Calculus of Variations

Spring 2019

Michael Struwe

Introduction: A bit of history

Greeks' search for "optimal form" (as in isoperimetric problems, e.g. problem of Dido) is part of our classical heritage.

Fermat's postulate that light rays travel along an optically shortest path is the first analytic expression of variational principle, allowing to deduce the laws of reflection and refraction

In 1744 Euler published the first textbook on the Calculus of Variations, where he expresses his belief that "every effect in nature follows a maximum or minimum rule";

at the same time Maupertuis published his famous "least action principle" with the same claim.

However, extremal properties of solutions to the equations considered were usually deduced "a posteriori" after solving the Euler-Lagrange equations.

Conversely, in the 19th century a solution of the Dirichlet problem

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \subset \mathbb{R}^n \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
$$

arising in Riemann's proof of his famous uniformization theorem ("Riemann mapping thm.") was obtained by invoking Dirichlet's principle, assuming the existence of a minimizer of the associated energy

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f u \, dx.$$

This latter reasoning was finally validated by Weierstrass, Hilbert, and others.

The search for closed geodesics led to the development of tools for finding also non-minimizing critical points of variational integrals in the work of Birkhoff and later Lusternik-Schnirelman and Morse.

But a rigorous analytic foundation for a general critical point theory only became available in the 1960's through the work of Palais-Smale, which then opened the way to a very fruitful and increasingly successful development.

Contents.

References

Hildebrandt–Tromba: 'Mathematics and optimal form', Scientific American Books

M.S.: "Variational methods", Springer

and a number of further references mentioned in the lectures.

1. The direct method

1.1 A general existence result

"Dirichlet's principle" can be deduced from the following result.

Satz 5.4.1, FA I

Theorem 1.1.1. Let X be a reflexive Banach space, $M \subset X$ a weakly sequentially (w.s.) closed subset, and let $E : M \to \mathbb{R} \cup \{\infty\}$, $E \not\equiv \infty$. Also suppose that E is coercive in the sense

$$E(u) \to \infty \quad (\|u\|_X \to \infty,\ u \in M).$$

Finally, assume that E is weakly sequentially lower semi-continuous (w.s.l.s.c.), that is,

$$E(u) \leq \liminf_{k \to \infty} E(u_k)$$

whenever $(u_k) \subset M$ satisfies $u_k \xrightarrow{w} u$ $(k \to \infty)$.

Then there exists $u \in M$ such that

$$E(u) = \inf_{v \in M} E(v) < \infty .$$

Proof: Let $(u_k) \subset M$ be a minimizing sequence, satisfying

$$E(u_k) \to \inf_{v \in M} E(v) =: \alpha \geq -\infty.$$

By coerciveness of E, (u_k) is bounded.

Since X is reflexive, a sub-sequence $u_k \xrightarrow{w} u$ $(k \to \infty, k \in \Lambda)$.

Since $(u_k) \subset M$, and since M is w.s. closed, $u \in M$, and

$$\alpha \leq E(u) \leq \liminf_{k \to \infty} E(u_k) = \alpha$$

by w.s.l.s.c. of E. Thus,

$$\alpha = \inf_{v \in M} E(v) = E(u) > -\infty.$$

$\square$

Remarks.

1.2 Dirichlet's principle

Let $\Omega \subset \mathbb{R}^n$ be smoothly bounded, and let $f \in C^\infty(\bar{\Omega})$ (for simplicity).

We seek a (sufficiently smooth) function $u : \Omega \to \mathbb{R}$ (say, $u \in C^2(\bar{\Omega})$) such that

$$(1.2.1) \quad -\Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{(\partial x_i)^2} = f \quad \text{in } \Omega,$$

$$(1.2.2) \qquad u = 0 \quad \text{on } \partial\Omega.$$

Introduce the energy

$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v \, dx, \quad v \in H_0^1(\Omega).$$

Note that we can expand

$$(1.2.3) \quad E(v+\varphi) = E(v) + \int_\Omega (\nabla v \cdot \nabla\varphi - f\varphi)\, dx + \frac{1}{2}\int_\Omega |\nabla\varphi|^2 dx$$

for any $v, \varphi \in H_0^1(\Omega)$, so that

$$\int_\Omega (\nabla v \cdot \nabla \varphi - f\varphi)\, dx = \langle dE(v), \varphi \rangle_{H^{-1}\times H_0^1(\Omega)}$$

is the Fréchet derivative of E at $v \in H_0^1(\Omega)$, evaluated on $\varphi \in H_0^1(\Omega)$.

Proposition 1.2.1. Suppose that $u \in C^2(\bar{\Omega}) \hookrightarrow H_0^1(\Omega)$ solves (1.2.1), (1.2.2). Then there holds

$$E(u) = \min_{v \in H_0^1(\Omega)} E(v).$$

Proof. For any $\varphi \in H_0^1(\Omega)$, upon integrating by parts from (1.2.1) we have

$$\int_\Omega \left(\nabla u \cdot \nabla \varphi - f\varphi\right) dx = -\int_\Omega (\Delta u + f)\varphi\, dx = 0,$$

and (1.2.3) gives

$$E(u+\varphi) = E(u) + \frac{1}{2}\int_\Omega |\nabla \varphi|^2 dx \geq E(u),$$

as claimed. $\square$

The argument to conversely solve (1.2.1), (1.2.2) by minimizing E is more subtle.

In a first step we employ Theorem 1.1.1 to find a "weak solution" of (1.2.1), (1.2.2).

<u>Proposition</u> 1.2.2. For any $f \in L^2(\Omega)$ there exists $u \in H^1_0(\Omega)$ such that

$$E(u) = \min_{v \in H^1_0(\Omega)} E(v).$$

<u>Proof</u>. We verify the conditions in Thm. 1.1.1.

i) The space $X = H^1_0(\Omega)$ is a Hilbert space, hence also a reflexive Banach space;

ii) The set $M = X$ is w.s. closed;

iii) Since $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$, by Hölder's inequality for any $v \in H^1_0(\Omega)$ we have

$$\left|\int_\Omega f v \, dx\right| \leq \|f\|_{L^2} \|v\|_{L^2} \leq L \|f\|_{L^2} \|\nabla v\|_{L^2},$$

and $E(v)$ is finite for every $v \in H^1_0(\Omega)$, where we use that by <u>Poincaré's inequality</u>

$$\|v\|_{L^2} \leq L \|\nabla v\|_{L^2}, \quad v \in H^1_0(\Omega),$$

with $L > 0$ such that $\Omega \subset]0, L[\times \mathbb{R}^{n-1}$.

iv) For $v \in H^1_0(\Omega)$ then also $\|v\|_{H^1_0} \leq (1+L)\|\nabla v\|_{L^2}$ and

$$E(v) \geq \frac{1}{2}\|\nabla v\|^2_{L^2} - L\|f\|_{L^2}\|\nabla v\|_{L^2} \to \infty$$

if $\|v\|_{H^1_0} \to \infty$

~~we have (1.2.4)~~, and E is coercive.

v) If $u_k \overset{w}{\to} u$ in $H^1_0(\Omega)$ as $k \to \infty$,

by Rellich's theorem we may deduce

that $u_k \to u$ in $L^2(\Omega)$ and

$$\int_\Omega u_k f\,dx \to \int_\Omega u f\,dx \quad (k \to \infty).$$

Moreover, expanding with $\varphi_k := u_k - u \overset{w}{\to} 0$ $(k \to \infty)$, we have

$$\int_\Omega |\nabla u_k|^2 dx = \int_\Omega |\nabla(u + \varphi_k)|^2 dx$$

$$= \int_\Omega |\nabla u|^2 dx + \underbrace{2\int_\Omega \nabla u \nabla \varphi_k\,dx}_{= o(1) \to 0 \ (k \to \infty)} + \underbrace{\int_\Omega |\nabla \varphi_k|^2 dx}_{\geq 0}$$

$$\geq \int_\Omega |\nabla u|^2 dx + o(1),$$

and E is (w.s.l.s.c.).

Theorem 1.1.1 now yields the claim $\square$

Proposition 1.2.3. A minimizer $u \in H_0^1(\Omega)$ weakly solves (1.2.1), (1.2.2) in the sense that

$$\forall \varphi \in H_0^1(\Omega): \int_\Omega (\nabla u \cdot \nabla \varphi - f\varphi)\,dx = 0.$$

Proof: By (1.2.3) for any $\varphi \in H_0^1(\Omega)$ and any $\varepsilon \in \mathbb{R}$ we have

$$E(u) \leq E(u+\varepsilon\varphi) = E(u) + \varepsilon \int_\Omega (\nabla u\, \nabla\varphi - f\varphi)\,dx + \frac{\varepsilon^2}{2}\int_\Omega |\nabla\varphi|^2 dx.$$

Differentiating, we find

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u+\varepsilon\varphi) = \int_\Omega (\nabla u \cdot \nabla\varphi - f\varphi)\,dx,$$

as claimed. □

The second step in solving (1.2.1), (1.2.2) via Dirichlet's principle then is regularity theory, allowing to assert that the above weak solution actually solves (1.2.1), (1.2.2) classically.

Satz 9.1.2. FA I

1.3 Coercive nonlinearities

For smoothly bounded $\Omega \subset\subset \mathbb{R}^n$, $f \in C^\infty(\overline{\Omega})$ and any $\overset{1}{\cancel{2}} < p < \infty$ consider the problem

$$(1.3.1) \qquad -\Delta u + u|u|^{p-2} = f \quad \text{in } \Omega,$$

$$(1.3.2) \qquad u = 0 \quad \text{on } \partial\Omega$$

with associated energy

$$E(u) = \int_\Omega \left(\frac{|\nabla u|^2}{2} + \frac{|u|^p}{p} - fu \right) dx, \quad u \in H_0^1 \cap L^p(\Omega).$$

Recall that $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \le q \le 2^* := \frac{2n}{n-2}$, if $n \ge 3$. Thus, if $n \ge 3$, when $p > 2^*$ we may either regard $E : H_0^1 \cap L^p(\Omega) \to \mathbb{R}$, or view E as $E : H_0^1(\Omega) \to \mathbb{R} \cup \{\infty\}$.

With the latter choice $X = H_0^1(\Omega)$ we may invoke Thm. 1.1.1 most easily, since a large part of the proof of Prop. 1.2.2 still applies in the present context.

Proposition 1.3.1. There exists $u \in H_0^1(\Omega)$ such that

$$E(u) = \inf_{v \in H_0^1(\Omega)} E(v),$$

and $u \in C^{2,\gamma}(\bar{\Omega})$ for ~~every~~ some $0 < \gamma < 1$ is a classical solution of (1.3.1), (1.3.2).

Proof. We verify the conditions of Thm. 1.1.1.
$X = H_0^1(\Omega)$ is reflexive, $M = X$ is w.s.c.,
and $E : H_0^1(\Omega) \to \mathbb{R} \cup \{\infty\}$ is finite for $v \in H_0^1 \cap L^p(\Omega)$,
hence $E \not\equiv \infty$.

Moreover, by Step iv) of the proof of Prop. 1.2.2

$$E(v) \geq \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v\, dx \to \infty \quad (\|v\|_{H_0^1} \to \infty),$$

and E is coercive.

Finally, if $u_k \xrightarrow{w} u$ in $H_0^1(\Omega)$ as $k \to \infty$, by Rellich's theorem a subsequence $u_k \to u$ in $L^2(\Omega)$ and almost everywhere as $k \to \infty$, $k \in \Lambda$, and Fatou's lemma gives

$$\int_\Omega |u|^p dx \leq \liminf_{k \to \infty,\, k \in \Lambda} \int_\Omega |u_k|^p dx.$$

Together with Step v) of the proof of Prop. 1.2.2 this gives that E is w.s.l.s.c. on $H_0^1(\Omega)$.

By Thm. 1.1.1 there exists $u \in H_0^1(\Omega)$ such that

$$E(u) = \inf_{v \in H_0^1(\Omega)} E(v) < \infty;$$

in particular, $u \in L^p(\Omega)$ as well.

For any $\varphi \in \cancel{C_c^\infty(\Omega)}$ $H_0^1 \cap L^p(\Omega)$ then

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u+\varepsilon\varphi)$$

(1.3.3)
$$= \int_\Omega \left(\nabla u \nabla\varphi + \left(u|u|^{p-2} - f\right)\varphi\right) dx;$$

that is, $u \in H_0^1 \cap L^p(\Omega)$ solves (1.3.1) in the sense of distributions.

<u>Claim</u>: $u \in L^\infty$, $\|u\|_{L^\infty}^{p-1} \le \|f\|_{L^\infty}$.

<u>Proof</u>: Fix $L \geq 0$ with $L^{p-1} = \|f\|_{L^\infty}$.

Note

$$\varphi := (u-L)_+ = \max\{u-L, 0\} \in H_0^1 \cap L^p(\Omega).$$

By density of $C_c^\infty(\Omega)$ in $H_0^1 \cap L^p(\Omega)$, there is $(\varphi_k) \subset C_c^\infty(\Omega)$ with $\varphi_k \to \varphi$ in $H_0^1 \cap L^p(\Omega)$ as $k \to \infty$, and

$$\int_\Omega \left(\nabla u \, \nabla \varphi + u|u|^{p-2}\varphi - f\varphi \right) dx$$

$$= \lim_{k\to\infty} \int_\Omega \left(\nabla u \, \nabla \varphi_k + u|u|^{p-2}\varphi_k - f\varphi_k \right) dx = 0 .$$

Thus, φ is admissible in (1.3.3), and with

$$\nabla \varphi(x) = \begin{cases} \nabla u(x) , & \text{for a.e. } x \in \Omega \text{ with } u(x) > L \\ 0 , & \text{else} \end{cases}$$

we obtain

$$I := \int_\Omega \left(u|u|^{p-2} - f \right) (u-L)_+ \, dx$$

$$\leq \int_\Omega \left(\nabla u \, \nabla \varphi + u|u|^{p-2}\varphi - f\varphi \right) dx = 0 .$$

But $I > 0$, if we assume that $u(x) > L$ in a set of positive measure; so $u(x) \leq L$ almost everywhere. Similarly, $u(x) \geq -L$ a.e., and we obtain the claim. $\square$

In view of the claim now

$$-\Delta u = f - u|u|^{p-2} \in L^\infty;$$

hence, $u \in W^{2,q}(\Omega)$ for any $q < \infty$, and $u \in C^{1,\alpha}(\bar{\Omega})$ for every $0 < \alpha < 1$ by the Calderón-Zygmund estimate and Sobolev's embedding.

Since $p > 2$, then $u|u|^{p-2} \in C^\alpha$ for every $0 < \alpha < 1$,[1] and Schauder estimates give $u \in C^{2,\alpha}(\bar{\Omega})$.

$\square$

---

[1] $$\frac{|(u|u|^{p-2})(x) - (u|u|^{p-2})(y)|}{|x-y|^\gamma} \le \underbrace{\frac{|(u|u|^{p-2})(x) - (u|u|^{p-2})(y)|}{|u(x) - u(y)|^\gamma}}_{\le C < \infty,\ \gamma = p-1} \cdot \underbrace{\left(\frac{|u(x) - u(y)|}{|x-y|}\right)^\gamma}_{\le \|\nabla u\|_{L^\infty}}$$

1.4 A non-coercive model problem, constraints

For smoothly bounded $\Omega \subset\subset \mathbb{R}^n$, $2 < p < \infty$, now consider the problem

$$(1.4.1) \qquad -\Delta u = u|u|^{p-2} \text{ in } \Omega,$$

$$(1.4.2) \qquad u = 0 \text{ on } \partial\Omega$$

with associated energy

$$E(u) = \int_\Omega \left(\frac{|\nabla u|^2}{2} - \frac{|u|^p}{p}\right) dx, \quad u \in H_0^1 \cap L^p(\Omega).$$

Note that $u_0 \equiv 0$ is a non-degenerate solution of (1.4.1), (1.4.2) in the sense that the linearized operator $L = -\Delta - g'(0) = -\Delta$, where

$$g(u) = u|u|^{p-2}, \quad u \in \mathbb{R},$$

has positive spectrum.

However, u_0 does not achieve $\inf E$; in fact,

$$(1.4.3) \qquad E(su) = \frac{s^2}{2}\|\nabla u\|_{L^2}^2 - \frac{s^p}{p}\|u\|_{L^p}^p \xrightarrow{(s \to \infty)} -\infty$$

for every $u \in H_0^1 \cap L^p(\Omega) \setminus \{0\}$, and $\inf E = -\infty$.

ii) If $n \geq 3$ and $p > 2^*$, there holds

$$\inf_{u \in C_c^\infty(\Omega)} \|\nabla u\|_{L^2} / \|u\|_{L^p} = 0 .$$

Consequently, $u_0 \equiv 0$ is not a local E-minimizer in $H_0^1(\Omega)$ and it is unlikely that a non-trivial solution exists.

Note that by (1.4.3) $u_0 = 0$ is a local minimizer of E on any line $\{su;\ s \in \mathbb{R}\}$, $u \in H_0^1 \cap L^p(\Omega)$, and achieves a maximum when $s = s_u$, where

$$\langle dE(s_u u), u \rangle = s_u \|\nabla u\|_{L^2}^2 - s_u^{p-1} \|u\|_{L^p}^p = 0,$$

with

$$E(s_u u) = s_u^2 \left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla u\|_{L^2}^2 = \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{\|\nabla u\|_{L^2}}{\|u\|_{L^p}}\right)^{\frac{2p}{p-2}}.$$

i) see below

2 cases: ii) If $p \leq 2^* = \frac{2n}{n-2}$ for $n \geq 3$, by Sobolev's embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ there is $C_p > 0$ with

$$\|u\|_{L^p} \leq C_p \|\nabla u\|_{L^2}, \quad u \in H_0^1(\Omega),$$

and

$$E(s_u u) \geq \left(\frac{1}{2} - \frac{1}{p}\right) C_p^{-\frac{2p}{p-2}} =: \beta_p > 0.$$

Moreover, $u_0 \equiv 0$ is a local minimizer of E in the H_0^1-topology.

We may then hope to find a non-trivial solution $u \neq 0$ of (1.4.1), (1.4.2) along a ray $\{su;\ s \in \mathbb{R}\}$ where $u \in H_0^1(\Omega)$ minimizes the Sobolev quotient $\|\nabla u\|_{L^2} / \|u\|_{L^p}$.

Theorem 1.4.1. Let $\Omega \subset\subset \mathbb{R}^n$, $2 < p < 2^* = \frac{2n}{n-2}$ if $n \geq 3$, $2 < p < \infty$, if $n \leq 2$. Then there exists $0 < u \in C^2(\bar{\Omega})$ solving (1.4.1), (1.4.2).

For the proof we apply Thm. 1.1.1 with
$X = H_0^1(\Omega)$, $D(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx$, and

$$M = \{u \in H_0^1(\Omega);\ \|u\|_{L^p} = 1\}.$$

Note that M is (w.s.c.) by Rellich's thm, and D is (w.s.l.s.c.) and coercive on $H_0^1(\Omega)$.

By Thm. 1.1.1 thus there exists $u \in M$ with

$$D(u) = \inf_{v \in M} D(v).$$

Moreover, since $\|\nabla |u|\|_{L^2} = \|\nabla u\|_{L^2}$ and since $|u|$ and u have identical L^p-norm, we may assume that $u \geq 0$.

Claim 1: The function $\bar{u} = \alpha u$ weakly solves (1.4.1), (1.4.2), where

$$\alpha = \left(2D(u)\right)^{\frac{1}{p-2}} > 0 .$$

Proof: Given $0 \neq \varphi \in H_0^1(\Omega)$ fix $\varepsilon_0 = \frac{1}{2\|\varphi\|_{L^p}} > 0$.
Then for $|\varepsilon| < \varepsilon_0$ we have

$$\|u + \varepsilon\varphi\|_{L^p} > \frac{1}{2},$$

and

$$u_\varepsilon := \frac{u + \varepsilon\varphi}{\|u + \varepsilon\varphi\|_{L^p}} \in M, \quad |\varepsilon| < \varepsilon_0 .$$

Note that $\varepsilon \mapsto u_\varepsilon$ is of class C^1 with

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} u_\varepsilon = \varphi - u \frac{1}{p} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left(\int_\Omega |u + \varepsilon\varphi|^p dx \right)$$

$$= \varphi - u \int_\Omega u|u|^{p-2} \varphi \, dx$$

By minimality of $D(u)$ then

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} D(u_\varepsilon) = \int_\Omega \nabla u \cdot \nabla\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} u_\varepsilon\right) dx$$

$$= \int_\Omega \nabla u \cdot \nabla\varphi \, dx - \underbrace{\|\nabla u\|_{L^2}^2}_{= \alpha^{p-2}} \int_\Omega u|u|^{p-2} \varphi \, dx ;$$

that is, u weakly solves

$$-\Delta u = \|\nabla u\|_{L^2}^2 \, u|u|^{p-2} = u|\bar{u}|^{p-2} \quad \text{in } \Omega,$$

and our claim follows after multiplying with α. $\square$

For ease of notation, in the following we write u instead of $\bar{u}$.

Claim 2. $u \in C^2(\bar{\Omega})$ classically solves (1.4.1), (1.4.2); moreover, $u > 0$ in Ω.

Proof: Set $a := |u|^{p-2} \in L^{2^*/p-2}(\Omega)$, $n \geq 3$, where

$$s := \frac{2^*}{p-2} > \frac{2^*}{2^*-2} = \frac{n}{2} .$$

By Claim 1 then $u \in H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ weakly solves

$$-\Delta u = a u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Suppose $u \in L^{q_1}(\Omega)$ for some $q_1 \geq 2^*$. Then $au \in L^r(\Omega)$, where

$$\frac{1}{r} = \frac{1}{s} + \frac{1}{q_1},$$

and the Calderón-Zygmund inequality together with Sobolev's embedding yields $u \in W^{2,r}(\Omega) \hookrightarrow L^{q_2}$, with

$$\frac{1}{q_2} = \frac{1}{r} - \frac{2}{n} = \frac{1}{q_1} - \underbrace{\left(\frac{2}{n} - \frac{1}{s}\right)}_{=: \delta > 0},$$

or for any $q_2 < \infty$, if $\frac{1}{q_1} \leq \delta$.

Thus, either $q_1 \geq \frac{1}{8}$, or

$$q_2 = \frac{1}{\frac{1}{q_1} - 8} = \frac{q_1}{1 - 8q_1},$$

and after a finite number of iterations starting with $q_1 = 2^*$ we will have achieved that $u \in L^q(\Omega)$ for any $q < \infty$ as for $n \leq 2$;
hence $-\Delta u = au \in L^t(\Omega)$ for any $t < s$.

Choosing $\frac{n}{2} < t < s$, we arrive at

$$u \in W^{2,t}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$$

for some $0 < \alpha < 1$, and $-\Delta u = u|u|^{p-2} \in C^{0,\alpha}(\bar{\Omega})$.
Schauder theory then yields that $u \in C^{2,\alpha}(\bar{\Omega})$, proving the first claim.

Since in addition $u \geq 0$, $-\Delta u = u|u|^{p-2} \geq 0$, strict positivity of u follows from the strong maximum principle.

$\square$

Is the restriction $p < 2^*$ needed?

Thm. 1.4.2 (Pohožaev) Let $\Omega \subset\subset \mathbb{R}^n$, $n \geq 3$, be strictly star-shaped with respect to $x_0 = 0$ in the sense that

$$\nu(x) \cdot x > 0 \ , \quad x \in \partial\Omega,$$

where $\nu(x)$ is the outward unit normal to Ω at $x \in \partial\Omega$.

Then for $p \geq 2^*$ problem (1.4.1), (1.4.2) only admits $u_0 \equiv 0$ as classical solution.

Proof: Multiplying (1.4.1) with u and $x \cdot \nabla u$ we obtain the identities

$$0 = (\Delta u + u|u|^{p-2})u = \operatorname{div}(u \nabla u) - |\nabla u|^2 + |u|^p$$

and

$$0 = (\Delta u + u|u|^{p-2})\, x \cdot \nabla u$$

$$= \underbrace{\operatorname{div}(\nabla u\, x \cdot \nabla u)}_{= \sum_{i,j} \partial_i (\partial_i u\, x^j \partial_j u)} - |\nabla u|^2 - x \cdot \nabla \left(\frac{|\nabla u|^2}{2} - \frac{|u|^p}{p} \right)$$

$$= \operatorname{div}\left(\nabla u\, x \cdot \nabla u - x \left(\frac{|\nabla u|^2}{2} - \frac{|u|^p}{p} \right) \right) + \frac{n-2}{2} |\nabla u|^2 - \frac{n}{p} |u|^p,$$

respectively

Thus, we find

$$0 = \left(\Delta u + u|u|^{p-2}\right)\left(x\cdot\nabla u + \frac{n-2}{2}u\right)$$

$$= \operatorname{div}\left(\frac{n-2}{2}u\nabla u + \nabla u\, x\cdot\nabla u - x\left(\frac{|\nabla u|^2}{2} - \frac{|u|^p}{2}\right)\right)$$

$$+ \left(\frac{n-2}{2} - \frac{n}{p}\right)|u|^p .$$

Integrating, and using that (1.4.2) implies that

$$\nabla u = \nu\, \nu\cdot\nabla u \quad \text{on } \partial\Omega,$$

we obtain

$$0 = n\left(\frac{1}{2^*} - \frac{1}{p}\right)\int_\Omega |u|^p dx + \int_{\partial\Omega} \underbrace{\nu\cdot x}_{>0} \frac{|\nabla u|^2}{2} d\sigma,$$

where $\nu\cdot x > 0$ by assumption.

If $p > 2^*$, then $\|u\|_{L^p}^p = 0$, and $u \equiv 0$.

If $p = 2^*$, we find $\nabla u \equiv 0$ on $\partial\Omega$ and then $u \equiv 0$ by Heinz' theorem on unique continuation (1955).

$\square$

1.5 Concentration - Compactness

P.L. Lions' "Concentration-compactness principle" may be invoked to (partially) understand what happens in the case of problem (1.4.1), (1.4.2) when $p = 2^*$. We start with the locally compact setting.

1.5.1 The locally compact case

As a model problem consider

$$(1.5.1) \qquad -\Delta u + au = u|u|^{p-2} \text{ in } \mathbb{R}^n,$$

$$(1.5.2) \qquad u(x) \to 0 \quad (|x| \to \infty),$$

where $2 < p < 2^* = \frac{2n}{n-2}$ if $n \geq 3$, $2 < p < \infty$ for $n \leq 2$, and with $a \in C^0(\mathbb{R}^n)$ satisfying

$$(1.5.3) \qquad a(x) \to a_\infty > 0 \quad (|x| \to \infty).$$

Following the approach to (1.4.1), (1.4.2) in Section 1.4 we seek to find a solution $u > 0$ to (1.5.1), (1.5.2) by minimizing the energy

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^n}\left(|\nabla u|^2 + a u^2\right)dx$$

in $M = \{u \in H^1(\mathbb{R}^n);\ \|u\|_{L^p} = 1\}$.

Recall that for $n \geq 3$ we have

$$H^1(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n),\ L^{2^*}(\mathbb{R}^n)$$

and hence $H^1(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for every $2 < p < 2^*$ by interpolation, using Hölder's inequality.

Likewise, for $n \leq 2$ we have $H^1(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for every $2 < p < \infty$.

<u>Lemma</u> 1.5.1. $\mathcal{E}$ is (w.s.l.s.c) and coercive with respect to $H^1(\mathbb{R}^n)$.

<u>Proof</u>: Let

$$\Omega = \{x \in \mathbb{R}^n;\ a(x) < \tfrac{1}{2} a_\infty\},$$

Observe that by (1.5.3) the set Ω is bounded.

For $u \in M$ then

$$E(u) \geq \frac{1}{2} \|\nabla u\|^2_{L^2} + \frac{1}{4} a_\infty \|u\|^2_{L^2}$$

$$+ \frac{1}{2} \int_\Omega \left(a - \frac{1}{2} a_\infty\right) \underbrace{u^2}_{\in L^{p/2}} dx$$

$$\geq \min\left\{\frac{1}{2}, \frac{a_\infty}{4}\right\} \|u\|^2_{H^1} - C \mu(\Omega)^{\frac{p-2}{p}} \underbrace{\|u\|^2_{L^p}}_{=1}$$

$$\longrightarrow \infty \quad (\|u\|_{H^1} \to \infty,\ u \in M),$$

and E is coercive.

Moreover, splitting

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^n \setminus \Omega} a u^2 dx + \frac{1}{4} \int_\Omega a_\infty u^2 dx$$

$$+ \frac{1}{2} \int_\Omega \left(a - \frac{1}{2} a_\infty\right) u^2 dx$$

and observing that

$$(u, v)_a := \int_{\mathbb{R}^n} \nabla u \nabla v \, dx + \int_{\mathbb{R}^n \setminus \Omega} a u v \, dx + \int_\Omega \frac{a_\infty}{2} u v \, dx$$

defines an equivalent inner product on $H^1(\mathbb{R}^n)$, for $u_k \xrightarrow{w} u$ in $H^1(\mathbb{R}^N)$ we have

$$\|u_k\|_a^2 = (u_k, u_k)_a = \underbrace{(u_k - u, u_k - u)_a}_{\geq 0}$$

$$+ \underbrace{2(u, u_k - u)_a}_{\to 0 \ (k \to \infty)} + (u, u)_a \geq \|u\|_a^2 + o(1),$$

with $o(1) \to 0$ $(k \to \infty)$. Since by Rellich's theorem we may assume that $u_k \to u$ in $L^2(\Omega)$, we also have

$$\int_\Omega \left(a - \tfrac{1}{2} a_\infty\right) u_k^2 \, dx \xrightarrow{(k \to \infty)} \int_\Omega \left(a - \tfrac{1}{2} a_\infty\right) u^2 \, dx,$$

and we conclude that

$$\begin{aligned} E(u) &= \tfrac{1}{2} \|u\|_a^2 + \tfrac{1}{2} \int_\Omega \left(a - \tfrac{1}{2} a_\infty\right) u^2 \, dx \\ &\leq \liminf_{k \to \infty} \left(\tfrac{1}{2} \|u_k\|_a^2 + \tfrac{1}{2} \int_\Omega \left(a - \tfrac{1}{2} a_\infty\right) u_k^2 \, dx \right) \\ &= \liminf_{k \to \infty} E(u_k). \end{aligned}$$

$\square$

However, the set M is not (w.s.c.).
To see this, for $u \in M$, $x_0 \in \mathbb{R}^n$ let

$$u_{x_0}(x) = u(x + x_0), \quad x \in \mathbb{R}^n.$$

Then $u_{x_0} \in M$ for every $x_0 \in \mathbb{R}^n$, but $u_{x_0} \xrightarrow{w} 0 \notin M$,
$(|x_0| \to \infty)$

Moreover, we have after substituting $y = x + x_0$

$$E(u_{x_0}) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + a(x - x_0)\, u^2 \right) dx$$

$$\text{(1.5.4)} \qquad \underset{(|x_0| \to \infty)}{\longrightarrow} \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + a_\infty u^2 \right) dx =: E_\infty(u).$$

We call E_∞ the "functional at infinity".

Define

$$I := \inf_{u \in M} E(u), \quad I_\infty := \inf_{u \in M} E_\infty(u).$$

Thm 1.5.1 (P.L. Lions). i) There holds $I \leq I_\infty$,
ii) The condition $I < I_\infty$ is necessary and sufficient for the rel. compactness of all minimizing sequences $(u_k)_{k \in \mathbb{N}} \subset M$ for E.
iii) In particular, if $I < I_\infty$ there exists a solution $0 < u \in H^1(\mathbb{R}^n)$ of (1.5.1).

Preparations for the proof. For $0 \le \lambda \le 1$ let

$$M_\lambda = \{u \in H^1(\mathbb{R}^n);\ \|u\|_{L^p}^p = \lambda\}$$

and set

$$I_\lambda = \inf_{u \in M_\lambda} E(u), \quad I_{\lambda,\infty} = \inf_{u \in M_\lambda} E_\infty(u).$$

Note that by homogeneity there holds

$$E(u) = \lambda^{2/p} E\left(\frac{u}{\|u\|_{L^p}}\right), \quad u \in M_\lambda,$$

and likewise for E_∞, $0 < \lambda \le 1$. Hence

$$I_\lambda = \lambda^{2/p} I, \quad I_{\lambda,\infty} = \lambda^{2/p} I_\infty, \quad 0 \le \lambda \le 1,$$

Lemma 1.5.2. The condition $I < I_\lambda$ is equivalent to the condition

$$(1.5.5) \quad \forall\, 0 \le \lambda < 1:\ I < I_\lambda + I_{1-\lambda,\infty}.$$

Proof: Suppose (1.5.5) holds. Set $\lambda = 0$ to obtain $I < I_{1,\infty} = I_\infty$.

Conversely, suppose $I < I_\infty$ and fix $0 \leq \lambda < 1$.
Note that $I_\infty > 0$. If $I \leq 0$ then

$$I - I_\lambda = \underbrace{(1-\lambda^{2/p})}_{>0} I \leq 0 < (1-\lambda)^{2/p} I_\infty = I_{1-\lambda,\infty}.$$

If $I > 0$, observing that the condition $p > 2$ implies

$$1-\lambda \leq (1-\lambda)^{2/p}, \quad \lambda \leq \lambda^{2/p}$$

we have

$$I - I_\lambda = (1-\lambda^{2/p}) I \leq (1-\lambda)^{2/p} I$$
$$< (1-\lambda)^{2/p} I_\infty = I_{1-\lambda,\infty},$$

as claimed. $\square$

<u>Proof of Thm. 1.5.1.</u> i) We show $I \leq I_\infty$.
Suppose by contradiction that $I > I_\infty$. Choose $u \in M$ with

$$E_\infty(u) < I.$$

But then by (1.5.4) we have

$$I \leq E(u_{x_0}) \longrightarrow E_\infty(u) < I \quad (|x_0| \to \infty). \;\lightning$$

ii) Necessity. Suppose $I = I_\infty$, and let $(u_k) \subset M$ with

$$E_\infty(u_k) \longrightarrow I_\infty = I \quad (k \to \infty).$$

By (3) there is $(x_k) \subset \mathbb{R}^n$ such that

$$\left| E(u_{k,x_k}) - E_\infty(u_k) \right| \leq \frac{1}{k}, \quad \forall k;$$

in addition, we may assume that

$$v_k := u_{k,x_k} \xrightarrow{w} 0 \quad \text{in } H^1(\mathbb{R}^n).$$

Then

$$E(v_k) = E(u_{k,x_k}) \leq E_\infty(u_k) + \frac{1}{k} \xrightarrow{(k\to\infty)} I_\infty = I,$$

but (v_k) is not rel. compact.

Sufficiency. Suppose $I < I_\infty$, and let $(u_k) \subset M$ be an arbitrary minimizing sequence for E. Since E is coercive on M, (u_k) is bounded, and we may assume that

$$u_k \xrightarrow{w} u \quad \text{in } H^1(\mathbb{R}^n) \text{ as } k \to \infty$$

for some $u \in H^1(\mathbb{R}^n)$, and pointwise a.e.

