves which identify the maps, the singular points, and the marked points. Now every complex automorphism of the 2-sphere has the form of a fractional linear transformation (also called **Möbius transformation**)

$$\varphi(z) = \frac{az + b}{cz + d}, \qquad ad - bc = 1.$$

Here the numbers a, b, c, d form a complex 2×2-matrix with determinant 1. Since the Möbius transformations associated to (a, b, c, d) and $(-a, -b, -c, -d)$ are equal, the group of Möbius transformations can be identified with the group

$$\mathrm{G} = \mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C}) / \{\pm \mathbb{1}\}.$$

[6]Strictly speaking, trees do not form a set but a category. So the collection of tuples $(\mathbf{u}, \mathbf{z})$ with the stated properties, which are modelled over trees, is not actually a set. However, if we restrict our definition of trees with m vertices to meaning a relation on the set $\{1, \ldots, m\}$ with the stated properties then the stable maps in M form a set.

Definition 4.4 (Equivalence) *Two stable J-holomorphic curves*

$$(\mathbf{u},\mathbf{z}) = (\{u_\alpha\}_{\alpha\in T}, \{z_{\alpha\beta}\}_{\alpha E\beta}, \{\alpha_i, z_i\}_{1\le i\le k}),$$

$$(\tilde{\mathbf{u}},\tilde{\mathbf{z}}) = (\{\tilde{u}_\alpha\}_{\alpha\in\tilde{T}}, \{\tilde{z}_{\alpha\beta}\}_{\alpha\tilde{E}\beta}, \{\tilde{\alpha}_i, \tilde{z}_i\}_{1\le i\le k})$$

of genus zero in M with k marked points are called **equivalent** *if there exists a tree isomorphism $f : T \to \tilde{T}$ and a collection of Möbius transformations $\varphi = \{\varphi_\alpha\}_{\alpha\in T}$ such that the following holds.*

(i) *For all $\alpha \in T$, $\tilde{u}_{f(\alpha)} = u_\alpha \circ {\varphi_\alpha}^{-1}$.*

(ii) *For all $\alpha, \beta \in T$ with $\alpha E\beta$, $\tilde{z}_{f(\alpha)f(\beta)} = \varphi_\alpha(z_{\alpha\beta})$.*

(iii) *For $i = 1,\dots,k$, $\tilde{\alpha}_i = f(\alpha_i)$ and $\tilde{z}_i = \varphi_{\alpha_i}(z_i)$.*

Definition 4.5 (Gromov convergence) *A sequence*

$$(\mathbf{u}^\nu,\mathbf{z}^\nu) = (\{u^\nu_\alpha\}_{\alpha\in T^\nu}, \{z^\nu_{\alpha\beta}\}_{\alpha E^\nu\beta}, \{\alpha^\nu_i, z^\nu_i\}_{1\le i\le k})$$

of stable J-holomorphic curves with k marked points is said to **Gromov converge** *to a stable J-holomorphic curve $(\mathbf{u},\mathbf{z}) = (\{u_\alpha\}_{\alpha\in T}, \{z_{\alpha\beta}\}_{\alpha E\beta}, \{\alpha_i, z_i\}_{1\le i\le k})$ if, for ν sufficiently large, there exists a surjective tree homomorphism $f^\nu : T \to T^\nu$ and a collection of Möbius transformations $\{\varphi^\nu_\alpha\}_{\alpha\in T}$ such that the following holds.*

(i) *For every $\alpha \in T$ the sequence $u^\nu_{f^\nu(\alpha)} \circ \varphi^\nu_\alpha : S^2 \to M$ converges to u_α, uniformly with all derivatives on compact subsets of $S^2 - Z_\alpha$. Moreover, if $\beta \in T$ such that $\alpha E\beta$, then*

$$m_{\alpha\beta}(\mathbf{u}) = \lim_{\varepsilon\to 0}\lim_{\nu\to\infty} E_{f^\nu(\alpha)}(\mathbf{u}^\nu, \varphi^\nu_\alpha(B_\varepsilon(z_{\alpha\beta}))).$$

(ii) *Let $\alpha, \beta \in T$ such that $\alpha E\beta$ and let ν_j be some subsequence. If $f^{\nu_j}(\alpha) = f^{\nu_j}(\beta)$ for all j then $(\varphi^{\nu_j}_\alpha)^{-1} \circ \varphi^{\nu_j}_\beta$ converges to $z_{\alpha\beta}$, uniformly on compact subsets of $S^2 - \{z_{\beta\alpha}\}$. If $f^{\nu_j}(\alpha) \neq f^{\nu_j}(\beta)$ for all j then $z_{\alpha\beta} = \lim_{j\to\infty}(\varphi^{\nu_j}_\alpha)^{-1}(z^{\nu_j}_{f^{\nu_j}(\alpha)f^{\nu_j}(\beta)})$.*

(iii) *For $i = 1,\dots,k$, $\alpha^\nu_i = f^\nu(\alpha_i)$ and $z_i = \lim_{\nu\to\infty}(\varphi^\nu_{\alpha_i})^{-1}(z^\nu_i)$.*

The previous definition is somewhat complicated and it is useful to record its meaning in the case where the trees T^ν all consist of single points. This means that each $u^\nu : S^2 \to M$ is a single J-holomorphic sphere equipped with k distinct marked points $z^\nu_1,\dots,z^\nu_k$. Such a sequence Gromov converges to a stable map $(\mathbf{u},\mathbf{z}) = (\{u_\alpha\}_{\alpha\in T}, \{z_{\alpha\beta}\}_{\alpha E\beta}, \{\alpha_i, z_i\}_{1\le i\le k})$ iff there exist sequences $\varphi^\nu_\alpha \in \mathrm{G}$ such that the following holds.

(i) For every $\alpha \in T$ the sequence $u^\nu \circ \varphi^\nu_\alpha : S^2 \to M$ converges to u_α uniformly with all derivatives on compact subsets of $S^2 - Z_\alpha$. Moreover, if $\beta \in T$ such that $\alpha E\beta$, then

$$m_{\alpha\beta}(\mathbf{u}) = \lim_{\varepsilon\to 0}\lim_{\nu\to\infty} E(u^\nu \circ \varphi^\nu_\alpha, B_\varepsilon(z_{\alpha\beta})).$$

(ii) If $\alpha, \beta \in T$ with $\alpha E\beta$ then $(\varphi^\nu_\alpha)^{-1} \circ \varphi^\nu_\beta$ converges to $z_{\alpha\beta}$, uniformly on compact subsets of $S^2 - \{z_{\beta\alpha}\}$.

(iii) For $i = 1,\dots,k$, $z_i = \lim_{\nu\to\infty}(\varphi^\nu_{\alpha_i})^{-1}(z^\nu_i)$.

Note that the first two conditions here summarize the convergence behaviour of sequences as discussed in Section 4.1. In that case there are no marked points and the third condition can be ignored. Definition 4.5 is the natural generalization of this concept of convergence to sequences of stable maps modelled on more complicated trees. Here we do not assume that the maps in the sequence are all modelled on the same tree. Although it is always possible to choose a subsequence modelled on the same tree, the definition should allow for sequences of stable map which do not all belong to the same stratum (tree structure) in the space of stable maps. Moreover, if the limit curve has nontrivial automorphisms, then the tree homomorphisms f^ν may not be uniquely determined by the sequence $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ and its limit $(\mathbf{u}, \mathbf{z})$.

Fix a spherical homology class $A \in H_2(M, \mathbb{Z})$ and denote by

$$\widetilde{\mathcal{M}}_{0,k,A} = \widetilde{\mathcal{M}}_{0,k,A}(M, J)$$

the set of stable J-holomorphic curves $(\mathbf{u}, \mathbf{z})$ of genus zero in M with k marked points which represent the class A. The quotient space will be denoted by

$$\mathcal{M}_{0,k,A} = \mathcal{M}_{0,k,A}(M, J) = \widetilde{\mathcal{M}}_{0,k,A}(M, J)/\sim.$$

Thus the elements of $\mathcal{M}_{0,k,A}(M, J)$ are equivalence classes of stable maps in M under the equivalence relation of Definition 4.4.

Definition 4.5 defines a topology on this quotient space, called the **Gromov topology**. A sequence of equivalence classes $[\mathbf{u}^\nu, \mathbf{z}^\nu]$ converges to $[\mathbf{u}, \mathbf{z}]$ in this topology if $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ Gromov converges to $(\mathbf{u}, \mathbf{z})$. A subset $F \subset \mathcal{M}_{0,k,A}(M, J)$ is called **Gromov closed** if the limit of every Gromov convergent sequence in F lies again in F. A subset $U \subset \mathcal{M}_{0,k,A}(M, J)$ is called **Gromov open** if its complement is Gromov closed. That this defines a topology on $\mathcal{M}_{0,k,A}(M, J)$ is obvious. However, that convergence with respect to this topology is equivalent to Gromov convergence is not immediately obvious.

Remark 4.6 Let $(X, \mathcal{U})$ be a topological space in which limits are unique and, for every $x \in X$ and every $A \subset X$, we have $x \in \mathrm{cl}(A)$ if and only if there exists a sequence $x_n \in A$ converging to x. Consider the space

$$\mathcal{C} \subset X \times X^{\mathbb{N}}$$

of all pairs $(x_0, (x_n)_n)$ of elements $x_0 \in X$ and sequences $x_n \in X$ such that x_n converges to x_0. Then the collection $\mathcal{C}$ of convergent sequences has the following properties.

(Constant) If $x_n = x_0$ for all $n \in \mathbb{N}$ then $(x_0, (x_n)_n) \in \mathcal{C}$.

(Subsequence) If $(x_0, (x_n)_n) \in \mathcal{C}$ and $g : \mathbb{N} \to \mathbb{N}$ is strictly increasing, then $(x_0, (x_{g(n)})_n) \in \mathcal{C}$.

(Subsubsequence) If, for every strictly increasing function $g : \mathbb{N} \to \mathbb{N}$, there exists a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $(x_0, (x_{g\circ f(n)})_n) \in \mathcal{C}$, then $(x_0, (x_n)_n) \in \mathcal{C}$.

(Diagonal) If $(x_0, (x_k)_k) \in \mathcal{C}$ and $(x_k, (x_{k,n})_n) \in \mathcal{C}$ for every k then there exist sequences $k_i, n_i \in \mathbb{N}$ such that $(x_0, (x_{k_i,n_i})_i) \in \mathcal{C}$.

(Uniqueness of limits) If $(x_0, (x_n)_n) \in \mathcal{C}$ and $(y_0, (x_n)_n) \in \mathcal{C}$ then $x_0 = y_0$.

The topology can be recovered from the collection $\mathcal{C}$ of convergent sequences, as the collection $\mathcal{U} \subset 2^X$ of all subsets $U \subset X$ such that, for all $x_0 \in U$ and all sequences $x_n \in X$ with $(x_0, (x_n)_n) \in \mathcal{C}$, there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in U$. Starting with a collection of sequences $\mathcal{C} \subset X \times X^{\mathbb{N}}$ one obtains a topology without imposing any axioms. The *"Subsequence"* axiom guarantees that, for every subset

$F \subset X$, $X - F \in \mathcal{U}$ if and only if $x_0 \in F$ whenever there exists a sequence $x_n \in F$ such that $(x_0, (x_n)_n) \in \mathcal{C}$. The *"Constant"* and *"Diagonal"* axioms are needed to show that the closure of a subset $A \subset X$ consists of all elements $x_0 \in X$ for which there exists a sequence $x_n \in A$ such that $(x_0, (x_n)_n) \in \mathcal{C}$. All five axioms are needed to prove that a sequence $x_n \in X$ converges to x_0 with respect to the topology $\mathcal{U}$ if and only if $(x_0, (x_n)_n) \in \mathcal{C}$. Details are left as an exercise. □

It is obvious from the definitions that the collection of all Gromov convergent sequences in $\mathcal{M}_{0,k,A}(M, J)$ satisfies the *"Constant"* and *"Subsequence"* axioms.

Exercise 4.7 Let $(\mathbf{u}, \mathbf{z}) = (\{u_\alpha\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{1 \le i \le k})$ be a stable J-holomorphic curve of genus zero with k marked points and fix a sufficiently small constant $\varepsilon > 0$. For any other stable map $(\mathbf{u}', \mathbf{z}') = (\{u'_\alpha\}_{\alpha' \in T'}, \{z'_{\alpha'\beta'}\}_{\alpha' E' \beta'}, \{\alpha'_i, z'_i\}_{1 \le i \le k})$ with k marked points define the real number

$$
\begin{aligned}
\theta_{\varepsilon,\mathbf{u},\mathbf{z}}(\mathbf{u}', \mathbf{z}') \quad = \quad & \inf_{\substack{f:T \to T' \\ f(\alpha_i) = \alpha'_i}} \inf_{\{\varphi_\alpha\}_{\alpha \in T}} \left\{ \sup_{\alpha} \sup_{S^2 - B_\varepsilon(Z_\alpha)} d\left(u'_{f(\alpha)} \circ \varphi_\alpha, u_\alpha\right) \right. \\
& + \sup_{\alpha E \beta} \left| E_\alpha\left(\mathbf{u}, B_\varepsilon(z_{\alpha\beta})\right) - E_{f(\alpha)}\left(\mathbf{u}', \varphi_\alpha\left(B_\varepsilon(z_{\alpha\beta})\right)\right) \right| \\
& + \sup_{\substack{\alpha E \beta \\ f(\alpha) = f(\beta)}} \sup_{S^2 - B_\varepsilon(z_{\alpha\beta})} d\left(\varphi_\beta^{-1} \circ \varphi_\alpha, z_{\beta\alpha}\right) \\
& \left. + \sup_{\substack{\alpha E \beta \\ f(\alpha) \ne f(\beta)}} d\left(\varphi_\beta^{-1}\left(z'_{f(\beta) f(\alpha)}\right), z_{\beta\alpha}\right) + \sup_{1 \le i \le k} d\left(\varphi_{\alpha_i}^{-1}(z'_i), z_i\right) \right\}.
\end{aligned}
$$

Here the infimum runs over all surjective tree homomorphisms $f : T \to T'$ which satisfy $f(\alpha_i) = \alpha'_i$ for all i and all tuples $\{\varphi_\alpha\}_{\alpha \in T} \in \mathrm{G}^T$. The number $\varepsilon > 0$ is chosen such that $E(u_\alpha, B_\varepsilon(z_{\alpha\beta})) \le \hbar/2$ and $B_\varepsilon(z_{\alpha\beta}) \cap B_\varepsilon(z_{\alpha\gamma}) = \emptyset$ whenever $\alpha E \beta$, $\alpha E \gamma$ and $\beta \ne \gamma$. Note that $E_\alpha(\mathbf{u}, B_\varepsilon(z_{\alpha\beta})) = E(u_\alpha, B_\varepsilon(z_{\alpha\beta})) + m_{\alpha\beta}(\mathbf{u})$.

Prove that a sequence $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ Gromov converges to $(\mathbf{u}, \mathbf{z})$ if and only if the sequence of real numbers $\theta^\nu = \theta_{\varepsilon,\mathbf{u},\mathbf{z}}(\mathbf{u}^\nu, \mathbf{z}^\nu)$ converges to zero. Prove that the function $\theta_{\varepsilon,\mathbf{u},\mathbf{z}}$ is lower semi-continuous with respect to the Gromov topology in the domain $\theta_{\varepsilon,\mathbf{u},\mathbf{z}} < \delta$ for $\delta > 0$ sufficiently small. Deduce that Gromov convergence satisfies the *"Subsubsequence"* and *"Diagonal"* axioms. □

Theorem 4.8 (Uniqueness of limits) *Let $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ be a sequence of stable J-holomorphic curves of genus zero with k marked points which Gromov converges to two stable maps $(\mathbf{u}, \mathbf{z})$ and $(\tilde{\mathbf{u}}, \tilde{\mathbf{z}})$. Then $(\mathbf{u}, \mathbf{z})$ is equivalent to $(\tilde{\mathbf{u}}, \tilde{\mathbf{z}})$.*

Theorem 4.9 (Gromov compactness) *Every sequence $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ of stable J-holomorphic curves of genus zero with k marked points with $\sup_\nu E(\mathbf{u}^\nu) < \infty$ has a Gromov convergent subsequence.*

Theorem 4.10 (Second countable) *The topology of $\mathcal{M}_{0,k,A}(M, J)$ has a countable basis.*

Corollary 4.11 *$\mathcal{M}_{0,k,A}(M, J)$ is a compact metrizable space.*

Proof: By Theorem 4.10, each point in $\mathcal{M}_{0,k,A}(M,J)$ has a countable neighbourhood basis and, by Theorem 4.8, limits are unique. Hence $\mathcal{M}_{0,k,A}(M,J)$ is a Hausdorff space.[7] By Theorem 4.9, the space is sequentially compact. Now every sequentially compact topological space with a countable basis is compact.[8] Hence $\mathcal{M}_{0,k,A}(M,J)$ is a compact Hausdorff space. Hence the result follows from the fact that a compact Hausdorff space is metrizable iff it has a countable basis (cf. Kelley [24]). □

The proof of Theorem 4.9 follows essentially from the arguments in section 4.1 which, in a slightly different form are also contained in [19, 31, 37]. Full details of the proofs of all three theorems 4.8, 4.9, and 4.10 can be found in [21]. It is interesting to discuss the special case, where the target space M is a single point, in more detail. In this case $\mathcal{M}_{0,k,A}(M,J) = \mathcal{M}_{0,k}$ is the Deligne-Mumford compactification of the moduli space of k distinct marked points on the 2-sphere. In this situation Corollary 4.11 can be strengthened to the assertion that $\mathcal{M}_{0,k}$ naturally admits the structure of a compact smooth manifold. This will be discussed in the next section.

Exercise 4.12 Consider the target space $M = S^2$ with the standard complex structure and denote by $L \in H^2(S^2;\mathbb{Z})$ the positive generator. Prove that the moduli space $\mathcal{M}_{0,0,L}(S^2, i)$ is a single point. Prove that there is a bijection

$$\mathcal{M}_{0,0,2L}(S^2, i) \cong \mathbb{C}P^2.$$

Hint: The singular stratum (consisting of equivalence classes of stable maps modelled over a tree with 2 vertices) can be naturally identified with $\mathbb{C}P^1$ via the image of the unique double point under the stable map. The open stratum is the space of equivalence classes of rational maps $u : S^2 \to S^2$ of degree 2 under the equivalence relation

$$u_1 \sim u_2 \qquad \Longleftrightarrow \qquad \exists \varphi \in \mathrm{G} \ni u_2 = u_1 \circ \varphi.$$

Prove that this open stratum can be identified with $\mathbb{C}^2$. Namely, every rational function of degree 2 has two distinct critical values, and this pair of critical values (up to reordering) determines the equivalence class. □

Exercise 4.13 As in the previous exercise consider the target space $M = S^2$ with the standard complex structure. Examine the limit behaviour of the sequence of rational maps

$$u_n(z) = \frac{(z-1+\frac{1}{n})^2}{(z-1)^2}\frac{(z+1+\frac{1}{n})^2}{(z+1)^2}\frac{(z-\frac{1}{n}+\frac{1}{n^2})}{(z-\frac{1}{n})}\frac{(z+\frac{1}{n}+\frac{1}{n^2})}{(z+\frac{1}{n})}$$

of degree six. Find the limit stable map and prove that u_n Gromov converges to your limit. **Hint:** The limit is a stable map, modelled over a tree with six vertices, two each of degree 0, 1, and 2. □

[7] Let x and y be distinct points in a topological space with countable neighbourhood bases and unique limits. Let $\{U_n\}_n$ and $\{V_n\}_n$ be countable neighbourhood bases for x and y, respectively. If $U_n \cap V_n \neq \emptyset$ for all n then any sequence $z_n \in U_n \cap V_n$ converges to both x and y. Hence uniqueness of limits implies that $U_n \cap V_n = \emptyset$ for some n.

[8] Every open cover has a countable refinement $\{U_n\}_{n\in\mathbb{N}}$, consisting of all those elements from the basis which are contained in some element of the cover. Suppose that this refinement does not have a finite subcover. Then there exists a sequence $x_n \in U_n$ such that $x_n \notin U_m$ for $m < n$. Choose a convergent subsequence $x_{n_i} \to x$. Then $x \in U_m$ for some m. Hence there exists an $i_0 \in \mathbb{N}$ such that $x_{n_i} \in U_m$ for $i \geq i_0$. Hence $x_{n_i} \in U_m$ for some $n_i > m$, a contradiction.

4.3 Deligne-Mumford compactification

We begin by repeating the formal definition of a stable map in the simplified context where the target space is a point.

Definition 4.14 *A* **stable Riemann surface of genus zero with** n **marked points, modelled over a tree** (T,E), *is a tuple* $\mathbf{z} = (\{z_{\alpha\beta}\}_{\alpha E\beta}, \{\alpha_i, z_i\}_{1\le i\le n})$ *consisting of points* $z_{\alpha\beta} \in S^2$ *for* $\alpha,\beta \in T$ *with* $\alpha E\beta$ *and pairs* $(\alpha_i, z_i) \in T \times S^2$ *for* $i = 1,\dots,n$ *such that the following holds.*

(i) *If* $\alpha E\beta$, $\alpha E\gamma$, *and* $\beta \ne \gamma$, *then* $z_{\alpha\beta} \ne z_{\alpha\gamma}$. *If* $\alpha_i = \alpha_j$ *with* $i \ne j$ *then* $z_i \ne z_j$. *If* $\alpha_i = \alpha$ *and* $\alpha E\beta$ *then* $z_i \ne z_{\alpha\beta}$.

(ii) *For each* $\alpha \in T$ *the set*

$$Z_\alpha = Z_\alpha(\mathbf{z}) = \{z_{\alpha\beta} \,:\, \beta \in T,\, \alpha E\beta\} \cup \{z_i \,:\, 1 \le i \le n,\, \alpha_i = \alpha\}$$

contains at least three elements.

Definition 4.15 *Two stable Riemann surfaces*

$$\mathbf{z} = (\{z_{\alpha\beta}\}_{\alpha E\beta}, \{\alpha_i, z_i\}_{1\le i\le n}), \qquad \tilde{\mathbf{z}} = (\{\tilde z_{\alpha\beta}\}_{\alpha \tilde E\beta}, \{\tilde\alpha_i, \tilde z_i\}_{1\le i\le n})$$

of genus zero with n marked points are called **equivalent** *if there exists a tree isomorphism* $f : T \to \tilde T$ *and a collection of Möbius transformations* $\varphi = \{\varphi_\alpha\}_{\alpha\in T}$ *such that the following holds.*

(i) *If* $\alpha,\beta \in T$ *with* $\alpha E\beta$ *then* $\tilde z_{f(\alpha)f(\beta)} = \varphi_\alpha(z_{\alpha\beta})$.

(ii) *For* $i = 1,\dots,n$, $\tilde\alpha_i = f(\alpha_i)$ *and* $\tilde z_i = \varphi_{\alpha_i}(z_i)$.

Definition 4.16 *A sequence* $\mathbf{z}^\nu = (\{z^\nu_{\alpha\beta}\}_{\alpha E^\nu\beta}, \{\alpha^\nu_i, z^\nu_i\}_{1\le i\le n})$ *of stable Riemann surfaces of genus zero with n marked points is said to* **DM-converge** *to a stable Riemann surface* $\mathbf{z} = (\{z_{\alpha\beta}\}_{\alpha E\beta}, \{\alpha_i, z_i\}_{1\le i\le n})$ *if, for* ν *sufficiently large, there exists a surjective tree homomorphism* $f^\nu : T \to T^\nu$ *and a collection of Möbius transformations* $\{\varphi^\nu_\alpha\}_{\alpha\in T}$ *such that the following holds.*

(i) *Let* $\alpha,\beta \in T$ *with* $\alpha E\beta$. *If* $f^{\nu_j}(\alpha) \ne f^{\nu_j}(\beta)$ *for some subsequence* ν_j *then*

$$z_{\alpha\beta} = \lim_{j\to\infty} (\varphi^{\nu_j}_\alpha)^{-1}(z^{\nu_j}_{f^{\nu_j}(\alpha)f^{\nu_j}(\beta)}).$$

If $f^{\nu_j}(\alpha) = f^{\nu_j}(\beta)$ *for some subsequence* ν_j *then* $(\varphi^{\nu_j}_\alpha)^{-1} \circ \varphi^{\nu_j}_\beta$ *converges to* $z_{\alpha\beta}$, *uniformly on compact subsets of* $S^2 - \{z_{\beta\alpha}\}$.

(ii) *For* $i = 1,\dots,n$, $\alpha^\nu_i = f^\nu(\alpha_i)$ *and* $z_i = \lim_{\nu\to\infty}(\varphi^\nu_{\alpha_i})^{-1}(z^\nu_i)$.

Consider the moduli space $\mathcal{M}_{0,n}$ of equivalence classes $[\mathbf{z}]$ of stable Riemann surfaces of genus zero with n marked points under the equivalence relation of Definition 4.15. This quotient space inherits a topology from Definition 4.16 as described in the previous section. The goal of this section is to explain how $\mathcal{M}_{0,n}$ naturally admits the structure of a compact smooth manifold. To begin with let us denote by

$$\mathcal{M}^*_{0,n} = \frac{S^2 \times \cdots \times S^2 - \Delta}{\mathrm{G}}$$

the space of equivalence classes of ordered n-tuples of distinct points in S^2 under the diagonal action of the conformal group $\mathrm{G} = \mathrm{PSL}(2,\mathbb{C})$. This quotient is the open stratum in $\mathcal{M}_{0,n}$.[9] We shall use cross ratios to construct an embedding of $\mathcal{M}_{0,n}$ into $(S^2)^N$ for $N = \binom{n}{4}$. Our embedding is reminiscent of a construction by Fulton and MacPherson in [15] for higher dimensional varieties. That paper also contain more references about the moduli space $\mathcal{M}_{0,n}$.

The basic observation is that the relative position of four distinct points on the 2-sphere is, up to complex isomorphisms, determined by the cross ratio. To see this note that, for any three distinct points $z_0, z_1, z_2 \in S^2 = \mathbb{C} \cup \{\infty\}$ there is a unique fractional linear transformation $\varphi \in \mathrm{G}$ which sends z_0 to 0, z_1 to 1, and z_2 to ∞. This transformation is given by

$$\varphi(z_3) = w(z_0, z_1, z_2, z_3) = \frac{(z_1 - z_2)(z_3 - z_0)}{(z_0 - z_1)(z_2 - z_3)}.$$

The cross ratio is well defined for $(z_0, z_1, z_2, z_3) \in (S^2)^4 - \Delta_3$ and satisfies

$$w(z_0, z_1, z_2, z_3) = \begin{cases} \infty, & \text{if } z_0 = z_1 \text{ or } z_2 = z_3, \\ 1, & \text{if } z_0 = z_2 \text{ or } z_3 = z_1, \\ 0, & \text{if } z_0 = z_3 \text{ or } z_1 = z_2. \end{cases} \tag{40}$$

Here $\Delta_3 = \{(z_0, z_1, z_2, z_3) : \exists i < j < k \ni z_i = z_j = z_k\}$ denotes the set of quadruples with three equal points. Moreover, the cross ratio is invariant under the diagonal action of G, meaning that $w(\varphi(z_0), \varphi(z_1), \varphi(z_2), \varphi(z_3)) = w(z_0, z_1, z_2, z_3)$ for $\varphi \in \mathrm{G}$.

Let us now introduce the maps $w_{ijk\ell} : \mathcal{M}_{0,n}^* \to S^2$ given by

$$w_{ijk\ell}(\mathbf{z}) = w(z_i, z_j, z_k, z_\ell) \tag{41}$$

for any four distinct integers $i, j, k, \ell \in \{1, \ldots, n\}$. We claim that these maps extend continuously to $\mathcal{M}_{0,n}$, that they collectively form an injection of $\mathcal{M}_{0,n}$ into $(S^2)^N$, that this injection is a homeomorphism onto its image, and that this image is a smooth (in fact algebraic) submanifold of $(S^2)^N$.

Exercise 4.17 Prove that the maps $w_{ijk\ell} : \mathcal{M}_{0,n}^* \to S^2$ defined by (41) satisfy

$$w_{jik\ell} = w_{ij\ell k} = 1 - w_{ijk\ell}, \qquad w_{ikj\ell} = \frac{w_{ijk\ell}}{w_{ijk\ell} - 1}, \tag{42}$$

$$(1, \infty, w_{ijk\ell}, w_{ijkm}) \notin \Delta_3 \quad \Longrightarrow \quad w_{jk\ell m} = \frac{w_{ijkm} - 1}{w_{ijkm} - w_{ijk\ell}} \tag{43}$$

for any five distinct integers $i, j, k, \ell, m \in \{1, \ldots, n\}$. □

It is useful to introduce the notation

$$\mathcal{I}_n = \left\{(i, j, k, \ell) \in \mathbb{N}^4 \, : \, i, j, k, \ell \text{ are pairwise distinct and } \le n\right\},$$

and write the elements of $(S^2)^{\mathcal{I}_n}$ in the form $w = \{w_I\}_{I \in \mathcal{I}_n}$.

Proposition 4.18 *The maps $w_{ijk\ell} : \mathcal{M}_{0,n}^* \to S^2$ defined by (41) extend to continuous functions $\mathcal{M}_{0,n} \to S^2$, still denoted by $w_{ijk\ell}$. The resulting map*

$$\mathcal{M}_{0,n} \to (S^2)^{\mathcal{I}_n} : \mathbf{z} \mapsto \{w_I(\mathbf{z})\}_{I \in \mathcal{I}_n}$$

is bijective and its image is the set $M_n \subset (S^2)^{\mathcal{I}_n}$ of all tuples $w = \{w_I\}_{I \in \mathcal{I}_n}$ which satisfy (42) and (43).

[9] We denote by $\mathcal{M}_{0,n}$ what others denote by $\overline{\mathcal{M}_{0,n}}$ and by $\mathcal{M}_{0,n}^*$ what others denote by $\mathcal{M}_{0,n}$.

Proposition 4.19 *The map $\mathcal{M}_{0,n} \to (S^2)^{\mathcal{I}_n}$ of Proposition 4.18 is a homeomorphism onto its image.*

Proposition 4.20 *The image of the map $\mathcal{M}_{0,n} \to (S^2)^{\mathcal{I}_n}$ of Proposition 4.18 is a smooth submanifold of $(S^2)^{\mathcal{I}_n}$.*

Below we sketch the main ideas of the proofs. Full details are given in [20]. For any stable Riemann surface $\mathbf{z}$, any $\alpha \in T$, and any $i \in \{1, \ldots, n\}$, denote

$$z_{\alpha i} = \begin{cases} z_i, & \text{if } \alpha_i = \alpha, \\ z_{\alpha\beta}, & \text{if } \alpha_i \in T_{\alpha\beta}. \end{cases} \tag{44}$$

If $\alpha \neq \alpha_i$, then one can think of $z_{\alpha i}$ as the unique singular point on the α-sphere, through which it is connected to the sphere on which z_i lies, namely, the α_i-sphere (see Figure 15).

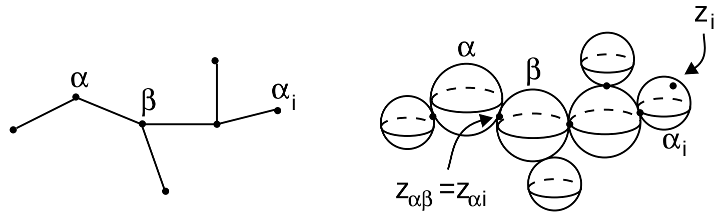

Figure 15: Marked points

Exercise 4.21 Let $\mathbf{z} = (\{z_{\alpha\beta}\}_{\alpha E\beta}, \{\alpha_i, z_i\}_{1\le i\le k})$ be a stable Riemann surface with n marked points. Prove that, if $\alpha, \beta \in T$ with $\alpha E \beta$ and $z_{\alpha i} \neq z_{\alpha\beta}$, then $z_{\beta i} = z_{\beta\alpha}$. □

Proof of Proposition 4.18: The proof consists of four steps. The first two steps are the definition of the map $\mathcal{M}_{0,n} \to (S^2)^{\mathcal{I}_n}$ and their proofs are easy exercises. The third step is injectivity. The fourth step identifies the image with M_n.