Lemma 1.5.3 Let $u_k \xrightarrow{w} u$ in $H^1(\mathbb{R}^n)$ and a.e. With error $o(1) \to 0$ $(k \to \infty)$ then we have

$$1 = \|u_k\|_{L^p}^p = \|u_k - u\|_{L^p}^p + \|u\|_{L^p}^p + o(1).$$

Proof. Since $u_k \to u$ a.e., for any $0 \le t \le 1$ we also have

$$f_k(t,\cdot) := |u_k - tu|^{p-2}(u_k - tu)u \overset{(k\to\infty)}{\longrightarrow} (1-t)^{p-1}|u|^p \text{ a.e.}$$

Using that pointwise a.e. we can bound

(1.5.6) $|f_k(t,\cdot)| \le (|u_k| + |u|)^{p-1}|u|$, uniformly in $0 \le t \le 1$,

by dominated convergence then a.e. we have

$$|u_k|^p - |u_k - u|^p = -\int_0^1 \frac{d}{dt}|u_k - tu|^p dt$$

$$= p\int_0^1 |u_k - tu|^{p-2}(u_k - tu)u\, dt = p\int_0^1 f_k(t,\cdot)dt$$

$$\overset{(k\to\infty)}{\longrightarrow} p\int_0^1 (1-t)^{p-1}|u|^p dt = |u|^p.$$

But by (1.5.6) the functions $\int_0^1 f_k(t,\cdot)dt$ are also equi-integrable. Indeed for any measurable $\Omega \subset \mathbb{R}^n$

$$\int_\Omega \left|\int_0^1 f_k(t,x)dt\right| dx \le \int_\Omega (|u_k| + |u|)^{p-1}|u|dx$$

$$\le C\left(\|u_k\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1}\right)\left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}} < \varepsilon,$$

if $\mathcal{L}^n(\Omega) < \delta(\varepsilon)$ or if $\Omega \subset \mathbb{R}^n \setminus B_R(0)$, $R \ge R(\varepsilon)$. The claim then follows by Vitali's theorem. $\square$

Lemma 1.5.4. Let $u_k \overset{w}{\to} u$ in $H^1(\mathbb{R}^n)$. With error $o(1) \to 0$ we have

$$E(u_k) = E(u) + E_\infty(u_k - u) + o(1).$$

Proof. Note the identities

$$\|\nabla u_k\|_{L^2}^2 = \|\nabla((u_k-u)+u)\|_{L^2}^2$$

$$= \|\nabla(u_k-u)\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \underbrace{2\int_{\mathbb{R}^n} \nabla(u_k-u)\,\nabla u\,dx}_{\to 0\ (k\to\infty)}$$

and

$$\int_{\mathbb{R}^n} a u_k^2\,dx = \int_{\mathbb{R}^n} a\big((u_k-u)+u\big)^2 dx$$

$$= \int_{\mathbb{R}^n} a(u_k-u)^2 dx + \int_{\mathbb{R}^n} a u^2 dx + \underbrace{2\int_{\mathbb{R}^n} \underbrace{au}_{\in L^2}(u_k-u)\,dx}_{\to 0\ (k\to\infty)},$$

respectively. Set $v_k = u_k - u$, and for $\varepsilon > 0$ let

$$\Omega_\varepsilon = \{x \in \mathbb{R}^n;\ |a(x) - a_\infty| > \varepsilon\} \subset\subset \mathbb{R}^n.$$

By Rellich's theorem we ~~may assume~~ have that $v_k \to 0$ in $L^2(\Omega_\varepsilon)$ for every $\varepsilon > 0$ as $k \to \infty$.

Hence

$$\left|\int_{\mathbb{R}^n} a v_k^2 dx - \int_{\mathbb{R}^n} a_\infty v_k^2 dx\right| = \left|\int_{\mathbb{R}^n} (a-a_\infty) v_k^2 dx\right|$$

$$\le \left|\int_{\mathbb{R}^n\setminus\Omega_\varepsilon} (a-a_\infty) v_k^2 dx\right| + o(1) \le \varepsilon \|v_k\|_{L^2}^2 + o(1).$$

Letting $\varepsilon = \varepsilon_k \downarrow 0$ suitably, we obtain the claim. □

Returning to the proof of sufficiency of the condition $I < I_\infty$, resp. (1.5.5), for rel. compactness of (u_k), from Lemmas 1.5.3 and 1.5.4 we conclude that with $o(1) \to 0$ $(k \to \infty)$

$$I = E(u_k) + o(1) = E(u) + E_\infty(u_k - u) + o(1)$$

$$\geq I_\lambda + I_\infty(u_k - u) + o(1),$$

where

$$\lambda = \|u\|_{L^p}^p, \quad \|u_k - u\|_{L^p}^p \to 1 - \lambda \quad (k \to \infty)$$

for some $0 \leq \lambda \leq 1$.

Suppose $\lambda < 1$. Replacing $u_k - u$ by

$$w_k = \frac{u_k - u}{\|u_k - u\|_{L^p}} (1-\lambda)^{1/p} \in M_{1-\lambda},$$

we have

$$E_\infty(w_k) = E_\infty(u_k - u) + o(1)$$

and hence

$$I \geq I_\lambda + E_\infty(w_k) + o(1) \geq I_\lambda + I_{1-\lambda,\infty} + o(1),$$

contradicting (1.5.5) for large k.

Hence $\lambda = 1$, and $u \in M = M_1$. But then

$$I \leq E(u) \leq \lim_{k\to\infty} E(u_k) = E(u) + \lim_{k\to\infty} E_\infty(u_k - u) = I$$

shows that $E(u) = I$ and $\|u_k - u\|_{H^1}^2 \leq C E_\infty(u_k - u) \to 0$ $(k \to \infty)$.

Hence $u_k \to u$ in $H^1(\mathbb{R}^n)$, as claimed.

iii) Assume $I < I_\infty$ and let $(u_k) \subset M$ be a minimizing sequence. By part ii) we may assume that $u_k \xrightarrow{(k\to\infty)} u$ in $H^1(\mathbb{R}^n)$, where $u \in M$, and

$$E(u) = \lim_{k\to\infty} E(u_k) = \inf_{v\in M} E(v).$$

Thus, following the argument for the proof of Thm. 1.4.1, u weakly solves (1.5.1) and achieves (1.5.2) in the sense that $u \in L^2(\mathbb{R}^n)$. Since $p < 2^*$, also the argument for regularity may be carried over from Thm. 1.4.1, and $u \in C^2(\mathbb{R}^n)$ solves (1.5.1) classically.

Observing that $E(v) = E(|v|)$, $v \in M$, we may again assume that $u \geq 0$, and strict positivity follows from the maximum principle, applied on any ball $B_R(0)$, $R > 0$.

$\square$

Remark 2. If $a = a(|x|)$, we may restrict our attention to functions $u = u(|x|) \in H^1(\mathbb{R}^n)$. If $n \geq 3$, then for $2 < p < 2^*$ the set

$$M_{rad} = \{ u = u(|x|) \in H^1(\mathbb{R}^n) ; \ \|u\|_{L^p} = 1 \}$$

is (w.s.c.). Hence there always exists a (positive) minimizer of E in M_{rad}, corresponding to a solution $0 < u = u(|x|) \in H^1(\mathbb{R}^n)$ of (2).

In particular, for $a \equiv a_\infty = 1$ on $\mathbb{R}^3$, $p = 4$, there exists a ground state solution $u > 0$ of the static nonlinear Schrödinger equation

$$-\Delta u + u = u^3 \quad \text{on } \mathbb{R}^3 .$$

Proof. For $u = u(|x|) \in C_c^\infty(\mathbb{R}^n)$ we have

$$|u(x)| = \left| \int_{|x|}^\infty u_r(s)\, ds \right| \leq \left(\int_{|x|}^\infty |u_r(s)|^2 s^{n-1} ds \cdot \int_{|x|}^\infty \frac{ds}{s^{n-1}} \right)^{1/2}$$

$$\leq C \|\nabla u\|_{L^2} |x|^{\frac{2-n}{2}} , \quad \forall x \neq 0 .$$

Hence, $H^1_{rad}(\mathbb{R}^n) = \{ u \in H^1(\mathbb{R}^n) ; \ u = u(|x|) \} \hookrightarrow L^p(\mathbb{R}^n)$ compactly for any $p \in]2, 2^*[$. Indeed, let $u_k \xrightarrow{w} u$ in $H^1_{rad}(\mathbb{R}^n)$, and let $\varepsilon > 0$ be given. Then

$$\|u_k - u\|^p_{L^p(\mathbb{R}^n)} = \int_{B_R(0)} |u_k - u|^p dx + \int_{\mathbb{R}^n \setminus B_R(0)} |u_k - u|^p dx$$

$$\leq o(1) + \|u_k - u\|^2_{L^2(\mathbb{R}^n)} \|u_k - u\|^{p-2}_{L^\infty(\mathbb{R}^n \setminus B_R(0))} \leq C R^{-\frac{(n-2)(p-2)}{2}} + o(1)$$

$$\leq \varepsilon , \quad \text{if } R \geq R_0(\varepsilon), \ k \geq k_0(R, \varepsilon) .$$

$\square$

($a \equiv 1 \rightarrow$ transl. invariance, previous thm. does not apply) $I \equiv I_\infty$ removed in radial setting

1.5.2 The locally non-compact case.

As an illustration of the additional difficulties arising in the limit case $p = 2^* = \frac{2n}{n-2}$ when $n \geq 3$ consider the following.

Example. Let $0 \neq u \in H^1_0(B_1(0))$, $n \geq 3$.

For $k \in \mathbb{N}$ scale

$$u_k(x) = k^{\frac{n-2}{2}} u(kx), \quad x \in B_1(0),$$

where we extend $u(x) = 0$ for $|x| \geq 1$.

Note that $u_k \in H^1_0(B_1(0))$ with

$$\|\nabla u_k\|^2_{L^2} = \|\nabla u\|^2_{L^2} =: A,$$

$$\|u_k\|^{2^*}_{L^{2^*}} = \|u\|^{2^*}_{L^{2^*}} =: B,$$

for all $k \in \mathbb{N}$, and $u_k \xrightarrow[(k\to\infty)]{w} 0$ in $H^1_0(B_1(0))$

with

$$|\nabla u_k|^2 dx \xrightarrow{w^*} A\,\delta_{\{x=0\}}, \quad |u_k|^{2^*} dx \xrightarrow{w^*} B\,\delta_{\{x=0\}}$$

weakly in the sense of measures. Moreover, Sobolev's inequality

$$S\,\|u\|^2_{L^{2^*}} \leq \|\nabla u\|^2_{L^2}, \quad u \in C^\infty_c(\mathbb{R}^n)$$

gives the bound $S B^{2/2^*} \leq A$.

<u>Sobolev's constant</u>. The number

$$\mathcal{S} = \inf_{0 \neq u \in H_0^1(\Omega)} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^{2^*}}^2}$$

is the best constant for the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ for any open $\Omega \subset \mathbb{R}^n$.
By scaling as above, we see that $\mathcal{S}$ is independent of the domain Ω.

Similar to Section 1.5.1 we now again let $\Omega = \mathbb{R}^n$. Recall the definition

$$\dot{H}^1(\mathbb{R}^n) = clos\left(C_c^\infty(\mathbb{R}^n)\right)$$

with respect to the norm

$$\|u\|_{\dot{H}^1} = \|\nabla u\|_{L^2}.$$

Theorem 1.5.3 (P.-L. Lions) Let $(u_k) \subset \dot H^1(\mathbb{R}^n)$ with $u_k \xrightarrow{w} u$ in $\dot H^1(\mathbb{R}^n)$ and suppose that

$$\mu_k = |\nabla u_k|^2 dx \xrightarrow{w^*} \mu, \quad \nu_k = |u_k|^{2^*} dx \xrightarrow{w^*} \nu \quad (k \to \infty).$$

Then there is an at most countable set $J \subset \mathbb{N}$ and concentration points $x^{(j)} \in \mathbb{R}^n$ with weights $\mu^{(j)}, \nu^{(j)} > 0$, $j \in J$, such that

$$\mu \geq |\nabla u|^2 dx + \sum_{j \in J} \mu^{(j)} \delta_{\{x = x^{(j)}\}},$$

$$\nu = |u|^{2^*} dx + \sum_{j \in J} \nu^{(j)} \delta_{\{x = x^{(j)}\}}.$$

Moreover, we have

$$S \left(\nu^{(j)}\right)^{2/2^*} \leq \mu^{(j)}, \quad \forall j \in J,$$

and

$$\sum_{j \in J} \left(\nu^{(j)}\right)^{2/2^*} < \infty.$$

Remark. An analogous result holds true for the embeddings

$$\dot W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \quad kp < n, \quad \frac{1}{q} = \frac{1}{p} - \frac{k}{n}, \quad p > 1,$$

where $\dot W^{k,p}(\mathbb{R}^n)$ is the homogeneous Sobolev space obtained as closure of $C_c^\infty(\mathbb{R}^n)$ w.r.t.

$$\|u\|_{\dot W^{k,p}} = \sum_{|\alpha| = k} \|\partial^\alpha u\|_{L^p}, \quad u \in C_c^\infty(\mathbb{R}^n).$$

Proof. i) It suffices to consider $u = 0$.
Indeed, let $v_k = u_k - u$ with $v_k \xrightarrow{w} 0$ in $\dot{H}^1(\mathbb{R}^n)$.
By Lemma 1 with error $o(1) \xrightarrow{w^*} 0$ we have

$$|v_k|^{2^*} dx = \left(|u_k|^{2^*} - |u|^{2^*}\right) dx + o(1)$$

$$= \nu_k - |u|^{2^*} dx + o(1) \xrightarrow{w^*} \nu - |u|^{2^*} dx \geq 0,$$

$$|\nabla v_k|^2 dx = \left(|\nabla u_k|^2 - |\nabla u|^2\right) dx + o(1)$$

$$= \mu_k - |\nabla u|^2 dx + o(1) \xrightarrow{w^*} \mu - |\nabla u|^2 dx \geq 0$$

and it suffices to consider the case $u = 0$.

ii) Suppose now that $u_k \xrightarrow{w} 0$ in $\dot{H}^1(\mathbb{R}^n)$, and $u_k \to 0$ in $L^2_{loc}(\mathbb{R}^n)$ by Rellich's thm.
For any $\xi \in C_c^\infty(\mathbb{R}^n)$ then we obtain

$$S\left(\int_{\mathbb{R}^n} |\xi|^{2^*} d\nu\right)^{2/2^*} = S \lim_{k\to\infty} \left(\int_{\mathbb{R}^n} |\xi|^{2^*} d\nu_k\right)^{2/2^*}$$

$$= S \lim_{k\to\infty} \left(\int_{\mathbb{R}^n} |u_k \xi|^{2^*} dx\right)^{2/2^*} \underset{(1.5.6)}{\leq} \liminf_{k\to\infty} \left(\int_{\mathbb{R}^n} |\nabla(u_k \xi)|^2 dx\right)$$

$$= \liminf_{k\to\infty} \left(\int_{\mathbb{R}^n} |\xi|^2 |\nabla u_k|^2 dx\right) = \int_{\mathbb{R}^n} |\xi|^2 d\mu .$$

Let
$$\nu = \nu_0 + \sum_{j \in J} \nu^{(j)} \delta_{\{x = x^{(j)}\}},$$
with ν_0 free of atoms and $\nu^{(j)} > 0$, $j \in J$.

Since
$$\int_{\mathbb{R}^n} d\nu < \infty,$$
the set J is at most countable.

For any $j \in J$, upon choosing $\xi \in C_c^\infty(\mathbb{R}^n)$ with $0 \leq \xi \leq 1$, $\xi(x^{(j)}) = 1$, from (1.5.6) we obtain
$$S (\nu^{(j)})^{2/2^*} \delta_{\{x = x^{(j)}\}} \leq \mu;$$
hence
$$\mu \geq \sum_{j \in J} \mu^{(j)} \delta_{\{x = x^{(j)}\}}$$
with
$$S (\nu^{(j)})^{2/2^*} \leq \mu^{(j)}, \quad j \in J.$$
In particular, we have
$$S \sum_{j \in J} (\nu^{(j)})^{2/2^*} \leq \sum_{j \in J} \mu^{(j)} \leq \int_{\mathbb{R}^n} d\mu$$
$$\leq \limsup_{k \to \infty} \int_{\mathbb{R}^n} d\mu_k < \infty.$$

The Theorem then follows once we prove that $\nu_0 = 0$.

Claim. $\nu_0 = 0$.

Proof. Let $\Omega \subset \mathbb{R}^n$ be open, and let $(\xi_\ell) \subset C_c^\infty(\Omega)$ with $0 \leq \xi_\ell \leq 1$ satisfy

$$\xi_\ell \to \chi_\Omega \quad \text{a.e. } (k \to \infty).$$

From we ~~then~~ obtain

$$S\left(\int_\Omega d\nu\right)^{2/2^*} = S \lim_{\ell \to \infty} \left(\int_{\mathbb{R}^n} |\xi_\ell|^{2^*} d\nu\right)^{2/2^*}$$

(1.5.7)
$$\leq \lim_{\ell \to \infty} \left(\int_{\mathbb{R}^n} |\xi_\ell|^2 d\mu\right) = \int_\Omega d\mu .$$

In particular, $\nu_0 \leq \nu$ is absolutely continuous w.r.t. μ. By Radon-Nikodym Theorem there exists $f \in L^1(\mathbb{R}^n, d\mu)$ such ~~that~~

$$d\nu_0 = f \, d\mu,$$

and
$$f(x) = \lim_{r \to 0} \left(\frac{\int_{B_r(x)} d\nu_0}{\int_{B_r(x)} d\mu} \right) \qquad \mu\text{-a.e. } x \in \mathbb{R}^n .$$

But by (1.5.7) and since ν_0 is free of atoms

$$\frac{\int_{B_r(x)} d\nu_0}{\int_{B_r(x)} d\mu} \leq \frac{\left(\int_{B_r(x)} d\nu\right)^{2/2^*} \left(\int_{B_r(x)} d\nu_0\right)^{\frac{2^*-2}{2^*}}}{\int_{B_r(x)} d\mu}$$

$$\leq \frac{1}{S}\left(\int_{B_r(x)} d\nu_0\right)^{\frac{2^*-2}{2^*}} \to 0 \quad \text{as } r \to 0$$

for every $x \in \mathbb{R}^n$. Thus $f = 0$ μ-a.e., and $\nu_0 = 0$. □

Application. We use Theorems 1.3.2–3 to show that the (best) Sobolev constant for the embedding $\dot{H}^1(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$, $n \geq 3$, is attained.

Theorem 1.5.4. $\exists\, 0 < u \in \dot{H}^1(\mathbb{R}^n)$: $S\|u\|^2_{L^{2^*}} = \|\nabla u\|^2_{L^2}$.

Proof (following P.-L. Lions). Let

$$M = \{u \in \dot{H}^1(\mathbb{R}^n);\ \|u\|_{L^{2^*}} = 1\},$$

and let $(u_k) \subset M$ be a minimizing sequence for

$$E(u) = \int_{\mathbb{R}^n} |\nabla u|^2 dx.$$

Normalization. Scale

$$\tilde{u}_k(x) = R_k^{\frac{n-2}{2}} u_k(x_k + R_k x)$$

with suitable $x_k \in \mathbb{R}^n$, $R_k > 0$. Note that

$$\|\tilde{u}_k\|_{L^{2^*}} = \|u_k\|_{L^{2^*}} = 1, \quad k \in \mathbb{N},$$

while

$$E(\tilde{u}_k) = E(u_k), \quad k \in \mathbb{N}$$

For suitably chosen x_k and R_k we can achieve

$$(1.5.8) \quad \int_{B_1(0)} |\tilde{u}_k|^{2^*} dx = \sup_{x_0 \in \mathbb{R}^n} \int_{B_1(x_0)} |\tilde{u}_k|^{2^*} dx = 1/2.$$

Replacing u_k by $\tilde{u}_k$, if necessary, we may assume $u_k = \tilde{u}_k$.

Convergence. We may assume $u_k \overset{w}{\rightharpoonup} u$ in $\dot{H}^1(\mathbb{R}^n)$. Let $v_k = u_k - u \overset{w}{\rightharpoonup} 0 \ (k\to\infty)$. By Lemma 1 with error $o(1) \to 0 \ (k \to \infty)$ we have

$$E(u_k) = E(u) + E(v_k) + o(1),$$

and

$$1 = \|u_k\|^{2^*}_{L^{2^*}} = \|u\|^{2^*}_{L^{2^*}} + \|v_k\|^{2^*}_{L^{2^*}} + o(1).$$

Set $\lambda = \|u\|^{2^*}_{L^{2^*}} \in [0,1]$. If $\lambda = 1$, we have $u \in M$, and hence

$$S \leq E(u) \leq \liminf_{k\to\infty} E(u_k) = S,$$

and the proof is complete.

If $0 < \lambda < 1$, by strict concavity of the map $f(s) = s^{2/2^*}$ we have

$$\lambda^{2/2^*} + (1-\lambda)^{2/2^*} > 1$$

and hence

$$S < S \liminf_{k\to\infty} \left(\|u\|^2_{L^{2^*}} + \|v_k\|^2_{L^{2^*}} \right)$$

$$\leq \liminf_{k\to\infty} \left(\|\nabla u\|^2_{L^2} + \|\nabla v_k\|^2_{L^2} \right) \leq \liminf_{k\to\infty} \|\nabla u_k\|^2_{L^2} = S;$$

Contradiction!

Finally, if $\lambda = 0$ we have $u = 0$; that is, $u_k \overset{w}{\rightharpoonup} 0$ in $\dot{H}^1(\mathbb{R}^n)$.

By Thm. 1.5.3 we may assume that

$$|u_k|^{2^*} dx \overset{w^*}{\rightharpoonup} \nu = \sum_{i \in J} \nu_i \, \delta_{\{x = x^{(i)}\}}$$

with weights $0 < \nu_i < 1$, $i \in J$. Our normalization (1.5.8) implies

$$\sum_{|x^{(i)}| \leq 1} \nu_i \geq \frac{1}{2} \quad , \quad 0 < \nu_i \leq \frac{1}{2}, \quad \forall i \in J.$$

For a suitable $\varepsilon > 0$, as will be determined below, choose $R > 1$ such that

$$\sum_{|x^{(i)}| > R} \nu_i \leq \varepsilon .$$

Fix a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_R(0)$, $|\nabla \varphi| < 1$.

Decompose $u_k = v_k + w_k$, where

$$v_k = u_k \varphi \, , \quad w_k = u_k (1 - \varphi) \overset{w}{\rightharpoonup} 0 \text{ in } \dot{H}^1(\mathbb{R}^n).$$

We can estimate $\varphi^{2^*} + (1-\varphi)^{2^*} \leq 1$ and thus also

$$0 \leq \|u_k\|_{L^{2^*}}^{2^*} - \left(\|v_k\|_{L^{2^*}}^{2^*} + \|w_k\|_{L^{2^*}}^{2^*}\right)$$

(1.5.9)

$$= \int_{\mathbb{R}^n} |u_k|^{2^*} \underbrace{\left(1 - \left(\varphi^{2^*} + (1-\varphi)^{2^*}\right)\right)}_{\in C_0^\infty(\mathbb{R}^n \setminus B_R(0))} dx$$

$$\xrightarrow{(k\to\infty)} \sum_{0<\varphi(x^{(i)})<1} \nu_i \left(1 - \left(\varphi^{2^*}(x^{(i)}) + (1-\varphi(x^{(i)}))^{2^*}\right)\right)$$

$$\leq \sum_{|x^{(i)}|>R} \nu_i \leq \varepsilon,$$

Let

$$\omega_0 = \liminf_{k\to\infty} \|w_k\|_{L^{2^*}}^{2^*} \leq \frac{1}{2}$$

(by (1.5.8)) and recall that

$$|v_k|^{2^*} dx = |u_k|^{2^*} \varphi^{2^*} dx \xrightarrow{w^*} \nu\varphi^{2^*} =: \omega$$

$$= \sum_{i\in J} \underbrace{\nu_i \varphi^{2^*}(x^{(i)})}_{=:\omega_i} \delta_{\{x=x^{(i)}\}}$$

with

$$\sum_{i\in J} \omega_i \geq \sum_{|x^{(i)}|\leq R} \nu_i \geq \frac{1}{2} \geq \nu_i \geq \omega_i, \quad \forall i \in J$$

The estimate (1.5.9) implies that

$$0 \leq 1 - \left(\sum_{i\in J} \omega_i + \omega_0\right) \leq \varepsilon;$$

that is

$$\Omega := \omega_0 + \sum_{i\in J} \omega_i \in [1-\varepsilon, 1].$$

By Thm. 1.5.3 we also may assume that

$$|\nabla v_k|^2 dx \overset{w^*}{\rightharpoonup} \lambda \geq \sum_{i \in J} \lambda_i \, \delta_{\{x = x^{(i)}\}},$$

where

$$S \, \omega_i^{2/2^*} \leq \lambda_i, \quad \forall i \in J.$$

It follows that, with $J_0 := J \cup \{0\}$, we have

$$S \geq E(v_k) + E(w_k) + o(1)$$

$$\geq \sum_{i \in J} \lambda_i + S \, \|w_k\|^2_{L^{2^*}} + o(1)$$

$$\geq S \sum_{i \in J_0} \omega_i^{2/2^*} + o(1) \geq S (1-\varepsilon)^{2/2^*} \sum_{i \in J_0} \left(\frac{\omega_i}{\Omega}\right)^{2/2^*} + o(1);$$

that is,

$$S \geq S (1-\varepsilon)^{2/2^*} \sum_{i \in J_0} \left(\frac{\omega_i}{\Omega}\right)^{2/2^*}.$$

But for $0 < \varepsilon < \frac{1}{3}$ we can split any sum of numbers $0 < \alpha_i \leq \frac{1}{2(1-\varepsilon)} < \frac{3}{4}$, $i \in J_0$, with

$$\sum_{i \in J_0} \alpha_i = 1$$

into two separate sums with

$$\frac{1}{4} < \sum_{i \in I} \alpha_i, \ \sum_{i \in J_0 \setminus I} \alpha_i < \frac{3}{4}.$$

(This is clear if there exists i_0 with $\alpha_{i_0} > \frac{1}{4}$; otherwise we collect indices i until $\sum_{i \in I} \alpha_i$ first exceeds $\frac{1}{4}$.)

By strict convexity then

$$\sum_{i\in J_0} \alpha_i^{2/2^*} = \sum_{i\in I} \alpha_i^{2/2^*} + \sum_{i\in J_0\setminus I} \alpha_i^{2/2^*}$$

$$\geq \Big(\sum_{i\in I} \alpha_i\Big)^{2/2^*} + \Big(\sum_{i\in J_0\setminus I} \alpha_i\Big)^{2/2^*}$$

$$\geq \inf_{\frac{1}{4}<s<\frac{3}{4}} \left(s^{2/2^*} + (1-s)^{2/2^*}\right) > 1 .$$

We conclude that for sufficiently small $0<\varepsilon<\frac{1}{3}$ we have

$$(1-\varepsilon)^{2/2^*} \sum_{i\in J_0} \left(\frac{\omega_i}{\Omega}\right)^{2/2^*} > 1 .$$

The contradiction rules out the case $\lambda = 0$.

Thus $\lambda = 1$, and the proof is complete. $\square$

Remark. i) The Sobolev constant S is never attained on a domain $\Omega \subsetneq \mathbb{R}^n$, $n \geq 3$.

If we suppose that for some such domain Ω there exists $0 < u \in H_0^1(\Omega)$ with $S\|u\|^2_{L^{2^*}} = \|\nabla u\|^2_{L^2}$ a suitable multiple $\tilde{u} = \alpha u$, $\alpha > 0$, solves

$$
\begin{aligned}
-\Delta \tilde{u} &= \tilde{u}^{2^*-1} && \text{in } \Omega, \\
\tilde{u} &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

From this fact, and from the fact that by scale-invariance $S(\Omega) = S(\mathbb{R}^n) = S$ is independent of the domain we derive a contradiction, as follows.

Case 1. If Ω is bounded, $\Omega \subset B_R(0)$ for some $R > 0$, extending $u(x) = 0$ for $x \in B_R(0) \setminus \Omega$, from u we obtain a minimizer $0 \neq u \in H_0^1(B_R(0))$ of Sobolev's ratio, which is impossible by Pohozaev's theorem 1.4.2.

Case 2. If Ω is unbounded, $\Omega \neq \mathbb{R}^n$, by extending $u(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$ we obtain a $0 \leq \tilde{u} \neq 0$ solving $-\Delta \tilde{u} = \tilde{u}^{2^*-1}$ in $\mathbb{R}^n$ vanishing outside Ω, and the strong maximum principle then gives a contradiction.

ii) On $\mathbb{R}^n$, $n \geq 3$, the Sobolev constant is attained by the Talenti functions

$$u(x) = \left(\frac{1}{1+|x|^2}\right)^{\frac{n-2}{2}}$$

and their rescalings

$$u_{\varepsilon,x_0}(x) = \varepsilon^{\frac{2-n}{2}} u\left(\frac{x-x_0}{\varepsilon}\right)$$

$$= \left(\frac{\varepsilon}{\varepsilon^2 + |x-x_0|^2}\right)^{\frac{n-2}{2}},$$

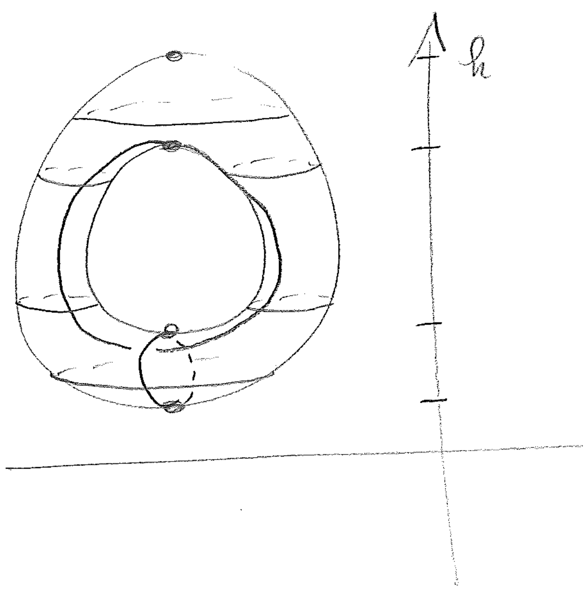
which we will re-encounter later in the context of Yamabe's theorem.

iii) Lions' proof of Thm. 1.5.4 also works in the case of the embedding $\mathring{W}^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ when $\frac{1}{q} = \frac{1}{p} - \frac{k}{n} > 0$ and yields a minimizer of the Sobolev ratio in this case, as well.

2. Saddle points

2.1 Finite-dimensional examples

Example 1. The height function h on a torus as in the illustration



has exactly 4 critical points p_i where

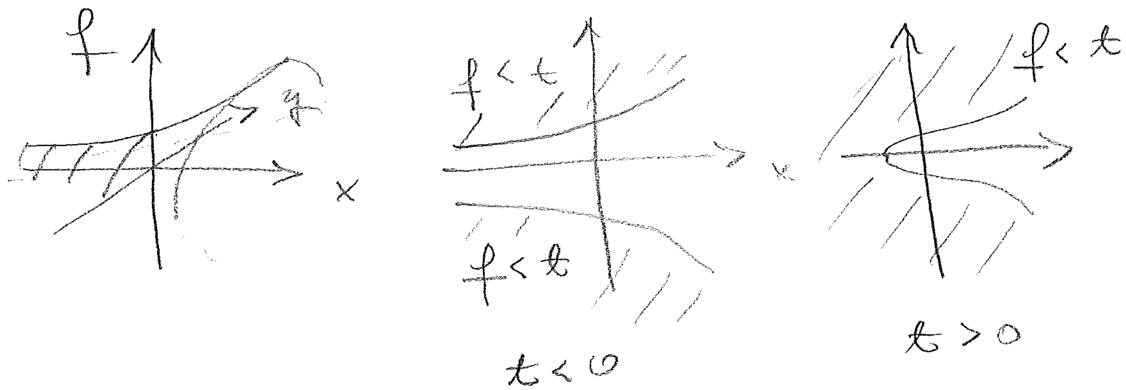
$$dh(p_i) = 0, \quad 1 \leq i \leq 4,$$

reflecting a change of topology of the level surfaces $h^{-1}(t)$, $t \in \mathbb{R}$, or corresponding to the generators of $\pi_k(T^2)$, $k = 0, 1, 2$.

<u>Example</u> 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = e^x - y^2, \quad (x,y) \in \mathbb{R}^2.$$

Note that for $t < 0$ the level set $f^{-1}(t)$ has 2 components while for $t > 0$ it has only one component.



However, in view of

$$df(x,y) = (e^x, -2y) \neq 0, \quad (x,y) \in \mathbb{R}^2,$$

the function f does not have any critical points! - This shows that some <u>compactness</u> is needed if we want to carry over the ideas from Example 1 to the general case.

Example 3: A finite-dimensional analogue of the famous theorem by Lusternik-Schnirelman.

In 1929 Lusternik-Schnirelman published the following result.

Theorem 2.1.1 (L-S). On any surface $S \subset \mathbb{R}^3$ diffeomorphic to S^2 there are at least 3 geometrically distinct, non-constant closed geodesics with self-intersections ("prime").

An analogous result can be obtained for a flat, convex, smoothly bounded billard. We may view this as a special limit case of the above theorem, for a sequence of convex surfaces S_k that are symmetric with respect to the (x,y)-plane with fixed intersection

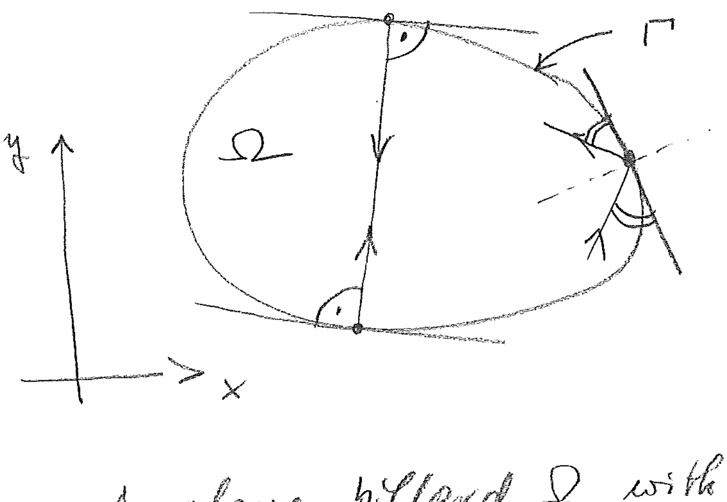
$$\Gamma = S_k \cap \{(x,y,z) \in \mathbb{R}^3;\ z=0\},\ k \in \mathbb{N}$$

and height above $\{z=0\}$ tending to zero $(k \to \infty)$.

Def. 2.1.1. i) A (convex) billard is a smoothly bounded, strictly convex domain $\Omega \subset\subset \mathbb{R}^2$ with boundary Γ diffeomorphic to S^1.

ii) Geodesics in Ω are straight lines that are reflected with equal angles in $\Gamma = \partial\Omega$.

iii) Prime (non self-intersecting) closed geodesics are lines through Ω that are reflected in themselves



A plane billard Ω with a prime closed geodesic

The analogue of Theorem 2.1.1 for a billard then may be stated, as follows.

<u>Theorem</u> 2.1.2. Any plane, convex billard Ω contains (at least) 2 distinct prime closed geodesics.

<u>Remark</u> 2.1.1. Regarding Ω as a (collapsed) surface $S \subset \mathbb{R}^3$, doubly covered by Ω, we may regard $\Gamma = \partial\Omega$ as a first prime closed geodesic in S and the 2 closed geodesics guaranteed by Thm. 2.1.2 as the remaining 2 prime closed geodesics as in Thm 2.1.1 by Lusternik - Schnirelman.

<u>Analytic formulation</u>. Parametrize Γ by $\gamma \in C^2(\mathbb{R}/\mathbb{Z}; \mathbb{R}^2)$ and let

$$f(s,t) := \frac{1}{2} |\gamma(s) - \gamma(t)|^2, \ (s,t) \in \mathbb{R}^2.$$

A line ℓ connecting $\gamma(s)$ with $\gamma(t)$ is reflected in itself at the point $\gamma(s)$ iff

$$0 = \langle \dot\gamma(s), \gamma(s)-\gamma(t)\rangle_{\mathbb{R}^2} = \frac{\partial}{\partial s} f(s,t),$$

and at the point $\gamma(t)$ iff

$$0 = \langle \dot\gamma(t), \gamma(s)-\gamma(t)\rangle_{\mathbb{R}^2} = -\frac{\partial}{\partial t} f(s,t).$$

<u>Conclusion.</u> A line $\ell = \overline{\gamma(s)\,\gamma(t)}$ is a prime closed geodesic in Ω iff

$$df(s,t) = 0;$$

that is, prime closed geodesics uniquely correspond to critical points of f.

<u>Obvious critical points</u>. Since our domain $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \cong S^1 \times S^1 \cong T^2$ is compact, f achieves its maximum and its minimum. However, since $f(s,t) \geq 0$, $f(s,t) = 0 \Longleftrightarrow s = t \mod \mathbb{Z}$ minimizers are related to the constant curves and do not give rise to prime closed geodesics in Ω.

On the other hand, there exists $(\bar{s}, \bar{t})$ with

$$f(\bar{s}, \bar{t}) = \max_{0 \le s, t \le 1} f(s,t) = \bar{\beta} > 0,$$

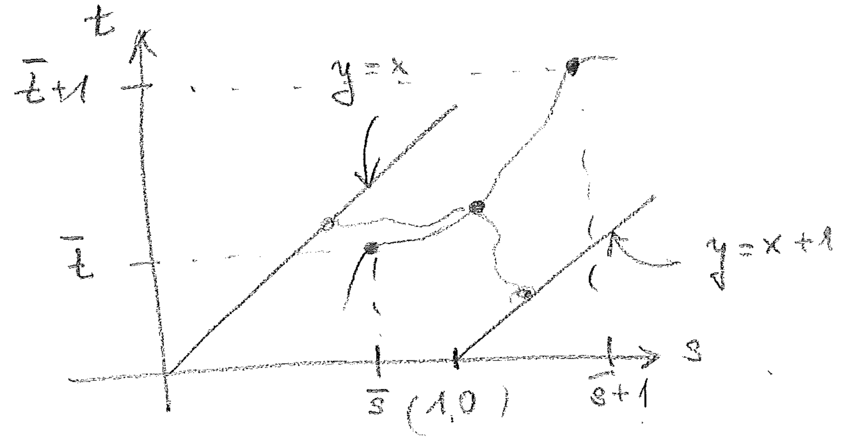
corresponding to a "longest" closed prime geodesic: the <u>diameter</u> of Ω.

<u>Saddle points</u>. In the "energy landscape" defined by f, the "valleys"

$$\{(s,s);\ s \in \mathbb{R}\}, \quad \{(s+1, s);\ s \in \mathbb{R}\}$$

thus are separated by a "mountain ridge" with peaks at the points

$$\{(\bar{s}+k, \bar{t}+k);\ k \in \mathbb{Z}\}.$$



We hope to find a second critical point at a "mountain-pass".

For this we seek a "path" p connecting two points x_0, x_1 in different "valleys" and crossing the dividing "mountain ridge" at point of lowest maximal "elevation".

In the present finite-dimensional setting, "paths" may be simply compact connected sets.

<u>Def</u>. 2.1.1. A set $\phi \neq p \subset T^2$ is <u>connected</u> if for any open sets $O_1, O_2 \subset T^2$ there holds

$$p \subset O_1 \cup O_2, \quad O_1 \cap p \neq \phi = O_2 \cap p \Rightarrow O_1 \cap O_2 \neq \phi.$$

Recall that an interval $\phi \neq I \subset \mathbb{R}$ is connected; moreover, continuous functions map connected sets to connected sets.

Proof of Thm. 2.1.2. Let

$$\beta = \inf_{p \in P} \sup_{(s,t)\in p} f(s,t) > 0,$$

where

$$P = \{ p \subset T^2 : x_0 := (0,0) \in p,\ x_1 := (1,0) \in p,\ p \text{ compact and connected} \}.$$

Claim 1. $\exists p \in P : \sup_{(s,t)\in p} f(s,t) = \beta.$

Proof: Let $(p_k) \subset P$ with

$$\sup_{(s,t)\in p_k} f(s,t) < \beta + \frac{1}{k}, \quad k \in \mathbb{N},$$

and set

$$K_\ell = \overline{\bigcup_{k \geq \ell} p_k} \subset T^2.$$

Then K_ℓ is compact and connected,

and $x_{0,1} \in K_{\ell+1} \subset K_\ell \neq \emptyset$, $\ell \in \mathbb{N}$.

It follows that

$$x_{0,1} \in K := \bigcap_{\ell \in \mathbb{N}} K_\ell \subset K_\ell, \quad \ell \in \mathbb{N},$$

is compact and connected, and

$$\sup_K f \leq \sup_{K_\ell} f \leq \beta + \frac{1}{\ell}, \quad \ell \in \mathbb{N}.$$

Set $p := K$. $\square$

Claim 2. $\exists X^* = (s^*, t^*) \in p : f(s^*,t^*) = \beta,\ df(s^*,t^*) = 0.$

Proof: (indirect) Suppose by contradiction that for any $x = (s,t) \in p$ with $f(x) = \beta$ there holds $df(x) \neq 0$.

By compactness of ϕ then there exists $\varepsilon > 0$ such that

$$\forall x \in T^2:\ \operatorname{dist}(x,p) = \inf\{|x-y|;\ y \in p\} < \varepsilon,$$

$$(2.1.1)\qquad |f(x) - \beta| < \varepsilon \Rightarrow |df(x)| > \varepsilon.$$

NLoG: $\varepsilon < \beta$.

Define the negative gradient flow $\Phi : T^2 \times [0,1] \to T^2$, as follows.

Let $\varphi \in C^\infty(\mathbb{R})$ with $0 \leq \varphi \leq 1$, $\varphi(s) = 1$ for $s \geq 1$ and for $s \leq 1/2$, $\varphi(s) = 0$ for any $x \in T^2$ let $\Phi(x,t)$ solve the initial value problem

$$(2.1.2)\qquad \frac{d}{dt}\Phi(x,t) = -\varphi\left(\frac{\beta - f(x)}{\varepsilon}\right) df\big(\Phi(x,t)\big),$$

$$(2.1.3)\qquad \Phi(x,0) = x.$$

Since $\gamma \in C^2$, $f \in C^2$ the vector field on the right of (2.1.2) is of class C^1. Hence there exists a solution $\Phi \in C^1(T^2 \times [0,1], T^2)$ of (2.1.2), (2.1.3). (The solution exists for $0 \leq t \leq 1$ by compactness of T^2.)

Moreover, for any $x \in p$, any $0 < t < 1$ such that $\beta \geq f(\Phi(x,t)) > \beta - \varepsilon/2$

we have $\varphi\left(\dfrac{\beta - f(\Phi(x,t))}{\varepsilon}\right) = 1$

and hence

$$\frac{d}{dt} f(\Phi(x,t)) = \langle df(\Phi(x,t)), \frac{d}{dt}\Phi(x,t) \rangle$$

(2.1.4)

$$= - |df(\Phi(x,t))|^2;$$

in addition, letting

$$C_1 = \sup_{x \in T^2} |df(x)| < \infty$$

there holds

$$\operatorname{dist}(\Phi(x,t), p) \leq |\Phi(x,t) - x| \leq \int_0^t \left|\frac{d}{dt}\Phi(x,s)\right| ds \leq Ct$$

$$\leq C_1 t < \varepsilon,$$

if $0 < t < \varepsilon/C_1$, and hence (2.1.1) and (2.1.4) give

$$\frac{d}{dt} f(\Phi(x,t)) < -\varepsilon^2.$$

In conclusion, at time $t_1 = \min\{1, \varepsilon/C_1\} > 0$ for any $x \in p$ we either have

$$f(\Phi(x, t_1)) \leq \beta - \frac{\varepsilon}{2} < \beta,$$

or there holds

$$\beta \geq f(x) \geq f(\Phi(x,t)) \geq \beta - \frac{\varepsilon_1}{2}, \quad 0 \leq t \leq t_1,$$

and thus

$$\frac{d}{dt} f(\Phi(x,t)) < -\varepsilon^2, \quad 0 \leq t \leq t_1,$$

leading to the bound

$$f(\Phi(x, t_1)) \leq f(x) - \varepsilon^2 t_1 \leq \beta - \frac{\varepsilon^3}{C_1} < \beta.$$

Finally, since $0 < \varepsilon < \beta$ by assumption, we have $\beta - f(x_{0,1}) = \beta > \varepsilon$, and $\Phi(x_{0,1}, t) = x_{0,1}$ for all $0 \leq t \leq t_1$.

Thus, $p_1 := \Phi(p, t_1) \in P$.

But

$$\sup_{p_1} f \leq \max\left\{\beta - \frac{\varepsilon}{2}, \beta - \frac{\varepsilon^3}{C_1}\right\} < \beta,$$

contradicting the definition of β. $\square$

If $\beta < \bar{\beta}$ clearly $x^* \neq \bar{x}$, and the proof is complete.

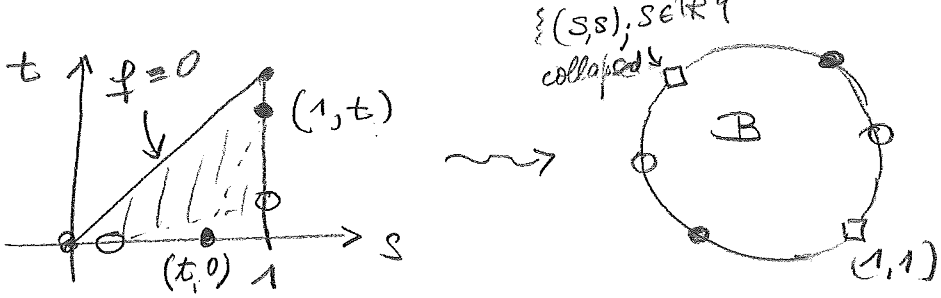
If $\beta = \bar{\beta}$ then <u>every</u> $p \in P$ contains a point $x = (s, t)$ of maximal length, corresponding to a prime closed geodesic. Thus, in this case there even exist infinitely many such lines. $\square$

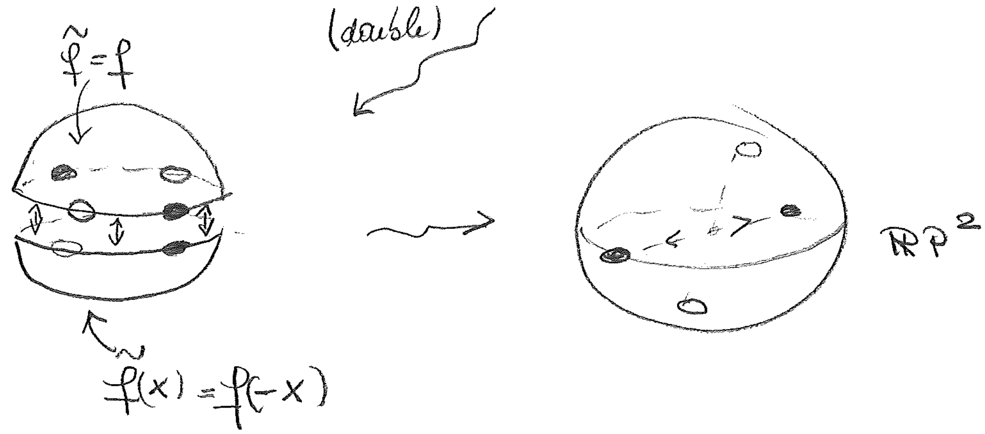
Remarks. i) The existence of 3 critical points in Thm. 2.1.2 again reflects the topology of the underlying space.

To see this, note that the symmetry

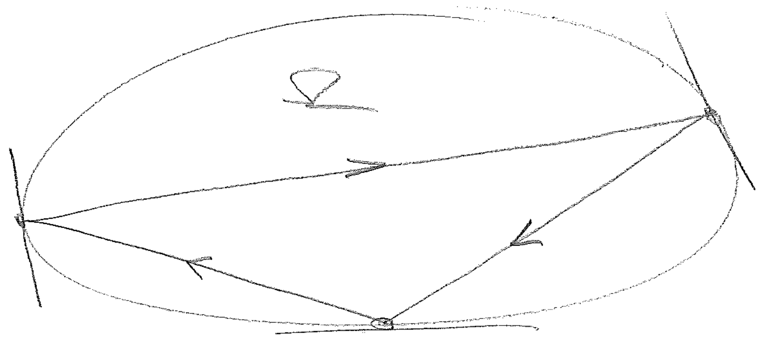
$$f(s,t) = f(t,s) = f(t,s-1), \quad 0 \le s,t \le 1,$$

allows to regard f as a function on $\mathbb{R}P^2$ by means of the identifications indicated in the picture below.





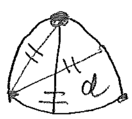
ii) There are other kinds of "closed geodesics" in a billard, as in the picture below.



These (and more complicated patterns) can also be studied with tools from the theory of dynamical systems.

iii) If $\beta = \beta_0$ the billard Ω has the same width, viewed from any direction. Is Ω a ball?

No! There are many curves Γ of constant width, including for instance, regular polygons with circular boundary segments, such as the Reuleaux triangles.



Reuleaux triangle.

iv) Whereas Claim 1 in the proof of Thm. 2.1.2 is finite-dimensional in nature, the proof of Claim 2 is sufficiently flexible to be generalized to the infinite-dimensional case, provided a compactness condition holds.

2.2 Pseudo-gradient vector fields

Let X be a Banach space, $E : X \to \mathbb{R}$.

Def. 2.2.1. i) E is Fréchet differentiable at $u \in X$ if there exists a linear map $dE(u) \in X^*$ such that

$$E(u+v) - E(u) - \langle dE(u), v \rangle_{X^* \times X} = o(\|v\|_X)$$

for $v \in X$.

ii) E is of class C^1, if E is Fréchet differentiable at every $u \in X$ and if the map $X \ni u \mapsto dE(u) \in X^*$ is continuous.

Examples 2.2.1. i) Let $\Omega \subset\subset \mathbb{R}^n$,

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega f u \, dx, \quad u \in H_0^1(\Omega),$$

with

$$\langle dE(u), v \rangle_{H^{-1} \times H_0^1} = \int_\Omega (\nabla u \nabla v - f v) \, dx$$

for every $u, v \in H_0^1(\Omega)$; hence $E \in C^1(H_0^1(\Omega))$.

ii) For $\Omega \subset\subset \mathbb{R}^n$, $2 \le p \le 2^* = \frac{2n}{n-2}$ if $n \ge 3$ let

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{p} \int_\Omega |u|^p dx \ , \quad u \in H_0^1(\Omega),$$

with

$$\langle dE(u), v \rangle_{H^{-1} \times H_0^1} = \int_\Omega \left(\nabla u \nabla v - u|u|^{p-2} v \right) dx$$

for $u, v \in H_0^1(\Omega)$. By Hölder's inequality, the map $H_0^1(\Omega) \ni u \mapsto u|u|^{p-2} \in H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)^*$ is continuous; thus, $E \in C^1(H_0^1(\Omega))$.

iii) Let $\Omega \subset\subset \mathbb{R}^n$, $a \in C^1(\mathbb{R})$ with $|a|, |a'| \le 1$, and let

$$E(u) = \frac{1}{2} \int_\Omega a(u) |\nabla u|^2 dx , \quad u \in H_0^1(\Omega).$$

Then for $v \in C_c^\infty(\Omega)$ there holds

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u + \varepsilon v) = \int_\Omega \left(a(u) \nabla u \nabla v + a'(u) |\nabla u|^2 v \right) dx ,$$

but

$$v \mapsto \int_\Omega a'(u) |\nabla u|^2 v \, dx$$

in general does not extend to all $v \in H_0^1(\Omega)$ unless $a \equiv \text{const}$, $a' \equiv 0$ or $n = 1$.

iv) Let $\Omega \subset\subset \mathbb{R}^n$ and let

$$\mathcal{E}(u) = \int_\Omega \sqrt{1+|\nabla u|^2}\,dx\,, \quad u \in W_0^{1,1}(\Omega).$$

Then for any $u, v \in W_0^{1,1}(\Omega)$ the Gateaux (directional) derivative

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}(u+\varepsilon v) = \int_\Omega \frac{\nabla u \nabla v}{\sqrt{1+|\nabla u|^2}}\,dx$$

exists, but $\mathcal{E}$ is nowhere Fréchet differentiable.

Take e.g. $u \equiv 0$ and note

$$\mathcal{E}(0+v) - \mathcal{E}(0) = \int_\Omega \left(\sqrt{1+|\nabla v|^2} - 1\right) dx$$

with

$$\sup_{\|v\|_{W_0^{1,1}} = \varepsilon} \frac{\int_\Omega \left(\sqrt{1+|\nabla v|^2} - 1\right) dx}{\int_\Omega |\nabla v|\,dx} = 1$$

for every $\varepsilon > 0$. (Take $v = v_k = \varepsilon \varphi_k$, where $0 \leq \varphi_k \leq 1$, $\varphi_k \xrightarrow[\text{a.e.}]{(k\to\infty)} \chi_B$ for a ball $B \subset\subset \Omega$.)

Def. 2.2.2. Let $E \in C^1(X)$.

i) A point $u \in X$ is critical if $dE(u) = 0$; otherwise u is regular.

ii) A number $\beta \in \mathbb{R}$ is a critical value of E, if there exists a critical point $u \in X$ with $E(u) = \beta$; otherwise, β is a regular value.

For $E \in C^1(X)$ let

$$\tilde{X} = \{u \in X;\ dE(u) \neq 0\}$$

be the set of regular points.

Def. 2.2.3. Let $E \in C^1(X)$. A vector field $\tilde{e} : \tilde{X} \to X$ is a pseudo-gradient vector field (p.-g. v. f.) provided $\tilde{e}$ is locally Lipschitz continuous and satisfies the bounds

i) $\|\tilde{e}(u)\|_X < 1$,

ii) $\langle dE(u), \tilde{e}(u)\rangle > \frac{1}{2}\|dE(u)\|_{X^*}$

for any $u \in \tilde{X}$.

Theorem 2.2.1 (Palais, 1966) Every $E \in C^1(X)$ admits a pseudo-gradient vector field.

Proof. i) Let $u_0 \in \tilde{X}$. By definition of $\|dE(u_0)\|_{X^*}$, there is $v_0 = v(u_0) \in X$ such that

(2.2.1) $\|v_0\|_X < 1, \quad \langle dE(u_0), v_0 \rangle > \frac{1}{2} \|dE(u_0)\|_{X^*}$.

Moreover, since dE is continuous, (2.2.1) holds true (with the same v_0) for all u in a neighborhood $U_0 = U(u_0)$ of u_0 in $\tilde{X}$.

ii) The above family $(U(u))_{u \in \tilde{X}}$ clearly is an open cover of $\tilde{X}$. Since $\tilde{X} \subset X$ is metric and hence (Stone) paracompact, there exists a locally finite refinement $(U_\iota)_{\iota \in I}$ of this cover so that $U_\iota \subset U(u_\iota)$ for some $u_\iota \in \tilde{X}$, $\iota \in I$.

Let $(\varphi_\iota)_{\iota \in I}$ be a partition of unity subordinate to $(U_\iota)_{\iota \in I}$ with Lipschitz continuous $0 \leq \varphi_\iota \leq 1$ satisfying

$$supp(\varphi_\iota) \subset U_\iota, \quad \sum_{\iota \in I} \varphi_\iota(x) = 1, \quad \forall x \in \tilde{X}.$$

(Note that the sum is locally finite.)

For instance, take

$$\psi_\iota(x) = \operatorname{dist}(x, X \setminus \mathcal{U}_\iota)$$
$$= \inf\{\|y-x\|_X ;\ y \in X \setminus \mathcal{U}_\iota\} \in C_0^{0,1}(\mathcal{U}_\iota)$$

and let

$$\varphi_\iota(x) = \frac{\psi_\iota(x)}{\sum\limits_{\kappa \in I} \psi_\kappa(x)}, \quad \iota \in I.$$

Since the sum near any $x \in \tilde{X}$ involves only a fixed finite set of indices $\kappa \in I$, and is uniformly positive, $\varphi_\iota \in C_0^{0,1}(\mathcal{U}_\iota)$ has the desired properties.

iii) Set

$$\tilde{e}(u) = \sum_{\iota \in I} \varphi_\iota(u)\, v(u_\iota), \quad u \in \tilde{X}.$$

Then $\tilde{e}$ is locally Lipschitz and satisfies

$$\|\tilde{e}(u)\|_X \leq \max_{u \in \mathcal{U}_\iota} \|v(u_\iota)\|_X < 1,$$

$$\langle dE(u), \tilde{e}(u)\rangle_{X^* \times X} = \sum_{\iota \in I} \varphi_\iota(u) \underbrace{\langle dE(u), v(u_\iota)\rangle}_{> \frac{1}{2}\|dE(u)\|_{X^*},\ \forall \iota : u \in \mathcal{U}_\iota}$$

$$> \frac{1}{2}\|dE(u)\|_{X^*}.$$

□

2.3 Compactness, Palais-Smale condition

Let $E \in C^1(X)$, $\beta \in \mathbb{R}$, and set

$$E_\beta = \{u \in X;\ E(u) < \beta\},$$

$$K_\beta = \{u \in X;\ E(u) = \beta,\ dE(u) = 0\},$$

and for any $\delta > 0$ let

$$N_{\beta,\delta} = \{u \in X;\ |E(u) - \beta| < \delta,\ \|dE(u)\|_{X^*} < \delta\}.$$

Def. 2.3.1. i) A sequence $(u_k) \subset X$ is a $(P.-S.)_\beta$-sequence if

$$E(u_k) \to \beta,\quad \|dE(u_k)\|_{X^*} \to 0\ (k \to \infty).$$

ii) E satisfies the Palais-Smale condition at energy β ($(P.-S.)_\beta$ for short) if any $(P.-S.)_\beta$-sequence is relatively compact.

Note that when $E \in C^1(X)$ satisfies $(P.-S.)_\beta$ then as an analogue of (2.1.1) – where compactness of T^2 was crucial – we obtain the following conclusion.

<u>Remark</u> 2.3.1. Let $E \in C^1(X)$, $\beta \in \mathbb{R}$, and suppose E satisfies $(P.-S.)_\beta$. Then

$$K_\beta = \emptyset \Rightarrow \exists\, \delta > 0 : N_{\beta,\delta} = \emptyset.$$

<u>Proof</u>: Suppose by contradiction that for every $k \in \mathbb{N}$ there exists $u_k \in X$ with

$$|E(u_k) - \beta| < \frac{1}{k}, \quad \|dE(u_k)\|_{X^*} < \frac{1}{k}.$$

Then (u_k) is a $(P.-S.)_\beta$ - sequence.

Since E by assumption satisfies $(P.-S.)_\beta$ a subsequence $u_k \rightarrow u$ $(k \rightarrow \infty, k \in \Lambda)$.

By continuity of E and dE then

$$E(u) = \beta, \quad dE(u) = 0,$$

and $u \in K_\beta$; contradiction! $\square$

A number of interesting examples of C^1-functionals satisfying the Palais-Smale condition can be characterized, as follows.

Theorem 2.3.1. Let $E \in C^1(X)$, $\beta \in \mathbb{R}$, and suppose

i) any $(P.-S.)_\beta$-sequence is bounded,

ii) there is a (fixed) linear isomorphism $L: X \to X^*$ such that

$$dE(u) = Lu - K(u), \quad u \in X,$$

where $K: X \to X^*$ is continuous and compact; that is, K maps bounded sets to relatively compact sets.

Then E satisfies $(P.-S.)_\beta$.

Proof: Let $(u_k) \subset X$ be a $(P.-S.)_\beta$-sequence. By i) (u_k) is bounded; hence by ii) a subsequence $K(u_k) =: v_k \to v \in X^*$ $(k \to \infty,\ k \in \Lambda)$. From

$$w_k := dE(u_k) = Lu_k - v_k \to 0 \text{ in } X^*$$

then we obtain $u_k = L^{-1}(v_k + w_k) \to L^{-1}v =: u$, $\square$
$(k \to \infty,\ k \in \Lambda)$

Examples 2.3.1. i) Let $\Omega \subset\subset \mathbb{R}^n$, $f \in L^2(\Omega)$,

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f u \, dx , \quad u \in H_0^1(\Omega),$$

with

$$dE(u) = -\Delta u - f .$$

Since

$$L : H_0^1(\Omega) \ni u \mapsto -\Delta u \in H^{-1}(\Omega)$$

is a linear isomorphism with

$$\|Lu\|_{H^{-1}} = \sup_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{H_0^1} = 1}} \overbrace{\langle Lu, v\rangle_{H^{-1}\times H_0^1}}^{= \int_\Omega \nabla u \nabla v \, dx} = \|\nabla u\|_{L^2} = \|u\|_{H_0^1}$$

for $u \in H_0^1(\Omega)$, and since $K(u) \equiv f$ defines a continuous, compact operator $K : H_0^1(\Omega) \to H^{-1}(\Omega)$, $(P.-S.)_\beta$-sequences for any $\beta \in \mathbb{R}$ are bounded and E satisfies $(P.-S.)_\beta$ for any $\beta \in \mathbb{R}$ by Thm. 2.3.1.

ii) Let $\Omega \subset\subset \mathbb{R}^n$, $2 < p < 2^* = \frac{2n}{n-2}$, and consider

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx, \quad u \in H_0^1(\Omega),$$

with

$$dE(u) = -\Delta u - u|u|^{p-2} = Lu - K(u), \quad u \in H_0^1(\Omega)$$

where

$$L = -\Delta, \quad K : H_0^1(\Omega) \ni u \mapsto u|u|^{p-2} \in H^{-1}(\Omega).$$

For $p < 2^*$ the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact. Thus, if $(u_k) \subset H_0^1(\Omega)$ is bounded, a subsequence $u_k \to u$ in $L^p(\Omega)$ and

$$K(u_k) \underset{(k\to\infty,\, k\in\Lambda)}{\longrightarrow} K(u) \text{ in } L^{p/p-1}(\Omega) \subset L^{2^*/2^*-1}(\Omega) \cong (L^{2^*}(\Omega))^* \subset H^{-1}(\Omega).$$

Thus, condition ii) of Thm. 2.3.1 holds true.

<u>Claim</u>: Any $(P.-S.)_\beta$-sequence for E is bounded.

<u>Proof</u>: Let $(u_k) \subset H_0^1(\Omega)$ be a $(P.-S.)_\beta$-sequence. Compute

$$p\,E(u_k) - \langle dE(u_k), u_k \rangle_{H^{-1}\times H_0^1}$$

$$= \left(\frac{p}{2} - 1\right) \|\nabla u_k\|_{L^2}^2 = \frac{p-2}{2} \|u_k\|_{H_0^1}^2$$

$$= p\beta + o(1)\left(1 + \|u_k\|_{H_0^1}\right).$$

The claim follows. $\square$

Thus, E satisfies $(P.-S.)_\beta$ for any $\beta \in \mathbb{R}$.

iii) Let $\Omega \subset\subset \mathbb{R}^n$, $n \geq 3$, and let $p = 2^* = \frac{2n}{n-2}$,

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx, \quad u \in H^1_0(\Omega).$$

Claim: There is $\beta > 0$ such that E does not satisfy $(P.-S.)_\beta$.

Proof: NLoG there exists $R > 0$ such that

$$B_R(0) \subset \Omega.$$

Choose a cut-off function $\varphi \in C^\infty_c(B_R(0))$ with $0 \leq \varphi \leq 1$, $\varphi = 1$ on $B_{R/2}(0)$ and let

$$u_k(x) = \varphi(x)\left(\frac{c\,k}{1+k^2|x|^2}\right)^{\frac{n-2}{2}} = \varphi(x)\, u^*_{1/k}(x),$$

where

$$u^*(x) = \left(\frac{c}{1+|x|^2}\right)^{\frac{n-2}{2}}, \quad x \in \mathbb{R}^n,$$

solves

(2.3.1) $$-\Delta u^* = (u^*)^{2^*-1} \quad \text{on } \mathbb{R}^n$$

and $u^*_\varepsilon(x) = \varepsilon^{\frac{2-n}{2}} u^*(x/\varepsilon)$ for any $\varepsilon > 0$ as in Section 1.5.

Then, as $k \to \infty$,

$$E(u_k) \to \frac{1}{2}\|\nabla u^*\|^2_{L^2} - \frac{1}{2^*}\|u^*\|^{2^*}_{L^{2^*}}$$

$$= \left(\frac{1}{2} - \frac{1}{2^*}\right)\|\nabla u^*\|^2_{L^2} =: \beta^*.$$

Moreover, by Hölder's inequality and (2.3.1),

$$\langle dE(u_k), v\rangle_{H^{-1}\times H^1_0} = \int_\Omega \left(\nabla u_k \nabla v - u_k |u_k|^{2^*-2} v\right) dx$$

$$= \int_\Omega \Big(\varphi \, \nabla u^*_{1/k} \nabla v + u^*_{1/k} \nabla\varphi \nabla v$$

$$- \varphi^{2^*-1} (u^*_{1/k})^{2^*-1} v\Big) dx$$

$$= \underbrace{\int_{\mathbb{R}^n} \left(\nabla u^*_{1/k} \nabla(v\varphi) - (u^*_{1/k})^{2^*-1} v\varphi\right) dx}_{=0}$$

$$+ O\left(\int_{B_R \setminus B_{R/2}(0)} \left(|\nabla u^*_{1/k}| + |u^*_{1/k}|\right)\left(|\nabla v| + |v|\right) dx\right)$$

$$+ O\left(\int_{B_R \setminus B_{R/2}(0)} |u^*_{1/k}|^{2^*-1} |v| \, dx\right)$$

$$\leq C \left(\int_{B_R \setminus B_{R/2}(0)} \left(|\nabla u^*_{1/k}|^2 + |u^*_{1/k}|^2\right) dx\right)^{1/2} \cdot \|v\|_{H^1_0}$$

$$+ C \left(\int_{B_R \setminus B_{R/2}(0)} |u^*_{1/k}|^{2^*} dx\right)^{\frac{2^*-1}{2^*}} \|v\|_{L^{2^*}}$$

$$= o(1) \|v\|_{H^1_0}, \quad o(1) \to 0 \quad (k \to \infty).$$

As a refinement of Remark 2.3.1 we now obtain the following result.

For $\beta \in \mathbb{R}$, $\rho > 0$ also let

$$\mathcal{U}_{\beta,\rho} = \bigcup_{u \in K_\beta} B_\rho(u; X)$$

be the norm-neighborhood of K_β.

<u>Lemma</u> 2.3.1. Suppose $E \in C^1(X)$ satisfies $(P.-S.)_\beta$ for some given $\beta \in \mathbb{R}$.

Then

i) K_β is compact;

ii) the families $(N_{\beta,\delta})_{\delta>0}$, $(\mathcal{U}_{\beta,\rho})_{\rho>0}$ each constitute a fundamental system of neighborhoods of K_β; that is, given any open $N \supset K_\beta$ there exist $\rho > 0$, $\delta > 0$ with

$$\mathcal{U}_{\beta,\rho} \subset N, \quad N_{\beta,\delta} \subset N.$$

iii) In particular, if $K_\beta = \phi = N$, there is $\delta > 0$ such that $N_{\beta,\delta} = \phi$.

Proof. i) Let $(u_k) \subset K_\beta$. Then (u_k) is a $(P.-S.)_\beta$-sequence. Since E satisfies $(P.-S.)_\beta$, a subsequence $u_k \to u$ $(k \to \infty, k \in \Lambda)$, where $u \in K_\beta$ by continuity of E and dE.

Thus, K_β is (sequentially) compact.

ii) Let $N \supset K_\beta$ be open. Suppose by contradiction that $N_{\beta,\delta} \setminus N \neq \phi$ for every $\delta > 0$ or that $\mathcal{U}_{\beta,\rho} \setminus N \neq \phi$ for every $\rho > 0$.

Let $u_k \in N_{\beta,1/k} \setminus N$, $v_k \in \mathcal{U}_{\beta,1/k} \setminus N$, resp., $k \in \mathbb{N}$.

Then (u_k) is a $(P.-S.)_\beta$-sequence. By $(P.-S.)_\beta$, a subsequence $u_k \to u \in K_\beta$ $(k \to \infty, k \in \Lambda)$; hence $u_k \in N$ for sufficiently large $k \in \Lambda$, contrary to our choice of (u_k).

Similarly, for $v_k \in \mathcal{U}_{\beta,1/k} \setminus N$ let $w_k \in K_\beta$ with
$$\| v_k - w_k \|_X < 2/k \quad , k \in \mathbb{N}.$$

By i), a subsequence $w_k \to w \in K_\beta$ $(k \to \infty, k \in \Lambda)$, and
$$\| v_k - w \|_X \le \| v_k - w_k \|_X + \| w_k - w \|_X \xrightarrow{(k\to\infty,\, k\in\Lambda)} 0$$
shows that $v_k \in N$ for sufficiently large $k \in \Lambda$. $\square$

2.4 Deformation Lemma, Minimax Principle

With the help of Lemma 2.3.1 we now obtain the following result.

Theorem 2.4.1 (Deformation Lemma)
Let $E \in C^1(X)$, $\beta \in \mathbb{R}$, N an open neighborhood of K_β, and let $\bar{\varepsilon} > 0$.

Then, if E satisfies $(P.\text{-}S.)_\beta$, there exist $0 < \varepsilon < \bar{\varepsilon}$ and $\Phi \in C^0(X \times [0,1]; X)$ such that

i) $\Phi(\cdot, t) : X \to X$ is a homeomorphism, $0 \leq t \leq 1$;

ii) $\Phi(u, t) = u$, if either $t = 0$, or if $dE(u) = 0$, or if $|E(u) - \beta| \geq \bar{\varepsilon}$;

iii) $t \longmapsto E(\Phi(u, t))$ is non-increasing, $u \in X$;

iv) $\Phi(E_{\beta+\varepsilon}, 1) \subset E_{\beta-\varepsilon} \cup N$, and
$\Phi(E_{\beta+\varepsilon} \setminus N, 1) \subset E_{\beta-\varepsilon}$.

Proof. By Lemma 2.3.1 we can find numbers $0 < \delta, \rho < 1$ such that

$$N \supset \mathcal{U}_{\beta,3\rho} \supset \mathcal{U}_{\beta,\rho} \supset N_{\beta,\delta} \supset K_\beta .$$

Fix
$$\varepsilon = \frac{1}{4} \min \{ \bar{\varepsilon}, \delta\rho \} > 0 .$$

Let $\varphi \in C^\infty(\mathbb{R})$ with $0 \leq \varphi \leq 1$ and such that

$$\varphi(s) = \begin{cases} 1, & \text{if } |s| \leq 1/2, \\ 0, & \text{if } |s| \geq 1 . \end{cases}$$

Also let

$$\eta(u) = \min \{ \rho^{-1} \operatorname{dist}(u, \mathcal{U}_{\beta,\rho}), 1 \}$$

so that $\eta : X \to \mathbb{R}$ with $0 \leq \eta \leq 1$ is Lipschitz continuous with

$$\eta(u) = \begin{cases} 1, & \text{if } u \notin \mathcal{U}_{\beta,2\rho}, \\ 0, & \text{if } u \in \mathcal{U}_{\beta,\rho} . \end{cases}$$

With $\tilde{X}$, $\tilde{e}: \tilde{X} \to X$ as in Thm. 2.2.1

then let

$$e(u) = \begin{cases} \eta(u) \, \overbrace{\varphi\left(\frac{E(u)-\beta}{\varepsilon}\right)}^{=:\, \varphi_{\varepsilon,\beta}(E(u))} \tilde{e}(u), & u \in \tilde{X}, \\ 0, & \text{else.} \end{cases}$$

<u>Claim 1.</u> e is well-defined and locally Lipschitz continuous with $e(u) = 0$ for $u \in X$ with $dE(u) = 0$, and $\|e(u)\|_X < 1$, $u \in X$.

<u>Proof.</u> Note that the φ-factor vanishes if

$$|E(u) - \beta| \geq \varepsilon,$$

while

$$\eta(u) = 0, \quad u \in N_{\beta,\delta}.$$

Let $u \in X$ be critical for E.