Step 1: *Let $\mathbf{z}$ be a stable Riemann surface of genus zero with n marked points. Then for any three distinct indices $i, j, k \in \{1, \ldots, n\}$ there exists a unique vertex $\alpha \in T$ such that $z_{\alpha i} \neq z_{\alpha j} \neq z_{\alpha k} \neq z_{\alpha i}$ (see Figure 16).*

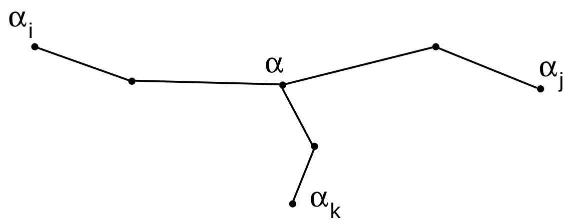

Figure 16: A tree triangle

Step 2: *Let $\mathbf{z}$ be a stable Riemann surface of genus zero with n marked points and suppose that the integers $i, j, k, \ell \in \{1, \ldots, n\}$ are pairwise distinct. Then there exists an $\alpha \in T$ such that $(z_{\alpha i}, z_{\alpha j}, z_{\alpha k}, z_{\alpha \ell}) \notin \Delta_3$. The number*

$$w_{ijk\ell}(\mathbf{z}) := w(z_{\alpha i}, z_{\alpha j}, z_{\alpha k}, z_{\alpha \ell})$$

is independent of the choice of this α.

The existence of α is obvious from Step 1. Moreover, if the points $z_{\alpha i}$, $z_{\alpha j}$, $z_{\alpha k}$, $z_{\alpha \ell}$ are all distinct, then α is unique. If not then $w(z_{\alpha i}, z_{\alpha j}, z_{\alpha k}, z_{\alpha \ell}) \in \{0, 1, \infty\}$ and a simple combinatorial argument shows that this number is independent of α.

Step 3: *Two stable Riemann surfaces $\mathbf{z}$ and $\tilde{\mathbf{z}}$ of genus zero with n marked points are equivalent if and only if $w_I(\mathbf{z}) = w_I(\tilde{\mathbf{z}})$ for all $I = (i, j, k, \ell) \in \mathcal{I}_n$.*

This is proved by induction over the number of vertices. The induction step is to remove an endpoint from the tree. A set of indices $I \subset \{1, \dots, n\}$, corresponding to the marked points on a given endpoint of the tree, can be characterized by the conditions

$$i, i' \in I,\ j, j' \notin I \qquad \Longrightarrow \qquad w_{ii'jj'} = \infty, \tag{45}$$

$$i, i', i'' \in I,\ j \notin I \qquad \Longrightarrow \qquad w_{ii'i''j} \neq \infty. \tag{46}$$

Given any such set one can reduce the number of vertices by replacing $\{1, \dots, n\}$ with $\{1, \dots, n\} - I$ and reordering.

Step 4: *Let $w = \{w_{ijk\ell}\} \in (S^2)^{\mathcal{I}_n}$ be given. Then there exists a stable Riemann surface $\mathbf{z}$ of genus zero with n marked points such that $w_{ijk\ell} = w_{ijk\ell}(\mathbf{z})$ for all i, j, k, ℓ if and only if w satisfies (42) and (43).*

This is again proved by induction over the set of vertices. The key point in the induction is to prove that, for every tuple $w = \{w_{ijk\ell}\}$ which satisfies (42) and (43), there exists a nontrivial subset $I \subset \{1, \dots, n\}$ which satisfies (45) and (46). Here nontrivial means that $I \neq \emptyset$ and $I \neq \{1, \dots, n\}$. For more details see [20]. □

Proof of Proposition 4.19: For any sequence $\mathbf{z}^\nu$ of stable Riemann surfaces of genus zero with n marked points and any $\mathbf{z}$ one proves that the following are equivalent.

(a) $\mathbf{z}^\nu$ DM-converges to $\mathbf{z}$.

(b) For ν sufficiently large there exists a surjective tree homomorphisms $f^\nu : T \to T^\nu$ and Möbius transformations $\varphi_\alpha^\nu \in \mathrm{G}$ such that $f^\nu(\alpha_i) = \alpha_i^\nu$ and

$$z_{\alpha i} = \lim_{\nu \to \infty} (\varphi_\alpha^\nu)^{-1}(z_{f^\nu(\alpha)i}^\nu)$$

for $\alpha \in T$ and $i \in \{1, \dots, n\}$.

(c) For any four distinct integers $i, j, k, \ell \in \{1, \dots, n\}$, $w_{ijk\ell}(\mathbf{z}) = \lim_{\nu \to \infty} w_{ijk\ell}(\mathbf{z}^\nu)$.

The proof of *(a)* $\Longleftrightarrow$ *(b)* $\Longrightarrow$ *(c)* is fairly straight forward. The hard part is to show that (c) implies (b). The key point here is the observation that, for any two stable Riemann surfaces $\mathbf{z}$ and $\mathbf{z}'$ of genus zero with n marked points, there exists a surjective tree homomorphism $f : T \to T'$ with $f(\alpha_i) = \alpha_i'$ for all i if and only if

$$w_{ijk\ell}(\mathbf{z}') = \infty \qquad \Longrightarrow \qquad w_{ijk\ell}(\mathbf{z}) = \infty$$

for all $i, j, k, \ell \in \{1, \dots, n\}$. For more details see [20]. □

Proof of Proposition 4.20: The proof is based on an explicit construction of coordinate charts. Given a point $[\mathbf{z}] \in \mathcal{M}_{0,n}$ we must pick out $n - 3$ of the cross ratios from which all the others can be reconstructed (in a neighbourhood of the point $\{w_I(\mathbf{z})\}_{I \in \mathcal{I}_n}$) as smooth functions of the given $n - 3$ cross ratios. The choice of the $n - 3$ coordinates is geometric. If there exists a marked point z_m such that the corresponing vertex α_m has at least four special points, then there is a crossratio

$w_{i_m j_m k_m m}$ which completely determines the position of the point z_m and hence all other cross ratios involving the point z_m. Now remove the point z_m and proceed by induction until there is no marked point left which lies on a sphere with at least four special points. Next choose a marked point z_m lying on an endpoint $\alpha_m = \alpha$ of the tree. Then there is precisely one other marked point z_{k_m} with $\alpha_{k_m} = \alpha$ and precisely one vertex β with $\alpha E \beta$. The marked points z_{i_m} and z_{j_m} should be chosen such that $z_{\beta i_m} \neq z_{\beta j_m}$ (see Figure 17). Then the cross ratio $w_{i_m j_m k_m m}$ uniquely determines the position of z_m in a neighbourhood of the given stable Riemann surface. Now proceed again by induction to find the required $n-3$ coordinates. For more details see [20]. □

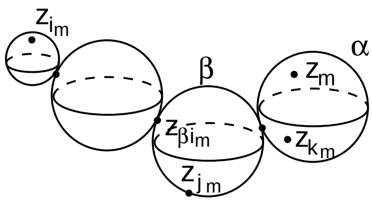

Figure 17: Coordinates for $\mathcal{M}_{0,n}$

Exercise 4.22 Prove that $\mathcal{M}_{0,4} \cong \mathbb{C}P^1$ and

$$\mathcal{M}_{0,5} \cong \mathbb{C}P^2 \# 4\overline{\mathbb{C}P}^2 \cong (S^2 \times S^2) \# 3\overline{\mathbb{C}P}^2.$$

Examine the natural projection $\mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$ and describe it with the above identifications in the case $n = 5$. **Hint 1:** The fiber of the projection is the curve corresponding to the point in $\mathcal{M}_{0,n-1}$. In the case $n = 5$ there are three exceptional fibers, corresponding to the three special points in $\mathcal{M}_{0,4}$. Think of the fibration as a family of quadrics passing through four generic points in the (complex projective) plane. The three singular fibers correspond to the three pairs of lines passing through these points. Blow up the four points to obtain $\mathcal{M}_{0,5}$ (see Figure 18). **Hint 2:** Fix three points at $z_0 = 0$, $z_1 = 1$, $z_2 = \infty$. Show that $\mathcal{M}_{0,5}$ consists of pairs (z_3, z_4) which are not equal to the three pairs $(0,0)$, $(1,1)$, (∞, ∞), together with three 2-spheres representing the possible configurations which arise from a collision of both z_3 and z_4 with z_i for $i = 0, 1, 2$. Examine neighbourhoods of these 2-spheres to obtain the product $S^2 \times S^2$ with three points on the diagonal blown up. □

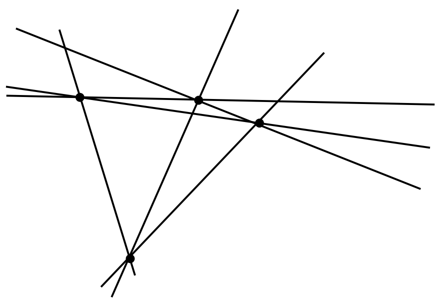

Figure 18: Three pairs of lines determined by four generic points

5 Multi-valued perturbations

In this lecture we return to the proof of the Arnold conjecture. We shall assume throughout that (M, ω) satisfies (4) with a negative factor, i.e.

$$c_1(v) = \tau\omega(v)$$

for all smooth maps $v : S^2 \to M$ and some constant $\tau < 0$, where $c_1(v) = \int v^*c_1$ and $\omega(v) = \int v^*\omega$. This negative monotonicity condition implies that every J-holomorphic sphere in M has negative Chern number. The first section explains the difficulties which arise in the compactness theorem from the presence of multiply covered J-holomorphic spheres with negative Chern number. In the spring of 1996 several groups of researches found methods to overcome these difficulties. The first paper which appeared was by Fukaya-Ono [14]. Other approaches are due to Liu-Tian [28], Ruan [42], Siebert [51], and Hofer-Salamon [20, 21, 22, 23]. The approach discussed here was developed in [22, 23].

Section 5.2 describes an axiomatic setup for multi-valued perturbations of the $\bar{\partial}$-equation, which destroy multiply covered J-holomorphic spheres. The existence of such pertubations is proved in Section 5.3. Section 5.4 deals with the resulting moduli spaces. They are no longer smooth manifolds, but instead are *branched manifolds with rational weights*. Section 5.5 deals with the compactness problem for these moduli spaces. Section 5.6 discusses the definition of the Gromov-Witten invariants, and Section 5.7 describes how the constructions of this lecture lead to Floer homology over the rationals. The analytical details are given in [22, 23].

5.1 J-holomorphic spheres with negative Chern number

Fix an almost complex structure $J \in \mathcal{J}(M, \omega)$ and denote by $\mathcal{M}(k; J)$ the moduli space of J-holomorphic spheres $v : S^2 \to M$ with Chern number $c_1(v) = k$. Under our assumptions this moduli space is only nonempty when $k < 0$. For a generic almost complex structure J the subset

$$\mathcal{M}^s(k; J) \subset \mathcal{M}(k; J)$$

of simple spheres is a smooth manifold of dimension

$$\dim \mathcal{M}^s(k; J) = 2n + 2k.$$

Let us now fix a smooth time dependent Hamiltonian $H_t = H_{t+1} : M \to \mathbb{R}$ and consider a sequence

$$u^\nu \in \mathcal{M}(x^-, x^+; H, J)$$

of Floer connecting orbits, i.e. solutions of (7) and (8). Let us suppose that the index of each u^ν is one, i.e.

$$\mu(u^\nu, H) = 1 = \eta_H(x^-) - \eta_H(x^+) + 2\tau E(u^\nu).$$

This formula shows that the energy $E(u^\nu)$ is independent of ν and hence there exists a subsequence which converges modulo bubbling (Proposition 3.3). Let us consider the simplest nontrivial limit configuration with a single J-holomorphic sphere $v : S^2 \to M$ bubbling off, and the sequence of connecting orbits splitting into a connected sum $u\#v$ where $u \in \mathcal{M}(x^-, x^+; H, J)$ (see Figure 19).

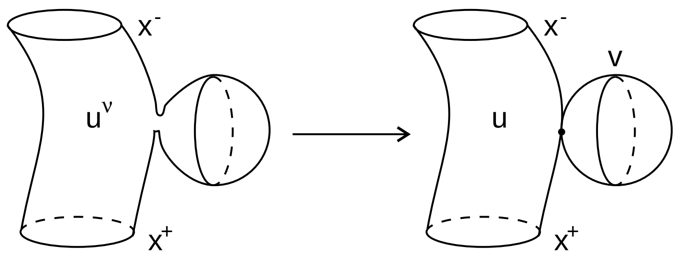

Figure 19: Bubbling for Floer's connecting orbits

The homotopy class of the limit configuration $u\#v$ must agree with that of u^ν for large ν, and hence we obtain

$$\mu(u,H) + 2c_1(v) = \mu(u^\nu, H) = 1.$$

It follows from the Floer-Gromov compactness theorem that the image of v intersects the image of u. Suppose, for example, that $\mu(u,H) = 2k+1$ and $c_1(v) = -k$. Then the set of points lying on simple spheres of Chern number $-k$ is the image of the evaluation map $\mathrm{ev} : \mathcal{M}^s(-k;J) \times S^2/\mathrm{G} \to M$ and hence has codimension $2k+4$ for a generic J. On the other hand the moduli space $\mathcal{M}(x^-,x^+;H,J)/\mathbb{R}$ has dimension $2k$ near u and hence the points on connecting orbits in $\mathcal{M}(x^-,x^+;H,J)$ near u form a subset of M of dimension $2k+2$. Comparing this with the above statement about codimension $2k+4$, we find that, for a generic pair (J,H), the connecting orbits of index $2k+1$ and the simple J-holomorphic spheres of Chern number $-k$ will never meet. Hence the above bubbling cannot occur, generically, provided that v is a simple J-holomorphic sphere. Similar arguments work for arbitrarily complicated bubble trees, again under the assumption that the bubble tree is simple.

Unfortunately, this argument breaks down completely in the case where v is a multiply covered sphere with negative Chern number. In such a case the *actual* dimension of the moduli space $\mathcal{M}(-k;J)$ will be much bigger than the *virtual* dimension, predicted by the index theorem. As a result, we can no longer argue that the image of the connecting orbit u must be disjoint from the image of v. This problem can be resolved by means of a perturbation which destroys the multiply covered J-holomorphic spheres with negative Chern number. The construction of such perturbations will be explained in the next four sections.

Example 5.1 Suppose that (M,ω) is an 8-dimensional symplectic manifold which satisfies (4) with $\tau < 0$ and has minimal Chern number $N = 1$. The simplest such example is a hypersurface of degree 7 in $\mathbb{C}P^5$ (see Exercise 1.5). In this case J-holomorphic spheres of Chern number -1 are isolated and, for a generic almost complex structure $J \in \mathcal{J}(M,\omega)$, there will be finitely many such spheres which cannot be destroyed by a perturbation of J. Suppose that $v : S^2 \to M$ is such a curve and that $f : S^2 \to S^2$ is a rational map of degree $k \geq 2$. Then $v \circ f$ is a J-holomorphic curve of Chern number $c_1(v \circ f) = -k$. Thus the J-holomorphic curves of Chern number $-k$, modulo reparametrization, form a moduli space of dimension $\dim \mathrm{Rat}_k - \dim \mathrm{G} = 4k-4$ while the virtual dimension is $2-2k < 0$. □

5.2 Multi-valued perturbations

A smooth map $v : S^2 \to M$ is a J-holomorphic sphere if $\bar{\partial}_J(v) = 0$, where

$$\bar{\partial}_J(v) = \frac{1}{2}(dv + J \circ dv \circ i) \in \Omega^{0,1}(S^2, v^*TM).$$

From an abstract point of view there is an infinite dimensional vector bundle $\mathcal{E} \to \mathcal{B}$ over the space $\mathcal{B} = \mathrm{Map}(S^2, M)$ with fibers $\mathcal{E}_v = \Omega^{0,1}(S^2, v^*TM)$, the nonlinear operator $\bar{\partial}_J : \mathcal{B} \to \mathcal{E}$ is a section of this bundle, and the J-holomorphic spheres are the zeros of this section. This section has the following crucial properties.