<u>Case i)</u> $u \in N_{\beta,\delta/2}$. Then $\eta \equiv 0$ in a neighborhood of u.

<u>Case ii)</u> $u \notin N_{\beta,\delta/2}$. Then $|E(u) - \beta| \geq \delta/2 > 2\varepsilon$, and the φ-factor vanishes in a neighborhood of u.

Thus, e is well-defined near u and locally Lipschitz. Moreover, $\|e(u)\|_X \leq \|\tilde{e}(u)\|_X$ for $u \in \tilde{X}$, or $e(u) = 0$. $\square$

In view of Claim 1 then the vector field e induces a flow $\Phi : X \times \mathbb{R} \to X$ with

$$(2.4.1) \quad \frac{d}{dt}\Phi(u,t) = -e(\Phi(u,t))$$

$$(2.4.2) \quad \Phi(u,0) = u \quad , \quad u \in X.$$

i) By the semi-group property

$$\Phi(\cdot, s+t) = \Phi(\cdot, s) \circ \Phi(\cdot, t)$$

for all $s, t \in \mathbb{R}$, each $\Phi(\cdot, t)$ is a homeomorphism with $\left(\Phi(\cdot, t)\right)^{-1} = \Phi(\cdot, -t)$, $t \in \mathbb{R}$.

ii) Clearly, $\Phi(u,t) = u$ for $t=0$ by (2.4.2) or when $e(u) = 0$ and any $t \in \mathbb{R}$. By construction, $e(u) = 0$ for u with $|E(u) - \beta| \geq \bar{\varepsilon} > 2\varepsilon$, or when $dE(u) = 0$ by Claim 1.

iii) For $u \in X$, $t \in \mathbb{R}$ compute

$$\frac{d}{dt} E(\Phi(u,t)) = \langle dE(\Phi(u,t)), \frac{d}{dt}\Phi(u,t)\rangle_{X^* \times X}$$

$$= - \eta(\Phi(u,t))\, \varphi\left(\frac{E(u) - \beta}{\varepsilon}\right) \langle dE(\Phi(u,t)), \tilde{e}(\Phi(u,t))\rangle_{X^* \times X} \leq 0,$$

iv) Suppose for $u \in E_{\beta+\varepsilon}$ there holds $\Phi(u,1) \notin E_{\beta-\varepsilon}$ and therefore by iii)

$$\left| E\left(\Phi(u,t)\right) - \beta \right| \leq \varepsilon, \quad 0 \leq t \leq 1.$$

By choice of φ then $\varphi\left(\frac{E(\Phi(u,t)) - \beta}{\varepsilon}\right) = 1$ for $0 \leq t \leq 1$.

<u>Claim 2</u>: $\Phi(u,t) \in N$ for all $0 \leq t \leq 1$.

<u>Proof</u>. Suppose $\Phi(u,t) \notin U_{\beta,2\rho} \supset N_{\beta\delta}$ for all $0 \leq t \leq 1$. Then

$$\eta\left(\Phi(u,t)\right) = 1, \| dE(\Phi(u,t)) \|_{X^*} \geq \delta$$

for all $0 \leq t \leq 1$, and

$$\frac{d}{dt} E\left(\Phi(u,t)\right) \leq -\langle dE(\Phi(u,t)), \tilde{e}(\Phi(u,t)) \rangle_{X^* \times X}$$

(2.4.3)
$$\leq -\frac{1}{2} \| dE\left(\Phi(u,t)\right) \|_{X^*} \leq -\frac{\delta}{2}$$

for $0 \leq t \leq 1$, so that

$$E\left(\Phi(u,1)\right) \leq E(\underbrace{\Phi(u,0)}_{=u}) - \frac{\delta}{2} < \beta + \varepsilon - 2\varepsilon = \beta - \varepsilon,$$

contrary to our assumption.

Thus, there is $t_0 \in [0,1]$ with $\Phi(u,t_0) \in \mathcal{U}_{\beta,2\rho}$.
Suppose that there is $t_1 \in [0,1]$ such that
$\Phi(u,t_1) \notin \mathcal{U}_{\beta,3\rho}$. Note that $\|e(v)\|_X < 1$
implies

$$\|\Phi(u,s) - \Phi(u,t)\|_X \leq |t-s|, \quad 0 \leq s,t \leq 1.$$

Hence there is time interval $I \subset [0,1]$ of
length $|I| \geq \rho$ between t_0 and t_1 such that

$$\Phi(u,t) \in \mathcal{U}_{\beta,3\rho} \setminus \mathcal{U}_{\beta,2\rho}, \quad t \in I,$$

and (2.4.3) implies

$$E\left(\Phi(u,1)\right) \leq E(u) - \frac{\delta}{2}|I| < \beta + \varepsilon - \frac{\delta\rho}{2} \leq \beta - \varepsilon,$$

again contradicting our assumption.
Hence $\Phi(u,t) \in \mathcal{U}_{\beta,3\rho} \subset N$ for all $t \in [0,1]$.

$\square$

By Claim 2, if $\Phi(u,1) \notin E_{\beta-\varepsilon}$ thus it
follows that $\Phi(u,1) \in N$.
Conversely, if $u \notin N$ we must have
$\Phi(u,1) \in E_{\beta-\varepsilon}$, and the proof is complete.

$\square$

<u>Minimax</u> strategies. With the help of the pseudo-gradient flows Φ constructed in Thm. 2.4.1 we can now characterize "minimax" critical values, as follows.

<u>Def.</u> 2.4.1. Let M be a topological space, $(\Phi(\cdot,t))_{0\le t\le 1}$ a continuous family of homeomorphisms $\Phi(\cdot,t)$ of M with $\Phi(\cdot,0)=id$. A family $\mathcal{F}\subset\mathcal{P}(M)$ is (forward) Φ-<u>invariant</u>, if

$$\forall F\in\mathcal{F},\ 0\le t\le 1:\ \Phi(F,t)\in\mathcal{F}.$$

<u>Example</u> 2.4.1. i) $\mathcal{F}=\{M\}$, or $\mathcal{F}=\mathcal{P}(M)$ are Φ-invariant for any Φ as in Def. 2.4.1.

ii) For any map $\alpha\in C^0(S^k,M)$ the family

$$[\alpha]=\{\alpha'(S^k)\subset M;\ \alpha' \text{ is homotopic to } \alpha\}$$

is Φ-invariant for any Φ as above; similarly for maps from any topological space N into M.

Theorem 2.4.2 (Minimax principle, Palais)

Let $E \in C^1(X)$, $\beta_0 \geq -\infty$. Suppose that $\mathcal{F}$ is forward Φ-invariant for any of the flows Φ constructed in Thm. 2.4.1 that fixes E_{β_0}, and suppose that

$$\beta := \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u) > \beta_0 .$$

Then, if E satisfies $(P.S.)_\beta$, the value β is critical.

Proof. Suppose by contradiction that $K_\beta = \phi$. For $N = \phi$, $\bar{\varepsilon} = \beta - \beta_0 > 0$ let $\Phi : X \times [0,1] \to X$, $\varepsilon > 0$ be a constructed in Thm. 2.4.1, and let $F \in \mathcal{F}$ with

$$\sup_{u \in F} E(u) < \beta + \varepsilon$$

Then $\Phi(\cdot, t)$ fixes E_{β_0} for $0 \leq t \leq 1$

and $$F_1 := \Phi(F, 1) \subset E_{\beta - \varepsilon}, \quad F_1 \in \mathcal{F},$$

contradicting the definition of β. $\square$

As an example of a concrete implementation of Thm. 2.4.2 we consider the following.

<u>Theorem</u> 2.4.3 (Mountain pass lemma)
Let $E \in C^1(X)$ with $E(0) = 0$, and suppose there exist $\rho > 0$, $a > 0$ such that

$$\inf_{\|u\|_X = \rho} E(u) \geq a.$$

Moreover, suppose that there exists $u_1 \in X$ with $\|u_1\|_X > \rho$ and $E(u_1) \leq 0$.
Then, if E satisfies $(P.-S.)_\beta$ for every $\beta \geq a$, there is a critical value $\beta \geq a$ of E.

<u>Proof</u>: Let

$$\mathcal{F} = \{F \subset X;\ F \text{ connected},\ u_0 = 0,\ u_1 \in F\}$$

and set

$$\beta = \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u) \geq a > \beta_0 = 0.$$

Then $\mathcal{F}$ is Φ-invariant for every $(\Phi(\cdot,t))_{0 \leq t \leq 1}$ that fixes E_0, and Thm. 2.4.2 gives the claim.
$\square$

Example 2.4.2. i) Let $\Omega \subset\subset \mathbb{R}^n$, $2 < p < 2^* = \frac{2n}{n-2}$ if $n \geq 3$,

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{1}{p} \int_\Omega |u|^p dx, \quad u \in H_0^1(\Omega).$$

Then $E(0) = 0$, $E(\lambda u) = \frac{\|u\|^2_{H_0^1}}{2} \cdot \lambda^2 - \frac{\|u\|^p_{L^p}}{p} \cdot \lambda^p \to -\infty$ $(\lambda \to \infty)$

for every $u \in H_0^1(\Omega) \setminus \{0\}$. Moreover, by

Sobolev's embedding we can bound

$$E(u) \geq \frac{1}{2} \|u\|^2_{H_0^1} - c(p,\Omega) \|u\|^p_{H_0^1} \geq a > 0$$

for

$$\|u\|_{H_0^1} = \left(\frac{1}{4c(p,\Omega)}\right)^{\frac{1}{p-2}} =: \rho \ , \quad a = \frac{1}{4} \rho^2 .$$

Finally, E satisfies $(P.-S.)_\beta$ for every $\beta \in \mathbb{R}$ by Ex. 2.3.1. ii), and Thm. 2.4.3 applies.

ii) A more general case of i). Let $\Omega \subset\subset \mathbb{R}^n$, $g \in C^0(\mathbb{R})$ and consider the problem

$$(2.4.4) \quad -\Delta u = g(u) \quad \text{in } \Omega,$$

$$(2.4.5) \quad u = 0 \quad \text{on } \partial\Omega.$$

Suppose that g satisfies the following conditions:

a) $\exists\, C > 0,\ 0 \leq s < 2^* - 1 = \frac{n+2}{n-2}$ if $n \geq 3$ $\forall u \in \mathbb{R}$:

$$|g(u)| \leq C\left(1 + |u|^s\right);$$

b) $g(u)/u \to 0 \quad (0 \neq |u| \to 0)$;

c) $\exists\, \theta > 2,\ R > 0\ \forall |u| \geq R$:

$$0 < \theta\, G(u) \leq g(u)\, u,$$

where

$$G(u) = \int_0^u g(s)\, ds$$

is the primitive of g with $G(0) = 0$.

Then there exists $0 \neq u \in H_0^1(\Omega)$ solving (2.4.4), (2.4.5).

To see this, let

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega G(u)\, dx, \quad u \in H_0^1(\Omega).$$

By a) the functional $E \in C^1(H_0^1(\Omega))$ with

$$\langle dE(u), v \rangle_{H^{-1} \times H_0^1} = \int_\Omega (\nabla u \nabla v - g(u) v)\, dx$$

for any $u, v \in H_0^1(\Omega)$.

Moreover, for any $\beta \in \mathbb{R}$ let $(u_k) \subset H_0^1(\Omega)$ be a $(P.-S.)_\beta$ - sequence. Then

$$\theta E(u_k) - \langle dE(u_k), u_k \rangle_{H^{-1} \times H_0^1} = \theta\beta + o(1)(1 + \|u_k\|_{H_0^1})$$

$$= \frac{\theta - 2}{2} \|\nabla u_k\|_{L^2}^2 - \underbrace{\int_\Omega (\theta G(u_k) - g(u_k) u_k)\, dx}_{\leq 0, \text{ if } |u_k| \geq R}$$

$$\geq \frac{\theta - 2}{2} \|u_k\|_{H_0^1}^2 - C,$$

and (u_k) is bounded. Since $s < 2^* - 1$, by Rellich's thm. the operator

$$K : H_0^1(\Omega) \ni u \mapsto g(u) \in H^{-1}(\Omega)$$

is compact. Hence, E satisfies $(P.-S.)_\beta$ for every $\beta \in \mathbb{R}$ by Thm. 2.3.1.

Next, we note that c) implies

$$\frac{d}{du}\left(\log u^\theta\right) = \frac{\theta}{u} \leqslant \frac{g(u)}{G(u)} = \frac{d}{du}\left(\log G(u)\right), \quad u \geq R,$$

and similarly for $u \leq -R$. Hence

$$G(u) \geq G(R)\,|u|^\theta / R^\theta = C|u|^\theta, \quad |u| \geq R.$$

In particular, $s+1 \geq \theta > 2$.
Thus, we have $E(0)=0$, and for every

$u \neq 0$, $u \in H_0^1(\Omega)$

$$E(\lambda u) = \frac{\lambda^2}{2}\|u\|^2_{H_0^1} - C\lambda^\theta \|u\|^\theta_{H_0^1} \xrightarrow{(\lambda\to\infty)} -\infty,$$

so that $E(\lambda u) \leq 0$ for $\lambda \geq \lambda_1 > 0$.

Finally, let $\lambda_1 > 0$ be the first Dirichlet eigenvalue of $-\Delta$ on Ω. By b) there exists $r_0 > 0$ such that

$$|g(u)| \leq \frac{\lambda_1}{2}|u| \text{ for } |u| \leq r_0,$$

and hence

$$G(u) = \int_0^u g(v)\,dv \leq \frac{\lambda_1}{4}|u|^2 \text{ for } |u| \leq r_0.$$

By Tchebychev's inequality, for any $u \in H_0^1(\Omega)$ we can bound the measure

$$A(r_0) = \mathcal{L}^n\left(\{x \in \Omega;\ |u(x)| \geq r_0\}\right) \leq r_0^{-2^*} \int_\Omega |u|^{2^*} dx$$

$$\leq C r_0^{-2^*} \|u\|_{H_0^1}^{2^*}$$

It follows that for any $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1} = \rho$ we can estimate

$$I(u) = \frac{1}{2}\|u\|_{H_0^1}^2 - \int_\Omega \underbrace{\left(G(u) - \frac{\lambda_1}{4} u^2\right)}_{\leq 0, \text{ if } |u| \leq r_0} dx - \frac{\lambda_1}{4}\|u\|_{L^2}^2$$

$$\geq \frac{1}{4} \underbrace{\|u\|_{H_0^1}^2}_{= \rho^2} - \int_{\{x;\ |u(x)| \geq r_0\}} G(u)\, dx,$$

where

$$\int_{\{x;\ |u(x)| \geq r_0} G(u)\, dx \leq C \int_{\{x;\ |u(x)| \geq r_0\}} \left(1 + |u|^{s+1}\right) dx$$

$$\leq C A(r_0) + C\left(A(r_0)\right)^{\frac{2^* - (s+1)}{2^*}} \|u\|_{L^{2^*}}^{s+1}$$

$$\leq C r_0^{-2^*} \|u\|_{H_0^1}^{2^*} + C r_0^{(s+1) - 2^*} \|u\|_{H_0^1}^{2^*}$$

$$\leq C \|u\|_{H_0^1}^{2^*} \leq C \rho^{2^*},$$

and again Thm 2.4.3 can be applied.

2.5 Index theory

Briefly returning to the finite-dimensional setting, recall that for a symmetric matrix $A \in \mathbb{R}^{n\times n}$ the eigenvalues $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$ are characterized by the minimax principle

$$\lambda_i = \inf_{\substack{V \subset \mathbb{R}^n \\ \dim V \ge i}} \ \sup_{\substack{x \in V \\ |x| = 1}} \langle Ax, x\rangle_{\mathbb{R}^n}, \quad 1 \le i \le n.$$

Similarly, for the Laplace operator on $\Omega \subset\subset \mathbb{R}$ with homogeneous Dirichlet boundary condition we have the minimax-characterization

$$\lambda_i = \inf_{\substack{V \subset H^1_0(\Omega) \\ \dim V \ge i}} \ \sup_{\substack{u \in V \\ \|u\|_{L^2} = 1}} \|\nabla u\|^2_{L^2}, \quad i \in \mathbb{N},$$

of the i^{th} Dirichlet eigenvalue.

However, these characterizations strongly use the fact that the operators involved are linear.

In the nonlinear case, for instance, for the equation

$$-\Delta u = u|u|^{p-2} \text{ in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

is there a concept that generalizes the notion of dimension?

In fact, in the case of symmetry with respect to some compact group action such a concept can be defined; in particular, when we are dealing with an <u>even</u> functional $\Phi \in C^1(X)$ with $\Phi(-u) = \Phi(u)$, $u \in X$, with symmetry group $\mathbb{Z}_2 = \{id, -id\}$, or when there is an $O(2)$-symmetry with S^1-orbits on which the energy is constant.

For simplicity, in the following we only consider the case of a $\mathbb{Z}_2$-symmetry, that is, the case when Φ is even.

$\underline{A}$ $\mathbb{Z}_2$-index: The Krasnoselskii genus.

Let $M \subset X$ be a submanifold of class $C^{1,1}$; for instance, $M = \{u \in X; \|u\|_X = 1\}$.

Suppose $M = -M = \{-u; u \in M\}$ and set

$$\mathcal{A} = \{A \subset X;\ A \text{ closed},\ A = -A\}.$$

<u>Def</u>. 2.5.1 (Coifman (1969)) For $\phi \neq A \in \mathcal{A}$ let

$$\gamma(A) = \begin{cases} \inf\{m; \exists\, h \in C^0(A, \mathbb{R}^m \setminus \{0\}),\ h(-u) = -h(u)\}, \\ \infty, \text{ if } \{\dots\} = \phi; \text{ in particular, if } A \ni 0, \end{cases}$$

and set $\gamma(\phi) = 0$.

<u>Remark</u> 2.5.1. By the Tietze extension thm. any odd map $h \in C^0(A; \mathbb{R}^m)$ may be extended to a map $\tilde{h} \in C^0(X; \mathbb{R}^m)$. Letting

$$h(u) = \frac{1}{2}\left(\tilde{h}(u) - \tilde{h}(-u)\right)$$

the extension may be chosen to be odd.

The index $\gamma(A)$ achieves the desired generalisation of the concept of dimension.

To see this, we need certain facts from degree theory.

Theorem 2.5.1:
For any open $\Omega \subset \mathbb{R}^n$, $h \in C^0(\bar{\Omega}; \mathbb{R}^n)$, any $y \in \mathbb{R}^n \setminus h(\partial\Omega)$ there exists a number

$$d(h, \Omega, y) \in \mathbb{Z},$$

the topological degree of h in y w.r.t Ω, having the properties

i) $d(id, \Omega, y) = 1$, $\forall y \in \Omega$ (normalization)

ii) $d(h, \Omega, y) \neq 0 \Rightarrow \exists x \in \Omega : h(x) = y$ (sol. prop.)

iii) $d(h(t), \Omega, y(t)) = \text{const.}$ for any $h \in C^0(\bar{\Omega} \times [0,1], \mathbb{R}^n)$, $y \in C^0([0,1], \mathbb{R}^n)$ with $y(t) \notin h(\partial\Omega, t)$, $0 \leq t \leq 1$ (homotopy invariance)

Confer Deimling: Nonlinear funct. analysis.

Theorem 2.5.2 (Borsuk-Ulam). Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with $\Omega = -\Omega$, $0 \in \Omega$, and let $h \in C^0(\bar{\Omega}; \mathbb{R}^n)$ be odd with

$$0 \notin h(\partial\Omega), \quad h(-u) = -h(u), \; u \in \Omega.$$

Then $d(h, \Omega, 0)$ is odd;

in particular, $d(h, \Omega, 0) \neq 0$.

Thm. 2.5.3. Let $0 \in \mathcal{U} = -\mathcal{U} \subset \mathbb{R}^m \subset X$ be open. Then $\gamma(\partial \mathcal{U}) = m$.

Proof. Clearly, $id : \partial\mathcal{U} \to \mathbb{R}^m \setminus \{0\}$ is odd, hence $\gamma(\partial \mathcal{U}) \leq m$.

Suppose $\gamma(\partial \mathcal{U}) = k \leq m$, and let

$$h : \partial\mathcal{U} \to \mathbb{R}^k \setminus \{0\} \hookrightarrow \mathbb{R}^m$$

be odd. We extend h to an odd map $h \in C^0(X; \mathbb{R}^k)$. Since $0 \notin h(\partial\mathcal{U})$, the top. degree of h in 0 w.r.t. $\mathcal{U}$ is well-defined, and

$$d(h, \mathcal{U}, 0) = 1 \mod 2$$

by Borsuk-Ulam thm. in view of oddness of h.

But by continuity of the degree in $y \notin \partial\mathcal{U}$ then we also have

$$d(h, \mathcal{U}, y) = 1 \mod 2$$

for all $y \in \mathbb{R}^m$ close to 0. By the solution property of the degree for any such $y \in \mathbb{R}^m$ there exists $x \in \mathcal{U}$ with $h(x) = y$; hence $k = m$.

$\square$

The index has the following properties.

<u>Thm.</u> 2.5.4. Let $A, A_1, A_2 \in \mathcal{A}$, $h: X \to X$ continuous and odd. Then we have

i) $\gamma(A) \geq 0$, $\gamma(A) = 0$ iff $A = \phi$ (definiteness)

ii) $A_1 \subset A_2 \Rightarrow \gamma(A_1) \leq \gamma(A_2)$ (monotonicity)

iii) $\gamma(A_1 \cup A_2) \leq \gamma(A_1) + \gamma(A_2)$ (sub-additivity)

iv) $\gamma(A) \leq \gamma(\overline{h(A)})$ (supervariance)

v) A compact, $0 \notin A \Rightarrow \gamma(A) < \infty$ and there is a nbhd. $N \supset A$ with $\overline{N} \in \mathcal{A}$ s.t.

$$\gamma(\overline{N}) = \gamma(A) \qquad \text{(continuity)}.$$

vi) $0 \notin A$ finite $\Rightarrow \gamma(A) \leq 1$.

<u>Proof</u>. i) immediate from the definition

ii) We may assume $\gamma(A_2) = m < \infty$. Let $h: A_2 \to \mathbb{R}^m \setminus \{0\}$ be continuous and odd. Then $h|_{A_1}: A_1 \to \mathbb{R}^m \setminus \{0\}$ has these properties, and $\gamma(A_1) \leq m$.

iii) Again we may assume that $\gamma(A_1) = m_1 < \infty$, $\gamma(A_2) = m_2 < \infty$ and choose $h_i : A_i \to \mathbb{R}^{m_i} \setminus \{0\}$ continuous and odd, $i = 1,2$. Extend h_i to an odd map $h_i \in C^0(X, \mathbb{R}^{m_i})$ for $i = 1, 2$. Then

$$h = (h_1, h_2) \in C^0(X, \mathbb{R}^{m_1 + m_2})$$

is continuous and odd, and satisfies

$$h(A_1 \cup A_2) \not\ni 0.$$

iv) If $\gamma(\overline{h(A)}) = m < \infty$, we may choose an odd, continuous $l : \overline{h(A)} \to \mathbb{R}^m \setminus \{0\}$. With $l \circ h : A \to \mathbb{R}^m \setminus \{0\}$ as compositio map, we conclude that $\gamma(A) \leq m$.

v) For any $u \in A$ let $0 < r = r(u) < \|u\|_X$ and define $h_u : \overline{B_r(u; X) \cup B_r(-u; X)} \to \mathbb{R} \setminus \{0\}$ by letting $h_u = \pm 1$ in $B_r(\pm u; X)$.

Finitely many sets $N_{u_1}, \dots, N_{u_I}$, where

$$N_u = B_r(u; X) \cup B_r(-u, X), \quad u \in A$$

cover A; hence $\gamma(A) < \infty$ by iii).

Let $m = \gamma(A)$, and choose an odd, continuous $h : A \to \mathbb{R}^m \setminus \{0\}$. Since is compact, also $h(A)$ is compact.

Hence
$$2\rho := \operatorname{dist}(0, h(A)) > 0.$$
Let $\tilde{h} \in C^0(X, \mathbb{R}^m)$ be an odd extension of h and set
$$N = \tilde{h}^{-1}(U_\rho(h(A))).$$
Then $\overline{N} \in \mathcal{A}$, and $\operatorname{dist}(0, \tilde{h}(\overline{N})) \geq \rho > 0$ shows that $\gamma(\overline{N}) \leq m$. Equality follows by ii).

vi) If $A = \{u_i, -u_i;\ 1 \leq i \leq I\} \neq \emptyset$, choose $h(\pm u_i) = \pm 1$. Then h is odd and continuous with $h : A \to \mathbb{R} \setminus \{0\}$.

$\square$

Equivariant pseudo-gradient vector fields and flows

For applications, we now also need variants of Thm. 2.2.1 and Thm. 2.4.1 that respect the symmetry. We phrase these results for an arbitrary compact symmetry group G.

Let $E \in C^1(X)$, $M \subset X$ a $C^{1,1}$-submanifold, G a compact group acting linearly on X, as a group of isometries with

$$g M = \{gu;\, u \in M\} = M, \quad g \in G,$$

and suppose E is G-equivariant in the sense

$$E(gu) = E(u), \quad u \in X, \; g \in G.$$

Note that then we also have

$$\langle dE(gu), gv \rangle_{X^* \times X} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(g(u+\varepsilon v))$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u+\varepsilon v) = \langle dE(u), v \rangle_{X^* \times X};$$

that is,

$$(2.5.1) \quad g^* dE(gu) = dE(u), \quad u \in X, \; g \in G.$$

Similarly to Def. 2.3.1 we let

Def. 2.5.3. i) $(u_k) \subset M$ is a $(P.-S.)_\beta$-sequence for $E|_M$ if

$$E(u_k) \to \beta, \quad \|dE|_M(u_k)\|_{T^*_{u_k}M} \to 0 \quad (k \to \infty).$$

ii) $E|_M$ satisfies $(P.-S.)_\beta$ if every $(P.-S.)_\beta$-sequence (u_k) is relatively compact.

Setting, for $\beta \in \mathbb{R}$, $\delta > 0$, $\rho > 0$ the sets

$$E_\beta = \{u \in M;\ E(u) < \beta\},$$

$$K_\beta = \{u \in M;\ E(u) = \beta,\ dE|_M(u) = 0\},$$

$$N_{\beta,\delta} = \{u \in M;\ |E(u) - \beta| < \delta,\ \|dE|_M(u)\|_{T^*_u M} < \delta\},$$

$$\mathcal{U}_{\beta,\rho} = \bigcup_{u \in K_\beta} B_\rho(u) \cap M,$$

then the statements of Lemma 2.3.1 hold.

Note that if E and M are G-equivariant, then also the sets E_β, K_β, $N_{\beta,\delta}$, and $\mathcal{U}_{\beta,\rho}$ are.

Let $$\tilde{M} = \{u \in M;\ dE(u)|_{T_u M} \neq 0\}$$

be the regular set of $E|_M$.

<u>Def.</u> 2.5.2. A locally Lipschitz vector field $\tilde{e}: \tilde{M} \to X$ is a pseudo-gradient vector field for E on M if for any $u \in \tilde{M}$ there holds

i) $\tilde{e}(u) \in T_u M$, $u \in \tilde{M}$

ii) $\|\tilde{e}(u)\|_X < 1$, $u \in \tilde{M}$,

iii) $$\langle dE(u), \tilde{e}(u)\rangle_{X^* \times X} > \frac{1}{2} \sup_{\substack{v \in T_u M \\ \|v\|_X < 1}} \langle dE(u), v\rangle_{X^* \times X'}$$
$$=: \frac{1}{2} \|dE|_M(u)\|_{T_u^* M}, \quad u \in \tilde{M}.$$

In the same way as Thm. 2.2.1 we then can show the following result.

<u>Theorem 2.5.5</u>. For any $E \in C^1(X)$, $M \subset X$ as above there exists a pseudo-gradient vector field $\tilde{e}: \tilde{M} \to X$. The vector field $\tilde{e}$ may be chosen to be G-equivariant with

(2.5.2) $\tilde{e}(gu) = g\,\tilde{e}(u)$, $u \in \tilde{M}$, $g \in G$,

if E and M are, in accordance with (2.5.1).

Proof: The construction of $\tilde{e}$ with properties i) – iii) in Def. 2.5.2 procede exactly as in Thm. 2.2.1.

To obtain a G-equivariant vector field, for any $u \in \tilde{M}$ let

$$\bar{e}(u) = \int_G g^{-1} \tilde{e}(gu)\, dg,$$

where dg is a Haar measure on G. For any $h \in G$ then we have

$$\bar{e}(h(u)) = \int_G g^{-1} \tilde{e}(ghu)\, dg$$

$$= h \int_G \underbrace{h^{-1} g^{-1}}_{=(gh)^{-1}} \tilde{e}((gh)u)\, \underbrace{dg}_{=d(gh)} = h\, \bar{e}(u),$$

as desired. $\square$

We then have the analogue of Thm. 2.4.1.

<u>Theorem</u> 2.5.6. Let $E \in C^1(X)$, $M \subset X$ as above, $\beta \in \mathbb{R}$, and suppose $E|_M$ satisfies $(P.S.)_\beta$. Then for any open $N \supset K_\beta$, any $\bar{\varepsilon} > 0$ there exist $\varepsilon > 0$ and a flow $\Phi \in C^0(M \times [0,1]; M)$ with the properties i) – iv) of Thm. 2.4.1.

If E, M, and N are G-equivariant, the flow Φ may be chosen to be equivariant, as well, satisfying

$$\Phi(gu, t) = g\,\Phi(u,t) \tag{2.5.3}$$

for all $g \in G$, $u \in M$, $0 \le t \le 1$.

<u>Proof</u>: With $\tilde{e}: \tilde{M} \to X$ as in Thm. 2.5.5, $0 < \rho, \delta \le 1$, $\varepsilon > 0$, $\varphi_{\varepsilon,\beta}$, and η as in the proof of Thm. 2.4.1, let $\Phi: M \times \mathbb{R} \to M$ solve

$$\frac{d}{dt}\Phi(u,t) = -e(\Phi(u,t)),$$
$$\Phi(u,0) = u,$$

where

$$e(u) = \eta(u)\,\varphi_{\varepsilon,\beta}(E(u))\,\tilde{e}(u) \in T_u M, \quad u \in \tilde{M},$$

and $e(u) = 0$ if $dE_{|M}(u) = 0$. The unique, global solution $\Phi \in C^0(M \times \mathbb{R}; X)$ to this initial value problem defines a flow on M with the desired properties.

If E, M, and N are G-equivariant, we may chose $\tilde{e}$ satisfying (2.5.2), and (2.5.2) then also holds true for e. Since $g\Phi(u,t) =: \Phi_g(u,t)$ for any $g \in G$, $u \in M$ then also satisfies

$$\frac{d}{dt}\Phi_g(u,t) = g\frac{d}{dt}\Phi(u,t)$$

$$= -g\,e(\Phi(u,t)) = -e(\Phi_g(u,t)), \quad t \in \mathbb{R},$$

$$\Phi_g(u,0) = gu,$$

by uniqueness we have

$$\Phi_g(u,t) = \Phi(gu,t),$$

as desired. $\square$

Minimax-critical points of even functionals.

Combining the equivariant deformation lemma for an even functional with the notion of index, we then obtain the following result. (Now $G = \mathbb{Z}_2 = \{id, -id\}$.)

Theorem 2.5.7. Let $E \in C^1(X)$, $M \subset X \setminus \{0\}$ as above with $E(u) = E(-u)$, $u \in X$, $M = -M$, and suppose that $E|_M$ satisfies $(P.-S.)_\beta$ for every $\beta \in \mathbb{R}$. Suppose that for $K \leq \infty$ the numbers

$$\beta_k = \inf_{\substack{A \in \mathcal{A} \\ \gamma(A) \geq k}} \sup_{u \in A} E(u)$$

are well-defined with $\beta_k > -\infty$, $1 \leq k < K$. Then each $\beta = \beta_k$, $1 \leq k < K$, is critical for E, $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_k \leq \beta_{k+1} \leq \ldots$, and if

$$\beta_k = \beta_{k+1} = \ldots = \beta_{k+l} =: \beta$$

for some k and $l \in \mathbb{N}$ there holds $\# K_\beta = \phi$.

Proof: i) Suppose $K_\beta = \emptyset$ for some $\beta = \beta_k$, and let Φ, $\varepsilon > 0$ be as constructed in Thm. 2.5.6 for $N = \emptyset$, $\bar{\varepsilon} = 1$.

By definition of β_k there exists $A \in \mathcal{A}$ with $\gamma(A) \geq k$ such that

$$\sup_{u \in A} E(u) < \beta + \varepsilon .$$

By (2.5.2) and Thm. 2.5.4 then

$$A_1 := \Phi(A, 1) \in \mathcal{A}, \quad \gamma(A_1) \geq \gamma(A) \geq k,$$

and by Thm. 2.5.6 we have $A_1 \subset E_{\beta - \varepsilon}$, that is,

$$\sup_{u \in A_1} E(u) \leq \beta - \varepsilon ,$$

contradicting the definition of β_k.

ii) Let $\beta = \beta_k = \beta_{k+1} = \dots = \beta_{k+l}$ for some k and $l \in \mathbb{N}$. By Lemma 2.3.1, K_β is compact, and $0 \notin K_\beta \in \mathcal{A}$ by (2.5.1). By Thm. 2.5.4.v) there exists $N \supset K_\beta$ open with $\overline{N} \in \mathcal{A}$ and $\gamma(\overline{N}) = \gamma(K_\beta) < \infty$.

Let Φ, $\varepsilon > 0$ be as constructed in Thm. 2.5.5 for N and $\overline{\varepsilon} = 1$, and let $A \in \mathcal{A}$ with $\gamma(A) \geq k + l$ such that

$$\sup_{u \in A} E(u) < \beta + \varepsilon .$$

Then $A \subset E_{\beta+\varepsilon}$, and $A_1 = \Phi(A,1) \subset E_{\beta-\varepsilon} \cup N$ with

$$k + l \leq \gamma(A) \leq \gamma(A_1) \leq \gamma(\overline{E_{\beta-\varepsilon}}) \cup \gamma(\overline{N})$$

by Thm. 2.5.4. iii) and iv). But

$$\gamma(\overline{E_{\beta-\varepsilon}}) < k$$

by definition of β_k; hence

$$\gamma(K_\beta) = \gamma(\overline{N}) > l \geq 1 ,$$

and by Thm. 2.5.4. vi) it follows that $\# K_\beta = \infty$, as claimed.

$\square$

Application. As a model problem, we can apply Thm. 2.5.7 to obtain infinitely many solutions to the boundary value problem

$$(2.5.4) \qquad -\Delta u = u|u|^{p-2} \quad \text{in } \Omega \subset\subset \mathbb{R}^n,$$

$$(2.5.5) \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $2 < p < 2^* = \frac{2n}{n-2}$, if $n \geq 3$.

Theorem 2.5.8. Let $\Omega \subset\subset \mathbb{R}^n$, $2 < p < 2^* = \frac{2n}{n-2}$, if $n \geq 3$. Then there exist infinitely many distinct pairs $(u, -u)$ of solutions $u \in H_0^1(\Omega)$ to (2.5.4), (2.5.5).

Proof: Let $X = H_0^1(\Omega)$, and set

$$M = \{u \in H_0^1(\Omega);\ \|u\|_{L^p} = 1\}.$$

Moreover, let

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx, \quad u \in H_0^1(\Omega).$$

Letting $G = \{id, -id\}$, we then note that M and E are G-equivariant with $M = -M$ and

$$E(u) = E(-u), \quad u \in H_0^1(\Omega).$$

Moreover, $M = f^{-1}(\{0\})$, where

$$f(u) = \|u\|_{L^p}^p - 1, \quad u \in H_0^1(\Omega),$$

is of class C^2, with

$$T_u M = \ker df(u) = \left\{ v \in H_0^1(\Omega);\ \int_\Omega v u |u|^{p-2} dx = 0 \right\}$$

for $u \in M$. Note that the projection

$$\pi_u : H_0^1(\Omega) \ni v \longmapsto \pi_u(v) = v - \varepsilon_u(v) u \in T_u M$$

with

$$\varepsilon_u(v) = \int_\Omega v u |u|^{p-2} dx$$

is locally Lipschitz in $u \in M$, as we see from

$$\| \pi_{u_1}(v) - \pi_{u_2}(v) \|_{H_0^1} = \| \varepsilon_{u_1}(v) u_1 - \varepsilon_{u_2}(v) u_2 \|_{H_0^1}$$

$$\leqslant | \varepsilon_{u_1}(v) - \varepsilon_{u_2}(v) | \, \|u_1\|_{H_0^1} + |\varepsilon_{u_2}(v)| \, \|u_1 - u_2\|_{H_0^1}$$

with

$$|\varepsilon_{u_2}(v)| \leqslant \|u_2\|_{L^p}^{p-1} \|v\|_{L^p} \leqslant C \|u_2\|_{H_0^1}^{p-1} \|v\|_{H_0^1}$$

and with

$$|\varepsilon_{u_1}(v) - \varepsilon_{u_2}(v)| \leqslant \|v\|_{L^p} \, \| u_1 |u_1|^{p-2} - u_2 |u_2|^{p-2} \|_{L^{p/p-1}}$$

$$\leqslant C \|v\|_{L^p} \left(\|u_1\|_{L^p}^{p-2} + \|u_2\|_{L^p}^{p-2} \right) \|u_1 - u_2\|_{L^p} .$$

Finally, $E|_M$ satisfies $(P.-S.)_\beta$ for any $\beta \in \mathbb{R}$.
To see this, let $(u_k) \subset M$ be a $(P.-S.)_\beta$-sequence.

Then $(u_k) \subset M$ is bounded, and we may select a subsequence with $u_k \xrightarrow{w} u$ in $H_0^1(\Omega)$ and strongly in $L^p(\Omega)$.

Thus, also $v_k = \pi_{u_k}(u_k - u)$ is bounded, and with error $o(1) \to 0$ there holds

$$o(1) = \langle dE(u_k), \pi_{u_k}(u_k - u)\rangle$$

$$= \int_\Omega \nabla u_k \nabla(u_k - u)\, dx - \varepsilon_{u_k}(u_k - u)\int_\Omega |\nabla u_k|^2 dx$$

$$= \|\nabla(u_k - u)\|_{L^2}^2 - 2\underbrace{E(u_k)}_{\to \beta} \int_\Omega \underbrace{(u_k - u)}_{\to 0 \text{ in } L^p(\Omega)} u_k |u_k|^{p-2} dx + o(1)$$

$$= \|u_k - u\|_{H_0^1}^2 + o(1),$$

and $u_k \to u$ strongly in $H_0^1(\Omega)$, as desired.

Thm. 2.5.7 thus is applicable, and yields infinitely many distinct critical points u_k of $E|_M$ with critical values β_k, satisfying the equation

$$0 = \langle dE(u_k), \pi_{u_k}(v) \rangle_{H^{-1} \times H_0^1}$$

$$= \int_\Omega \nabla u_k \nabla v \, dx - \varepsilon_{u_k}(v) \underbrace{\int_\Omega |\nabla u_k|^2 dx}_{= 2\beta_k + \frac{2}{p}}$$

$$= \int_\Omega \left(\nabla u_k \nabla v - \alpha_k^{p-2} u_k |u_k|^{p-2} v \right) dx,$$

where $\alpha_k = \left(2\beta_k + \frac{2}{p}\right)^{\frac{1}{p-2}}$. Then

$$\tilde{u}_k = \alpha_k u_k = \left(2\beta_k + \frac{2}{p}\right)^{\frac{1}{p-2}} u_k \in H_0^1(\Omega)$$

solves (2.5.4), (2.5.5) with

$$\tilde{\beta}_k := E(\tilde{u}_k) = \frac{1}{2} \|\nabla \tilde{u}_k\|_{L^2}^2 - \frac{1}{p} \|\tilde{u}_k\|_{L^p}^p$$

$$= \alpha_k^2 \left(\beta_k + \frac{1}{p}\right) - \frac{1}{p} \alpha_k^p = \left(\frac{1}{2} - \frac{1}{p}\right) \alpha_k^p$$

$$= \left(\frac{1}{2} - \frac{1}{p}\right) \left(2\beta_k + \frac{2}{p}\right)^{\frac{p}{p-2}}, \quad k \in \mathbb{N}.$$

$\square$

Remark 2.5. 2. i) We can estimate the rate of growth of the sequence β_k, $k \in \mathbb{N}$.

Let $0 < \lambda_1 < \lambda_2 \le \dots \le \lambda_k \le \lambda_{k+1} \le \dots$

be the Dirichlet eigenvalues of $-\Delta$ on Ω with L^2-orthonormal eigenfunctions φ_k satisfying

$$-\Delta \varphi_k = \lambda_k \varphi_k \quad \text{in } \Omega,$$

$$\varphi_k = 0 \quad \text{on } \partial\Omega, \quad k \in \mathbb{N}.$$

For $k \in \mathbb{N}$ then also let

$$X_k = \operatorname{span}\{\varphi_1, \dots, \varphi_k\}$$

with L^2-orthonal projection

$$\pi_k : H_0^1(\Omega) \longrightarrow X_k \cong \mathbb{R}^k.$$

Note that for any $u \in \ker \pi_{k-1}$ there holds

$$\|\nabla u\|_{L^2}^2 = \sum_{i \ge k} (u, \varphi_i)_{L^2}^2 \overbrace{\|\nabla \varphi_i\|_{L^2}^2}^{= \lambda_i \|\varphi_i\|_{L^2}^2 = \lambda_i}$$

$$\ge \lambda_k \sum_{i \ge k} (u, \varphi_i)_{L^2}^2 = \lambda_k \|u\|_{L^2}^2 .$$

Interpolating, then for $2 < p < 2^*$ we find

$$\|u\|_{L^p} \le \|u\|_{L^2}^{\alpha} \|u\|_{L^{2^*}}^{1-\alpha} \le C \lambda_k^{-\frac{\alpha}{2}} \|\nabla u\|_{L^2},$$

where $\quad \frac{1}{p} = \frac{\alpha}{2} + \frac{1-\alpha}{2^*} = \frac{1}{2^*} + \frac{\alpha}{n}; \quad \alpha = n\left(\frac{1}{p} - \frac{1}{2^*}\right)$

$$= 1 - n\left(\frac{1}{p} - \frac{1}{2}\right).$$

With $M = \{u \in H_0^1(\Omega);\ \|u\|_{L^p} = 1\}$ as before, then for any $A \in \mathcal{A}$ there holds

$$\gamma(A) \geq k \implies \pi_{k-1}(A) \ni 0\,;$$

hence, with a constant $c_0 > 0$ we have

$$2\beta_k + \frac{2}{p} \geq \inf_{\substack{\pi_{k-1}(u)=0 \\ u \in M}} \|\nabla u\|_{L^2}^2 \geq c_0 \lambda_k^{\alpha} = c_0 \lambda_k^{\underbrace{1-n\left(\frac{1}{p}-\frac{1}{2}\right)}_{=\frac{p-2}{2p}}}$$

for any $k \in \mathbb{N}$.

The energies $E(\tilde{u}_k)$ of the corresponding solutions then are bounded from below by

$$\tilde{\beta}_k = \left(\frac{1}{2} - \frac{1}{p}\right)\left(2\left(\beta_k + \frac{2}{p}\right)\right)^{\frac{p}{p-2}}$$
$$\geq c_1 \lambda_k^{\frac{p}{p-2} - \frac{n}{2}}.$$

Finally, with the Weyl asymptotic formula

$$\lambda_k \cong c(\Omega)\, k^{2/n}, \quad k \in \mathbb{N},$$

we obtain the bound

$$\tilde{\beta}_k \geq c_2\, k^{\frac{2p}{n(p-2)} - 1}, \qquad k \in \mathbb{N},$$

with constants $c_1, c_2, c(\Omega) > 0$.

ii) Bahri - Lions (1988) improved the bound in i) to

$$\tilde{\beta}_k \geq c_0 \, k^{\frac{2p}{n(p-2)}}, \quad k \in \mathbb{N}.$$

iii) Letting, for $X = H^1_0(\Omega)$,

$$\Gamma = \{ h \in C^0(X, X);\ h(-u) = -h(u),$$
$$h(u) = u \text{ if } E(u) \leq 0 \},$$

we can compare $\tilde{\beta}_k$ with the numbers

$$\tilde{\delta}_k := \inf_{h \in \Gamma} \sup_{u \in X_k} E(h(u)) \geq \tilde{\beta}_k, \quad k \in \mathbb{N}.$$

Proof: Let

$$\tilde{M} := \{ u \in H^1_0(\Omega) \setminus \{0\};\ 0 = \langle d\tilde{E}(u), u \rangle_{H^{-1} \times H^1_0}$$
$$= \|u\|^2_{H^1_0} - \|u\|^p_{L^p} \}$$

and observe that $\tilde{M}$ is homeomorphic to M via

$$\pi : \tilde{M} \ni u \mapsto u / \|u\|_{L^p} \in M.$$

For any $h \in \Gamma$, any $k \in \mathbb{N}$, the set $h^{-1}(\tilde{M}) \cap X_k$ bounds an open neighborhood $U = -U$ of 0 in $X_k \cong \mathbb{R}^k$; hence

$$\gamma(h^{-1}(\tilde{M}) \cap X_k) = \gamma(\partial U) \geq k$$

by Thm. 2.5.3, and thus by Thm. 2.5.4, iii)

$$A := \pi\left(h(X_k) \cap \tilde{M} \right) \in \mathcal{A}$$

satisfies

$$\gamma(A) \geq k .$$

So

$$\sup_{v \in A} E(v) \geq \beta_k .$$

Moreover, for any $u \in h(X_k) \cap \tilde{M}$ with $\pi(u) = v \in M$ and $u = \alpha v$ we have

$$\alpha^2 \|\nabla v\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 = \|u\|_{L^p}^p = \alpha^p, \quad \alpha^{p-2} = \|\nabla v\|_{L^2}^2 ;$$

so

$$E(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla u\|_{L^2}^2 = \left(\frac{1}{2} - \frac{1}{p}\right) \alpha^2 \|\nabla v\|_{L^2}^2$$

$$= \left(\frac{1}{2} - \frac{1}{p}\right) \alpha^p = \left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla v\|_{L^2}^{\frac{2p}{p-2}}$$

$$= \left(\frac{1}{2} - \frac{1}{p}\right)\left(2 E(v) + \frac{2}{p}\right)^{\frac{p}{p-2}} ,$$

and there results

$$\sup_{u \in X_k} E(h(u)) \geq \left(\frac{1}{2} - \frac{1}{p}\right)\left(2\beta_k + \frac{2}{p}\right)^{\frac{p}{p-2}} = \tilde{\beta}_k .$$

$\square$

Perturbation theory. What happens, if we perturb the symmetry? Given $f \in L^2(\Omega)$, do we still find infinitely many solutions u of

(2.5.6) $\quad -\Delta u = u|u|^{p-2} + f$ in Ω,

(2.5.7) $\quad u = 0$ on $\partial\Omega$?

Theorem 2.5.9 (S. (1980), Bahri-Berestycki (1981), Bahri-Lions (1988)) Let $\Omega \subset\subset \mathbb{R}^n$, $f \in L^2(\Omega)$, and suppose $2 < p \left(< 2^* = \frac{2n}{n-2} \text{ if } n \geq 3\right)$ satisfies

(2.5.8) $\quad \dfrac{2p}{n(p-2)} > \dfrac{p}{p-1}$, i.e. $p < \dfrac{2(n-1)}{n-2}$;

Then (2.5.6), (2.5.7) admits infinitely many distinct solutions.

Remark. If $n=3$, condition (2.5.8) is satisfied for any $2 < p < 4$.

The proof of Thm. 2.5.9 may be considerably simplified with the help of the following abstract result.

<u>Theorem 2.5.10</u> (Rabinowitz (1982))

Suppose $E \in C^1(X)$ satisfies $(P.-S.)_\beta$ for all $\beta \in \mathbb{R}$. Let $W \subset X$ be a subspace with $\dim W < \infty$, $w^* \in X \setminus W$, and set $W^* = W \oplus \operatorname{span}\{w^*\}$. Also let

$$W^*_+ = \{w + tw^*;\ w \in W,\ t \geq 0\}.$$

Suppose

i) $E(0) = 0$,

ii) $\exists R^* > 0\ \forall u \in W^*: \|u\|_X \geq R^* \Rightarrow E(u) \leq 0$.

Let

$$\Gamma = \{h \in C^0(X, X);\ h(-u) = -h(u),$$
$$h(u) = u \text{ if } \max\{E(u), E(-u)\} \leq 0\}$$

and suppose

$$\beta^* := \inf_{h \in \Gamma} \sup_{u \in W^*_+} E(h(u))$$
$$> \beta := \inf_{h \in \Gamma} \sup_{u \in W} E(h(u)) > 0.$$

Then there is $\gamma^* \geq \beta^*$ with $K_{\gamma^*} \neq \emptyset$.

Proof. Fix some $\gamma \in]\beta, \beta^*[$ and let

$$\Lambda = \{h \in \Gamma;\ E(h(u)) \le \gamma,\ \forall u \in W\}.$$

Then $\Lambda \neq \emptyset$ by definition of β and

$$\gamma^* = \inf_{h \in \Lambda} \sup_{u \in W_+^*} E(h(u)) \ge \beta^* > \gamma$$

is well-defined.

Claim. $K_{\gamma^*} \neq \emptyset$.

Proof. Suppose $K_{\gamma^*} = \emptyset$. For γ^*, $\bar{\varepsilon} = \gamma^* - \gamma > 0$, $N = \emptyset$ let $0 < \varepsilon < \bar{\varepsilon}$, $\Phi \in C^0(X \times [0,1], X)$ be as in Thm. 2.~~4~~3. Choose $h \in \Lambda$ with

$$\sup_{u \in W_+^*} E(h(u)) < \gamma^* + \varepsilon.$$

Note that by choice of $\bar{\varepsilon}$ and since $h \in \Lambda$ we have

$$(2.5.9) \quad \Phi(h(-u), 1) = h(-u) = -h(u) = -\Phi(h(u), 1)$$

for any $u \in W$. Hence, letting

$$h_1(u) = \begin{cases} \Phi(h(u), 1), & \text{if } u \in W_+^*, \\ -\Phi(h(-u), 1), & \text{if } -u \in W_+^* \end{cases}$$

we obtain an odd map $h_1 \in C^0(W^*; X)$.
Moreover, $h_1(u) = u$ for any u with $\max\{E(u), E(-u)\} \le 0$, since $h \in \Lambda \subset \Gamma$ and $\Phi(\cdot, 1)$ both do, as well.
Hence h_1 may be extended to $h_1 \in \Gamma$, and $h_1 \in \Lambda$ by (2.5.9). But

$$\sup_{u \in W_+^*} E(h_1(u)) \le \gamma^* - \varepsilon. \quad ↯$$

$\square$

Proof of Thm. 2.5.9: Let

$$E_f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx - \int_\Omega f u \, dx, \quad u \in H_0^1(\Omega),$$

and let

$$E_0(u) = \frac{E_f(u) + E_f(-u)}{2} = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx .$$

Note that we have

$$d E_0(u) - d E_f(u) = f, \quad u \in H_0^1(\Omega).$$

Moreover, E_f satisfies $(P.-S.)_\beta$ for any $\beta \in \mathbb{R}$.
Suppose by contradiction that there is $\beta_0 > 1$ such that E_f has no critical value $\beta > \beta_0$.
Choose $k_0 \in \mathbb{N}$ such that

$$\tilde{\delta}_k = \inf_{h \in \Gamma} \sup_{u \in X_k} E_0(h(u)) > \beta_0$$

for $k \geq k_0$, where $X_k = \operatorname{span}\{\varphi_1, \dots, \varphi_k\}$ as in Remark 2.5.2, and where

$$\Gamma = \{ h \in C^0(X, X);\ h(-u) = -h(u),$$
$$h(u) = u \text{ if } E_0(u) \leq \max\{E_f(u), E_f(-u)\} \leq 0 \}.$$

Then for $k \geq k_0$ we also have

$$\delta_k := \inf_{h \in \Gamma} \sup_{u \in X_k} E_f(h(u)) \geq \tilde{\delta}_k > \beta_0 .$$

From our assumption and Thm. 2.5.10 we then conclude that for every $k \geq k_0$ there holds

$$\delta_k^* := \inf_{h \in \Gamma} \sup_{u \in X_{k+1}^+} E_f(h(u)) = \delta_k ,$$

where $$X_{k+1}^+ = \{ w + t\varphi_{k+1} ;\ w \in X_k,\ t \geq 0 \}.$$

Choose $h \in \Gamma$ so that

$$\sup_{u \in X_{k+1}^+} E_f(h(u)) \leq \delta_k + 1 .$$

Claim 1. We may assume that E_f achieves its maximum in $h(X_{k+1})$ in N_{12}, where $dE_0(u) - dE_f(u)$ $\parallel$

$$N_a = \{ u \in X ;\ \| dE_f(u) \|_{H^{-1}} + \| dE_f(-u) \|_{H^{-1}} \leq a \| f \|_{H^{-1}} \}.$$

Note that for $a > 2$ the set $N_a = -N_a$ is an open neighborhood of

$$K = \{ u \in X ;\ dE_f(u) = 0 \text{ or } dE_f(-u) = 0$$
$$\text{or } dE_0(u) = 0 \}.$$

Proof: Let $\tilde{e}$ be an odd p.g.v.f. for E_0 and set

$$e(u) = -\varphi\left(\max\{E_f(u), E_f(-u)\}\right)\eta(u)\,\tilde{e}(u),$$

if $u \notin K$, $e(u) = 0$ for $u \in K$, where $\varphi \in C^\infty$ satisfies $0 \leq \varphi \leq 1$, $\varphi(s) = \begin{cases} 0, & s \leq 0 \\ 1, & s \geq 1 \end{cases}$, and where η is locally Lipschitz continuous with $0 \leq \eta(u) = \eta(-u) \leq 1$ (so that e is odd) and

$$\eta(u) = \begin{cases} 1, & u \notin N_{12}, \\ 0, & u \in N_8. \end{cases}$$

Then, if $u \notin N_{12}$ with $\max\{E_f(u), E_f(-u)\} \geq 1$ we have

$$-\langle dE_f(u), e(u)\rangle_{H^{-1}\times H_0^1} = \langle dE_0(u), \tilde{e}(u)\rangle_{H^{-1}\times H_0^1} - \langle f, \tilde{e}(u)\rangle_{H^{-1}\times H_0^1}$$

$$\geq \frac{1}{2}\|dE_0(u)\|_{H^{-1}} - \|f\|_{H^{-1}} \geq \frac{1}{2}\|dE_f(\pm u)\|_{H^{-1}} - 2\|f\|_{H^{-1}}$$

$$\geq \frac{1}{4}\left(\|dE_f(u)\|_{H^{-1}} + \|dE_f(-u)\|_{H^{-1}} - 8\|f\|_{H^{-1}}\right)$$

$$\geq \|f\|_{H^{-1}}.$$ (Similarly, $\langle dE_f(u), e(u)\rangle_{H^{-1}\times H_0^1} \leq 0$, if $u \notin N_8$.)

That is, for the (even) flow Φ generated by e we have

$$\frac{d}{dt}E_f(\Phi(u,t)) \leq -\|f\|_{H^{-1}}$$ away from N_{12}.

Choosing $h_t = \Phi(\cdot, t)\circ h$ for large $t \geq 0$, we obtain our claim. □

In view of Claim 1 then we have

$$\delta_k + 1 \geq \sup_{u \in X^+_{k+1}} E_f(h(u)) \geq \sup_{u \in X_{k+1}} E_f(h(u)) - \sup_{\substack{u \in N_{12} \\ \min\{E_f(u), E_f(-u)\} \leq \delta_k + 1}} |\underbrace{E_f(u) - E_f(-u)}_{= 2\int_\Omega f u \, dx}|$$

$$\geq \delta_{k+1} - C \sup_{\substack{u \in N_{12} \\ \min\{E(u), E(-u)\} \leq \delta_k + 1}} \|u\|_{L^2} .$$

Claim 2. For $u \in N$ with $\min\{E_f(u), E_f(-u)\} \leq \delta_k + 1$ there holds $\|u\|_{L^2} \leq C \delta_k^{1/p} + C$.

Proof. Compute, using Young's inequality

$$\|u\|^p_{L^2} \leq C \|u\|^p_{L^p} = C\left(2 E_0(u) - \langle dE_0(u), u\rangle_{H^{-1} \times H^1_0}\right)$$

$$\leq C\Big(2 \min\{E_f(u), E_f(-u)\} - \langle dE_f(u), u\rangle_{H^{-1} \times H^1_0}$$

$$+ 3\|f\|_{L^2}\|u\|_{L^2}\Big)$$

$$\leq C(\delta_k + 1) + C\|f\|_{L^2}\|u\|_{L^2}$$

$$\leq C\,\delta_k + \frac{1}{2}\|u\|^p_{L^2} + C .$$

The claim follows. $\square$

Since $\delta_k \geq \beta_k \longrightarrow \infty$ $(k \to \infty)$ we conclude the bound

$$\delta_{k+1} \leqslant \delta_k + C\delta_k^{1/p} + C$$

$$\leqslant \delta_k + C\delta_k^{1/p} = \delta_k\left(1 + C\delta_k^{\frac{1-p}{p}}\right), \quad k \geq k_0.$$

By iteration, Remark 2.5.2, and (2.5.8) then

$$\delta_{k_0+l} \leqslant \delta_{k_0+l-1}\left(1 + C\delta_{k_0+l-1}^{\frac{1-p}{p}}\right)$$

$$\leqslant \dots \leqslant \delta_{k_0}\prod_{j=0}^{l-1}\left(1 + C\delta_{k_0+j}^{\frac{1-p}{p}}\right)$$

$$= \delta_{k_0}\exp\left(\sum_{j=0}^{l-1}\log\left(1 + C\delta_{k_0+j}^{\frac{1-p}{p}}\right)\right)$$

$$\leqslant \delta_{k_0}\exp\left(\sum_{j=0}^{\infty} C\delta_{k_0+j}^{\frac{1-p}{p}}\right)$$

$$\leqslant \delta_{k_0}\exp\left(C\sum_{k=k_0}^{\infty} k^{\frac{2p}{n(p-2)}\cdot\frac{1-p}{p}}\right)$$

$$\leqslant C\delta_{k_0}, \quad \text{uniformly in } l \in \mathbb{N},$$

if $\dfrac{2p}{n(p-2)} < \dfrac{p}{p-1}$.

Contradiction! $\square$

<u>Remark.</u> A. Bahri (J. Funct. Analysis (1981)) showed that for any $p \in]2, 2^*[$ the equation (2.5.6), (2.5.7) has infinitely many solutions for generic $f \in L^2(\Omega)$.

2.6 Lusternik-Schnirelman theory on convex sets – The classical Plateau problem

2.6.1 Abstract setting.

Let X be a Banach space, $E \in C^1(X)$, and let $M \subset X$ be closed and convex.

Def. 2.6.1. For any $u \in M$ the number

$$g(u) := \sup_{\substack{v \in M \\ \|u-v\|_X \leq 1}} \langle dE(u), u-v \rangle_{X^* \times X}$$

is the slope of E at u relative to M.

Remark 2.6.1. i) If $M = X$ there holds

$$g(u) = \|dE(u)\|_{X^*}, \quad u \in M = X.$$

ii) The map $M \ni u \mapsto g(u) \in \mathbb{R}$ is continuous.

Proof: For any $u_0 \in M$, any $\varepsilon > 0$ there is an open $U_0 \subset B_\varepsilon(u_0; X)$ such that $u_0 \in U_0$ and

$$\|dE(u) - dE(u_0)\|_{X^*} < \varepsilon, \quad u \in U_0 \cap M.$$

For $u \in U_0 \cap M$ let $v = v(u) \in M$ such that

$$g(u) \leq \langle dE(u), u-v \rangle_{X^* \times X} + \varepsilon, \quad \|u-v\|_X < 1-\varepsilon.$$

Then for $u_1, u_2 \in U_0 \cap M$ with $v_i = v(u_i) \in M$

$$g(u_1) - g(u_2) \leq \langle dE(u_1), u_1 - v_1 \rangle_{X^* \times X} - \langle dE(u_2), u_2 - v_1 \rangle_{X^* \times X} + \varepsilon$$

$$\leq \|dE(u_1) - dE(u_2)\|_{X^*} + \|dE(u_2)\|_{X^*} \|u_1 - u_2\|_X + \varepsilon \leq C\varepsilon,$$

□

Def. 2.6.2. i) A point $u \in M$ is <u>critical</u> for E <u>in</u> M, if
$$g(u) = 0;$$
otherwise, u is called <u>regular</u>.

ii) A number $\beta \in \mathbb{R}$ is a <u>critical value</u> of E <u>in</u> M, if there exists a critical point $u \in M$ with
$$E(u) = \beta;$$
otherwise, $\beta \in \mathbb{R}$ is called a <u>regular value</u>.

As before, for $\beta \in \mathbb{R}$, $\delta > 0$, $\rho > 0$ we let
$$E_\beta = \{u \in M;\ E(u) < \beta\},$$
$$K_\beta = \{u \in M;\ g(u) = 0,\ E(u) = \beta\},$$
$$N_{\beta,\delta} = \{u \in M;\ |E(u) - \beta| < \delta,\ g(u) < \delta\},$$
$$U_{\beta,\rho} = \{u \in M;\ \exists v \in K_\beta : \|u - v\|_X < \rho\}.$$

Also let
$$\tilde{M} = \{u \in M;\ g(u) \neq 0\}$$
be the set of regular points.

Def. 2.6.3. i) A sequence $(u_k) \subset M$ is a $(P.-S.)_\beta$-sequence for E in M, if

$$E(u_k) \to \beta, \quad g(u_k) \to 0 \quad (k \to \infty).$$

ii) E satisfies $(P.-S.)_\beta$ in M, if every $(P.-S.)_\beta$-sequence in M is relatively compact.

Then we have the analogue of Lemma 2.3.

Lemma 2.6.1. Suppose E satisfies $(P.-S.)_\beta$ in M. Then there holds

i) $K_\beta \subset M$ is compact;

ii) The families $(N_{\beta,\delta})_{\delta>0}$, $(U_{\beta,\rho})_{\rho>0}$ each are fundamental systems of neighborhoods of K_β; in particular,

iii) if $K_\beta = \phi$ there exists $\delta > 0$ with $N_{\beta,\delta} = \phi$.

Proof: identical with the proof of Lemma 2.3.

Def. 2.6.4. A locally Lipschitz continuous map $\tilde{e} : \tilde{M} \to X$ defines a (negative) <u>pseudo-gradient vector field</u> on M, if

i) $v(u) = u + \tilde{e}(u) \in M, \quad u \in \tilde{M},$

and if

ii) $\|\tilde{e}(u)\|_X = \|u - v(u)\|_X < 1, \quad u \in \tilde{M},$

iii) $\frac{1}{2} g(u) < \langle dE(u), u - v(u) \rangle_{X^* \times X}, \quad u \in \tilde{M}.$

Similar to Thm. 2.3.1 we can then show the following result.

<u>Theorem 2.6.1.</u> There exists a p.g.v.f. $\tilde{e} : \tilde{M} \to X$.

<u>Proof</u>: For any $u_0 \in \tilde{M}$ choose $v_0 = v(u_0) \in M$ with $\|u_0 - v_0\|_X < 1$ and

$$\frac{1}{2} g(u_0) < \langle dE(u_0), u_0 - v_0 \rangle_{X^* \times X} .$$

By continuity of dE and g there exists an open $U_0 = U(u_0) \ni u_0$ such that

$$\forall u \in U_0 \cap M : \frac{1}{2} g(u) < \langle dE(u), u - v_0 \rangle_{X^* \times X},$$

$$\|u - v_0\|_X < 1 .$$

The family $(\mathcal{U}(u))_{u \in \tilde{M}}$ admits a locally finite refinement $\mathcal{U}_\iota \subset \mathcal{U}(u_\iota)$, $\iota \in I$, still covering $\tilde{M}$. With a locally Lipschitz continuous partition of unity $(\varphi_\iota)_{\iota \in I}$ subordinate to $(\mathcal{U}_\iota)_{\iota \in I}$ then let

$$v(u) = \sum_{\iota \in I} \varphi_\iota(u)\, v(u_\iota), \quad u \in \tilde{M}.$$

By convexity of M, then $v(u) \in M$ for every $u \in \tilde{M}$, and

$$\tilde{e}(u) := v(u) - u$$

has the desired properties. $\square$

<u>Remark</u> 2.6.2. Since we assume that $E \in C^1(X)$ is globally defined, we can extend the definition of g to the open neighborhood

$$M_1 = \bigcup_{u \in M} B_1(u; X) \subset X$$

of M and proceed as in Thm. 2.6.1 to obtain $\tilde{e}_1 : \tilde{M}_1 \to X$ with the properties i) - iii) in Def. 2.6.4, extending $\tilde{e}$ to $\tilde{M}_1$.

Theorem 2.6.2. Suppose $E \in C^1(X)$ satisfies $(P.-S.)_\beta$ on M for some $\beta \in \mathbb{R}$. Then for every $\bar{\varepsilon} > 0$, every open $N \supset K_\beta$ in M there exists $0 < \varepsilon < \bar{\varepsilon}$ and a negative pseudo-gradient flow $\Phi \in C^0(M \times [0,1]; M)$ such that

i) $\Phi(\cdot, t) : M \to M$ is continuous, $0 \le t \le 1$;

ii) $\Phi(u,t) = u$ if either $t = 0$, or if $g(u) = 0$, or if $|E(u) - \beta| \ge \bar{\varepsilon}$;

iii) the map $t \mapsto E(\Phi(u,t))$ is non-increasing for any $u \in M$;

iv) $\Phi(E_{\beta+\varepsilon}, 1) \subset E_{\beta-\varepsilon} \cup N$, and

$$\Phi(E_{\beta+\varepsilon} \setminus N, 1) \subset E_{\beta-\varepsilon} .$$

Proof: Given $N \supset K_\beta$, with the help of Lemma 2.6.1 we find $0 < \delta, \rho \le 1$ with

$$N \supset \mathcal{U}_{\beta, 3\rho} \supset \mathcal{U}_{\beta, \rho} \supset N_{\beta, \delta} \supset K_\beta$$

as in the proof of Thm. 2.4.1. With

$$0 < \varepsilon = \frac{1}{4} \min\{\bar{\varepsilon}, \rho\,\delta\} < \frac{\bar{\varepsilon}}{2}$$

and locally Lipschitz $0 \leq \varphi, \eta \leq 1$ such that

$$\varphi_{\beta,\varepsilon}(E(u)) = \varphi\left(\frac{|E(u)-\beta|}{\varepsilon}\right),$$

$$\eta(u) = \begin{cases} 0, & u \in U_{\beta,\rho}, \\ 1, & u \notin U_{\beta,2\rho} \end{cases}$$

we then let

$$e(u) = \begin{cases} \eta(u)\,\varphi_{\beta,\varepsilon}(E(u))\,\tilde{e}(u), & u \in \tilde{M}, \\ 0, & \text{if } g(u) = 0, \end{cases}$$

to obtain a locally Lipschitz continuous vector field $e : M \to X$ with

$$\|e(u)\|_X < 1, \quad e(u) + u \in M, \quad u \in M,$$

and such that

$$\langle dE(u), e(u) \rangle_{X^* \times X} \leq 0, \quad u \in M.$$

Claim: There exists a unique solution $\Phi \in C^0(M \times [0,\infty[; M)$ of the initial value problem

$$\frac{d}{dt}\Phi(u,t) = e(\Phi(u,t)), \quad t \geq 0,$$

$$\Phi(u,0) = u$$

for every $u \in M$.

Proof. With the extension $\tilde{e}_1 : \tilde{M}_1 \to X$ we can extend e as as in Rem. 2.6.2

$$e_1(u) = (1 - \operatorname{dist}(u,M))\,\eta(u)\,\varphi_{\beta,\varepsilon}(\Xi(u))\,\tilde{e}_1(u)$$

to a locally Lipschitz continuous vector field $e_1 : X \to X$. The solution $\Phi_1 \in C^0(X \times \mathbb{R}; X)$ to

$$\frac{d}{dt}\Phi_1(u,t) = e_1(\Phi_1(u,t))$$

$$\Phi_1(u,0) = u$$

then defines a flow of homeomorphisms $\Phi_1(\cdot,t)$ of X. Since for any $u \in M$ we have

$$u + \frac{d}{dt}\Big|_{t=0}\Phi_1(u,t) = u + e_1(u) \in M,$$

the set M is forward Φ_1-invariant and $\Phi := \Phi_1\big|_{M\times[0,1]}$ satisfies

$$\Phi(\cdot,t): M \to M, \quad 0\le t\le 1.$$

The remaining properties follow from the properties of e as in the proof of Thm. 2.4.1. $\square$

The following is a variant of the "mountain-pass" lemma,

Theorem 2.6.3. Suppose $E\in C^1(X)$ satisfies $(P.-S.)_\beta$ for any $\beta\in\mathbb{R}$ and assume that $u_0\neq u_1$ are relative minimizers of E in M with $E(u_1)\le E(u_0)=\beta_0$.

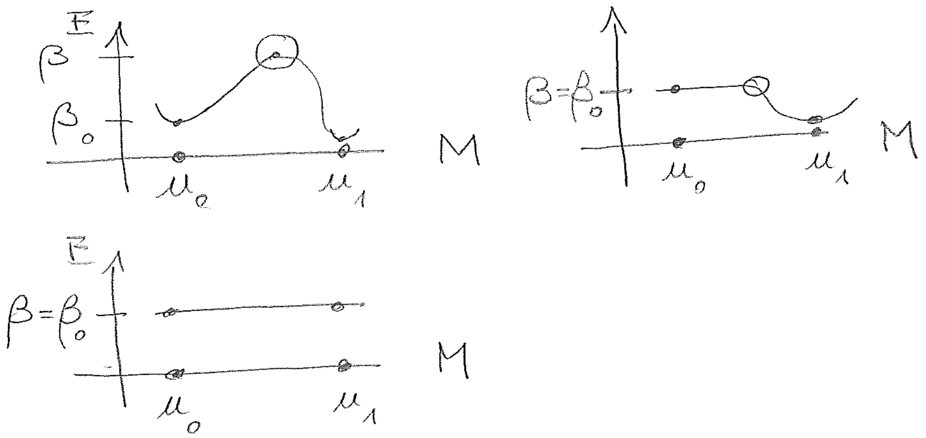
Let

$$\beta = \inf_{\gamma\in\Gamma}\sup_{0\le s\le 1} E(\gamma(s)) \ge \beta_0,$$

where $\Gamma = \{\gamma\in C^0([0,1];M);\ \gamma(0)=u_0,\ \gamma(1)=u_1\}$.

Then either there exists $u \in K_\beta$ which is not a relative minimizer of E, or $E(u_1) = \beta_0 = \beta$, and for any open $N_1 \supset K_\beta$ there exists $\gamma_1 \in \Gamma$ with $\gamma_1([0,1]) \subset N_1$.

<u>Remark</u> 2.6.3. Theorem 2.6.3 covers the following 3 cases, illustrated in the pictures below.