(i) $\bar{\partial}_J$ is a Fredholm section. This means that the vertical differential

$$D_v = D\bar{\partial}_J(v) : T_v\mathcal{B} = C^\infty(S^2, v^*TM) \longrightarrow E_v = \Omega^{0,1}(S^2, v^*TM)$$

is a Fredholm operator between appropriate Sobolev completions. Its Fredholm index is given by $\mathrm{index}\, D_v = 2n + 2c_1(v)$, where $2n = \dim M$.

(ii) For a generic almost complex structure J the restriction of $\bar{\partial}_J$ to the subset $\mathcal{B}^s$ of simple maps is transverse to the zero section. This means that the vertical differential D_v is surjective for every simple J-holomorphic sphere v.

(iii) There is an action of the group $\mathrm{G} = \mathrm{PSL}(2, \mathbb{C})$ of Möbius transformations on both $\mathcal{B}$ and $\mathcal{E}$. The section $\bar{\partial}_J$ is equivariant under this action, i.e.

$$\bar{\partial}_J(v \circ \varphi) = \varphi^* \bar{\partial}_J(v)$$

for $v \in \mathcal{B}$ and $\varphi \in \mathrm{G}$.

As noted in the previous section, the transversality statement does not extend to all of $\mathcal{B}$. Hence a different perturbation $\gamma : \mathcal{B} \to \mathcal{E}$ must be found to make $\bar{\partial}_J - \gamma$ transverse to the zero section over all of $\mathcal{B}$. This perturbation must satisfy all three requirements of transversality, equivariance, and of being *"lower order"*, so that the perturbed equation is still Fredholm with the same index. Unfortunately, all three requirements cannot be fulfilled, in general, by single-valued perturbations. This phenomenon is already apparent in the finite dimensional case where transversality cannot, in general, be achieved while preserving equivariance. It will, however, be possible to achieve transversality by choosing an equivariant multi-valued perturbation

$$\Gamma : \mathcal{B} \to 2^{\mathcal{E}}.$$

Thus, for each $v \in \mathcal{B}$, $\Gamma(v)$ is a finite subset of $\mathcal{E}_v = \Omega^{0,1}(S^2, v^*TM)$. The perturbed Cauchy-Riemann equations take the form

$$\bar{\partial}_J(v) \in \Gamma(v). \tag{47}$$

The solutions of this differential inequality will be called **(J, Γ)-holomorphic curves**. To ensure that the space of such curves is still invariant under G we shall assume that Γ is equivariant.

Suppose, for example, that an arbitrarily small multi-valued perturbation of this form is introduced in a neighbourhood of a perfectly regular J-holomorphic curve. Then this curve will split up into finitely many solutions of (47), depending on the number of branches of Γ, and the union of these curves should be counted as one curve. Alternatively, each of the individual solutions should be counted with some weight so that the sum of the weights is 1. This motivates the introduction of **positive rational weights** $\lambda(v,\eta) > 0$ for $\eta \in \Gamma(v)$ such that

$$\sum_{\eta \in \Gamma(v)} \lambda(v,\eta) = 1$$

for every $v \in \mathcal{B}$. Call the pair (Γ, λ) a **weighted multi-valued G-perturbation**. Such perturbations form a semigroup with sum $(\Gamma,\lambda) = (\Gamma_1 + \Gamma_2, \lambda_1 * \lambda_2)$ defined by

$$\Gamma(v) = \{\eta_1 + \eta_2 \,:\, \eta_i \in \Gamma_i(v)\}, \qquad \lambda(v,\eta) = \sum_{\eta_1+\eta_2=\eta} \lambda_1(v,\eta_1)\cdot\lambda_2(v,\eta_2).$$

Here we use the convention $\lambda(v,\eta) = 0$ for $\eta \notin \Gamma(v)$. With this convention, Γ is uniquely determined by λ. In fact, one can think of (Γ, λ) as a collection of discrete measures on the fibers $\mathcal{E}_v$ of our infinite dimensional vector bundle. We shall impose the following conditions on the multi-valued perturbations. The first two were already mentioned above.

(Finiteness) $\Gamma(v)$ is a finite subset of $\mathcal{E}_v$ for every $v \in \mathcal{B}$. The weight function $\Gamma(v) \to \mathbb{Q} : \eta \mapsto \lambda(v,\eta)$ is positive and satisfies $\sum_{\eta\in\Gamma(v)} \lambda(v,\eta) = 1$ for every $v \in \mathcal{B}$.

(Conformality) For $v \in \mathcal{B}$, $\eta \in \mathcal{E}_v$, and $\varphi \in \mathrm{G}$

$$\Gamma(v\circ\varphi) = \varphi^*\Gamma(v), \qquad \lambda(v,\eta) = \lambda(v\circ\varphi, \varphi^*\eta).$$

(Energy) There is a constant $c = c(\Gamma) > 0$ such that, for all $v \in \mathcal{B}$ and all $\eta \in \mathcal{E}_v$,

$$\eta \in \Gamma(v) \qquad \Longrightarrow \qquad \int_{S^2} |\eta|^2 \mathrm{dvol}_{S^2} \le c.$$

(Local structure) For every $u \in \mathcal{B}$ there exists a C^0-neighbourhood $\mathcal{U}$ of u, finitely many continuous sections $\gamma_i : \mathcal{U} \to \mathcal{E}$, $i = 1, \ldots, m$, and positive rational numbers $\lambda_1, \ldots, \lambda_m$ such that $\lambda_1 + \cdots + \lambda_m = 1$ and

$$\Gamma(v) = \{\gamma_1(v), \ldots, \gamma_m(v)\}, \qquad \lambda(v,\eta) = \sum_{\gamma_i(v)=\eta} \lambda_i$$

for $v \in \mathcal{U}$ and $\eta \in \mathcal{E}_v$. The γ_i are called the **branches** of Γ and λ_i is called the **weight** of γ_i. We assume that the branches satisfy the following.

- Each γ_i restricts to a C^ℓ-function from $\mathcal{U}^{k,p} \to \mathcal{E}^{k-\ell,p}$ for $0 \le \ell \le k$.[10]
- The vertical differential $D\gamma_i(v) : T_v\mathcal{B}^{k,p} \to \mathcal{E}_v^{k-1,p}$ is a compact operator for $v \in \mathcal{U}^{k,p}$ and $i = 1, \ldots, m$.[11]

[10]The superscripts denote Sobolev completions. Thus $\mathcal{E}_v^{k,p} = W^{k,p}(S^2, \Lambda^{0,1}T^*S^2 \otimes v^*TM)$, and $\mathcal{U}^{k,p} = \mathcal{U} \cap \mathcal{B}^{k,p}$ is an open subset of $\mathcal{B}^{k,p} = W^{k,p}(S^2, M)$.

[11]The **vertical differential** of a continuously differentiable section $\gamma : \mathcal{B}^{k,p} \to \mathcal{E}^{j,p}$ (with $j \le k$) is defined as the linear operator $D\gamma(v) : T_v\mathcal{B}^{k,p} \to \mathcal{E}_v^{j,p}$ defined by

$$D\gamma(v)\xi = \left.\frac{d}{dt}\right|_{t=0} \Phi_v(t\xi)^{-1}\gamma(\exp_v(t\xi))$$

for $\xi \in W^{k,p}(S^2, v^*TM)$. Here $\Phi_v(\xi) : v^*TM \to \exp_v(\xi)^*TM$ denotes parallel transport with respect to the Hermitian connection induced by J. If ∇ denotes the Levi-Civita connection of the metric $\langle\cdot,\cdot\rangle = \omega(\cdot, J\cdot)$ then the Hermitian connection is defined by $\nabla^J = \nabla - \frac{1}{2}J\nabla J$.

(Transversality) If $v : S^2 \to M$ is a (J,Γ) holomorphic curve and $\bar\partial_J(v) = \gamma_i(v)$ then the vertical differential $D_{i,v} = D\bar\partial_J(v) - D\gamma_i(v)$ is surjective.

(Free) If $v : S^2 \to M$ is a (J,Γ) holomorphic curve and $\varphi \in \mathrm{G}$ such that $v \circ \varphi = v$ then $\varphi = \mathrm{id}$.

Perhaps the most surprising condition here is the relatively complicated form of the *"local structure"* axiom. The reason for this formulation is the fact that the action of G on $\mathcal{B}^{k,p}$ is not smooth. To be more precise, let $v : S^2 \to M$ be a $W^{k,p}$-function and $t \mapsto \varphi_t$ be a smooth path in G. Then the function $\mathbb{R} \to \mathcal{B}^{k,p} : t \mapsto v_t = v \circ \varphi_t$ is only continuous but not differentiable, because differentiating with respect to t we obtain a vector field along v_t of class $W^{k-1,p}$ but not in $W^{k,p}(S^2, {v_t}^*TM) = T_{v_t}\mathcal{B}^{k,p}$. In other words, the function

$$\mathcal{B}^{k,p} \times \mathrm{G} \to \mathcal{B}^{k,p} : (v, \varphi) \mapsto v \circ \varphi$$

is only continuous, but the function $\mathcal{B}^{k,p} \times \mathrm{G} \to \mathcal{B}^{k-\ell,p} : (v,\varphi) \mapsto v \circ \varphi$ is of class C^ℓ for $0 \le \ell \le k$. This differentiability of the action at the expense of the loss of differentibility of v is inherited by our perturbation. The compactness of the vertical differential $D\gamma_i(v)$ follows essentially from the fact that the group G is finite dimensional.

Remark 5.2 Since the space $\mathcal{B} = \mathrm{Map}(S^2, M)$ is separable (with the C^0-topology) it can be covered by countably many open sets $\mathcal{U}(j)$ which satisfy the requirements of the *"local structure"* axiom. Hence there exists a collection of local sections $\gamma_i : \mathcal{U}_i \to \mathcal{E}$ and rational numbers $\lambda_i > 0$, indexed by a countable set I, and a decomposition of the index set $I = \bigcup_j I(j)$ into finite sets, such that each γ_i satisfies the smoothness requirement of the *"local structure"* axiom, $\mathcal{U}_i = \mathcal{U}(j)$ for $i \in I(j)$, $\sum_{i \in I(j)} \lambda_i = 1$, and $\Gamma(v) = \{\gamma_i(v) \,:\, i \in I(j)\}$, $\lambda(v,\eta) = \sum_{\substack{i\in I(j)\\ \gamma_i(v)=\eta}} \lambda_i$ for $v \in \mathcal{U}(j)$ and $\eta \in \mathcal{E}_v$. □

We shall have to deal with three problems: the existence of perturbations which satisfy the above axioms, the properties of the resulting moduli spaces, and the compactness problem for sequences of (J,Γ)-holomorphic curves. These will be discussed in the next three sections.

5.3 Local slices

The goal of this section is to prove the existence of perturbations Γ which satisfy the above requirements. Our construction is based on local slices for the G-action on $\mathcal{B}$. Let us fix a smooth function $u : S^2 \to M$ with finite isotropy subgroup

$$\mathrm{G}_u = \{\varphi \in \mathrm{G} \,:\, u \circ \varphi = u\}\,.$$

The tangent space of the (6-dimensional) orbit $u \cdot \mathrm{G} = \{u \circ \varphi \,:\, \varphi \in \mathrm{G}\}$ at u is given by

$$\mathrm{Vert}_u = \{du \circ \theta \,:\, \theta \in \mathrm{Lie}(\mathrm{G})\}\,.$$

Here we think of θ as a vector field on S^2, tangent to a 1-parameter subgroup of G. The space Vert_u is invariant under the obvious action of the isotropy subgroup G_u by $\xi \mapsto \xi \circ \varphi$. We shall choose a complement of Vert_u which is also invariant under the action of G_u, by fixing a volume form $\omega_u \in \Omega^2(S^2)$ which is invariant under G_u, and defining

$$\mathrm{Hor}_u = \left\{\xi \in C^\infty(S^2, u^*TM) \,:\, \int_{S^2} \langle \xi, du \circ \theta \rangle \omega_u = 0 \,\forall \theta \in \mathrm{Lie}(\mathrm{G})\right\}.$$

If we extend the notation ω_u to the G-orbit of u via $\omega_{u\circ\varphi} = \varphi^*\omega_u$ for $\varphi \in \mathrm{G}$ then

$$\mathrm{Hor}_{u\circ\varphi} = \mathrm{Hor}_u \circ \varphi, \qquad \mathrm{Vert}_{u\circ\varphi} = \mathrm{Vert}_u \circ \varphi.$$

Here we have introduced the horizontal spaces as Frechét spaces. In the following we shall denote the relevant Sobolev completions by $\mathrm{Hor}_u^{k,p}$. The next Lemma asserts the existence of multi-valued local slices for the G-action.

Lemma 5.3 *Let $u : S^2 \to M$ be a smooth function with finite isotropy subgroup G_u of order m. Then there exists a G-invariant open neighbourhood $\mathcal{U} = \mathcal{U}^{k,p} \subset W^{k,p}(S^2, M)$ of u and a continuous map*

$$\rho : \mathcal{U}^{k,p} \to 2^{\mathrm{Hor}_u^{k,p} \times \mathrm{G}}$$

with the following properties (see Figure 20).

(i) *For each $v \in \mathcal{U}^{k,p}$ the set $\rho(v)$ consists of precisely m elements and the union of the sets $\rho(v)$ over all $v \in \mathcal{U}^{k,p}$ is an open neighbourhood of $\{0\} \times \mathrm{G}$ in $\mathrm{Hor}_u^{k,p} \times \mathrm{G}$.*

(ii) *If $v \in \mathcal{U}^{k,p}$ and $\xi \in \mathrm{Hor}_u^{k,p}$ is sufficiently small then*

$$(\xi, \psi) \in \rho(v) \qquad \Longleftrightarrow \qquad v = \exp_u(\xi) \circ \psi.$$

(iii) *Locally near every $v \in \mathcal{U}^{k,p}$ the branches ρ_i of ρ are ℓ times continuously differentiable as maps from (a neighbourhood ov v in) $\mathcal{U}^{k,p}$ to $\mathrm{Hor}_u^{k-\ell,p} \times \mathrm{G}$. Moreover, the differential*

$$d\rho_i(v) : W^{k,p}(S^2, v^*TM) \to \mathrm{Hor}_u^{k-1,p} \times T_{\psi_i}\mathrm{G}$$

is a compact operator for every $v \in \mathcal{U}^{k,p}$ and every i. Here $\psi_i \in \mathrm{G}$ is defined by $\rho_i(v) \in \mathrm{Hor}_u^{k-1,p} \times \{\psi_i\}$.

(iv) *If $v \in \mathcal{U}^{k,p}$ and $(\xi, \psi) \in \rho(v)$ then*

$$\mathrm{G}_v = \left\{\psi^{-1} \circ \varphi \circ \psi \,:\, \varphi \in \mathrm{G}_u,\, \xi \circ \varphi = \xi\right\}.$$

Proof: This result follows from the the implicit function theorem for the map

$$\mathrm{Hor}_u^{k,p} \times \mathrm{G} \to \mathcal{B}^{k,p} : (\xi, \psi) \mapsto \exp_u(\xi) \circ \psi$$

which maps $\{0\} \times \mathrm{G}_u$ to u. Details can be found in [22]. □

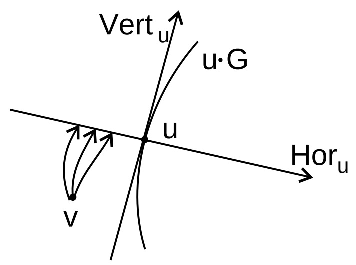

Figure 20: Local slices for the G-action on $\mathrm{Map}(S^2, M)$

Exercise 5.4 Let $\rho : \mathcal{U}^{k,p} \to 2^{\mathrm{Hor}_u^{k,p} \times \mathrm{G}}$ be the local slice of Lemma 5.3, and fix an element $v \in \mathcal{U}^{k,p}$. If $(\xi, \psi) \in \rho(v)$, prove that

$$\rho(v) = \left\{ (\xi \circ \varphi^{-1}, \varphi \circ \psi) \; : \; \varphi \in \mathrm{G}_u \right\}. \qquad \square$$

Corollary 5.5 *Let $u \in \mathrm{Map}(S^2, M)$ be a smooth function with finite isotropy subgroup G_u of order m. Then for every $\eta \in \Omega^{0,1}(S^2, u^*TM)$ there exists a weighted multi-valued G-section (Γ, λ) which satisfies the "finiteness", "conformality", "energy", and "local structure" axioms, as well as*

$$\Gamma(u) = \{\varphi^*\eta \; : \; \varphi \in \mathrm{G}_u\}, \qquad \lambda(u, \eta) = \frac{\#\{\varphi \in \mathrm{G}_u \; : \; \varphi^*\eta = \eta\}}{m}.$$

Proof: Let $\rho : \mathcal{U}^{k,p} \to 2^{\mathrm{Hor}_u^{k,p} \times \mathrm{G}}$ be the local slice of Lemma 5.3, fix an element $v \in \mathcal{U}^{k,p}$, and write

$$\rho(v) = \{(\xi_1, \psi_1), \ldots, (\xi_m, \psi_m)\}.$$

Then $v = \exp_u(\xi_i) \circ \psi_i$ for all i. Let $\Phi_u(\xi) : u^*TM \to \exp_u(\xi)^*TM$ be given by parallel transport as in the footnote on page 60. Then

$$(\Phi_u(\xi_i)\eta) \circ d\psi_i \in \mathcal{E}_v$$

for each i. Now choose a smooth cutoff function $\beta : \mathbb{R} \to [0, 1]$ which is equal to 1 in $[-1/2, 1/2]$ and equal to zero outside $[-1, 1]$. Then define $\beta_{\varepsilon,p} : \mathrm{Hor}_u \to [0, 1]$ by

$$\beta_{\varepsilon,p}(\xi) = \beta\left(\int_{S^2} \left(\left(\varepsilon^{-2}|\xi|^2 + \varepsilon\right)^{p/2} + \sum_{\varphi \in \mathrm{G}_u} \left(\varepsilon^{-2}|\nabla(\xi \circ \varphi)|^2 + \varepsilon\right)^{p/2} \right) \omega_u \right).$$

This is a smooth G_u-invariant cutoff function on $\mathrm{Hor}_u^{1,p}$ vanishing outside the ε-neighbourhood of zero. Next define

$$\Gamma(v) = \{\zeta_1, \ldots, \zeta_m\}, \qquad \lambda(v, \zeta) = \frac{\#\{i \; : \; \zeta_i = \zeta\}}{m},$$

where $\zeta_i = \beta_{\varepsilon,p}(\xi_i)(\Phi_u(\xi_i)\eta) \circ d\psi_i \in \mathcal{E}_u$ for $i = 1, \ldots, m$. This perturbation has the required properties. In particular, note that $\zeta_i = \zeta_{i'}$ if and only if $\varphi^*\eta = \eta$ where $\varphi = \psi_{i'} \circ {\psi_i}^{-1} \in \mathrm{G}_u$. Hence the number of distinct branches is the quotient m/m_η, where $m_\eta = \#\{\varphi \in \mathrm{G}_u \; : \; \varphi^*\eta = \eta\}$, and two distinct branches agree precisely on the zero set of the cutoff function $\beta_{\varepsilon,p}$ (which is connected and contained in the closure of its interior). $\square$

So far we have constructed a perturbation which satisfies the *"finiteness"*, *"conformality"*, *"energy"*, and *"local structure"* axioms, but not the *"transversality"* and *"free"* axioms. To find a perturbation which also satisfies the *"transversality"* axiom it is useful to construct a sufficiently large family of perturbations, parametrized by a separable Hilbert space H, by superposition. This is possible because of the semigroup property of the class of perturbations considered above. The family of perturbations takes the form of a pair of maps

$$H \times \mathcal{B} \to 2^{\mathcal{E}} : (h, v) \mapsto \Gamma_h(v), \qquad H \times \mathcal{E} \to \mathbb{Q} : (h, v, \eta) \mapsto \lambda_h(v, \eta),$$

such that, for each h, the pair (Γ_h, λ_h) satisfies the *"finiteness"*, *"conformality"*, *"energy"*, and *"local structure"* axioms, In addition, the branches $H \times \mathcal{U}_i \to \mathcal{E} : (h, v) \mapsto$

$\gamma_{i,h}(v)$ are required to be linear in h for each $v \in \mathcal{U}_i$, The crucial condition is that the linear operator

$$T_v\mathcal{B}^{k,p} \times H \to \mathcal{E}_v^{k-1,p} : (\xi, \hat{h}) \mapsto D\bar{\partial}_J(v)\xi - D\gamma_{i,h}(v)\xi - \gamma_{i,\hat{h}}(v)$$

is surjective whenever $\bar{\partial}_J(v) = \gamma_{i,h}(v)$ and $\|h\| < \varepsilon$ (for a sufficiently small number $\varepsilon > 0$). Under these assumptions the universal moduli spaces

$$\mathcal{M}_{i,\varepsilon}(J, \{\Gamma_h\}_h) = \left\{(v, h) \in \mathcal{U}_i \times H \,:\, \|h\| < \varepsilon,\, \bar{\partial}_J(v) = \gamma_{i,h}(v)\right\}$$

are all smooth Hilbert manifolds. Now choose a common regular value h of the projections

$$\mathcal{M}_{i,\varepsilon}(J, \{\Gamma_h\}_h) \to H$$

with $\|h\| < \varepsilon$ to obtain a perturbation (Γ_h, λ_h) which satisfies the *"transversality"* axiom. That such a regular value exists follows from the Sard-Smale theorem. That a family of perturbations $\{\Gamma_h\}_h$ with the above properties exists, follows from Corollary 5.5 via superposition.

To construct a perturbation which satisfies the *"free"* axiom we use a similar argument. We note that, if $v \circ \varphi = v$ for some $\varphi \in \mathrm{G} - \{\mathrm{id}\}$ then, for every point $z \in S^2$, there exists a point $z' \in S^2 - \{z\}$ with $v(z) = v(z')$. The goal is to show that this does not happen generically for the solutions of (47). To see this consider the universal spaces $\mathcal{X}_i$ consisting of tuples $(v, z_1, \ldots, z_m, z_1', \ldots, z_m', J, h)$ such that $v \in \mathcal{U}_i$, $J \in \mathcal{J}(M, \omega)$, $h \in H$, $z_j, z_j' \in S^2$ with $z_j \neq z_j'$, and

$$\bar{\partial}_J(v) = \gamma_{i,h}(v), \qquad v(z_1) = v(z_1'), \ldots, v(z_m) = v(z_m').$$

One can use the techniques of [31] to prove that the spaces $\mathcal{X}_i$ are Banach manifolds and the projections

$$\mathcal{X}_i \to \mathcal{J}(M, \omega) \times H$$

are Fredholm maps of index $2n + 2c_1(A_i) + m(4 - 2n)$, where $A_i \in H_2(M, \mathbb{Z})$ is the homology class of $v \in \mathcal{U}_i$. If $n \geq 3$ and $m = m_i$ is chosen sufficiently large, then this index is negative. Now let (J, h) be a common regular value of these projections. Then every solution $v \in \mathcal{U}_i$ of $\bar{\partial}_J(v) \in \Gamma_h(v)$ has less than m_i double points. This implies that $v \circ \varphi \neq v$ for every $\varphi \in \mathrm{G} - \{\mathrm{id}\}$. Hence, for any such common regular value (J, h), the pair (J, Γ_h) satisfies the *"free"* axiom. Full details of these arguments are given in [22].

5.4 Branched moduli spaces

Let (Γ, λ) be a weighted multi-valued perturbation with branches $\gamma_i : \mathcal{U}_i \to \mathcal{E}$ which satisfies all the axioms in Section 5.2. Fix a spherical homology class $A \in H_2(M, \mathbb{Z})$ and consider the moduli space

$$\mathcal{M} = \mathcal{M}(A; J, \Gamma) = \left\{v : S^2 \to M \,:\, v_*[S^2] = A,\, \bar{\partial}_J(v) \in \Gamma(v)\right\}.$$

This space carries a natural rational label $\mathcal{M}(A; J, \Gamma) \to \mathbb{Q} : v \mapsto \lambda(v) = \lambda(v, \bar{\partial}_J(v))$. It can be expressed as a union of countably many branches

$$\mathcal{M}_i = \mathcal{M}(A; J, \gamma_i) = \left\{v \in \mathcal{U}_i \,:\, v_*[S^2] = A,\, \bar{\partial}_J(v) = \gamma_i(v)\right\}.$$

In other worde, $\mathcal{M}_i$ is the zero set of the section $\bar{\partial}_J - \gamma_i$. The *"transversality"* axiom asserts that this section is transverse to the zero section of $\mathcal{E}$ and it follows from the implicit function theorem and the *"local structure"* axiom that $\mathcal{M}_i$ is a smooth manifold of dimension

$$\dim \mathcal{M}_i = \operatorname{index} D_{i,v} = 2n + 2c_1(A).$$

Here we abbreviate $D_{i,v} = D\bar{\partial}_J(v) - D\gamma_i(v) : T_v\mathcal{B}^{k,p} \to \mathcal{E}_v^{k-1,p}$. This operator is the vertical differential of the section $\bar{\partial}_J - \gamma_i$ at v (see the footnote on page 60). The *"local structure"* axiom guarantees that this operator is Fredholm and its index agrees with that of $D_v = D\bar{\partial}_J(v)$. In summary, the moduli space $\mathcal{M} = \mathcal{M}(A; J, \Gamma)$ is a union of countably many smooth manifolds, called the **branches of** $\mathcal{M}$. For a generic perturbation Γ it is a branched manifold in the following sense.

Definition 5.6 *A* **branched** m**-manifold** *is a pair (M, λ), consisting of a Hausdorff topological space M with a countable basis and a function $\lambda : M \to \mathbb{Q}$, together with a countable collection of triples $\{(M_i, \lambda_i, \varphi_i)\}_{i\in I}$ and a decomposition of the index set $I = \bigcup_j I(j)$ into finite sets such that the following holds.*

(i) *The sets $M(j) = \bigcup_{i\in I(j)} M_i$ form an open cover of M, for every $i \in I(j)$, M_i is closed relative to $M(j)$ and, for all $i, i' \in I$,*

$$\operatorname{int}_{M_{i'}}(M_i \cap M_{i'}) = \operatorname{int}_{M_i}(M_i \cap M_{i'}).$$

The λ_i are positive rational numbers such that

$$x \in M(j) \qquad \Longrightarrow \qquad \lambda(x) = \sum_{\substack{i\in I(j)\\ x\in M_i}} \lambda_i.$$

(ii) *For every $i \in I$ the map $\varphi_i : M_i \to \mathbb{R}^m$ is a homeomorphism onto an open subset of $\mathbb{R}^m$. For every pair $i, i' \in I$ the transition map*

$$\varphi_{i'} \circ {\varphi_i}^{-1} : \varphi_i(\operatorname{int}_{M_i}(M_i \cap M_{i'})) \to \varphi_{i'}(\operatorname{int}_{M_{i'}}(M_i \cap M_{i'}))$$

is smooth.

(iii) *For every $x \in M$ there exists a continuous function $\theta_x : M \to [0, 1]$ such that $\theta_x(x) = 0$, $\theta_x(y) > 0$ for $y \neq x$, and $\theta_x \circ {\varphi_i}^{-1} : \varphi_i(M_i) \to [0, 1]$ is smooth for every i.*

A **branched** m**-manifold with boundary** *is defined similarly. In this case the charts $\varphi_i : M_i \to \mathbb{H}^m$ are homeomorphisms onto open subsets of the upper half space $\mathbb{H}^m = \{x \in \mathbb{R}^m : x_m \geq 0\}$, we define $\partial M_i = {\varphi_i}^{-1}(\partial\mathbb{H}^m)$, and assume that*

$$M_{i'} \cap \partial M_i \subset \partial M_{i'}$$

for all $i, i' \in I$. The boundary of M is then defined by

$$\partial M = \bigcup_i \partial M_i.$$

A branched manifold (M, λ) (with or without boundary) is called **oriented** *if*

$$\det d(\varphi_{i'} \circ {\varphi_i}^{-1})(x) > 0$$

for all $i, i' \in I$ and all $x \in \varphi_i(\operatorname{int}_{M_i}(M_i \cap M_{i'}))$.

Remark 5.7 Let (M, λ) be a branched manifold with charts $\{(M_i, \lambda_i, \varphi_i)\}_{i\in I}$. Then the set

$$M_{\mathrm{reg}} = \{x \in M \,:\, x \in M_i \Longrightarrow x \in \mathrm{int}(M_i)\}$$

is open and dense in M (see Lemma 5.10 below). The definition shows that M has a well defined tangent space T_xM for $x \in M_{\mathrm{reg}}$. If $x \notin M_{\mathrm{reg}}$, then there is a tangent space T_xM_i for every $i \in I$ with $x \in M_i$ and these tangent spaces of the different branches need not be naturally isomorphic. □

Exercise 5.8 Let (M, λ) be a branched manifold with charts $\{(M_i, \lambda_i, \varphi_i)\}_{i\in I}$ and suppose that each set $I(j)$ in the partition of the index set consists of a single point. Prove that M is a smooth manifold and $\lambda : M \to \mathbb{Q}$ is constant on the components of M. **Hint:** Each M_i is an open subset of M. □

Exercise 5.9 Define $M = D \times \{\pm 1\}/\sim$ where $D \subset \mathbb{C}$ denotes the closed unit disc and the equivalence relation is given by $(z, -1) \sim (z^2, 1)$ for $|z| = 1$. Define $\lambda : M \to \mathbb{Q}$ by $\lambda([z, -1]) = 1/2$ for $|z| < 1$ and $\lambda([z, 1]) = 1$ for $|z| \le 1$. Prove that (M, λ) admits the structure of an oriented branched 2-manifold. Note that this space cannot be expressed as a union of closed 2-manifolds. □

Lemma 5.10 *Suppose that M is a metric space with a covering $\{M_i\}_{i\in I}$ and a decomposition $I = \bigcup_j I(j)$ of the index set into disjoint finite subsets $I(j)$ such that*

(i) *for every j, $M(j) = \bigcup_{i\in I(j)} M_i$ is an open subset of M,*

(ii) *for every $i \in I(j)$, M_i is a relatively closed subset of $M(j)$,*

(iii) *for all $i, i' \in I$, $\mathrm{int}_{M_{i'}}(M_i \cap M_{i'}) = \mathrm{int}_{M_i}(M_i \cap M_{i'})$.*

Then $M_{\mathrm{reg}} = \{x \in M \,:\, x \in M_i \Longrightarrow x \in \mathrm{int}(M_i)\}$ is a dense open subset of M.

Proof: The proof is in three steps.

Step 1: *M_{reg} is open.*

We prove that $M_{i'} \cap \mathrm{int}(M_i) \subset \mathrm{int}(M_{i'})$ for all $i, i' \in I$. To see this note that the set $M_{i'} \cap \mathrm{int}(M_i)$ is open relative to $M_{i'}$ and hence

$$M_{i'} \cap \mathrm{int}(M_i) \subset \mathrm{int}_{M_{i'}}(M_i \cap M_{i'}) = \mathrm{int}_{M_i}(M_i \cap M_{i'}).$$

Now let $x \in M_{i'} \cap \mathrm{int}(M_i)$. By what we have just proved, $x \in \mathrm{int}_{M_i}(M_i \cap M_{i'})$, and hence there exists an open neighbourhood $U \subset M$ of x such that $U \cap M_i \subset M_i \cap M_{i'}$. Hence $U \cap \mathrm{int}(M_i)$ is an open set in M which contains x and is contained in $M_{i'}$. Hence $x \in \mathrm{int}(M_{i'})$. Thus we have proved that

$$M_{i'} \cap \mathrm{int}(M_i) \subset \mathrm{int}(M_{i'})$$

for all $i, i' \in I$. This implies

$$M_{\mathrm{reg}} = \bigcup_{i\in I} \mathrm{int}(M_i),$$

and hence M_{reg} is open.