<u>Proof of Thm.</u> 2.6.3: In case $\beta > \beta_0$ let $\bar{\varepsilon} = \beta - \beta_0$; else let $\bar{\varepsilon} = 1$. Suppose that K_β consists only of relative minimizers of E in M. Then K_β is both open and closed in

$$E_\beta = \{u \in M;\ E(u) \le \beta\},$$

and there exists an open $N_0 \supset K_\beta$ such that

$$N_0 \cap E_\beta = \phi .$$

Given any open $N_1 \supset K_\beta$ we then let $0 < \varepsilon < \bar{\varepsilon}$, Φ be as given by Thm. 2.6.2 for $N = N_0 \cap N_1$. Choose $\gamma \in \Gamma$ with

$$\sup_{0 \le s \le 1} E(\gamma(s)) < \beta + \varepsilon .$$

Since $g(u_0) = 0 = g(u_1)$, then $\gamma_1 := \Phi(\cdot, 1) \circ \gamma \in \Gamma$ and

$$\gamma_1([0,1]) \subset \Phi(E_{\beta+\varepsilon}, 1) \subset E_{\beta-\varepsilon} \cup N .$$

But $$(E_{\beta-\varepsilon} \cap N) \subset (E_{\beta-\varepsilon} \cap N_0) = \phi ,$$

and we cannot have $\gamma_1([0,1]) \subset E_{\beta-\varepsilon}$ by definition of β.

Thus, $\gamma_1([0,1]) \subset N \subset N_1$, and the proof is complete.

$\square$

2.6.2 The classical Plateau problem.

Let

$$B = B_1(0) \subset \mathbb{R}^2 \cong \mathbb{C}$$

be the unit disc.

Def. 2.6.5. A map $u \in C^1(B; \mathbb{R}^3)$ is conformal if the Hopf differential

$$\Phi_u(z) = (\partial u)^2 = \frac{1}{4}\left(|u_x|^2 - |u_y|^2 - 2i\langle u_x, u_y\rangle_{\mathbb{R}^3}\right)$$

vanishes, where

$$z = x + iy \in B \subset \mathbb{C}, \quad \partial u = \frac{1}{2}(\partial_x u - i\partial_y u), \quad u_x = \partial_x u,$$

etc. for brevity.

For conformal $u \in C^2(B; \mathbb{R}^3)$ there holds

$$-\Delta u = 2H(u)\, u_x \wedge u_y \quad \text{in } B$$

away from branch points where $u_x = 0 = u_y$.

Here, $H = H(u(z))$ is the mean curvature of the surface $S = u(B)$ at $p = u(z)$.

Def. 2.6.6. $u \in C^2(B; \mathbb{R}^3)$ is minimal, if $H(u) \equiv 0$.

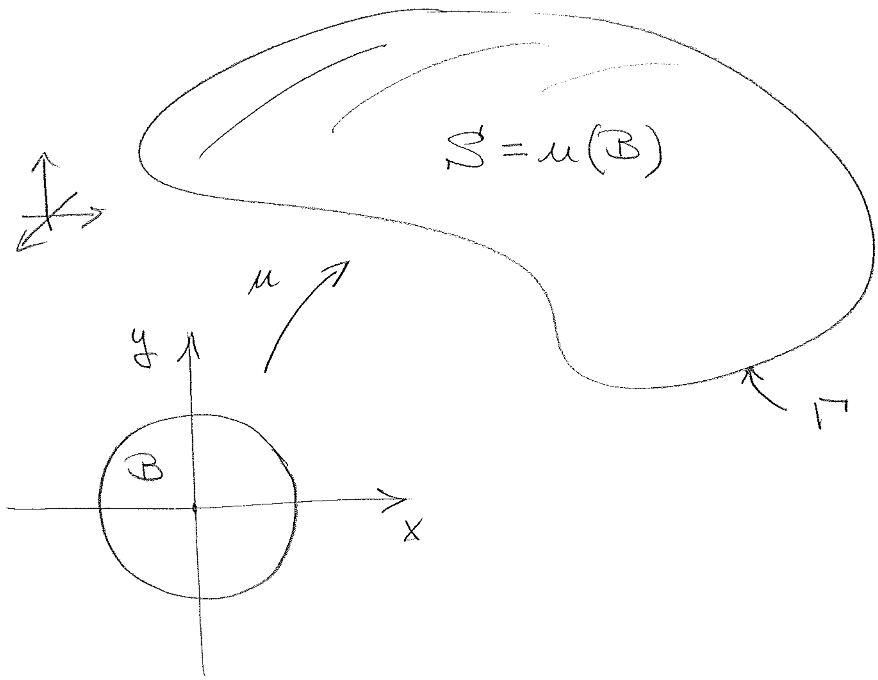
Since for every $u \in C^1(B;\mathbb{R}^3)$ we may introduce conformal parameters, Plateau's problem for disc-type minimal surfaces then is to determine all solutions $u \in C^2(B;\mathbb{R}^3) \cap C^1(\overline{B};\mathbb{R}^3)$ of the problem

(2.6.1) $-\Delta u = 0$ in B,

(2.6.2) $\Phi_u = \frac{1}{4}\left(|u_x|^2 - |u_y|^2 - 2i\langle u_x, u_y\rangle\right) = 0$ in B,

(2.6.3) $u|_{\partial B} : \partial B \to \Gamma$ is a homeomorphism,

where $\Gamma \subset \mathbb{R}^3$ is a given, smooth, closed curve in $\mathbb{R}^3$.



Remark 2.6.4. i) Conditions (2.6.1)-(2.6.3) are conformally invariant: If u satisfies these conditions then so does every map $u\circ g$, where $g \in C^\infty(B; B)$ is conformal.

ii) The Möbius group $\mathcal{G}$ of conformal transformations of B is given by

$$\mathcal{G} = \left\{ g(z) = e^{i\theta}\frac{a-z}{1-\bar{a}z};\ \theta\in\mathbb{R},\ a\in B \right\}.$$

iii) Fixing 3 distinct points $q_k \in \Gamma$, $k=1,2,3$, we may normalize admissible maps u by imposing the 3 point condition,

$$(2.6.4)\qquad u\left(e^{i\frac{2\pi k}{3}}\right) = q_k,\quad 1\le k\le 3.$$

Let

$$\mathcal{C}(\Gamma) = \{ u \in H^1(B;\mathbb{R}^3);\ u|_{\partial B}: \partial B \to \Gamma \text{ is a continuous, weakly monotone parametrization of } \Gamma \}$$

and set

$$\mathcal{C}^*(\Gamma) = \{ u\in\mathcal{C}(\Gamma);\ u \text{ satisfies } (2.6.4) \}.$$

The first important result (which was rewarded with a fields medal for J. Douglas) is the following theorem.

Theorem 2.6.4 (Douglas (1930), Radò (1931))

For every smoothly embedded closed curve $\Gamma \cong S^1$ in $\mathbb{R}^3$ there exists $u \in \mathcal{C}^*(\Gamma)$ with

$$D(u) = \frac{1}{2} \int_B |\nabla u|^2 dz = \min_{v \in \mathcal{C}^*(\Gamma)} D(v),$$

and u satisfies (2.6.1), (2.6.2).

The key for showing compactness of a minimizing sequence is the following result.

Lemma 2.6.2 (Courant-Lebesgue).

Any sequence $(u_k) \subset \mathcal{C}^*(\Gamma)$ with

$$\sup_k D(u_k) < \infty$$

has equi-continuous traces $(u_k|\partial B)$.

Proof. Let $(u_k) \subset \mathcal{E}^*(\Gamma)$ with $\sup_k D(u_k) = C_0 < \infty$. For $p = e^{i\theta} \in \partial B$ and $0 < \delta < 1/2$, by Hölder's and Fubini's theorem for any $u = u_k$ we can bound

$$\inf_{\delta^2 < r < \delta} \left(\int_{\partial B_r(p) \cap B} |\nabla u| \, ds \right)^2$$

$$\leq \frac{\int_{\delta^2}^{\delta} \left(\int_{\partial B_r(p) \cap B} |\nabla u| \, ds \right)^2 \frac{dr}{r}}{\int_{\delta^2}^{\delta} \frac{dr}{r}}$$

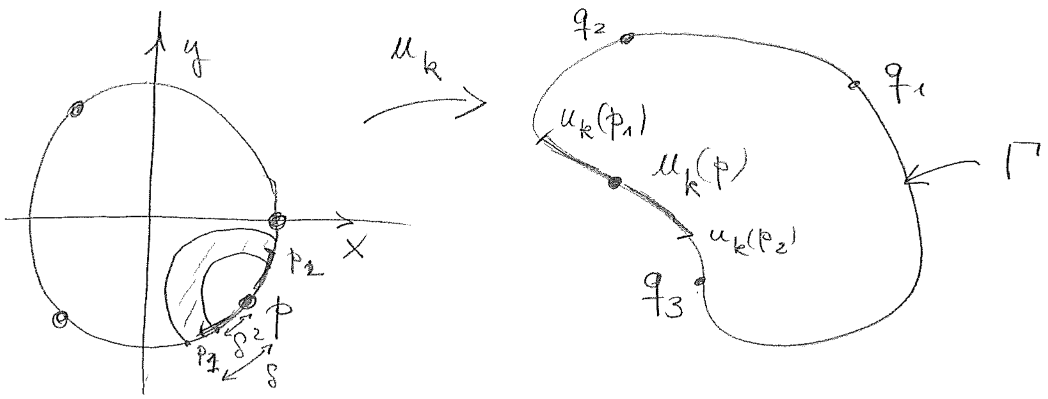
$$\leq \frac{2\pi \int_{\delta^2}^{\delta} \int_{\partial B_r(p) \cap B} |\nabla u|^2 \, ds \, dr}{\log\left(\frac{1}{\delta}\right)},$$

where ds is the element of length along $\partial B_r(p)$. Hence, letting $p_1(r), p_2(r)$ be the points of intersection in $\partial B_r(p) \cap \partial B$ we have

$$\inf_{\delta^2 < r < \delta} |u(p_1(r)) - u(p_2(r))|^2 \leq \inf_{\delta^2 < r < \delta} \left(\int_{\partial B_r(p) \cap B} |\nabla u| \, ds \right)^2$$

$$\leq \frac{4\pi D(u)}{\log\left(\frac{1}{\delta}\right)} \leq \frac{4\pi C_0}{\log\left(\frac{1}{\delta}\right)} \longrightarrow 0 \quad (\delta \downarrow 0).$$

Since the arc $\partial B \cap B_{1/2}(p)$ contains at most one point $e^{\frac{2\pi i l}{3}}$, $l \in \{1,2,3\}$, and $u_k \in \mathcal{C}^*(\Gamma)$, the arc $\partial B \cap B_{\delta^2}(p)$ is mapped by each u_k to a subarc of Γ containing at most one point q_l, whose diameter shrinks to 0 as $\delta \downarrow 0$, uniformly in k. The claim follows.



Proof of Thm. 2.6.4. Let $(u_k) \subset \mathcal{C}^*(\Gamma)$ be a minimizing sequence with

$$\mathcal{D}(u_k) \to \inf_{v \in \mathcal{C}^*(\Gamma)} \mathcal{D}(v) \quad (k \to \infty).$$

Then $(u_k) \subset H^1(B, \mathbb{R}^3)$ is bounded in view of

$$\|u\|_{L^2(B)} \leq C\left(\|\nabla u\|_{L^2} + \|u|_{\partial B}\|_{L^2(\partial B)}\right),$$

for $u \in H^1(B, \mathbb{R}^3)$, and $u_k \xrightarrow{w} u$ in $H^1(B, \mathbb{R}^3)$ as $k \to \infty$, $k \in \Lambda$.

Moreover, by Lemma 2.6.2 and Arzela-Ascoli's thm for a further subsequence we have $u_k|_{\partial B} \to u|_{\partial B}$ in $C^0(\partial B, \mathbb{R}^3)$), and $u \in \mathcal{C}^*(\Gamma)$ satisfies

$$\mathcal{D}(u) = \inf_{v \in \mathcal{C}^*(\Gamma)} \mathcal{D}(v).$$

For any $\varphi \in C_c^\infty(B, \mathbb{R}^3)$ then

$$\mathcal{D}(u) \leq \mathcal{D}(u + \varepsilon\varphi), \quad \varepsilon \in \mathbb{R}.$$

Hence

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{D}(u + \varepsilon\varphi) = \int_B \nabla u \nabla \varphi \, dz,$$

and

$$-\Delta u = 0 \quad \text{in } B.$$

In order to also see (2.6.2) let $\tau \in C^1(\overline{B}; \mathbb{R}^2)$ and for $|\varepsilon| << 1$ consider the diffeomorphism

$$id + \varepsilon\tau : B \to \Omega_\varepsilon := (id + \varepsilon\tau)(B).$$

For sufficiently small $|\varepsilon|$ then by the Riemann mapping theorem there exists a conformal diffeomorphism $\psi_\varepsilon : B \to \Omega_\varepsilon$. Composing with a suitable $g_\varepsilon \in G$, the conformal group of the disc, moreover, we may assume that

$$u_\varepsilon := u \circ (id + \varepsilon\tau)^{-1} \circ \psi_\varepsilon \circ g_\varepsilon \in \mathcal{C}^*(\Gamma).$$

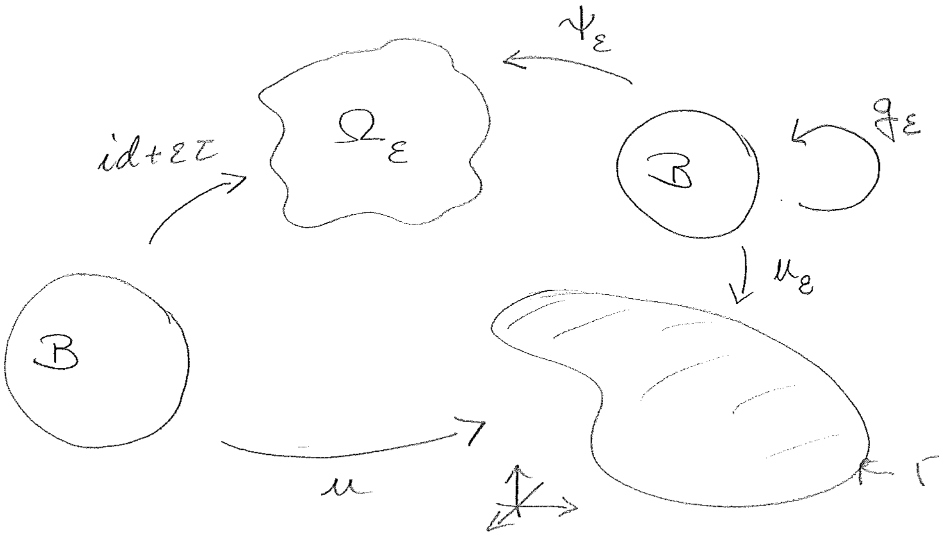
with

$$D(u_\varepsilon) = D(u \circ (id + \varepsilon\tau)^{-1}; \Omega_\varepsilon)$$

by conformal invariance of D.

By minimality of $D(u) = D(u_0)$ then we have

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon = 0} D(u \circ (id + \varepsilon\tau)^{-1}; \Omega_\varepsilon).$$



Changing variables $\zeta = (id + \varepsilon\tau)(z)$ and using

$$d\,(id+\varepsilon\tau)^{-1} = (\mathbb{1} + \varepsilon\, d\tau)^{-1} \circ (id+\varepsilon\tau)^{-1}$$
$$= \mathbb{1} - \varepsilon\, d\tau \circ (id+\varepsilon\tau)^{-1} + \mathcal{O}(\varepsilon^2)$$

as well as

$$\det(\mathbb{1} + \varepsilon\, d\tau) = 1 + \varepsilon\, \mathrm{trace}\, d\tau + \mathcal{O}(\varepsilon^2)$$

we find

$$\mathcal{D}\left(u \circ (id+\varepsilon\tau)^{-1}; \Omega_\varepsilon\right)$$
$$= \frac{1}{2} \int_{\Omega_\varepsilon} \left| du \cdot (\mathbb{1} - \varepsilon\, d\tau)) \circ (id+\varepsilon\tau)^{-1} \right|^2 d\zeta + \mathcal{O}(\varepsilon^2)$$
$$= \frac{1}{2} \int_{B} \left| du \cdot (\mathbb{1} - \varepsilon\, d\tau) \right|^2 \left|\det(\mathbb{1} + \varepsilon\, d\tau)\right| dz + \mathcal{O}(\varepsilon^2),$$

and these results

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{D}\left(u \circ (id+\varepsilon\tau)^{-1}; \Omega_\varepsilon\right)$$
$$= -\int_B \langle du \cdot d\tau, du \rangle_{\mathbb{R}^2 \times \mathbb{R}^3} dz + \frac{1}{2} \int_B |du|^2\, \mathrm{tr}\, d\tau\, dz$$

$$= \int_B \Big(\frac{1}{2} |du|^2 (\tau^1_x + \tau^2_y) - \langle u_x \tau^1_x + u_y \tau^2_x, u_x \rangle$$
$$- \langle u_x \tau^1_y + u_y \tau^2_y, u_y \rangle \Big) dz$$

$$= -\frac{1}{2}\int_{B}\Big(|u_x|^2(\tau^1_x - \tau^2_y) + |u_y|^2(\tau^2_y - \tau^1_x) + 2\langle u_x, u_y\rangle(\tau^1_y + \tau^2_x)\Big)\,dz$$

$$= \mathrm{Re}\int_{B} \Phi \overset{\mathbb{C}\text{-mult.}}{\cdot} \underbrace{\bar\partial\tau}_{=\frac{1}{2}(\partial_x + i\partial_y)\tau}\,dw \equiv 0$$

for any $\tau \in C^1(\bar{B};\mathbb{C})$. But for every $f \in C^\infty_c(B;\mathbb{C})$ there exists

$$\tau(z) = \int_{B}\frac{f(\xi)}{z-\xi}\,d\xi$$

with

$$\bar\partial\tau = f.$$

Hence $\Phi \equiv 0$, and the proof is complete.

$\square$

2.6.3 The mountain-pass lemma for minimal surfaces

There are curves $\Gamma \subset \mathbb{R}^3$ that span (at least) two different solutions of Plateau's problem, for instance, due to symmetry, as in the following

Example.

Already in 1939 Morse-Tompkins set up a Morse theory for the problem, predicting the existence of an unstable third solution of saddle-type in this case. Working in the C^0-topology, they sought to identify minimal surfaces with "homotopy critical" points in their theory. This approach was later critized by Tromba, who pointed out that even smoothly non-degenerate solutions of Plateau's problem might not be "homotopy-critical", similar to our observation in Section 1.4, and developed a version of degree theory to overcome this difficulty.

The full Morse theory was then derived by S.(1982) within the framework developed in 2.6.1.

Given a smoothly embedded closed curve $\Gamma \cong S^1$ in $\mathbb{R}^3$, we choose a smooth reference parametrization

$$\gamma : S^1 = \mathbb{R}/2\pi \to \Gamma .$$

For any smooth $\varphi : \partial B \to \mathbb{R}^3$ also let $h(\varphi) : B \to \mathbb{R}^3$ be the harmonic extension of φ to B. Note that the Dirichlet energy of $h(\varphi)$ may be expressed in terms of Douglas' integral

$$(2.6.5) \quad D(h(\varphi)) = \frac{1}{16\pi} \int_0^{2\pi}\int_0^{2\pi} \frac{|\varphi(\theta) - \varphi(\theta')|^2}{\sin^2\left(\frac{\theta-\theta'}{2}\right)} \, d\theta \, d\theta' ,$$

which is equivalent to the $H^{\frac{1}{2}}$-norm of φ on $\partial B \cong \mathbb{R}/2\pi$.

Hence, the map h extends to a continuous map

$$h : H^{\frac{1}{2}}(\partial B ; \mathbb{R}^3) \to H^1(B; \mathbb{R}^3).$$

We then may recast the setting for the solution of Plateau's problem, as follows.

Let

$$M = \{ u \in H^{\frac{1}{2}}_{loc} \cap C^0(\mathbb{R}) ;\ u(t+2\pi) = u(t) + 2\pi,\ t \in \mathbb{R},\ u \text{ non-decreasing} \}$$

be the set of admissible re-parametrizations such that the map $h(\gamma \circ u) \in H^1 \cap C^0(\overline{B}; \mathbb{R}^3)$ satisfies the Plateau boundary condition (2.6.3).

The set

$$M' = \{ v = u - id ;\ u \in M \}$$

then is a closed, convex subset of the Banach space

$$X = H^{\frac{1}{2}} \cap C^0(\mathbb{R}/2\pi) ;$$

for ease of notation, however, we prefer to work with the set M, which then is a closed, convex subset of the affine space $\{id\} + X$.

Also let

$$M^* = \{ u \in M ;\ u(\tfrac{2\pi k}{3}) = \tfrac{2\pi k}{3} ,\ k \in \mathbb{Z} \}$$

be the set of reparametrizations of γ such that $\gamma \circ u$ satisfies (2.6.4) for $q_k = \gamma(\frac{2\pi k}{3}) \in \Gamma$, $1 \le k \le 3$.

Finally, setting

$$E(u) = D(h(\gamma \circ u)) = \frac{1}{16\pi} \int_0^{2\pi}\int_0^{2\pi} \frac{|\gamma(u(s)) - \gamma(u(t))|^2}{\sin^2(\frac{t-s}{2})} \, ds\, dt$$

we obtain a functional of class C^1 on $\{id\} + X$ to which we can apply the abstract theory developed in Section 2.6.1 above.

In fact, for any $u \in M$, any $\xi \in X$, letting $\mathcal{U} = h(\gamma \circ u)$ be the harmonic extension of the parametrization $\gamma \circ u$ of Γ, we have

$$\langle dE(u), \xi \rangle_{X^* \times X} = \langle d_u D(h(\gamma \circ u)), \xi \rangle_{X^* \times X}$$

$$= \int_B \nabla \mathcal{U} \nabla \Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} h(\gamma \circ (u + \varepsilon \xi)) \Big) dz$$

$$= \int_{\partial B} \langle \partial_n \mathcal{U}, \gamma'(u) \xi \rangle_{\mathbb{R}^3} \, d\sigma ,$$

with

$$\gamma'(u)\xi \in H^{\frac{1}{2}} \cap C^0(\partial B; \mathbb{R}^3) \cong X$$

and outward normal derivative

$$\partial_n \mathcal{U} \in H^{-\frac{1}{2}}(\partial B; \mathbb{R}^3) \subset X^*.$$

Here we use that X is an algebra with

$$\|\xi \eta\|_X \leq \|\xi\|_X \|\eta\|_X , \quad \xi, \eta \in X ,$$

as is most easily seen by considering $\xi, \eta \in X$ as traces $\xi = \Xi|_{\partial B}$, $\eta = Y|_{\partial B}$ where $\Xi, Y \in H^1 \cap C^0(\bar{B})$, and observing that

$$\|\nabla(\Xi\, Y)\|_{L^2} \leq \|\nabla \Xi\|_{L^2} \|Y\|_{L^\infty} + \|\Xi\|_{L^\infty} \|\nabla Y\|_{L^2} .$$

In particular, as in Def. 2.6. we let

$$g(u) = \sup_{\substack{v \in M \\ \|u-v\|_X \leq 1}} \langle dE(u), u-v \rangle_{X^* \times X}, \quad u \in M.$$

Then we have the following key result.

<u>Proposition</u> 2.6.1. For any $u \in M$ there holds $g(u) = 0$ iff $u := h(\gamma \circ u)$ solves (2.6.1) – (2.6.3).

<u>Proof</u>: By definition of h and M, any surface $u = h(\gamma \circ u)$ with $u \in M$ solves (2.6.1) and satisfies the Plateau boundary condition (2.6.3). Thus, only condition (2.6.2) is an issue.

For this note the identity for $z = x + iy = re^{i\phi} \in B$

$$4z^2(\partial u)^2 = [(x+iy)(u_x - i u_y)]^2$$

$$(2.6.6) \qquad = (r u_r - i u_\phi)^2 = r^2|u_r|^2 - |u_\phi|^2 - 2ir\langle u_r, u_\phi \rangle_{\mathbb{R}^3}.$$

From

$$\bar\partial\left(z^2(\partial u)^2\right) = 2z^2 \overbrace{(\bar\partial \partial u)}^{=\frac{1}{4}\Delta u} \partial u = 0$$

we then conclude analyticity of

$$\Phi = r^2|u_r|^2 - |u_\phi|^2 - 2ir\langle u_r, u_\phi \rangle_{\mathbb{R}^3}.$$

Moreover, u satisfies (2.6.2) iff $\Phi \equiv 0$, and the latter holds iff $\langle \mathcal{U}_r, \mathcal{U}_\phi \rangle_{\mathbb{R}^3} \equiv 0$ on ∂B.

Indeed, if $\langle \mathcal{U}_r, \mathcal{U}_\phi \rangle_{\mathbb{R}^3} \equiv 0$ on ∂B, by harmonicity $\operatorname{Im}(\Psi) \equiv 0$ in B and $\Psi(re^{i\phi}) \equiv \text{const} = \Psi(0) = 0$ by the Cauchy-Riemann equations.

i) Suppose $\mathcal{U} = h(\gamma \circ u)$ solves (2.6.1) - (2.6.3). By the regularity theory of Hildebrandt and Nitsche, and since Γ is smooth, then $\mathcal{U} \in C^1(\overline{B}; \mathbb{R}^3)$ and (2.6.2) gives

$$\langle \mathcal{U}_r, \gamma'(u) \rangle_{\mathbb{R}^3} \equiv 0;$$

so

$$\langle dE(u), \xi \rangle_{X^* \times X} = \int_{\partial B} \langle \mathcal{U}_r, \gamma'(u) \rangle_{\mathbb{R}^3} \xi \, do = 0$$

for any $\xi \in X$. Hence $g(u) = 0$.

ii) To see the converse, we also require a regularity result.

Lemma 2.6. . Let $u \in M$ with $g(u) = 0$. Then $\mathcal{U} = h(\gamma \circ u) \in H^2(B; \mathbb{R}^3)$.

See S. (1982/1988); Prop. II.5.1, p. 66 ff., Imbusch - S. for the analogous result in M^*.

Proof of Prop. 2.6.1 (completed). Suppose $u \in M$ satisfies $g(u)=0$ and let $U = h(\gamma \circ u)$ as above. For any $\xi \in C^1(\mathbb{R}/2\pi)$ and any $|\varepsilon| < \|\xi'\|^{-1}_{L^\infty}$ the map $id + \varepsilon\xi \in C^1$ is a diffeomorphism on $\mathbb{R}$, and

$$u_\varepsilon = u \circ (id + \varepsilon\xi) \in M.$$

Moreover, regularity $U \in H^2(B, \mathbb{R}^3)$ implies

$$U_\phi = \gamma'(u) \cdot \frac{du}{d\phi} \in H^{\frac{1}{2}}(\partial B, \mathbb{R}^3),$$

and thus that $\frac{du}{d\phi} \in H^{\frac{1}{2}}$, since $|\gamma'(u)| \geq c_0 > 0$.
It then follows that $\|u_\varepsilon - u\|_X \to 0$ as $\varepsilon \to 0$, and we may choose $\varepsilon_0 > 0$ such that

$$\|u_\varepsilon - u\|_X < 1 \text{ for } |\varepsilon| < \varepsilon_0.$$

Thus, from our assumption that $g(u) = 0$ we deduce

$$0 = g(u) \geq \langle dE(u), u - u_\varepsilon \rangle_{X^* \times X}, \quad |\varepsilon| < \varepsilon_0.$$

It follows that for every $\xi \in C^1(\mathbb{R}/2\pi)$ there holds

$$0 \geq \lim_{\varepsilon \downarrow 0} \frac{\langle dE(u), u - u_\varepsilon \rangle_{X^* \times X}}{\varepsilon}$$

$$= \int_{\partial B} \langle \partial_n U, \gamma'(u) \rangle_{\mathbb{R}^3} \lim_{\varepsilon \downarrow 0} \frac{u - u_\varepsilon}{\varepsilon} \, do$$

$$= -\int_{\partial B} \langle \partial_n U, \gamma'(u) \rangle_{\mathbb{R}^3} \frac{du}{d\phi} \cdot \xi \, do = -\int_{\partial B} \langle \partial_r U, \partial_\phi U \rangle \cdot \xi \, do.$$

Replacing ξ with $-\xi$, we then see that the latter, in fact, equals 0.

But $\partial_\tau u$, $\partial_\phi u \in H^{\frac{1}{2}}(\partial B; \mathbb{R}^3) \hookrightarrow L^p(\partial B; \mathbb{R}^3)$ for every $p < \infty$. Choosing $p = 4$ and observing that $C^1(\mathbb{R}/2\pi)$ is dense in $L^2(\mathbb{R}/2\pi)$ we then obtain

$$\langle u_r, u_\phi \rangle_{\mathbb{R}^3} \equiv 0 \quad \text{on } \partial B,$$

and $\Psi = r^2|u_r|^2 - |u_\phi|^2 - 2ir\langle u_r, u_\phi\rangle_{\mathbb{R}^3} \equiv 0$, as shown above, which is equivalent to (2.6.2).

The proof is complete. $\square$

Finally, we verify the Palais-Smale condition.

Proposition 2.6.2. Let $(u_k) \subset M^*$ such that

$$E(u_k) \to \beta, \quad g(u_k) \to 0 \quad (k \to \infty).$$

Then there is $u \in K_\beta$ and a subsequence $\Lambda \subset \mathbb{N}$ such that $\|u_k - u\|_X \to 0$ $(k \to \infty, k \in \Lambda)$.

Proof: Let $\mathcal{U}_k = h(\gamma \circ u_k) \in \mathcal{C}^*(\Gamma)$, $k \in \mathbb{N}$.

Since $$D(\mathcal{U}_k) = E(u_k) \to \beta \quad (k \to \infty),$$

by Lemma 2.6.2 the family $(\mathcal{U}_k|_{\partial B} = \gamma \circ u_k)$ is equi-continuous and bounded in $H^{\frac{1}{2}}$ on ∂B. But then also

$$u_k = \gamma^{-1} \circ (\gamma \circ u_k)$$

is bounded in $H^{\frac{1}{2}}$, and a subsequence $u_k \overset{w}{\to} u \in M^*$ in $H^{\frac{1}{2}}$ and uniformly. For suitable $0 < s \le 1$ then

$$\|s(u_k - u)\|_X < 1, \quad k \in \mathbb{N},$$

and $$v_k = u_k + s(u - u_k) = (1-s)u_k + su \in M$$

satisfies $\|v_k - u_k\|_X < 1$; so

$$o(1) = g(u_k) \geq \langle dE(u_k), u_k - v_k \rangle_{X^* \times X}$$

$$= s \int_{\partial B} \langle \partial_n \mathcal{U}_k, \gamma'(u_k) \rangle_{\mathbb{R}^3} (u_k - u) \, d\sigma .$$

Expanding

$$U_k - U = \gamma\circ u_k - \gamma\circ u$$

$$= \gamma'(u_k)(u_k - u) - \int_0^1\int_0^t \gamma''(u_k + s(u-u_k))\,ds\,dt\,(u-u_k)^2$$

we then have

$$\Big\| \int_0^1\int_0^t \gamma''(u_k + s(u-u_k))\,ds\,dt\,(u-u_k)^2 \Big\|_{H^{\frac{1}{2}}}$$

$$\leq C\|u - u_k\|_{L^\infty} \to 0 \quad (k\to\infty,\ k\in\Lambda).$$

Thus, with error $o(1)\to 0$,

$$o(1) \geq \int_{\partial B} \langle \partial_n U_k, U_k - U\rangle_{\mathbb{R}^3}\,do - o(1)$$

$$= \int_B \nabla U_k \nabla(U_k - U)\,dz - o(1)$$

$$= \|\nabla(U_k - U)\|^2_{L^2(B)} - o(1),$$

and $U_k \to U$ in H^1, $U_k|_{\partial B} = \gamma\circ u_k \to \gamma\circ u$ in $H^{\frac{1}{2}}$.
Again using that γ is a diffeomorphism, we then obtain $u_k \to u$ in $H^{\frac{1}{2}}$, and thus

$$\|u_k - u\|_X \to 0 \quad (k\to\infty,\ k\in\Lambda).$$

$\square$

Remark. In order to see the relative compactness in C^0 of Palais-Smale sequences, we may also use the following observation.

Let $(u_k) \subset M^*$ be a $(P.-S.)_\beta$-sequence. By monotonicity we have the uniform $W^{1,1}$-bound

$$\int_0^{2\pi} |u_k'|\,ds = \int_0^{2\pi} u_k'\,ds = u_k(2\pi) - u_k(0) = 2\pi, \quad k \in \mathbb{N}.$$

Hence a subsequence $u_k \to u$ in $L^1_{loc}(\mathbb{R})$ and almost everywhere as $k \to \infty$, $k \in \Lambda$, where $u : [0, 2\pi] \to [0, 2\pi]$ is non-decreasing with

$$(2.6.7) \quad u\Big|_{\left[\frac{2\pi k}{3}, \frac{2\pi(k+1)}{3}\right]} \subset \left[\frac{2\pi k}{3}, \frac{2\pi(k+1)}{3}\right], \quad k \in \mathbb{Z}.$$

Moreover, by a theorem of Polya, the convergence is uniform if $u \in C^0$.

But if $u \notin C^0$, by (2.6.7) there would be a point s_0 with $u(s_0-) < u(s_0+) \leq u(s_0-) + \frac{4\pi}{3}$, and $D(h(\gamma \circ u)) = \infty$, contradicting

$$D(U) \leq \liminf_{k \to \infty} D(U_k) = \beta < \infty,$$

where $U = h(\gamma \circ u)$, $U_k = h(\gamma \circ u_k)$, $k \in \mathbb{N}$.

3. Limit cases

3.1 The Yamabe problem.

3.1.1 The Yamabe constant. Let (M, g_0) be a closed[1] Riemannian n-manifold, $n \geq 3$. A conformal metric

$$g = u^{\frac{4}{n-2}} g_0, \quad u > 0,$$

has scalar curvature

$$R_g = u^{1-2^*}\left(-c(n)\,\Delta_{g_0} u + R_{g_0} u\right),$$

where

$$\Delta_{g_0} u = \frac{1}{\sqrt{\det g_0}}\, \partial_i \left(g_0^{ij} \sqrt{\det g_0}\, \partial_j u\right), \quad (g_0^{ij}) = g_0^{-1},$$

is the Laplace-Beltrami operator in the metric g_0, and where

$$c(n) = \frac{4(n-1)}{n-2}.$$

Def. 3.1.1. The operator given by

$$L_g u = -c(n)\,\Delta_g u + R_g u$$

is the conformal Laplacian on (M, g).

---

[1] compact, $\partial M = \emptyset$

Note the conformal transformation rule:

For
$$h = v^{\frac{4}{n-2}} g = (uv)^{\frac{4}{n-2}} g_0 \,, \quad v > 0,$$
we have
$$R_h = v^{1-2^*} L_g v = (uv)^{1-2^*} L_{g_0}(uv);$$
that is, there holds

(3.1.1) $L_g v = u^{1-2^*} L_{g_0}(uv)$.

<u>Def.</u> 3.1.2. A conformal metric $g = u^{\frac{4}{n-2}} g_0$ is a <u>Yamabe metric</u> on (M, g_0) if $R_g \equiv$ const.; that is, if $u > 0$ solves the equation

(3.1.2) $L_{g_0} u = r u^{2^*-1}$ on M

for some constant $r \in \mathbb{R}$.