Step 2: For every $i \in I(j)$,

$$M_i - \mathrm{int}(M_i) = \bigcup_{i'\in I(j)-\{i\}} \Big(M_i \cap M_{i'} - \mathrm{int}_{M_{i'}}(M_i \cap M_{i'}) \Big).$$

Suppose first that $x \in M_i - \mathrm{int}(M_i)$. Then there exists a sequence $x_\nu \in M - M_i$ which converges to x. Since $M(j)$ is open, we may assume without loss of generality that $x_\nu \in M(j)$ for all ν. Since $I(j)$ is a finite set we may assume without loss of generality that there exists an $i' \in I(j) - \{i\}$ such that $x_\nu \in M_{i'}$ for all ν. Since $M_{i'}$ is closed, relative to $M(j)$, this implies that $x \in M_{i'}$. Moreover, x is the limit of a sequence in $M_{i'} - (M_i \cap M_{i'})$. Hence $x \in M_i \cap M_{i'} - \mathrm{int}_{M_{i'}}(M_i \cap M_{i'})$.

Conversely, suppose that $x \in M_i \cap M_{i'} - \mathrm{int}_{M_{i'}}(M_i \cap M_{i'})$ for some $i' \in I(j) - \{i\}$. Then there exists a sequence $x_\nu \in M_{i'} - (M_i \cap M_{i'})$ which converges to x. Since $x_\nu \notin M_i$ for all ν this implies $x \notin \mathrm{int}(M_i)$.

Step 3: *M_{reg} is dense in M.*

It suffices to prove that $\mathrm{int}(M_i)$ is dense in M_i. To see this we write $I(j) = \{i_0, \dots, i_m\}$ with $i_0 = i$ and prove by induction over ℓ that the following holds for $\ell = 1, \dots, m$. For every $x \in M_i$ and every $\varepsilon > 0$ there exists a $y \in M_i$ such that

(a) $d(x, y) < \varepsilon$.

(b) For every $k \in \{1, \dots, \ell\}$ either $y \notin M_{i_k}$ or $y \in \mathrm{int}_{M_i}(M_i \cap M_{i_k})$.

First suppose $\ell = 1$. If $x \in \mathrm{int}_{M_i}(M_i \cap M_{i_1})$ choose $y = x$. If $x \notin \mathrm{int}_{M_i}(M_i \cap M_{i_1})$ then there exists a sequence $x_\nu \in M_i - M_{i_1}$ converging to x and we can choose $y = x_\nu$ for ν sufficiently large. Now suppose that the assertion has been proved with ℓ replaced by $\ell - 1$ where $\ell \geq 2$. Choose $z \in M_i$ such that $d(x, z) < \varepsilon/2$ and, for $k \in \{1, \dots, \ell - 1\}$, either $z \notin M_{i_k}$ or $z \in \mathrm{int}_{M_i}(M_i \cap M_{i_k})$. If $z \in \mathrm{int}_{M_i}(M_i \cap M_{i_\ell})$ choose $y = z$. If $z \notin \mathrm{int}_{M_i}(M_i \cap M_{i_\ell})$ then there exists a sequence $z_\nu \in M_i - M_{i_\ell}$ converging to z. There exists an $N \in \mathbb{N}$ such that the following holds for $\nu \geq N$ and $k \in \{1, \dots, \ell - 1\}$.

- $d(z, z_\nu) < \varepsilon/2$.
- If $z \in \mathrm{int}_{M_i}(M_i \cap M_{i_k})$ then $z_\nu \in \mathrm{int}_{M_i}(M_i \cap M_{i_k})$.
- If $z \notin M_{i_k}$ then $z_\nu \notin M_{i_k}$.

Hence, for $\nu \geq N$, the point $y = z_\nu$ satisfies the conditions (a) and (b) above. This completes the induction. Thus we have proved that every point $x \in M_i$ can be approximated by a sequence

$$y_\nu \in \bigcap_{i' \in I(j) - \{i\}} \Big((M_i - M_{i'}) \cup \mathrm{int}_{M_i}(M_i \cap M_{i'}) \Big) = \mathrm{int}(M_i).$$

The last equality follows from (iii) and Step 2. Hence $\mathrm{int}(M_i)$ is dense in M_i, as claimed. This proves the lemma. □

Compact oriented branched 1-manifolds are also called **train-tracks** (see Figure 21). They were introduced by Thurston (with real weights) in connection with the study of laminations on surfaces. A number of mathematicians generalized these ideas and studied 3-manifolds by using branched 2-manifolds. The following result about the ends of compact oriented branched 1-manifolds is the analogue of the observation that every compact 1-manifold has an even number of ends. It plays the same role in the construction of rational invariants as the 1-manifold lemma plays in differential topology (cf. Milnor [33]). Train tracks also play a crucial role in the work of Fukaya-Ono [14].

Before stating the lemma we point out that every endpoint of a branched 1-manifold may belong to several branches and we do not assume here that $\partial M \subset M_{\mathrm{reg}}$.

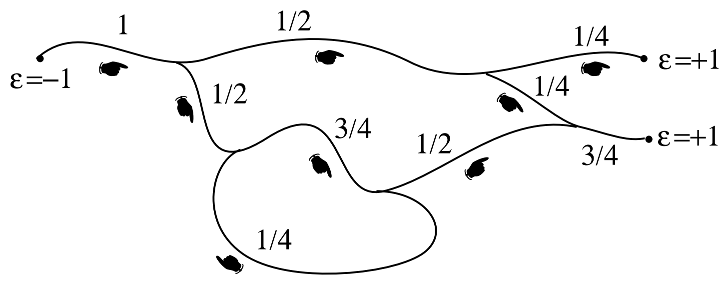

Figure 21: A compact oriented branched 1-manifold with $\partial M \subset M_{\mathrm{reg}}$

Let $x \in \partial M \cap M(j)$ and for every $i \in I(j)$ with $x \in M_i$ denote by $\nu_i \in T_{x_i}M$ the outward unit normal vector. This vector together with the orientation of M_i determines a sign

$$\varepsilon_i = \begin{cases} +1, & \text{if } \nu_i \text{ is positively oriented,} \\ -1, & \text{if } \nu_i \text{ is negatively oriented.} \end{cases}$$

Now consider the oriented sum of the weights λ_i and define

$$\rho(x) = \sum_{\substack{i \in I \\ x \in M_i}} \varepsilon_i \lambda_i. \tag{48}$$

Note that this number is not equal to $\lambda(x)$ unless the ε_i are all equal to 1. We leave it to the reader to prove that $\rho(x)$ is independent of the choice of j with $x \in M(j)$.

Lemma 5.11 *Let (M, λ) be a compact oriented branched 1-manifold with boundary. For each $x \in \partial M$ let $\rho(x) \in \mathbb{Q}$ be defined by (48). Then*

$$\sum_{x \in \partial M} \rho(x) = 0.$$

Proof: The proof is in three steps. The first step shows that M can be covered by sets M_i with endpoints in M_{reg}. The second step proves that (M, λ) admits the structure of a rational cycle. The last step proves the lemma.

Step 1: *We may assume without loss of generality that, for every i, the interval $\varphi_i(M_i)$ is one of $(-1,1)$, $[0,1)$, or $(-1,0]$, and that ${\varphi_i}^{-1}$ extends smoothly to the closure of this interval. Moreover, we may assume that the endpoints of M_i are regular, i.e.*

$$x_i^{\pm} = \lim_{t \to \pm 1} {\varphi_i}^{-1}(t) \in M_{\mathrm{reg}}$$

for every $i \in I$ with $\pm 1 \in \mathrm{cl}(\varphi_i(M_i))$.

For every $x \in M(j)$ choose a number $\varepsilon_x > 0$ such that

- For every $i \in I(j)$, ε_x is a regular value of $\theta_x \circ {\varphi_i}^{-1}$.
- $\varepsilon_x \notin \theta_x(M - M_{\mathrm{reg}})$.
- $\inf_{M - M(j)} \theta_x > \varepsilon_x$ and $\inf_{M_i} \theta_x > \varepsilon_x$ for every $i \in I(j)$ with $x \notin M_i$.

To see that such a number ε_x exists, note first that the set $\varphi_i(M_i - \mathrm{int}(M_i))$ has empty interior and is closed relative to $\varphi_i(M_i)$. Hence the image of this set under $\theta_x \circ {\varphi_i}^{-1}$ is of the first category in the sense of Baire, i.e. is a countable union of closed sets with empty interior.[12] The set $\theta_x(M - M_{\text{reg}})$ is the union of these images over all i, and hence is also a countable union of closed sets with empty interior. Hence the set $\mathbb{R} - \theta_x(M - M_{\text{reg}})$ is of the second category in the sense of Baire, i.e. is a countable intersection of dense open sets. Hence the set of all regular values of θ_x in the complement of $\theta_x(M - M_{\text{reg}})$ is also of the second category in the sense of Baire and, in particular, it is dense. This shows that there exists an $\varepsilon_x > 0$ which satisfies the first two conditions above and that ε_x can be chosen arbitrarily small. Next observe that if $x \in M(j) - M_i$ for some $i \in I(j)$ then $x \notin \text{cl}(M_i)$ and hence $\inf_{M_i} \theta_x > 0$. This shows that a sufficiently small number $\varepsilon_x > 0$ satisfies the third condition as well. Thus we have proved the existence of a constant $\varepsilon_x > 0$ which satisfies the above requirements.

Now define

$$U_x = \{y \in M \,:\, \theta_x(y) < \varepsilon_x\}.$$

Then U_x is an open neighbourhood of x and it is equal to the union of the branches $M_i \cap U_x$ over all $i \in I(j)$ with $x \in M_i$. Moreover, since $\varepsilon_x \notin \theta_x(M - M_{\text{reg}})$ is a regular value of the map $\theta_x \circ {\varphi_i}^{-1} : \varphi_i(M_i) \to [0,1]$ for every $i \in I(j)$, the set

$$\varphi_i(M_i \cap U_x) = \left\{t \in \varphi_i(M_i) \,:\, \theta_x \circ {\varphi_i}^{-1}(t) < \varepsilon_x\right\}$$

is a finite union of open intervals with boundary points in $\varphi_i(M_{\text{reg}})$. Now cover M by finitely many such sets U_x. This proves Step 1.

Step 2: *There exists a finite set of vertices $V \subset M_{\text{reg}} \cup \partial M$, a finite collection of continuous injections $\gamma_\alpha : [0,1] \to M$, indexed by $\alpha \in A$, and a map $A \to I : \alpha \mapsto i(\alpha)$ such that the following holds.*

- *For every α the endpoints $x_0 = \gamma_\alpha(0)$ and $x_1 = \gamma_\alpha(1)$ lie in the set V and*

$$\gamma_\alpha([0,1]) \subset M_{i(\alpha)}, \qquad \gamma_\alpha([0,1]) \cap V = \{x_0, x_1\}.$$

  Moreover, $\varphi_{i(\alpha)} \circ \gamma_\alpha$ is an orientation preserving diffeomorphism from $[0,1]$ onto the interval $[t_0, t_1]$, where $t_0 = \varphi_{i(\alpha)}(x_0)$ and $t_1 = \varphi_{i(\alpha)}(x_1)$.

- *For every $x \in M - V$,*

$$\lambda(x) = \sum_{x \in \gamma_\alpha([0,1])} \lambda_{i(\alpha)}.$$

- *If $x \in V - \partial M$,*

$$\lambda(x) = \sum_{\gamma_\alpha(0) = x} \lambda_{i(\alpha)} = \sum_{\gamma_\alpha(1) = x} \lambda_{i(\alpha)}.$$

---

[12]If $f : M \to N$ is a smooth map between manifolds of the same dimension and $A \subset M$ is a compact set with empty interior, then $f(A) \subset N$ is a compact set with empty interior. Since A is compact we may assume without loss of generality that the topology of M has a countable basis and so Sard's theorem applies. Now suppose, by contradiction, that $\text{int}(f(A)) \neq \emptyset$. By Sard's theorem, the function f has a regular value $y \in \text{int}(f(A))$. Since A is compact, $f^{-1}(y) \cap A$ is a finite set, and we denote $f^{-1}(y) \cap A = \{x_1, \ldots, x_m\}$. Choose compact neighbourhoods U_i of x_i such that, for every i, the restriction of f to U_i is a diffeomorphism onto some compact neighbourhood of y. Then there exists a compact neighbourhood $V \subset f(A)$ of y such that, for every $x \in A$ with $f(x) \in V$, there exists an $i \in \{1, \ldots, m\}$ such that $x \in U_i$. Otherwise there would be a sequence $x_\nu \in A - \bigcup_i U_i$ such that $f(x_\nu)$ converges to y and then any limit point of x_ν would be a preimage of y in A not equal to any of the x_i. This contradiction shows that the neighbourhood V exists. By definition, this neighbourhood satisfies $V \subset \bigcup_i f(U_i \cap A)$. But $f(U_i \cap A)$ is a compact set with empty interior for every i. By Baire's category theorem, V cannot be the union of finitely many such sets. This contradiction shows that $f(A)$ has empty interior.

The proof is by induction over the number of sets in the open cover $M = \bigcup_j M(j)$. Fix any index $j = j_1$. By Step 1, the closure of the set $M(j_1)$ can be covered by finitely many such paths $\gamma_\alpha : [0,1] \to M$. To see this simply choose the γ_α to be reparametrizations of the curves ${\varphi_i}^{-1} : \mathrm{cl}(\varphi_i(M_i)) \to \mathrm{cl}(M_i)$. Now under the assumptions of Step 1, one checks easily that the complement $\widehat{M} = M - M(j_1)$ is again a compact branched manifold covered by open sets $\widehat{M}(j) = M(j) - M(j_1)$, $j \neq j_1$, which still satisfy the requirements of Step 1. This completes the induction argument and the proof of Step 2.

Step 3: *We prove the lemma.*

Step 2 defines a directed graph with vertices $x \in V$ and edges γ_α. The edges carry rational weights $\lambda_\alpha = \lambda_{i(\alpha)} > 0$. Note that all the boundary points $x \in \partial M$ are vertices and that

$$\rho(x) = \sum_{\gamma_\alpha(1)=x} \lambda_\alpha - \sum_{\gamma_\alpha(0)=x} \lambda_\alpha$$

for $x \in \partial M$. On the other hand, by Step 2,

$$\lambda(x) = \sum_{\gamma_\alpha(0)=x} \lambda_\alpha = \sum_{\gamma_\alpha(1)=x} \lambda_\alpha$$

for $x \in V - \partial M$. Hence

$$\sum_{x\in\partial M} \rho(x) = \sum_{x\in V} \Bigg(\sum_{\gamma_\alpha(1)=x} \lambda_\alpha - \sum_{\gamma_\alpha(0)=x} \lambda_\alpha \Bigg) = 0.$$

This proves the lemma. □

Let us now return to the moduli space $\mathcal{M} = \mathcal{M}(A; J, \Gamma)$ of (J, Γ)-holomorphic spheres representing the homology class $A \in H_2(M, \mathbb{Z})$. The *"free"* axiom guarantees that the action of the reparametrization group $\mathrm{G} = \mathrm{PSL}(2, \mathbb{C})$ on $\mathcal{M}$ is free. However, the branches $\mathcal{M}_i = \mathcal{M}(A; J, \gamma_i)$ will not, in general, be invariant under G. Nevertheless, by using local slices (as in the case of principal bundles), one can show that the quotient $\mathcal{M}/\mathrm{G}$ is again a branched manifold of dimension

$$\dim\, \mathcal{M}(A; J, \Gamma)/\mathrm{G} = 2n + 2c_1(A) - 6.$$

So far we have not addressed the compactness question. As in the case of J holomorphic curves, the moduli space $\mathcal{M}/\mathrm{G}$ will not be compact, in general, but bubbling may occur. To obtain bubble trees of (J, Γ)-holomorphic curves in the limit, we must ensure the compatibility, under Gromov convergence, of our multi-valued perturbations correponding to different homotopy classes.