<u>Remarks</u> 3.1.1. i) Scaling $\tilde{u} = \alpha u$ with $\alpha = |r|^{1/(2^*-2)} > 0$ we can achieve $r \in \{-1, 0, 1\}$ in (3.1.2).

ii) Similar to Section 1.5 we can attempt to find a solution of (3.1.2) by minimizing

$$E_{(M,g_0)}(u) = \int_M \left(|\nabla u|^2_{g_0} + c(n)^{-1} R_{g_0} u^2 \right) d\mu_{g_0}$$

over the set

$$\mathcal{M} = \{ u \in H^1(M, g_0) ; \ \|u\|_{L^{2^*}(M, g_0)} = 1 \},$$

where, in local coordinates

$$|\nabla u|^2_{g_0} = g_0^{ij} \partial_i u \, \partial_j u \, , \quad d\mu_{g_0} = \sqrt{\det g_0} \, dx \, .$$

Note that for $g = u^{\frac{4}{n-2}} g_0$ we have

$$\sqrt{\det g} = u^{2^*} \sqrt{\det g_0} \, , \quad d\mu_g = u^{2^*} d\mu_{g_0} ,$$

so

$$\|u\|^{2^*}_{L^{2^*}(M, g_0)} = \int_M d\mu_g = vol(M, g) .$$

Equivalently, we can try to find a minimizer of the Sobolev-Yamabe ratio

$$S_{(M, g_0)}(u) = \frac{E_{(M,g_0)}(u)}{\|u\|^2_{L^{2^*}(M, g_0)}} .$$

iii) For any $u, v > 0$ with $g = u^{\frac{4}{n-2}} g_0$, $h = v^{\frac{4}{n-2}} g$ by (3.1.1) there holds

$$c(n)\, E_{(M,g)}(v) = \int_M \left(c(n)\, |\nabla v|_g^2 + R_g v^2 \right) d\mu_g$$

$$= \int_M L_g v \cdot v \, d\mu_g = \int_M u^{1-2^*} L_{g_0}(uv) \cdot v\, u^{2^*} d\mu_{g_0}$$

$$= \int_M (uv)\, L_{g_0}(uv)\, d\mu_{g_0} = c(n)\, E_{(M,g_0)}(uv);$$

moreover,

$$\|v\|_{L^{2^*}(M,g)}^{2^*} = \int_M v^{2^*} d\mu_g = \int_M (uv)^{2^*} d\mu_{g_0} = \|uv\|_{L^{2^*}(M,g_0)}^{2^*}.$$

In particular, the <u>Yamabe constant</u>

$$S_{(M,g_0)} = \inf_{(uv)>0} S_{(M,g_0)}(uv) = \inf_{v>0} S_{(M,g)}(v)$$

is a conformal invariant.

Example 3.1.1. Stereographic projection
$\pi : S^n \setminus \{p\} \to \mathbb{R}^n$ from $p = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} \in \mathbb{R}^{n+1}$ with inverse

$$\Phi : \mathbb{R}^n \ni x \mapsto \frac{1}{1+|x|^2} \begin{pmatrix} 2x \\ 1-|x|^2 \end{pmatrix} \in S^n \subset \mathbb{R}^{n+1}$$

induces the conformal metric

$$g = \Phi^* g_{S^n} = \left(\frac{2}{1+|x|^2}\right)^2 g_{\mathbb{R}^n} = u^{\frac{4}{n-2}} g_{\mathbb{R}^n}$$

with

$$u(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}}$$

having scalar curvature

$$R_g = R_{g_{S^n}} = n(n-1) = u^{1-2^*}\left(-c(n)\,\Delta u\right).$$

That is, u solves the equation

$$-\Delta u = \frac{n(n-1)}{c(n)}\, u^{2^*-1} = \frac{n(n-2)}{4}\, u^{2^*-1} \quad \text{on } \mathbb{R}^n.$$

The scaled function

$$u^*(x) = \left(\frac{n(n-2)}{4}\right)^{\frac{1}{2^*-2}} u(x) = \frac{(n(n-2))^{\frac{n-2}{4}}}{(1+|x|^2)^{\frac{n-2}{2}}}$$

then solves

$$-\Delta u^* = (u^*)^{2^*-1} \quad \text{on } \mathbb{R}^n.$$

Since by a theorem of Obata moreover we have

$$\mathcal{S}_{(S^n, g_{S^n})} = \mathcal{S}'_{(S^n, g_{S^n})}(1),$$

from Remark 3.1.1 we obtain

$$S = \mathcal{S}_{(\mathbb{R}^n, g_{\mathbb{R}^n})} = \inf_{0<v\in C_c^\infty(\mathbb{R}^n)} \mathcal{S}'_{(\mathbb{R}^n, g_{\mathbb{R}^n})}(uv)$$

$$= \inf_{0<v\in C_c^\infty(S^n\setminus\{p\})} \mathcal{S}'_{(S^n, g_{S^n})}(v) = \mathcal{S}'_{(S^n, g_{S^n})}$$

$$= \mathcal{S}_{(S^n, g_{S^n})}(1) = \mathcal{S}_{(\mathbb{R}^n, g_{S^n})}(u) = \frac{\|\nabla u\|^2_{L^2}}{\|u\|^2_{L^{2^*}}}.$$

Hence u (and u^*) achieve the best Sobolev constant, as claimed in Section 1.5.

In fact, the Yamabe invariant is maximal on S^n.

<u>Theorem 3.1.1.</u> For any closed (M^n, g_0), $n \geq 3$, we have
$$S_{(M,g_0)} \leq S_{(S^n, g_{S^n})} = S .$$

<u>Proof.</u> Fix geodesic normal coordinates in a ball $B_R(p_0)$ around $p_0 = 0 \in M$, and let $\varphi \in C_c^\infty(B_R(0))$ with $\varphi(x) = 1$ for $|x| < R/2$. Letting
$$u^*_\varepsilon(x) = \varepsilon^{\frac{2-n}{2}} u^*\left(\frac{x}{\varepsilon}\right) = \frac{[\varepsilon\, n(n-2)]^{\frac{n-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{n-2}{2}}}, \quad \varepsilon > 0,$$
as in Exampel 2.3.1. iii), and setting
$$u_k = \varphi\, u^*_{\varepsilon_k}, \quad \varepsilon_k \downarrow 0,$$
we obtain
$$S_{(M,g_0)} \leq S_{(M,g_0)}(u_k) \xrightarrow{(n \to \infty)} S_{(\mathbb{R}^n, g_{\mathbb{R}^n})} = S_{(S^n, g_{S^n})},$$
and the claim follows. $\square$

3.1.2 Local compactness.

In view of Thm. 3.1.1 the functional $E_{(S^n, g_{S^n})}$ – or the ratio $S_{(S^n, g_{S^n})}$ – plays the same role as E_∞ in Thm. 1.5.1.

Already in 1976 Aubin established the following result, anticipating P.L. Lions' Thm. 1.5.1 from 1984.

<u>Theorem</u> 3.1.2 (Aubin) Suppose $S_{(M,g_0)} < S_{(S^n, g_{S^n})}$. Then every minimizing sequence of normalized metrics $g_k = u_k^{\frac{4}{n-2}} g_0$, with $vol(M, g_k) = 1$, $k \in \mathbb{N}$, satisfying

$$S_{(M,g_0)}(u_k) \to S_{(M,g_0)} \quad (k \to \infty)$$

has an H^1-convergent subsequence $u_k \xrightarrow{(k\to\infty,\, k\in\Lambda)} u$ with limit metric $g = u^{\frac{4}{n-2}} g_0$ a Yamabe metric.

The case $S_{(M,g_0)} > 0$ is the most interesting. For this case let

$$F_{(M,g_0)}(u) = \frac{1}{2}\int_M \left(|\nabla u|_{g_0}^2 + c(n)^{-1} R_{g_0} u^2\right) d\mu_{g_0} - \frac{1}{2^*}\int_M u^{2^*} d\mu_{g_0}$$

be the "free" functional associated with (3.1.2).

As in Example 2.4.2.i), and using the correspondence between the constrained minimization problem for $\mathcal{S}'_{(M,g_0)}$ and saddle-type critical points of $F_{(M,g_0)}$ as in the proof of Thm. 1.4.1, the existence of a Yamabe metric also is an immediate consequence of our next result, provided that Aubin's condition

$$\mathcal{S}_{(M,g_0)} < \mathcal{S}'_{(S^n,g_{S^n})}$$

holds.

<u>Theorem</u> 3.1.3 (Trudinger (1968), Aubin (1976), Brezis-Nirenberg (1983), Cerami-Fortunato-S. (1984))
Let (M^n, g_0) be closed, $n \geq 3$. Then the functional $F_{(M,g_0)} \in C^1(H^1(M,g_0))$ satisfies $(P.-S.)_\beta$ for every

$$\beta < \frac{1}{n}\left(\mathcal{S}'_{(S^n,g_{S^n})}\right)^{n/2} = \frac{1}{n}\mathcal{S}'^{\,n/2} =: \beta^*.$$

Remark. A similar local compactness property was the basis for the proof of Rellich's conjecture (on the existence of disc-type surfaces of prescribed constant curvature), achieved independently by M. S. (1982/1986) and Brezis-Coron (1982/1983).

Proof of Thm. 3.1.3: i) Let $(u_k) \subset H^1(M, g_0)$ be a $(P.-S.)_\beta$-sequence, satisfying

$$F_{(M,g_0)}(u_k) \to \beta, \quad dF_{(M,g_0)}(u_k) \to 0 \quad (k \to \infty).$$

Computing, with error $o(1) \to 0 \ (k \to \infty)$,

$$2\beta + o(1)\left(\|u_k\|_{H^1} + 1\right) = 2F_{(M,g_0)}(u_k) - \langle dF_{(M,g_0)}(u_k), u_k \rangle_{H^{-1} \times H^1}$$

$$= \frac{2^*-2}{2^*} \|u_k\|_{L^{2^*}}^{2^*}$$

we find

$$\|u_k\|_{H^1}^2 \leq 2F_{(M,g_0)}(u_k) + \left(2c(n)^{-1}\|R_{g_0}\|_{L^\infty} + 1\right)\|u_k\|_{L^2}^2$$

$$+ \frac{2}{2^*}\|u_k\|_{L^{2^*}}^{2^*}$$

$$\leq 2\beta + C\|u_k\|_{L^{2^*}}^2 + \frac{2}{2^*}\|u_k\|_{L^{2^*}}^{2^*} + o(1)$$

$$\leq C(\beta) + o(1)\left(\|u_k\|_{H^1} + 1\right), \quad k \in \mathbb{N}.$$

Thus $(u_k) \subset H^1(M, g_0)$ is bounded and we may assume that $u_k \xrightarrow{w} u$ in H^1 and strongly in L^p for any $p < 2^*$ as $k \to \infty$.

In particular, for any $\varphi \in H^1(M, g_0)$ we have

$$o(1) = \langle dF_{(M,g_0)}(u_k), \varphi \rangle_{H^{-1} \times H^1}$$

$$= \int_M \left((\nabla u_k, \nabla \varphi)_{g_0} + c(n)^{-1} R_{g_0} u_k \varphi - |u_k|^{2^*-2} u_k \varphi \right) d\mu_{g_0}$$

$$\overset{(k \to \infty)}{\longrightarrow} \int_M \left((\nabla u, \nabla \varphi)_{g_0} + c(n)^{-1} R_{g_0} u \varphi - |u|^{2^*-2} u \varphi \right) d\mu_{g_0}$$

$$= \langle dF_{(M,g_0)}(u), \varphi \rangle_{H^{-1} \times H^1},$$

and u solves equation (3.1.2) with $r = 1$.

Choosing $\varphi = u$, moreover, we obtain

$$2F_{(M,g_0)}(u) = 2F_{(M,g_0)}(u) - \langle dF_{(M,g_0)}(u), u \rangle_{H^{-1} \times H^1}$$

$$= \frac{2^* - 2}{2^*} \|u\|^{2^*}_{L^{2^*}} \geq 0 .$$

ii) Let $v_k = u_k - u \xrightarrow{w} 0$ in $H^1(M, g_0)$ and strongly in $L^p(M, g_0)$ for any $p < 2^*$ as $k \to \infty$. Lemmas 1.5.3 and 1.5.4 give

$$F_{(M,g_0)}(u_k) = F_{(M,g_0)}(u) + F_{(M,g_0)}(v_k) + o(1)$$

$$= F_{(M,g_0)}(u) + F^{\infty}_{(M,g_0)}(v_k) + o(1),$$

and

$$F^{\infty}_{(M,g_0)}(v_k) = \frac{1}{2}\int_M |\nabla v_k|^2_{g_0} d\mu_{g_0} - \frac{1}{2^*}\int_M v_k^{2^*} d\mu_{g_0}$$

(3.1.3)

$$\leq F_{(M,g_0)}(u_k) - \underbrace{F_{(M,g_0)}(u)}_{\geq 0} + o(1) \leq \beta + o(1).$$

Similarly, we obtain

$$o(1) = \langle dF_{(M,g_0)}(u_k), v_k \rangle_{H^{-1}\times H^1}$$

$$= \int_M \Big((\nabla u_k, \nabla v_k)_{g_0} + c(n)^{-1} R_{g_0} u_k v_k - |u_k|^{2^*-2} u_k \overset{u_k - u}{\overset{\shortparallel}{v_k}} \Big) d\mu_{g_0}$$

(3.1.4)

$$= \|\nabla v_k\|^2_{L^2} - \Big(\|u_k\|^{2^*}_{L^{2^*}} - \|u\|^{2^*}_{L^{2^*}} \Big) + o(1)$$

$$= \|\nabla v_k\|^2_{L^2} - \|v_k\|^{2^*}_{L^{2^*}} + o(1),$$

and we find

$$F^{\infty}_{(M,g_0)}(v_k) = \frac{1}{2}\|\nabla v_k\|^2_{L^2} - \frac{1}{2^*}\|v_k\|^{2^*}_{L^{2^*}} = \underbrace{\Big(\frac{1}{2} - \frac{1}{2^*}\Big)}_{= \frac{1}{n}} \|v_k\|^{2^*}_{L^{2^*}} + o(1).$$

Together with (3.1.3) this gives the estimate

$$\frac{1}{n}\|v_k\|^{2^*}_{L^{2^*}} \leq c_0 < \beta^* = \frac{1}{n} S^{n/2}, \quad k \geq k_0 ;$$

that is, with $2^* \cdot \frac{2}{n} = \frac{4}{n-2} = 2^* - 2$, the bound

(3.1.5) $$\|v_k\|^{2^*-2}_{L^{2^*}} \leq c_1 < S, \quad k \geq k_0 .$$

iii) Finally, for any $\varphi \in C^\infty(M, g_0)$ compute

$$o(1) = \left\langle dF_{(M,g_0)}(u_k), v_k \varphi^2 \right\rangle_{H^{-1}\times H^1}$$

$$= \int_M \left(|\nabla (v_k \varphi)|^2_{g_0} - |v_k|^{2^*} \varphi^2 \right) d\mu_{g_0} + o(1)$$

$$\geq \|\nabla(v_k\varphi)\|^2_{L^2} - \|v_k\|^{2^*-2}_{L^{2^*}} \|v_k \varphi\|^2_{L^{2^*}} + o(1),$$

similar to (3.1.4), also using Hölder's inequality in the last step. With (3.1.5) we then obtain

$$\|\nabla(v_k\varphi)\|^2_{L^2} \leq c_1 \|v_k\varphi\|^2_{L^{2^*}} + o(1),$$

where $c_1 < S$. But if $\operatorname{diam}(\operatorname{supp}(\varphi)) < R_0$ for sufficiently small R_0 the metric g_0 on $\operatorname{supp}(\varphi)$ is sufficiently close to $g_{\mathbb{R}^n}$ to yield the bound

$$C_2 \, \| v_k \varphi \|^2_{L^{2^*}} \leqslant \| \nabla (v_k \varphi) \|^2_{L^2}$$

with a constant $C_2 > C_1$, and we conclude that $v_k \varphi \to 0$ in H^1 for any such φ.

Thus, if $(\varphi_\ell)_{1 \leq \ell \leq L} \subset C^\infty(M, g_0)$ is a partition of unity with $\operatorname{diam}(\operatorname{supp}(\varphi_\ell)) < R_0$ for each $\ell \in \{1, \dots, L\}$, we conclude that

$$\| v_k \|_{H^1} \leqslant \sum_{\ell=1}^{L} \| v_k \varphi_\ell \|_{H^1} \to 0 \quad (k \to \infty).$$

Thus, $u_k \to u$ strongly in $H^1(M, g_0)$, as desired. $\square$

Corollary 3.1.1: If $S_{(M,g_0)} < S$, there exists $u > 0$ with

$$S_{(M,g_0)}(u) = S_{(M,g_0)},$$

and u solves (3.1.2).

Proof: i) If $S_{(M,g_0)} > 0$, consider the functional $F_{(M,g_0)}$ and note that for each $u > 0$ the number

$$\beta(u) := \sup_{s>0} F_{(M,g_0)}(su) = \sup_{s>0}\left(\frac{s^2}{2} E_{(M,g_0)}(u) - \frac{s^{2^*}}{2^*}\|u\|^{2^*}_{L^{2^*}}\right)$$

is achieved for s with

$$s^{2^*-2}\|u\|^{2^*}_{L^{2^*}} = E_{(M,g_0)}(u),$$

so $s^{2^*-2}\|u\|^{2^*-2}_{L^{2^*}} = S_{(M,g_0)}(u)$, and

$$\beta(u) = s^2\underbrace{\left(\frac{1}{2} - \frac{1}{2^*}\right)}_{=\frac{1}{n}} E_{(M,g_0)}(u) = \frac{1}{n}\underbrace{s^2\|u\|^2_{L^{2^*}}}_{=\left(S_{(M,g_0)}(u)\right)^{2/2^*-2}} S_{(M,g_0)}(u)$$

$$= \frac{1}{n}\left(S_{(M,g_0)}(u)\right)^{n/2}.$$

Hence, if $S_{(M,g_0)} < S$ in view of Thm. 3.1.3 we may invoke the mountain-pass lemma, Thm. 2.4.3, with $0 < \beta < \beta^*$ to obtain the conclusion.

(by using $S_{(M,g_0)} > 0$)

ii) If $S_{(M,g_0)} = 0$ a minimizing sequence $(u_k) \subset H^1(M,g_0)$ for $E_{(M,g_0)}$ in

$$\Sigma_2 = \{ u \in H^1(M,g_0);\ \|u\|_{L^2} = 1 \}$$

accumulates at $u \in \Sigma_2$ with

$$0 \geq E_{(M,g_0)}(u) \geq S_{(M,g_0)} \|u\|^2_{L^{2^*}} = 0.$$

Hence

$$E_{(M,g_0)}(u) = 0 = \min_{v \in \Sigma_2} E_{(M,g_0)}(v),$$

and u solves

$$-c(n) \Delta_{g_0} u + R_{g_0} u = \lambda u$$

for some $\lambda \in \mathbb{R}$, where

$$\lambda = \lambda \|u\|^2_{L^2} = c(n)\, E_{(M,g_0)}(u) = 0.$$

Thus, u solves (3.1.2) with $r = 0$.

Moreover, u also minimizes $S_{(M,g_0)}$ with

$$S_{(M,g_0)}(u) = 0 = S_{(M,g_0)}.$$

iii) If $S_{(M,g_0)} < 0$, consider the coercive functional G on $H^1(M,g_0)$, given by

$$G_{(M,g_0)}(u) = \frac{1}{2} E_{(M,g_0)}(u) + \frac{1}{2^*} \|u\|^{2^*}_{L^{2^*}} .$$

By weak lower semi-continuity of Dirichlet's integral and $\|\cdot\|_{L^{2^*}}$, moreover, $G_{(M,g_0)}$ is w.s.l.s.c. and Thm. 1.1.1 applies to yield $u \in H^1(M,g_0)$ with

$$G_{(M,g_0)}(u) = \inf_{v \in H^1(M,g_0)} G_{(M,g_0)}(v) < 0,$$

satisfying (3.1.2) with $r = -1$.

Moreover, u achieves $\beta = \inf_{0 \neq v \in H^1} \beta(v)$, where

$$\beta(u) = \inf_{s>0} G_{(M,g_0)}(su) = \inf_{s>0} \left(\frac{s^2}{2} E_{(M,g_0)}(u) + \frac{s^{2^*}}{2^*} \|u\|^{2^*}_{L^{2^*}} \right)$$

$$= -\frac{1}{n} \left| S_{M,g_0}(u) \right|^{n/2}$$

is achieved for s with

$$s^{2^*-2} \|u\|^{2^*-2}_{L^{2^*}} = - \frac{E_{(M,g_0)}(u)}{\|u\|^2_{L^{2^*}}} = - S_{(M,g_0)}(u).$$

So u also minimizes $S_{(M,g_0)}(u)$. $\square$

The existence of a Yamabe metric on any closed (M^n, g_0), $n \geq 3$, then follows from the next result.

<u>Theorem</u> 3.1.4 (Aubin '76, Schoen '84)
Suppose (M^n, g_0), $n \geq 3$, is closed and not conformally diffeomorphic to (S^n, g_{S^n}).
Then $$S'_{(M, g_0)} < S'_{(S^n, g_{S^n})} = S'.$$

<u>Corollary</u> 3.1.2: On any closed (M^n, g_0), $n \geq 3$, there exists a Yamabe metric.

The proof of Thm. 3.1.4 is achieved via the explicit construction of suitable comparison functions by different methods in the cases

i) $n \geq 6$, and (M, g_0) not locally conformally flat (Aubin '76),

ii) $3 \leq n \leq 5$, or (M, g_0) locally conformally flat (Schoen '84).

We now sketch Schoen's proof in the case that either $n=3$ or that (M, g_0) is locally conformally flat. We may assume that $S_{(M,g_0)} > 0$.

The proof uses two key ingredients: the Green's function and the positive mass theorem.

<u>Green's function</u>: For any point $p_0 \in M$ there exists a unique function $G \in C^\infty(M \setminus \{p\}, g_0)$ with

$$L_{g_0} G = \gamma(n)\, \delta_{\{p=p_0\}},$$

where $\gamma(n) \in \mathbb{R}$ is such that $-\Delta(|x|^{2-n}) = \gamma(n)\, \delta_{\{x=0\}}$

and $G > 0$ since $S_{(M,g_0)} > 0$.

<u>Idea</u>: For $0 < \rho_k \in C^\infty(M, g_0)$ with $\rho_k\, dx \xrightarrow[(k\to\infty)]{w^*} \gamma(n)\, \delta_{\{p=p_0\}}$

let G_k solve

$$L_{g_0} G_k = \rho_k \quad \text{on } M.$$

Then

$$S_{(M,g_0)} \| G_k^- \|^2_{L^{2^*}} \leq E_{(M,g_0)}(G_k^-)$$

$$= \int_M L_{g_0} G_k \cdot G_k^- \, d\mu_{g_0} = \int_M \rho_k G_k^- \, d\mu_{g_0} \leq 0$$

implies that $G_k^- = \min\{G_k, 0\} = 0$.

Moreover, $G_k \xrightarrow[(k\to\infty)]{} G$ smoothly away from p_0;

see for instance Widman (1976).

Positive mass theorem (Schoen-Yau, Witten)

Suppose (M^n, g_0) is closed and not conformally diffeomorphic to (S^n, g_{S^n}), $n \geq 3$.

i) If (M, g_0) is locally conformally flat, in a suitable conformal chart around any point $p_0 \in M$ we may introduce Euclidean coordinates so that $g = g_{\mathbb{R}^n} = (\delta_{ij})$ near $p_0 = 0$, and there holds

(3.1.6) $\quad G(x) = |x|^{2-n} + A + \alpha(x)$,

where α is smooth and harmonic near $p_0 = 0$ with $\alpha(0) = 0$, and where

(3.1.7) $\qquad A > 0$.

ii) If $n = 3$, the expansion (3.1.6) holds in geodesic normal coordinates x near $p_0 = 0$ with $\alpha(x) = O(|x|)$ and with

$$A > 0.$$

Reference: Li-Parker, Bull. AMS

Proof of Thm. 3.1.4 when (M, g_0) is locally conformally flat.

Fix $p_0 \in M$ and introduce Euclidean coordinates x on a ball $B_R(0)$ around $p_0 = 0$ so that (3.1.6) holds, where $A > 0$.

For $\varepsilon > 0$ let

$$u_\varepsilon^*(x) = \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{n-2}{2}}}$$

as in Example 3.1.1. For suitable

$$0 < \varepsilon_0 << \rho_0 < R_0/2$$

choose $\varepsilon > 0$ so that

$$\varepsilon_0 \left(\rho_0^{2-n} + A\right) = u_\varepsilon^*(\rho_0) \quad \left(\sim \varepsilon^{\frac{n-2}{2}} \rho_0^{2-n}\right). \tag{3.1.8}$$

Let

$$\psi(x) = (2 - |x|/\rho_0)_+$$

and set

$$u(x) = \begin{cases} u_\varepsilon^*(x), & |x| \le \rho_0, \\ \varepsilon_0 \left(G(x) - \psi(x)\alpha(x)\right), & \rho_0 \le |x| \le 2\rho_0, \\ \varepsilon_0 G(x), & x \in M \setminus B_{2\rho_0}(0). \end{cases}$$

Then u is continuous and piecewise smooth, so $u \in H^1$. Using the equation

$$-\Delta u_\varepsilon^* = (u_\varepsilon^*)^{2^*-1}$$

we compute

$$S \, \|u_\varepsilon^*\|_{L^{2^*}}^2 = \|\nabla u_\varepsilon^*\|_{L^2}^2 = \|u_\varepsilon^*\|_{L^{2^*}}^{2^*} ;$$

so

$$\|u_\varepsilon^*\|_{L^{2^*}}^{2^*-2} = S ,$$

and Hölder's inequality yields

$$\int_{B_{\rho_0}(0)} |\nabla u|^2 dx = \int_{B_{\rho_0}(0)} |\nabla u_\varepsilon^*|^2 dx = \int_{B_{\rho_0}(0)} (u_\varepsilon^*)^{2^*-2+2} dx + \int_{\partial B_{\rho_0}(0)} \partial_r u_\varepsilon^* u_\varepsilon^* do$$

$$\leq S \, \|u\|_{L^{2^*}(M, g_0)}^2 + \int_{\partial B_{\rho_0}(0)} \partial_r u_\varepsilon^* \, u_\varepsilon^* \, do .$$

Moreover, since $R_{g_0} = 0$ in $B_{2\rho_0}(0)$, we have

$$\int_{M \setminus B_{\rho_0}(0)} \left(|\nabla u|^2 + c(n)^{-1} R_{g_0} u^2 \right) d\mu_{g_0}$$

$$= \varepsilon_0^2 \int_{M \setminus B_{\rho_0}(0)} \left(|\nabla G|^2 + c(n)^{-1} R_{g_0} G^2 \right) d\mu_0$$

$$+ \varepsilon_0^2 \int_{B_{2\rho_0} \setminus B_{\rho_0}(0)} \left(|\nabla(\psi \alpha)|^2 - 2 \nabla G \cdot \nabla(\psi \alpha) \right) d\mu_{g_0} .$$

Note that $|\alpha(x)| \leq C|x|$ implies the bound

$$|\nabla(\psi\alpha)| \leq C \quad \text{in } B_{2\rho_0} \setminus B_{\rho_0}(0).$$

Also observe that $L_{g_0} G = 0$ in $M \setminus B_{\rho_0}(0)$ gives

$$\int_{M \setminus B_{\rho_0}(0)} \left(|\nabla G|^2 + c(n)^{-1} R_{g_0} G^2\right) d\mu_{g_0} = \int_{M \setminus B_{\rho_0}(0)} L_{g_0} G \cdot G \, d\mu_{g_0}$$

$$- \int_{\partial B_{\rho_0}(0)} \partial_r G \cdot G \, do = - \int_{\partial B_{\rho_0}(0)} \partial_r G \cdot G \, do.$$

Therefore we find

$$E_{(M, g_0)}(u) \leq S \|u\|^2_{L^{2^*}} + \int_{\partial B_{\rho_0}(0)} \left(u_\varepsilon^* \partial_r u_\varepsilon^* - \varepsilon_0^2 G \partial_r G\right) do$$

$$+ C\rho_0 \varepsilon_0^2.$$

The Theorem then follows from

<u>Claim 1.</u> For suitable $0 < \varepsilon_0 << \rho_0 < R_0/2$ there holds

$$\int_{\partial B_{\rho_0}(0)} \left(u_\varepsilon^* \partial_r u_\varepsilon^* - \varepsilon_0^2 G \partial_r G\right) do + C\rho_0 \varepsilon_0^2 < 0.$$

Proof of Claim 1: Compute for $|x| = \rho_0$ and suitable $0 < \varepsilon_0 << \rho_0 << R_0$, using that $\varepsilon_0 \simeq \varepsilon^{\frac{n-2}{2}}$ by (3.1.8),

$$\partial_r u_\varepsilon^* - \varepsilon_0 \partial_r G = -(n-2)\Big(\underbrace{u_\varepsilon^*(\rho_0)}_{\overset{(3.1.8)}{=} \varepsilon_0(\rho_0^{2-n} + A)} \frac{\rho_0}{\varepsilon^2 + \rho_0^2} - \varepsilon_0 \rho_0^{1-n}\Big) + \varepsilon_0 \partial_r \alpha$$

$$= -(n-2)\,\varepsilon_0 \rho_0^{-1}\Big[(\rho_0^{2-n} + A)\underbrace{\frac{1}{1+(\varepsilon/\rho_0)^2}}_{\geq 1-(\varepsilon/\rho_0)^2} - \rho_0^{2-n}\Big] + \varepsilon_0 \underbrace{\partial_r \alpha}_{\leq C}$$

$$\leq -(n-2)\,\varepsilon_0 \rho_0^{-1} A + \underbrace{C\,\varepsilon_0 \rho_0^{1-n}\Big(\frac{\varepsilon}{\rho_0}\Big)^2}_{= \varepsilon_0 \varepsilon^2 \rho_0^{-1-n} \leq C\varepsilon_0, \text{ if } 0<\varepsilon_0<<\rho_0,} + C\varepsilon_0$$

$$\leq -\frac{n-2}{2}\,\varepsilon_0 \rho_0^{-1} A, \quad \text{if } 0<\rho_0<<R_0.$$

Thus, with constants $C, C_1 > 0$ there holds

$$\int_{\partial B_{\rho_0}(0)} \big(u_\varepsilon^* \partial_r u_\varepsilon^* - \varepsilon_0^2 G \partial_r G\big)\, do + C\varepsilon_0^2 \rho_0 = u_\varepsilon^*(\rho_0) \int_{\partial B_{\rho_0}(0)} \big(\partial_r u_\varepsilon^* - \varepsilon_0 \partial_r G\big)\, do - \varepsilon_0^2 \int_{\partial B_{\rho_0}(0)} \alpha\, \partial_r G\, do + C\varepsilon_0^2 \rho_0$$

$$\leq -C_1 \varepsilon_0^2 \rho_0^{(2-n)-1+(n-1)} A + C\varepsilon_0^2 \rho_0$$

$$\leq -C_1 \varepsilon_0^2 A + C\varepsilon_0^2 \rho_0 < 0, \quad \text{if } 0<\rho_0<<R_0.$$

$\square$

3.2 A global compactness result

Surprisingly, in Yamabe-type equations we can also obtain compactness at energies exceeding the threshold for concentration.

We illustrate this with the following model problem. Let $\Omega \subset\subset \mathbb{R}^n$ be a smoothly bounded domain in $\mathbb{R}^n$, $n \geq 3$. For $\lambda \in \mathbb{R}$ consider the boundary value problem

$$(3.2.1) \qquad -\Delta u - \lambda u = u|u|^{2^*-2} \quad \text{in } \Omega,$$

$$(3.2.2) \qquad u = 0 \quad \text{on } \partial\Omega$$

with associated energy

$$E_\lambda(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda u^2\right)dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx, \quad u \in H_0^1(\Omega).$$

We expect there to be an interplay between the solvability of (3.2.1), (3.2.2) and the Sobolev constant S which is attained on the solutions $u > 0$ of the equation

$$(3.2.3) \qquad -\Delta u = u|u|^{2^*-2} \quad \text{on } \mathbb{R}^n, \quad u \in \dot{H}^1(\mathbb{R}^n).$$

In fact, the following result holds true, which improves Thm. 3.1.3 beyond the threshold β^*.

Theorem 3.2.1 (S. '84). Let $(u_k) \subset H^1_0(\Omega)$ be a $(P.-S.)_\beta$-sequence for E_λ.

Then there exists a solution $u^{(0)}$ of (3.2.1), (3.2.2), and either i) a subsequence $u_k \to u^{(0)}$ strongly in $H^1_0(\Omega)$, or ii) a subsequence $u_k \xrightarrow{w} u^{(0)}$ in $H^1_0(\Omega)$, and there exist $I \in \mathbb{N}$ and solutions $u^{(i)} \in \dot{H}^1(\mathbb{R}^n)$ of (3.2.3) together with sequences $(x_k^{(i)}) \subset \Omega$, $(r_k^{(i)}) \subset \mathbb{R}_+$ satisfying $x_k^{(i)} \to x^{(i)} \in \bar{\Omega}$, $r_k^{(i)} \to 0$ $(k \to \infty)$

and with $\dfrac{\operatorname{dist}(x_k^{(i)}, \partial\Omega)}{r_k^{(i)}} \underset{(k\to\infty)}{\longrightarrow} \infty$, $1 \le i \le I$,

such that [1]

$$(3.2.4) \qquad u_k = u^{(0)} + \sum_{i=1}^{I} \underbrace{\left(r_k^{(i)}\right)^{\frac{2-n}{2}} u^{(i)}\left(\frac{x - x_k^{(i)}}{r_k^{(i)}}\right)}_{=:\, u^{(i)}_{r_k^{(i)}, x_k^{(i)}}(x)} + o(1)$$

where $o(1) \to 0$ in $\dot{H}^1(\mathbb{R}^n)$, and with

$$(3.2.5) \qquad E_\lambda(u_k) \to E_\lambda(u^{(0)}) + \sum_{i=1}^{I} E_0(u^{(i)}).$$

iii) If $u_k \ge o(1) \to 0$ in $H^1_0(\Omega)$, then $u^{(0)} \ge 0$ and also $u^{(i)} > 0$, $1 \le i \le I$.

---

[1] Extending any $u \in H^1_0(\Omega)$ by $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, we may regard $H^1_0(\Omega) \hookrightarrow \dot{H}^1(\mathbb{R}^n)$.

Remarks 3.2.1. i) If $\mu_k \geq o(1)$ and hence $u^{(i)} > 0$ we can precisely characterize all energy levels $\beta \in \mathbb{R}$ where $(P.-S.)_\beta$ fails.