5.5 Perturbations and stable maps

The goal of this section is to obtain the same kind of compactness results for the perturbed equations (47) as were discussed in Section 4 for J-holomorphic curves. For this it is useful to make sure that the perturbation vanishes in a neighbourhood of any point near which bubbling occurs. Moreover, we must match the perturbations on the components of a limiting bubble tree with the perturbation on the approximating curves. This requires a refinement of the construction in Section 5.2. Namely, we shall introduce perturbations which not only depend on the curve u but also on a finite set of marked points, and are required to vanish near the marked points.

Perturbations and marked points

Consider the space

$$\mathcal{B} = \mathcal{B}(A,k) = \mathrm{Map}_A(S^2, M) \times \big((S^2)^k - \Delta_k\big)$$

whose elements are tuples $(u, \mathbf{z}) = (u, z_1, \dots, z_k)$, where $u : S^2 \to M$ is a smooth map representing the homology class $A \in H_2(M, \mathbb{Z})$ and $z_1, \dots, z_k$ are pairwise distinct points in S^2. Denote by $\mathcal{E} \to \mathcal{B}$ vector bundle with fibers

$$\mathcal{E}_{u,\mathbf{z}} = \big\{\eta \in \Omega^{0,1}(S^2, u^*TM) \,:\, \eta = 0 \text{ near } z_i \text{ for } i = 1, \dots, k\big\}.$$

We shall consider multi-valued perturbations $\Gamma : \mathcal{B} \to 2^{\mathcal{E}}$ which satisfy all the axioms of Section 5.2. The *"equivariance"* axiom now takes the form

$$\Gamma(u \circ \varphi, \varphi^{-1}(z_1), \dots, \varphi^{-1}(z_k)) = \varphi^* \Gamma(u, z_1, \dots, z_k). \tag{49}$$

The definition of $\mathcal{E}$ guarantees that each solution $(u, z_1, \dots, z_k)$ of the equation

$$\bar{\partial}_J(u) \in \Gamma(u, z_1, \dots, z_k) \tag{50}$$

is an unperturbed J-holomorphic curve in some neighbourhood of the marked points.

Perturbations and stable maps

Fix a tree T and consider the space $\mathcal{B}(A; T, k)$ of all stable maps

$$(\mathbf{u}, \mathbf{z}) = (\{u_\alpha\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}, \{\alpha_i, z_i\}_{i=1,\dots,k}),$$

modelled over the tree T, which satisfy

$$\sum_{\alpha \in T} u_{\alpha *}[S^2] = A.$$

Here the $u_\alpha : S^2 \to M$ are arbitrary smooth functions, J-holomorphic or not, and the stability condition takes the form $\#Z_\alpha \geq 3$ whenever $u_{\alpha *}[S^2] = 0 \in H_2(M, \mathbb{Z})$. The space $\mathcal{B} = \mathcal{B}(A; T, k)$ is an infinite dimensional manifold whose tangent space $T_{(\mathbf{u},\mathbf{z})}\mathcal{B}$ consists of tuples

$$(\{\xi_\alpha\}_{\alpha \in T}, \{\zeta_{\alpha\beta}\}_{\alpha E \beta}, \{\zeta_i\}_{i=1,\dots,k})$$

with $\xi_\alpha \in C^\infty(S^2, {u_\alpha}^*TM)$, $\zeta_{\alpha\beta} \in T_{z_{\alpha\beta}} S^2$, $\zeta_i \in T_{z_i} S^2$, which satisfy

$$\xi_\alpha(z_{\alpha\beta}) + du_\alpha(z_{\alpha\beta})\zeta_{\alpha\beta} = \xi_\beta(z_{\beta\alpha}) + du_\beta(z_{\beta\alpha})\zeta_{\beta\alpha}.$$

For a tree T the perturbed equations will take the form

$$\bar{\partial}_J(u_\alpha) \in \Gamma(u_\alpha, Z_\alpha), \tag{51}$$

where

$$Z_\alpha = \{z_{\alpha\beta} \,:\, \alpha E \beta\} \cup \{z_i \,:\, \alpha_i = \alpha\}$$

(as in Section 4.2). We claim that Γ and J can be chosen such that the space of solutions of (51) is an oriented branched manifold of the predicted dimension. The details are as in [31] and are carried out in [22].

The space

$$\mathcal{B}(T,k) = \bigcup_A \mathcal{B}(A;T,k)$$

carries an action of the group $\mathrm{G}(T)$ of automorphisms of the bubble tree. The elements of $\mathrm{G}(T)$ are tuples $(f, \{\varphi_\alpha\}_{\alpha\in T})$ where $f: T\to T$ is a tree automorphism and $\varphi_\alpha \in \mathrm{G}$ for $\alpha \in T$. The group operation is given by

$$(g, \{\psi_\alpha\}_{\alpha\in T}) \circ (f, \{\varphi\}_{\alpha\in T}) = (g\circ f, \{\psi_{f(\alpha)} \circ \varphi_\alpha\}_{\alpha\in T}),$$

and the action on $\mathcal{B}(T,k)$ by $(f,\varphi)_*(\mathbf{u},\mathbf{z}) = (\tilde{\mathbf{u}}, \tilde{\mathbf{z}})$, where

$$\tilde{u}_{f(\alpha)} = u_\alpha \circ {\varphi_\alpha}^{-1}, \quad \tilde{z}_{f(\alpha)f(\beta)} = \varphi_\alpha(z_{\alpha\beta}), \quad \tilde{\alpha}_i = f(\alpha_i), \quad \tilde{z}_i = \varphi_{\alpha_i}(z_i)$$

(see Definition 4.4). It follows from (49) that the moduli space of solutions of (51) is invariant under the action of $\mathrm{G}(T)$. The *"free"* axiom can be extended to ensure that the action of $\mathrm{G}(T)$ on this moduli space is free. For details see [22].

Compatibility

The crucial compatibility condition for the perturbations in different homology classes is **continuity with respect to the Gromov topology**. More precisely this means the following.

(Compatibility) If a sequence $(u^\nu, z_1^\nu, \dots, z_k^\nu)$ Gromov converges to a stable map $(\mathbf{u},\mathbf{z}) \in \mathcal{B}(T,k)$, with corresponding sequences $\varphi_\alpha^\nu \in \mathrm{G}$ (see Definition 4.5), then

$$\Gamma(u_\alpha, Z_\alpha) = \lim_{\nu\to\infty} (\varphi_\alpha^\nu)^* \Gamma(u^\nu, z_1^\nu, \dots, z_k^\nu) \tag{52}$$

and

$$\lambda(u_\alpha, Z_\alpha; \eta) = \lim_{\nu\to\infty} \sum_{\substack{\eta^\nu \in \Gamma(u^\nu, z_1^\nu, \dots, z_k^\nu) \\ \|(\varphi_\alpha^\nu)^*\eta^\nu - \eta\|_{L^\infty} < \varepsilon}} \lambda(u^\nu, z_1^\nu, \dots, z_k^\nu; \eta^\nu) \tag{53}$$

for $\varepsilon > 0$ sufficiently small. In (52) the convergence is uniform on S^2. That $u^\nu \circ \varphi_\alpha^\nu$ only converges uniformly on compact subsets of $S^2 - Z_\alpha$ is immaterial because the perturbation vanishes in a neighbourhood of Z_α.

The existence of an almost complex structure J and a family of perturbations (Γ_A, λ_A), one for each homology class $A \in H_2(M,\mathbb{Z})$ (and each number k of marked points), which satisfy the axioms of Section 5.2 and the *"compatibility"* axiom, is proved by induction over $\omega(A)$. For $\omega(A) < 2\hbar$ all J-holomorphic curves are simple. Hence transversality can be achieved by a generic choice of J. Now suppose that the perturbations have been constructed for $\omega(A) < m\hbar$ for some integer $m \geq 2$, and fix a homology class B with $\omega(B) < (m+1)\hbar$. Then one can use the gluing construction, of Appendix A in [31], and the induction hypothesis, to define the perturbation in a Gromov neighbourhood of any given bubble tree, representing the class B. The estimate for the inverse in [31], Appendix A, then shows that the glued perturbations automatically satisfy the crucial *"transversality"* axiom. Away from the bubble trees one can use the methods of Section 5.3 to construct the perturbations. For more details of the induction argument see [22].

5.6 Rational Gromov-Witten invariants

The techniques explained so far suffice to define the Gromov-Witten invariants for general symplectic manifolds via solutions of the perturbed equations (50), following essentially the arguments in [31]. One considers the moduli space

$$\begin{aligned}\widetilde{\mathcal{M}}^*_{0,k,A}(M,\omega,J,\Gamma) &= \{(u,z_1,\dots,z_k)\in\mathcal{B}(A,k)\,:\,\bar\partial_J(u)\in\Gamma(u,z_1,\dots,z_k),\\ &\qquad u_*[S^2]=A,\ u \text{ is somewhere injective}\}.\end{aligned}$$

The group G acts freely on this space and, for a generic perturbation Γ, the quotient

$$\mathcal{M}^*_{0,k,A}(M,\omega,J,\Gamma)=\widetilde{\mathcal{M}}^*_{0,k,A}(M,\omega,J,\Gamma)/\mathrm{G}$$

is a branched manifold of dimension

$$\dim\,\mathcal{M}^*_{0,k,A}(M,\omega,J,\Gamma)=2n+2k+2c_1(A)-6,$$

with rational weight $\lambda:\mathcal{M}^*_{0,k,A}\to\mathbb{Q}$. Note that this space $\mathcal{M}^*_{0,k,A}$ is an open subset of the space $\mathcal{M}_{0,k,A}(M,\omega,J,\Gamma)$ of all stable (J,Γ)-holomorphic curves of genus zero with k marked points in M which represent the class A. The definitions and theorems are as in Section 4 for the unperturbed case. In particular, the moduli space $\mathcal{M}_{0,k,A}$ is a compact metrizable space, it is a union of branched manifolds corresponding to the different tree structures, and $\mathcal{M}^*_{0,k,A}$ is the *"top dimensional stratum"* of $\mathcal{M}_{0,k,A}$.

Given cohomology classes $\alpha_1,\dots,\alpha_k\in H^*(M,\mathbb{Z})$, choose submanifolds (or cycles) $N_j\subset M$ in general position which are Poincaré dual to the α_j, i.e. $\alpha_j=\mathrm{PD}([N_j])$. If

$$\sum_{j=1}^k\deg(\alpha_j)=2n+2k+2c_1(A)-6,$$

then the subset of all tuples $[u,z_1,\dots,z_k]\in\mathcal{M}^*_{0,k,A}$ with $u(z_j)\in N_j$ is a compact zero dimensional branched manifold with a natural orientations. Thus $\mathcal{M}^*_{0,k,A}$ is a finite set. But in the zero dimensional case a single isolated point may belong to several branches and each of the branches may carry different orientations, which in the zero dimensional case means different signs. Geometrically, this means that several branches of the multi-valued perturbation may have a common zero but not have the same differential at this zero. Given a solution $[u,z_1,\dots,z_k]\in\mathcal{M}^*_{0,k,A}$ with $u(z_j)\in N_j$, let $\gamma_1,\dots,\gamma_m$ denote the local branches of the perturbation Γ near $(u,z_1,\dots,z_k)$, and let $\lambda_1,\dots,\lambda_m$ be the corresponding weights. For each branch γ_i which satisfies $\gamma_i(u,z_1,\dots,z_k)=\bar\partial_J(u)$ there is a sign $\varepsilon_i\in\{\pm1\}$, determined by the orientation of this branch, and we define

$$\rho(u,z_1,\dots,z_k)=\sum_{\substack{i\in\{1,\dots,m\}\\ \bar\partial_J(u)=\gamma_i(u,z_1,\dots,z_k)}}\varepsilon_i\lambda_i.$$

Counting the solutions with these rational weights gives rise to the **rational Gromov-Witten invariant**

$$\Phi_A(\alpha_1,\dots,\alpha_k)=\sum_{\substack{[u,z_1,\dots,z_k]\in\mathcal{M}^*_{0,k,A}\\ u(z_j)\in N_j}}\rho(u,z_1,\dots,z_k). \tag{54}$$

One can prove, proceeding as in [31] and using Lemma 5.11, that the right hand side is independent of the cycles N_j, the almost complex structure J, and the perturbation Γ, used to define it. Details will be carried out in [22].

Remark 5.12 A crucial point in the construction of the Gromov-Witten invariants is the fact that, generically, there are only finitely many equivalence classes of solutions $(u, z_1, \ldots, z_k)$ of (50) which satisfy $u(z_i) \in N_i$, and that u is somewhere injective for each solution. The injective condition is redundant, by the *"free"* axiom. One can argue, as in [31], that if there were infinitely many solutions, then there would be a sequence of solutions which Gromov converges to a stable map modelled over a tree with more than one vertex. This limit curve would then belong to a moduli space of strictly lower dimension. But since we have started with a zero dimensional moduli space the limit moduli space has negative dimension and therefore must be empty. The key point is that for the perturbed equations the actual dimension always agrees with the virtual dimension, and hence the difficulties discussed in Section 5.1 do not arise. □

Other approaches to the construction of the Gromov-Witten invariants for general symplectic manifolds were developed by Li-Tian [27], Ruan [42], and Siebert [51]. These authors construct, for every $J \in \mathcal{J}(M, \omega)$, every $A \in H_2(M, \mathbb{Z})$, and every integer $k \geq 0$, a rational fundamental classes

$$[\mathcal{M}_{0,k,A}(M, \omega, J)] \in H_{2n+2c_1(A)+2k-6}(\mathcal{M}_{0,k,A}(M, \omega, J); \mathbb{Q})$$

on the compactified moduli space $\mathcal{M}_{0,k,A}(M, \omega, J)$ of stable J-holomorphic curves in the class A with k marked points. The Gromov-Witten invariants can then be defined by evaluating the cohomology class ${e_1}^*\alpha_1 \smile \cdots \smile {e_k}^*\alpha_k$ on the fundamental class:

$$\Phi_A(\alpha_1, \ldots, \alpha_k) = \int_{[\mathcal{M}_{0,k,A}(M,\omega,J)]} {e_1}^*\alpha_1 \smile \cdots \smile {e_k}^*\alpha_k.$$

Here the $e_i : \mathcal{M}_{0,k,A}(M, \omega, J) \to M$ denote the obvious evaluation maps.

The standard gluing techniques, as in [31, 26, 43], can be used to prove that the rational Gromov-Witten invariants satisfy the usual gluing rules. One version of these rules asserts the associativity of quantum cohomology and the WDVV equation. Another version involves the Chern classes of certain line bundles over the moduli space and plays a crucial role in the recent work of Givental on mirror symmetry [16, 17].

5.7 Rational Floer homology

The goal of this section is to explain how the above ideas can be used to define Floer homology groups for symplectic manifolds which satisfy (4) with $\tau < 0$. Fix a time dependent Hamiltonian $H_t = H_{t+1}$ with nondegenerate 1-periodic solutions $x \in \mathcal{P}(H)$ and, for each pair $x^\pm \in \mathcal{P}(H)$, consider the space $\mathcal{Z}(x^-, x^+)$ of smooth functions $u : \mathbb{R} \times S^1 \to M$ which satisfy (8). As in Exercise 2.10, abbreviate

$$\mathcal{Z} = \mathcal{Z}(H) = \bigcup_{x^\pm \in \mathcal{P}(H)} \mathcal{Z}(x^-, x^+)$$

and consider the vector bundle $\mathcal{E} \to \mathcal{Z}$ with fibers

$$\mathcal{E}_u = C_0^\infty(\mathbb{R} \times S^1, u^*TM).$$

For the purpose of this discussion the subscript zero can either denote compact support or some exponential decay condition. The precise notation is immaterial because, when it comes to the analysis, we will have to work with suitable Sobolev completions. To extend the above ideas to Floer homology we must choose a weighted multi-valued section

$$\Gamma : \mathcal{Z} \to 2^{\mathcal{E}}, \qquad \lambda : \mathcal{E} \to \mathbb{Q},$$

which is equivariant with respect to the $\mathbb{R}$-action on $\mathcal{Z}$ and $\mathcal{E}$. Instead of (7) we shall then consider the differential inequality

$$\bar{\partial}_{J,H}(u) = \partial_s u + J(u)\partial_t u - \nabla H_t(u) \in \Gamma(u). \tag{55}$$

The axioms for Γ are as before *"finiteness"*, *"conformality"* (here equivariance under the $\mathbb{R}$-action), *"local structure"*, and the energy bound

$$\eta \in \Gamma(u) \qquad \Longrightarrow \qquad \int_{-\infty}^{\infty}\int_0^1 |\eta|^2\, dtds \le c. \tag{56}$$

The *"transversality"* and *"free"* axioms are immaterial in this case. Transversality can be achieved by a generic perturbation of H (see [13] or [47]), and that the $\mathbb{R}$-action is free is obvious unless $x^- = x^+$, and this case can easily be dealt with separately. Instead, the crucial point is the *"compatibility"* axiom, which is needed to prevent the bubbling off of multiply covered J-holomorphic spheres with negative Chern number as in Section 5.1. The purpose of the perturbation is indeed, to make shure that what bubbles off are not J-holomorphic curves, but (J,Γ)-holomorphic curves, because these form moduli spaces of the predicted dimensions. To carry this out one must introduce a notion of **stable connecting orbits** in analogy to the notion of stable maps, by including finite chains of connecting orbits together with bubble trees. The stability condition allows the case of connecting orbits of the form $u(s,t) = x(t)$ for some $x \in \mathcal{P}(H)$, but such a onnecting orbit must contain at least one double point at which it intersects a J-holomorphic curve in the tree (see Figure 22).

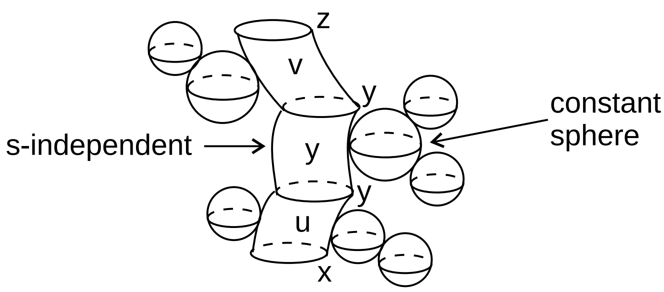

Figure 22: Stable connecting orbits

Then there is the notion of **Floer-Gromov convergence** for stable connecting orbits, and the compatibility condition has the form of continuity with respect to this **Floer-Gromov** topology. In other words, the perturbation for the stable connecting orbits will involve perturbations for the J-holomorphic curves in the bubble tree and if u^ν is a sequence of connecting orbits converging to such a stable connecting orbit in the Floer-Gromov topology, then the perturbations $\Gamma(u^\nu)$ must converge to those for the limit configuration. The details are as in Sections 4.2 and 5.2 and will not be discussed here. The reader is instead referred to [23].

It is important to note that the differential inequality (55) is no longer a gradient flow equation. Moreover, the energy identity of Exercise 1.23 now becomes an energy inequality.

Exercise 5.13 Prove that every solution $u \in \mathcal{Z}(x^-, x^+)$ of $\bar{\partial}_{J,H}(u) = \gamma(u)$ satisfies

$$E(u) = a_H(x^-, u^-) - a_H(x^+, u^+) + \int_{-\infty}^{\infty} \int_0^1 \langle \partial_s u, \gamma(u) \rangle \, dt ds,$$

where $u^\pm : B \to M$ are smooth maps which satisfy $u^\pm(s^{2\pi it}) = x^\pm(t)$ and $u^+ = u^- \# u$. Deduce that $(E(u) - c)/2 \leq a_H(x^-, u^-) - a_H(x^+, u^+) \leq 2(E(u) + c)$ where c is the constant of (56). Deduce further that, if (4) holds, then

$$E(u) \leq c + \frac{1}{\tau}\Big(\mu(u, H) + \eta_H(x^+) - \eta_H(x^-)\Big) \tag{57}$$

where $\eta_H : \mathcal{P}(H) \to \mathbb{R}$ is defined by (11). □

With the perturbation in place let us denote the space of perturbed connecting orbits by

$$\mathcal{M}(x^-, x^+; H, J, \Gamma) = \left\{ u \in \mathcal{Z}(x^-, x^+) \, : \, \bar{\partial}_{J,H}(u) \in \Gamma(u) \right\}.$$

For generic choices of H, J, and Γ, this is a finite dimensional branched manifold of local dimension $\mu(u, H)$ near u (Exercise 2.10) and it carries a weight function

$$\mathcal{M}(x^-, x^+; H, J, \Gamma) \to \mathbb{Q} : u \mapsto \lambda(u) = \lambda(u, \bar{\partial}_{J,H}(u)).$$

As before we denote by $\mathcal{M}^1(x^-, x^+; H, J, \Gamma)$ the 1-dimensional part of this moduli space. In view of the perturbation the difficulties of Section 5.1 disappear and, using (57), we find that the quotient

$$\widehat{\mathcal{M}}^1(x^-, x^+; H, J, \Gamma) = \mathcal{M}^1(x^-, x^+; H, J, \Gamma)/\mathbb{R}$$

is a finite set (compare with Remark 5.12). As in the case of the rational Gromov-Witten invariants, each perturbed connecting orbit $[u] \in \widehat{\mathcal{M}}^1(x^-, x^+; H, J, \Gamma)$ may belong to several branches of the zero dimensional moduli space and hence carries a rational weight $\rho(u) = \sum_i \varepsilon_i \lambda_i$, where the λ_i are the weights of the branches to which u belongs, and the $\varepsilon_i \in \{\pm 1\}$ are the signs determined by the coherent orientations. The Floer chain complex is now defined as the vector space over $\mathbb{Q}$, generated by the periodic solutions of (1):

$$CF_*(H) = \bigoplus_{x \in \mathcal{P}(H)} \mathbb{Q}\langle x \rangle.$$

The boundary operator $\partial = \partial^{J,H,\Gamma}$ is given by

$$\partial^{J,H,\Gamma}\langle y \rangle = \sum_{x \in \mathcal{P}(H)} \sum_{[u] \in \widehat{\mathcal{M}}^1(y,x;H,J,\Gamma)} \rho(u) \, \langle x \rangle.$$

Theorem 5.14 *Suppose that (M, ω) satisfies (4) with $\tau < 0$ and let H, J, Γ be chosen as above. Then $\partial \circ \partial = 0$ and the Floer homology groups*

$$HF_*(M, \omega; H, J, \Gamma) = \frac{\ker \partial^{J,H,\Gamma}}{\operatorname{im} \partial^{J,H,\Gamma}}$$

are naturally isomorphic to the singular homology of M with rational coefficients.

Proof: We only sketch the proof of $\partial \circ \partial = 0$. This is equivalent to the identity

$$\sum_{y \in \mathcal{P}(H)} \sum_{[v] \in \widehat{\mathcal{M}}^1(z,y;H,J,\Gamma)} \sum_{[u] \in \widehat{\mathcal{M}}^1(y,x;H,J,\Gamma)} \rho(v)\rho(u) = 0$$

for all $x, z \in \mathcal{P}(H)$. As in the monotone case, this follows by examining the moduli space $\widehat{\mathcal{M}}^2(z, x; H, J, \Gamma)$. For a generic perturbation Γ and a generic Hamiltonian H, this space is a regular oriented branched 1-manifold with rational weights. Via Floer's gluing theorem, it can be compactified by including as boundary points pairs $[v\#u]$ with $[u] \in \widehat{\mathcal{M}}^1(y, x; H, J, \Gamma)$ and $[v] \in \widehat{\mathcal{M}}^1(z, y; H, J, \Gamma)$. Associated to each boundary point is a rational weight $\rho([v\#u])$. One can show that these weights are given by

$$\rho([v\#u]) = \rho(v)\rho(u)$$

and hence the result follows from Lemma 5.11. This proves that $\partial \circ \partial = 0$.

The rest of the proof is a longer story, but the details are essentially as in the standard case. The easiest approach is to fix the perturbation Γ and the almost complex structure J, and only vary the Hamiltonian H. Then one can employ the methods of Section 3.4 to construct the isomorphism between Floer homology and ordinary homology. It is, however, more elegant to prove first that the Floer homology groups are independent of H, J, and Γ. For more details see [23]. □

The above construction can easily be extended to arbitrary compact symplectic manifold by using Novikov rings as in Section 3.7. Hence we have proved the following theorem, modulo some analytical details which are carried out in [20, 21, 22, 23]. Other proofs are given in [14, 28].

Theorem 5.15 *Suppose that (M, ω) is a compact symplectic manifold. Let $H_t = H_{t+1}$ be a smooth time dependent Hamiltonian on M such that all the 1-periodic solutions of (1) are nondegenerate. Then $\#\mathcal{P}(H) \geq \dim\, H_*(M, \mathbb{Q})$.*

References

[1] M.F. Atiyah, V.K. Patodi and I.M. Singer, Spectral asymmetry and Riemannian geometry, *Math. Proc. Camb. Phil. Soc.* **79** (1976), 79–99.

[2] C. Conley and E. Zehnder, The Birkhoff-Lewis fixed point theorem and a conjecture of V.I. Arnold, *Invent. Math.* **73** (1983), 33–49.

[3] S. Donaldson and M. Furuta, *Floer homology*, in preparation.

[4] S. Dostoglou and D.A. Salamon, Cauchy-Riemann operators, self-duality, and the spectral flow, in *First European Congress of Mathematics, Volume I, Invited Lectures (Part 1)*, edited by A. Joseph, F. Mignot, F. Murat, B. Prum, R. Rentschler, Birkhäuser Verlag, Progress in Mathematics, **Vol. 119**, 1994, pp. 511–545.

[5] S. Dostoglou and D.A. Salamon, Self-dual instantons and holomorphic curves, *Annals of Mathematics* **139** (1994), 581–640.

[6] Y. Eliashberg, Estimates on the number of fixed points of area preserving transformations, Syktyvkar University, Preprint, 1979.

[7] A. Floer, Morse theory for Lagrangian intersections, *J. Diff. Geom.* **28** (1988), 513–547.

[8] A. Floer, A relative Morse index for the symplectic action, *Comm. Pure Appl. Math.* **41** (1988), 393–407.

[9] A. Floer, The unregularized gradient flow of the symplectic action, *Comm. Pure Appl. Math.* **41** (1988), 775–813.

[10] A. Floer, Wittens complex and infinite dimensional Morse theory, *J. Diff. Geom.* **30** (1989), 207–221.

[11] A. Floer, Symplectic fixed points and holomorphic spheres, *Comm. Math. Phys.* **120** (1989), 575–611.

[12] A. Floer and H. Hofer, Coherent orientations for periodic orbit problems in symplectic geometry, *Math. Zeit.* **212** (1993), 13–38.

[13] A. Floer, H. Hofer, and D. Salamon, Transversality in elliptic Morse theory for the symplectic action, *Duke Math. Journal* **80** (1996), 251–292.

[14] F. Fukaya and K. Ono, Arnold conjecture and Gromov-Witten invariants for general symplectic manifolds, Preprint, June 1996.

[15] W. Fulton and R. MacPherson, A compactification of configuration space, *Annals of Mathematics* **139** (1994), 183–225.

[16] A. Givental, Equivariant Gromov-Witten invariants, UCB, Preprint, 1996.

[17] A. Givental, Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture, UCB, Preprint, 1996.

[18] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, *Invent. Math.* **82** (1985), 307–347.

[19] H. Hofer and D.A. Salamon, Floer homology and Novikov rings, in *The Floer Memorial Volume*, edited by H. Hofer, C. Taubes, A. Weinstein, and E. Zehnder, Birkhäuser, 1995, pp 483–524.

[20] H. Hofer and D.A. Salamon, Marked Riemann surfaces of genus zero, Preprint, August 1997.

[21] H. Hofer and D.A. Salamon, Gromov compactness and stable maps, Preprint, August 1997.

[22] H. Hofer and D.A. Salamon, J-holomorphic curves, multi-valued perturbations, and rational Gromov-Witten invariants, in preparation.

[23] H. Hofer and D.A. Salamon, Rational Floer homology and the general Arnold conjecture, in preparation.

[24] J. L. Kelley, *General Topology*, Van Norstrand, 1955.

[25] M. Kontsevich, Enumaration of rational curves via torus actions, Preprint, 1994.

[26] G. Liu, Associativity of quantum cohomology, Preprint, 1994.

[27] J. Li and G. Tian, Virtual moduli cycles and GW-invariants, Preprint, 1996.

[28] G. Liu and G. Tian, Floer Homology and Arnold Conjecture, Preprint, August 1996, revised May 1997.

[29] R.B. Lockhardt and R.C. McOwen, Elliptic operators on noncompact manifolds, it Ann. Sci. École Norm. Sup. Pisa **IV-12** (1985), 409–446.

[30] D. McDuff, Elliptic methods in symplectic geometry, *Bull. A.M.S.* **23** (1990), 311–358.

[31] D. McDuff and D. Salamon, *J-holomorphic Curves and Quantum Cohomology*, University Lecture Series **6**, American Mathematical Society, Providence, RI, 1994.

[32] D. McDuff and D. Salamon, *Introduction To Symplectic Topology*, Oxford University Press, 1995.

[33] J.W. Milnor, *Topology from the differential viewpoint*, The University Press of Virginia, 1965.

[34] S.P. Novikov, Multivalued functions and functionals - an analogue of the Morse theory, *Soviet Math. Dokl.* **24** (1981) 222–225.

[35] H. Ohta and K. Ono, Notes on symplectic 4-manifolds with $b^+ = 1$, I and II, Preprint, June 1995.

[36] K. Ono, The Arnold conjecture for weakly monotone symplectic manifolds, *Invent. Math.* **119** (1995), 519–537.

[37] T.H. Parker and J.G. Wolfson, Pseudoholomorphic maps and bubble trees, *Journ. Geom. Anal.* **3** (1993), 63–98.

[38] S. Piunikhin, D. Salamon, and M. Schwarz, Symplectic Floer-Donaldson theory and quantum cohomology, in *Contact and Symplectic Geometry*, edited by C.B. Thomas, Newton Institute Publications, Cambridge University Press, 1996, 171–200.

[39] M. Pozniak, Floer homology, Novikov rings, and clean intersections, PhD thesis, University of Warwick, 1994.

[40] J.W. Robbin and D.A. Salamon, The Maslov index for paths, *Topology* **32** (1993), 827–844.

[41] J.W. Robbin and D.A. Salamon, The spectral flow and the Maslov index, *Bulletin of the LMS* **27** (1995), 1–33.

[42] Y. Ruan, Virtual neighbourhoods and monopole equations, Preprint, March 1996.

[43] Y. Ruan and G. Tian, A mathematical theory of quantum cohomology, *J. Diff. Geom.* **42** (1995), 259–367.

[44] J. Sacks and K. Uhlenbeck, The existence of minimal immersions of 2-spheres, *Annals of Math.* **113** (1981), 1–24.

[45] D.A. Salamon, Morse theory, the Conley index and Floer homology, *Bull. L.M.S.* **22** (1990), 113–140.

[46] D.A. Salamon, *Spin Geometry and Seiberg-Witten invariants*, to appear in Birkhäuser Verlag.

[47] D.A. Salamon and E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, *Comm. Pure Appl. Math.* **45** (1992), 1303–1360.

[48] M. Schwarz, *Morse Homology*, Birkhäuser Verlag, 1993.

[49] M. Schwarz, PhD thesis, ETH-Zürich, (1995).

[50] M. Schwarz, In preparation.

[51] B. Siebert, Gromov-Witten invariants for general symplectic manifolds, Preprint, 1996.

[52] S. Smale, An infinite dimensional version of Sard's theorem, *Am. J. Math.* **87** (1973), 213–221.

[53] C. Viterbo, The cup-product on the Thom-Smale-Witten complex, and Floer cohomology, preprint (1993)

[54] E. Witten, Supersymmetry and Morse theory, *J. Diff. Geom.* **17** (1982), 661–692.