Indeed, by Gidas-Ni-Nirenberg ('79) if $u^{(i)} > 0$ solves (3.2.3) then up to scaling and translation

$$u^{(i)} \text{ equals } u^*(x) = \frac{[n(n-2)]^{\frac{n-2}{4}}}{[1+|x|^2]^{\frac{n-2}{2}}},$$

and

$$E_0(u^{(i)}) = E_0(u^*) = \beta^*, \quad 1 \leq i \leq I.$$

Hence, $(P.-S.)_\beta$ fails iff $I \geq 1$ iff

$$\beta = E_\lambda(u) + I\beta^*$$

for some solution $u \in H_0^1(\Omega)$ of (3.2.1), (3.2.2).

ii) Bahri-Coron ('88) note that the decomposition (3.2.4) is "orthogonal" in the sense that

$$\inf_{i \neq j} d_k^{(ij)} \to \infty \quad (k \to \infty),$$

where

$$d_k^{(ij)} = \frac{|x_k^{(i)} - x_k^{(j)}|}{r_k^{(j)}} + \frac{r_k^{(i)}}{r_k^{(j)}} + \frac{r_k^{(j)}}{r_k^{(i)}}.$$

Indeed, if for some $i < j$ there holds $d_k^{(ij)} \leq C < \infty$, with the notation

$$u^{r, x_0}(x) = r^{\frac{n-2}{2}} u(x_0 + rx), \quad r > 0, \ x_0 \in \mathbb{R}^n$$

we have

$$\tilde{u}_k^{(i)}(x) = (u_k)^{r_k^{(i)}, x_k^{(i)}}(x)$$

$$= u^{(i)}(x) + \left(\frac{r_k^{(i)}}{r_k^{(j)}}\right)^{\frac{n-2}{2}} u^{(j)}\left(\frac{x_k^{(i)} - x_k^{(j)}}{r_k^{(j)}} + \frac{r_k^{(i)}}{r_k^{(j)}} x\right)$$

$$+ \left(u^{(0)}\right)^{r_k^{(i)}, x_k^{(i)}}(x) + \sum_{\substack{l \neq i \\ l \neq j}} \left(u^{(l)}_{r_k^{(l)}, x_k^{(l)}}\right)^{r_k^{(i)}, x_k^{(i)}}(x)$$

$$\xrightarrow{w} \tilde{u}^{(i)} \quad \text{in } H^1_{loc}(\mathbb{R}^n) \text{ as } k \to \infty,$$

and for any $\varphi \in C_c^\infty(\mathbb{R}^n)$ there holds

$$\langle dE_0(\tilde{u}^{(i)}), \varphi \rangle_{\dot{H}^{-1} \times \dot{H}^1(\mathbb{R}^n)} = \lim_{k \to \infty} \langle dE_0(\tilde{u}_k^{(i)}), \varphi \rangle_{\dot{H}^{-1} \times \dot{H}^1(\mathbb{R}^n)}$$

$$= \lim_{k \to \infty} \langle \underbrace{dE_\lambda(u_k)}_{\|\cdot\|_{H^{-1}} \to 0}, \underbrace{\varphi_{r_k^{(i)}, x_k^{(i)}}}_{= \varphi_k \xrightarrow{w} 0} \rangle_{H^{-1} \times H^1(\Omega)} = 0,$$

but $\tilde{u}^{(i)} \neq u^{(i)}$ and $\tilde{u}^{(i)}$ does not satisfy (3.2.3).

iii) Thm. 3.2.1 (and the above remarks i), ii)) also holds true for other compact perturbation terms different from λu; moreover, it also holds true for the Yamabe energy $F_{(M, g_0)}$ and gives rise to a bubble-tree decomposition of $(P.-S.)_\beta$-sequences, arising, for instance, as $u(t_k)$ for a solution $u = u(t)$ of the Yamabe flow and $t_k \to \infty$, where it is the basic pillar for the proof of unconditional convergence of the flow in Schwetlick-S. (2003), Brendle (2005/06)

The Yamabe flow for conformal metrics $g = g(t) = u^{\frac{4}{n-2}} g_0$ on a closed n-manifold (M, g_0) with unit volume is given by

$$u_t = (r_g - R_g)u = \left(c(n)\Delta_{g_0} u - R_{g_0} u\right) u^{2-2^*} + r_g u,$$

where

$$r_g = \int_M R_g \underbrace{u^{2^*} d\mu_{g_0}}_{= d\mu_g}$$

so that

$$\frac{d}{dt} \operatorname{vol}(M, g) = \frac{d}{dt}\int_M u^{2^*} d\mu_{g_0} = 2^* \int_M (r_g - R_g)\, d\mu = 0.$$

That is, $u = u(t)$ satisfies

$$u^{2^*-2} u_t = c(n)\Delta_{g_0} u - R_{g_0} u + r_g u^{2^*-1},$$

and multiplying with u_t gives the energy identity

$$\int_M |u_t|^2 u^{2^*-2} d\mu_{g_0} = -c(n) \frac{d}{dt} E_{(M, g_0)}(u(t)).$$

Since $E_{(M,g_0)}(u(t)) = S'_{(M,g_0)}(u(t)) \geq S_{(M,g_0)}$ for all t, we then obtain the bound

$$\int_0^\infty \int_M |u_t|^2 u^{2^*-2} d\mu_{g_0} \leq S'_{(M,g_0)}(u(0)) - S'_{(M,g_0)}$$

and there exists $t_k \to \infty$ such that

$$(3.2.6)\quad \int_{M\times\{t_k\}} |u_t|^2 u^{2^*-2}\, d\mu_{g_0} = \int_{M\times\{t_k\}} |R_g - r_g|^2 u^{2^*}\, d\mu_{g_0} \longrightarrow 0 \ (k\to\infty),$$

which stronger than the condition

$$c(n)\,\| dS'_{(M,g_0)}(u_k)\|_{H^{-1}} = \sup_{\|\varphi\|_{H^1}\le 1} c(n)\langle dS'_{(M,g_0)}(u_k),\varphi\rangle_{H^{-1}\times H^1}$$

$$(3.2.7)\quad = \sup_{\|\varphi\|_{H^1}\le 1} \int_M \Big(\underbrace{L_{g_0}u_k}_{=R_{g_k}u_k^{2^*-1}} - r_{g_k} u_k^{2^*-1}\Big)\varphi\, d\mu_{g_0}$$

$$= \sup_{\|\varphi\|_{H^1}\le 1} \int_M \left(R_{g_k} - r_{g_k}\right) u_k^{2^*-1}\varphi\, d\mu_{g_0} \longrightarrow 0$$

for a (P.-S.)-sequence $u_k>0$, with $g_k = u_k^{\frac{4}{n-2}} g_0$, $k\in\mathbb{N}$,

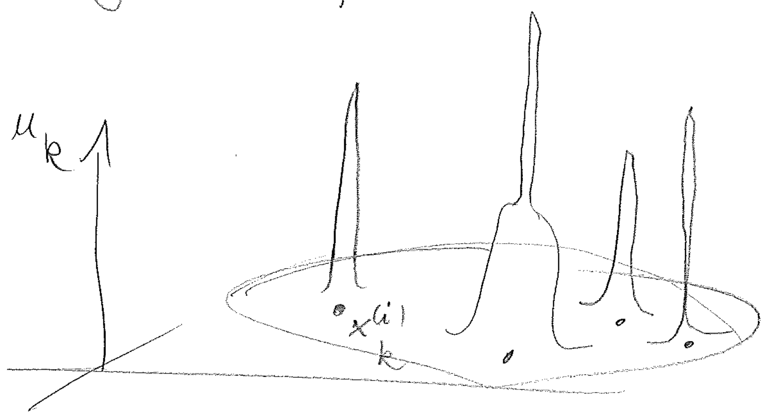
For $u_k>0$ satisfying the intermediate (between (3.2.6) and (3.2.7)) condition

$$\sup_{\|\varphi\|_{L^{2^*}}\le 1} \int_M \left(R_{g_k} - r_{g_k}\right) u_k^{2^*-1}\varphi\, d\mu_{g_0}$$

$$= \left(\int_M |R_{g_k} - r_{g_k}|^{2^+} u_k^{2^*}\, d\mu_{g_0}\right)^{\frac{1}{2^+}} \longrightarrow 0,$$

where $\frac{1}{2^+} = \frac{1}{2} + \frac{1}{n}$ with $2^+ = (2^*)' = \frac{2^*}{2^*-1}$, a better approximation estimate, improving (3.2.4) was obtained by Ciraolo-Figalli-Maggi, Figalli- in dimensions $3 \leq n \leq 5$.

Can one use this to simplify the proof of convergence of the Yamabe flow?



Bubble-tree (with "bubbles-on-bubbles")

iv) The first results on bubble-tree decompositions are due to Sacks-Uhlenbeck ('81) for harmonic maps of surfaces, and to Wente ('80) for disc-type surfaces of constant mean curvature (in the latter case only recovering one bubble). For the harmonic map heat flow, results analogous to (3.2.4), (3.2.5) were obtained by S.('85), Qing('95)/Ding-Tian('95), Wang('96), ... and for (P.-S.)-sequence satisfying the analogue of (3.2.6) by Qing-Tian ('97).

However, examples by Parker ('96) show that (3.2.4), (3.2.5) may fail for (P.-S.)-sequences for the harmonic map problem.

For surfaces of prescribed constant mean curvature the analogue of Thm. 3.2.1 (including (3.2.4), (3.2.5)) was established by Brezis-Coron ('85), S.('85).

v) If (μ_k) may be of different signs, also $u^{(i)}$ may be sign-changing. For such $u = u^{(i)}$ solving (3.2.3) we can estimate

$$E_0(u) > 2\beta^*.$$

Proof: Testing (3.2.3) with $u_\pm = \pm \max\{\pm u, 0\} \neq 0$ we find

$$\|\nabla u_\pm\|^2_{L^2} = \|u_\pm\|^{2^*}_{L^{2^*}} \geq S \|u_\pm\|^2_{L^{2^*}} > 0,$$

so $E_0(u_\pm) > \beta^*$ (as in the proof of Cor. 3.1.1), and

$$E_0(u) = E_0(u_+) + E_0(u_-) > 2\beta^*. \qquad \square$$

vi) Thus, if (3.2.1), (3.2.2) admits only the trivial solution $u^{(0)} \equiv 0$, condition $(P.-S.)_\beta$ holds for all $\beta \in]\beta^*, 2\beta^*[$ (regardless of the sign of (μ_k)), and for all $\beta \notin \beta^* \mathbb{N}$ if $\mu_k \geq o(1)$.

vii) The decomposition (3.2.4) and the original proof of Thm. 3.2.1 in (S, '84) inspired Bahouri - Gérard ('97) to derive an orthogonal (in the sense of iii)) nonlinear profile decomposition also for solutions of nonlinear wave equations

$$u_{tt} - \Delta u = u|u|^{p-2} \text{ on } \mathbb{R} \times \mathbb{R}^n;$$

however, the decomposition is exhaustive only in a weaker norm; see also Nakanishi - Schlag, ZLAM.

The proof of Thm. 3.2.1 may be broken up into

- a preparation step
- a key lemma
- the iteration argument and conclusion.

Proof of Thm. 3.2.1. i) Let $(u_k) \subset H_0^1(\Omega)$ be a (P.-S.)$_\beta$ - sequence. Computing, with error $o(1) \to 0$ $(k \to \infty)$, we obtain

$$2\beta + o(1)\left(1+\|u_k\|_{H^1}\right) = 2E_\lambda(u_k) - \langle dE_\lambda(u_k), u_k\rangle_{H^{-1}\times H_0^1}$$
$$= \frac{2^*-2}{2^*}\|u_k\|_{L^{2^*}}^{2^*}$$

and find

$$\|u_k\|_{H_0^1}^2 = 2E_\lambda(u_k) + |\lambda|\,\|u_k\|_{L^2}^2 + \frac{1}{2^*}\|u_k\|_{L^{2^*}}^{2^*}$$
$$\leq C(\beta) + o(1)\left(1+\|u_k\|_{H_0^1}\right),$$

similar to the proof of Thm. 3.1.3.

Thus, $(u_k) \subset H_0^1(\Omega)$ is bounded, and we may assume that $u_k \xrightarrow{w} u^{(0)}$ in $H_0^1(\Omega)$, and strongly in $L^p(\Omega)$, $p < 2^*$, where for any $\varphi \in H_0^1(\Omega)$ we have

$$\langle dE_\lambda(u^{(0)}), \varphi\rangle_{H^{-1}\times H_0^1} = \lim_{k\to\infty}\langle dE_\lambda(u_k), \varphi\rangle_{H^{-1}\times H_0^1} = 0,$$

and $u^{(0)}$ solves (3.2.1), (3.2.2).

Moreover, again similar to the proof of Thm. 3.1.3 we have

$$2E_\lambda(u^{(0)}) = 2E_\lambda(u^{(0)}) - \langle dE_\lambda(u^{(0)}), u^{(0)}\rangle_{H^{-1}\times H_0^1}$$
$$= \frac{2^*-2}{2^*} \|u^{(0)}\|_{L^{2^*}}^{2^*} \geq 0 .$$

Letting $v_k := u_k - u^{(0)} \overset{w}{\to} 0$ in $H_0^1(\Omega)$ then by Lemmas 1.5.3 and 1.5.4 we have

$$E_0(v_k) = E_\lambda(u_k) - E_\lambda(u^{(0)}) + o(1) \leq \beta + o(1).$$

The following result is basic for the iteration argument.

<u>Claim 1.</u> $dE_0(v_k) \xrightarrow{(k\to\infty)} 0$ in $H^{-1}(\Omega)$.

<u>Proof</u>: For any $\varphi \in H_0^1(\Omega)$ we have

$$\langle dE_\lambda(v_k), \varphi\rangle_{H^{-1}\times H_0^1} = \langle dE_\lambda(u_k), \varphi\rangle_{H^{-1}\times H_0^1} - \langle dE_\lambda(u^{(0)}), \varphi\rangle_{H^{-1}\times H_0^1}$$
$$+ \int_\Omega \left(u_k^{2^*-1} - v_k^{2^*-1} - (u^{(0)})^{2^*-1}\right)\varphi\, dx ,$$

where $u^{2^*-1} := u|u|^{2^*-2}$ for brevity.

But by Vitali's theorem we have

$$u_k^{2^*-1} - v_k^{2^*-1} = \int_0^1 \frac{d}{dt}\left(u_k - u^{(0)} + t\,u^{(0)}\right)^{2^*-1} dt$$

$$= (2^*-1)\int_0^1 u^{(0)}\,\left|u_k - u^{(0)} + t\,u^{(0)}\right|^{2^*-2} dt$$

$$\xrightarrow[(k\to\infty)]{} \left(u^{(0)}\right)^{2^*-1} \quad \text{in } L^{2^+}(\Omega),$$

where $2^+ = (2^*)'$ as above, and our claim follows. $\square$

For the iteration argument we then have the following key lemma.

Lemma 3.2.1. Let $v_k \overset{w}{\rightharpoonup} 0$ in $H_0^1(\Omega)$ be a $(P.-S.)_\beta$ - sequence for E_0. Then either

i) $\beta < \beta^* = \frac{1}{n} S^{n/2}$, and $\|v_k\|_{H^1} \to 0$ $(k \to \infty)$,

or ii) there exists a sub-sequence (v_k) and $(x_k) \subset \Omega$, $r_k \downarrow 0$ $(k \to \infty)$, a solution $0 \not\equiv v^{(0)} \in \dot{H}^1(\mathbb{R}^n)$ of (3.2.3), and a Palais-Smale sequence $w_k \overset{w}{\rightharpoonup} 0$ in $H_0^1(\Omega)$ for E_0 such that with error $o(1) \to 0$ in $\dot{H}^1(\mathbb{R}^n)$ there holds

$$v_k = w_k + r_k^{\frac{2-n}{2}} v^{(0)}\left(\frac{x - x_k}{r_k}\right) + o(1),$$

and

$$E_0(v_k) = E_0(w_k) + E_0(v^{(0)}) + o(1).$$

Moreover, we have

$$\frac{dist(x_k, \partial\Omega)}{r_k} \to \infty \quad (k \to \infty),$$

and $w_k \geqslant o(1)$ whenever $v_k \geqslant o(1) \to 0$ in $H_0^1(\Omega)$.

Proof: i) If $\beta < \beta^*$ the proof of Thm 3.1.3 gives convergence $v_k \xrightarrow{(k\to\infty)} 0$ in $H_0^1(\Omega)$.

ii) If $\beta \geq \beta^*$, computing

$$\frac{1}{n}\|\nabla v_k\|_{L^2}^2 = E_0(v_k) - \frac{1}{2^*}\langle dE_0(v_k), v_k\rangle_{H^{-1}\times H_0^1} \xrightarrow{(k\to\infty)} \beta,$$

we have

$$(3.2.7) \quad \|\nabla v_k\|_{L^2}^2 \to n\beta \geq n\beta^* = S^{n/2} \quad (k\to\infty).$$

Normalization: Fix $L \in \mathbb{N}$ so that $B_2(0)$ can be covered by L balls of radius 1.

Choose $x_k \in \overline{\Omega}$, $r_k > 0$ such that

$$\sup_{x_0} \int_{B_{r_k}(x_0)} |\nabla v_k|^2 dx = \int_{B_{r_k}(x_k)} |\nabla v_k|^2 dx = \frac{S^{n/2}}{2L}.$$

Note that by (3.2.7) this is possible, and $r_k < \mathrm{diam}(\Omega)$.

Scale

$$\tilde{v}_k(x) = (v_k)^{r_k, x_k}(x) = r_k^{\frac{n-2}{2}} v_k(x_k + r_k x) \in \dot{H}^1(\mathbb{R}^n)$$

with now

$$(3.2.8) \quad \sup_{x_0} \int_{B_1(x_0)} |\nabla \tilde{v}_k|^2 dx = \int_{B_1(0)} |\nabla \tilde{v}_k|^2 dx = \frac{S^{n/2}}{2L}.$$

Let

$$\Omega_k = \{ x \in \mathbb{R}^n ;\ x_k + r_k x \in \Omega \} , \quad k \in \mathbb{N} .$$

We may assume that (Ω_k) exhausts a limit domain Ω_∞, where either $\Omega_\infty = \mathbb{R}^n$ or Ω_∞ is a half-space.

In view of

$$\| \nabla \tilde{v}_k \|^2_{L^2(\Omega_k)} = \| \nabla v_k \|^2_{L^2(\Omega)} \to m\beta < \infty ,$$

$$\| dE_0(\tilde{v}_k \|_{H^{-1}(\Omega_k)} = \| dE_0(v_k) \|_{H^{-1}(\Omega)} \to 0$$

we may assume

$$\tilde{v}_k \xrightarrow{w} v^{(0)} \text{ in } \dot{H}^1(\mathbb{R}^n) ,$$

where $v^{(0)} \in \dot{H}^1(\Omega_\infty) = \|\cdot\|_{\dot{H}^1}\text{-clos}\big(C_c^\infty(\Omega_\infty)\big)$ solves (3.2.3) on Ω_∞.

Since any $\varphi \in C_c^\infty(\Omega_\infty)$ also belongs to $C_c^\infty(\Omega_k)$ for $k \geq k_0(\varphi)$, we may then also choose functions $\tilde{v}_k^{(0)} \in H_0^1(\Omega_k)$ such that

$$\tilde{v}_k^{(0)} \xrightarrow[(k\to\infty)]{} v^{(0)} \text{ strongly in } \dot{H}^1(\mathbb{R}^n) .$$

<u>Claim 1:</u> $\tilde{v}_k \xrightarrow[(k\to\infty)]{} v^{(0)}$ strongly locally in H^1.

Proof: It suffices to show

$$\forall x_0 \in \mathbb{R}^n : \quad \| \tilde{v}_k - \tilde{v}_k^{(0)} \|_{H^1(B_1(x_0))} \xrightarrow{(k\to\infty)} 0 .$$

Fix $x_0 \in \mathbb{R}^n$ and for $r>0$ let $B_r := B_r(x_0)$. Choose cut-off functions $\varphi_i \in C_c^\infty(\mathbb{R}^n)$ with $0 \le \varphi_i \le 1$, $i=1,2$, and such that

$$\varphi_1 \equiv 1 \text{ in } B_1, \quad \varphi_1 = 0 \text{ outside } B_{3/2},$$

$$\varphi_2 \equiv 1 \text{ in } B_{3/2}, \quad \varphi_2 = 0 \text{ outside } B_2 .$$

Let

$$\tilde{w}_k^{(i)} = (\tilde{v}_k - \tilde{v}_k^{(0)})\varphi_i \in H_0^1(\Omega_k) \subset \dot{H}^1(\mathbb{R}^n),$$

with

$$\| \tilde{w}_k^{(i)} \|_{H^1} \le C \| \tilde{v}_k - \tilde{v}_k^{(0)} \|_{H^1(B_2)} \le C,$$

uniformly in k, $i=1,2$. Since $\tilde{v}_k \xrightarrow{w} v^{(0)}$ in $\dot{H}^1$, and thus $\tilde{v}_k - \tilde{v}_k^{(0)} \xrightarrow{w} 0$ in $\dot{H}^1$, we also have

$$\tilde{w}_k^{(i)} \xrightarrow{w} 0 \quad \text{in } \dot{H}^1(\mathbb{R}^n)$$

and strongly in $L^p(B_2)$ for any $p < 2^*$.

With error $o(1) \to 0$ $(k\to\infty)$, from $dE_0(v^{(0)}) = 0$,

$$\| dE_0(\tilde{v}_k) \|_{H^{-1}(\Omega_k)} = \| dE_0(v_k) \|_{H^{-1}(\Omega)} \xrightarrow{(k\to\infty)} 0,$$

$$\| dE_0(\tilde{v}_k^{(0)}) \|_{H^{-1}(\Omega_k)} \longrightarrow \| dE_0(v^{(0)}) \|_{H^{-1}(\Omega_\infty)} = 0$$

we now have

$$o(1) = \langle dE_0(\tilde{v}_k) - dE_0(\tilde{v}_k^{(0)}), \overbrace{\tilde{w}_k^{(1)}\varphi_1}^{=(\tilde{v}_k - \tilde{v}_k^{(0)})\varphi_1^2} \rangle_{H^{-1}\times H_0^1(\Omega_k)}$$

$$= \int_\Omega \left(|\nabla(\tilde{v}_k - \tilde{v}_k^{(0)})|^2 - |\tilde{v}_k - \tilde{v}_k^{(0)}|^{2^*} \right) \underbrace{\varphi_1^2}_{1 = \varphi_2^{2^*-2} \text{ on } supp(\varphi_1)} dx + o(1)$$

(3.2.9)
$$= \int_\Omega \left(|\nabla \tilde{w}_k^{(1)}|^2 - |\tilde{w}_k^{(2)}|^{2^*-2} |\tilde{w}_k^{(1)}|^2 \right) dx + o(1)$$

$$\geq S \|\tilde{w}_k^{(1)}\|^2_{L^{2^*}} - \|\tilde{w}_k^{(2)}\|^{2^*-2}_{L^{2^*}} \|\tilde{w}_k^{(1)}\|^2_{L^{2^*}} + o(1)$$

similar to the preparation step.

But by (3.2.8) we can bound

$$S \|\tilde{w}_k^{(2)}\|^2_{L^{2^*}} \leq \|\nabla \tilde{w}_k^{(2)}\|^2_{L^2} = \int_{\mathbb{R}^n} |\nabla(\tilde{v}_k - \tilde{v}_k^{(0)})|^2 \varphi_2^2 dx + o(1)$$

$$= \int_{\mathbb{R}^n} \left(|\nabla \tilde{v}_k|^2 - |\nabla \tilde{v}_k^{(0)}|^2 \right) \varphi_2^2 dx + o(1) \leq \int_{B_2} |\nabla \tilde{v}_k|^2 dx + o(1)$$

$$\leq L \sup_{\tilde{x}} \int_{B_1(\tilde{x})} |\nabla \tilde{v}_k|^2 dx + o(1) \leq \frac{1}{2} S^{n/2} + o(1)$$

and
$$\|\tilde{w}_k^{(2)}\|^{2^*-2}_{L^{2^*}} \leq \left(\frac{1}{2} S^{\frac{n-2}{2}} \right)^{\frac{2^*-2}{2}} + o(1)$$

$$= \left(\frac{1}{2}\right)^{\frac{2}{n-2}} S + o(1) \leq c_0 S$$

for any $c_0 \in \left] \left(\frac{1}{2}\right)^{\frac{2}{n-2}}, 1 \right[$ and $k \geq k_0(c_0)$.

Hence from (3.2.9) we conclude that $\|\tilde{w}_k^{(1)}\|_{L^{2^*}} \to 0$ and then $\|\tilde{w}_k^{(1)}\|_{\dot{H}^1(\mathbb{R}^n)} \to 0$, proving our claim. □

Claim 1 and (3.2.8) now yield $v^{(0)} \not\equiv 0$.

Since $v^{(0)} \in \dot{H}^1_0(\Omega_\infty)$ solves (3.2.3), Pohozaev's Thm. 1.4.2 now implies that $\Omega_\infty = \mathbb{R}^n$, so

$$r_k^{-1}\, dist(x_k, \partial\Omega) \to \infty \quad (k \to \infty).$$

Moreover, letting $\tilde{w}_k = \tilde{v}_k - \tilde{v}^{(0)}_k$ and computing

$$\| dE_0(\tilde{w}_k)\|_{H^{-1}(\Omega_k)} = \sup_{\|\varphi\|_{H^1_0} \leq 1} \langle \underbrace{dE_0(\tilde{v}_k)}_{\downarrow\, 0} - \underbrace{dE_0(\tilde{v}^{(0)}_k)}_{\downarrow\, 0}, \varphi \rangle_{H^{-1}\times H^1_0(\Omega_k)}$$

$$+ \int_{\Omega_k} \left(\tilde{v}_k^{\,2^*-1} - \tilde{v}_k^{(0)\,2^*-1} - \tilde{w}_k^{\,2^*-1} \right) \varphi \, dx + o(1)$$

$$\to 0 \quad (k \to \infty)$$

as in the preparation step, upon scaling back to Ω we see that $w_k = v_k - v^{(0)}_{r_k, x_k} + o(1) \in H^1_0(\Omega_k)$ is a Palais-Smale sequence with energy

$$E_0(w_k) = E_0(v_k) - \underbrace{E_0(v^{(0)}_{r_k, x_k})}_{= E_0(v^{(0)})} + o(1),$$

Positivity. Suppose that $v_k \geq o(1) \to 0$ in $H_0^1(\Omega)$.
Then $\tilde{v}_k \geq o(1) \to 0$ in $H_0^1(\Omega_k)$ and Claim 1 gives

$$v^{(0)} = \lim_{k\to\infty} \tilde{v}_k \geq 0$$ locally (hence everywhere).

together with

$$\tilde{w}_k = \tilde{v}_k - v^{(0)} \geq o(1) \to 0 \text{ in } H_0^1(\Omega_k)$$

because $v^{(0)} \in \dot{H}^1(\mathbb{R}^n)$. Scaling back, we find

$$w_k \geq o(1) \to 0 \text{ in } H_0^1(\Omega).$$

□

Proof of Thm. 3.2.1 (completed). The Theorem follows by repeatedly applying Lemma 3.2.1 after the preparation step.
Since $E_0(v^{(0)}) \geq \beta^*$ for $v^{(0)}$ as in Lemma 3.2.1, the iteration stops after at most $[\beta/\beta^*]$ steps.

3.3 Coron's existence result

Thm. 3.2.1 may be applied to obtain solutions $u > 0$ of the problem

$$(3.3.1) \qquad -\Delta u = u^{2^*-1} \quad \text{in } \Omega,$$
$$(3.3.2) \qquad u = 0 \quad \text{on } \partial\Omega$$

for suitable domains $\Omega \subset\subset \mathbb{R}^n$, $n \geq 3$.

Recall that by Pohozaev's Thm. 1.4.2 on any star-shaped domain the problem (3.3.1), (3.3.2) only admits the trivial solution $u \equiv 0$.

On the other hand, when Ω is an "annulus" $\Omega = B_{R_2} \setminus B_{R_1}(0)$ for some $0 < R_1 < R_2$, by using compactness of the embedding $H^1_{rad}(\Omega) \hookrightarrow L^p(\Omega)$ for any $p < \infty$ as in the remark at the end of Section 1.5.1, we obtain a solution $u > 0$ to (3.3.1), (3.3.2).

With the help of Thm. 3.2.1 Coron was able to show that solutions $u > 0$ to (3.3.1), (3.3.2) also exist in a perturbed setting.

Thm. 3.3.1 (Coron '84)

Let $n \geq 3$. There is $R > 0$ such that for any $\Omega \subset\subset \mathbb{R}^n$ satisfying $0 \notin \Omega$, $B_{R_2} \setminus B_{R_1}(0) \subset \Omega$ for $0 < R_1 < R_2$ with

$$R_2 / R_1 \geq 16 R^2$$

there exists a sol. $u > 0$ of (1).

Proof. After scaling, we may assume

$$R_1 = \frac{1}{4R} < 1 < 4R = R_2,$$

where $R \geq 1$ is determined below.

Let $\Sigma = \{ x \in \mathbb{R}^n ; |x| = 1 \}$.

For $\sigma \in \Sigma$, $0 \leq t < 1$ define $u_t^\sigma \in \dot{H}^1(\mathbb{R}^n)$ with

$$u_t^\sigma(x) = \left(\frac{1-t}{(1-t)^2 + |x - t\sigma|^2} \right)^{\frac{n-2}{2}}, \quad x \in \mathbb{R}^n.$$

Recall that

$$S(u_t^\sigma) = \frac{\| \nabla u_t^\sigma \|_{L^2}^2}{\| u_t^\sigma \|_{L^{2^*}}^2} = S, \quad \forall \sigma \in \Sigma, 0 \leq t < 1.$$

Moreover, as $t \downarrow 0$

$$u_t^\sigma \rightarrow u_0 := \left(\frac{1}{1 + |x|^2} \right)^{\frac{n-2}{2}} \quad \text{in } \dot{H}^1(\mathbb{R}^n)$$

uniformly in $\sigma \in \Sigma$.

Let
$$M = \{ u \in H_0^1(\Omega);\ \|u\|_{L^{2^*}} = 1 \}$$

and set
$$v_t^\sigma = \frac{w_t^\sigma}{\|w_t^\sigma\|_{L^{2^*}}} \in M,\ \sigma \in \Sigma,\ 0<t<1.$$

Remark 3.2.1, vi) after Thm. 3.2.1 implies the following result.

Lemma 3.3.1. Suppose that (3.3.1), (3.3.2) only admits $u \equiv 0$ as a solution. Then the functional $S(u)$ satisfies $(P.-S.)_\gamma$ on M for every $\gamma \in]S, 2^{2/n} S[$.

Proof. By the proof of Cor. 3.1.1, if $(v_k) \subset M$ is a $(P.-S.)_\gamma$-sequence for $S(u)$, then the sequence
$$u_k = (S(v_k))^{\frac{1}{2^*-2}} v_k \in H_0^1(\Omega)$$
is a $(P.-S.)_\beta$-sequence for
$$E_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx,$$
with $\beta = \frac{1}{n}\gamma^{n/2}$. By Remark 3.2.1, vi) therefore $S(u)$ satisfies $(P.-S.)_\gamma$ for $(n\beta^*)^{2/n} = S < \gamma < (2n\beta^*)^{2/n} = 2^{2/n} S$.

□

Fix a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \varphi \le 1$ and

$$\varphi(x) = \begin{cases} 1, & \text{if } 1/2 < |x| < 2, \\ 0, & \text{if } |x| \le 1/4 \text{ or } |x| \ge 4. \end{cases}$$

Scale

$$\varphi_R(x) = \begin{cases} \varphi(Rx), & \text{if } |x| < R^{-1}, \\ 1, & \text{if } R^{-1} \le |x| \le R, \\ \varphi(x/R), & \text{if } |x| \ge R, \end{cases}$$

For any $\Omega \subset\subset \mathbb{R}^n$ with $B_{4R} \setminus B_{\frac{1}{4R}}(0) \subset \Omega$ then we have

$$w_t^\sigma := u_t^\sigma \varphi_R, \quad w_0 := u_0 \varphi_R \in H_0^1(\Omega).$$

<u>Claim 1.</u> $\sup_{\sigma \in \Sigma,\, 0 \le t < 1} S(w_t^\sigma) \to S \quad (R \to \infty).$

<u>Proof</u>. Estimate

$$\| \nabla (w_t^\sigma - u_t^\sigma) \|_{L^2}^2 \le$$

$$\le C \int_{\{x;\, |x| < \frac{1}{2R} \text{ or } |x| > 2R\}} |\nabla u_t^\sigma|^2 dx + \underbrace{C R^{-2} \int_{\{x;\, 2R < |x| < 4R\}} |u_t^\sigma|^2 dx}_{\le C \left(\int_{\{x;\, 2R < |x|\}} |u_t^\sigma|^{2^*} dx \right)^{2/2^*}} + \underbrace{C R^2 \int_{\{x;\, |x| < \frac{1}{2R}\}} |u_t^\sigma|^2 dx}_{\le C \left(\int_{\{x;\, |x| < \frac{1}{2R}\}} |u_t^\sigma|^{2^*} dx \right)^{2/2^*}}$$

$\to 0 \quad (R \to \infty),$

uniformly in $\sigma \in \Sigma$, $0 \le t < 1$. The claim follows. $\square$

Lemma 3.3.2. Suppose (3.3.1) only admits $u \equiv 0$ as sol. Then for any $\delta > 0$, any $S_1 < 2^{2/n} S$ there is a flow $\Phi \in C^0(M \times [0,1], M)$ of homeomorphisms $\Phi(\cdot, t): M \to M$ such that

i) $\Phi(v,t) = v$ if $t = 0$, or if $S(v) < S + \delta/2$,

ii) $\Phi(M_{S_1}, 1) \subset M_{S+\delta}$.

Proof. Cover $[S+\delta, S_1]$ with finitely many intervals $]\gamma - \varepsilon, \gamma + \varepsilon[$ as in Thm. 2.4.1 (or Thm. 2.5.6 for flows on M), where $0 < \varepsilon < \bar{\varepsilon} = \delta/2$, and compose the corresponding flows.

$\square$

For $v \in M$ define the center of v to be

$$F(v) := \int_\Omega x \, |v|^{2^*} dx.$$

Let $\rho > 0$ such ~~that~~

$$\Omega \cap B_\rho(0) = \emptyset,$$

and there exists a continuous nearest-neighbor proj. $\pi : U_\rho(\Omega) = \bigcup_{x \in \Omega} B_\rho(x) \to \Omega$.

Lemma 3.3.3. $\exists\ \delta > 0\ \forall v \in \mathcal{M}$:

$$S(v) < S' + \delta \implies F(v) \in U_\rho(\Omega).$$

Proof. (indirect) Let $(v_k) \subset \mathcal{M}$ satisfy

$$S'(v_k) \to S', \quad \operatorname{dist}(F(v_k), \Omega) \geq \rho.$$

As in Thm. 1.5.4 there is $(x_k) \subset \Omega$, $r_k > 0$ such that a subseq.

$$u_k = (v_k)_{x_k, r_k} \to \bar{u} \quad \text{in } \dot{H}^1(\mathbb{R}^n),$$

where $S(\bar{u}) = S'$.

By ~~Thm.~~ (3.2.3) then $|\bar{u}| > 0$; hence $r_k \to 0$.

It follows that

$$F(v_k) = F\Big(r_k^{\frac{2-n}{2}} u_k\big(\tfrac{x - x_k}{r_k}\big)\Big) = \underset{\overset{\cup}{\Omega}}{x_k} + o(1). \ ↯$$

$\square$

Proof of Thm. 3.3.1 (completed). Suppose (1) admits only $u \equiv 0$ as sol.

Choose $R > 0$ such that

$$S_1 := \sup_{\sigma \in \Sigma,\, 0 \le t < 1} S(v_t^\sigma) = \sup_{\sigma \in \Sigma,\, 0 \le t < 1} S(w_t^\sigma) < 2^{2/n} S.$$

For $\delta > 0$ given by Lemma 3.3.3 and S_1 let $\Phi \in C^0(M \times [0,1], M)$ be given by Lemma 2.

Then
$$F(\Phi(v_t^\sigma, 1)) \in U_\rho(\Omega), \quad \forall \sigma \in \Sigma,\ 0 \le t < 1.$$

Hence the map

$$h : \Sigma \times [0,1[\ni (\sigma, t) \mapsto \pi\left(F(\Phi(v_t^\sigma, 1))\right) \in \Omega$$

is well-defined and continuous.

Moreover, since
$$\sup_{\sigma \in \Sigma} S(v_t^\sigma) \to S \quad (t \uparrow 1)$$

we have $\quad \Phi(v_t^\sigma, 1) = v_t^\sigma$

for suff. large $t_1 \le t < 1$. Since clearly
$$F(v_t^\sigma) \to \sigma \quad (t \uparrow 1)$$
uniformly in σ, the map h continuously extends to $h \in C^0(\Sigma \times [0,1], \Omega)$ with
$$h(\sigma, 1) = \sigma, \quad \forall \sigma \in \Sigma.$$

Thus, h defines a contraction of Σ in Ω. But this is impossible, since $0 \notin \Omega$.

Thus, (3.3.1), (3.3.2) has a sol. $u \not\equiv 0$. $\square$

<u>Remark</u>. By working in the set

$$M_+ = \{ u \in M ; \ u \geq 0 \}$$

one can obtain a sol. $u > 0$ to (1).

More generally, one can obtain the following result:

<u>Thm.</u> 3.3.2 (Bahri - Coron '88)

Let $n = 3$, and suppose that Ω is not contractible. Then (1) admits a sol. $u > 0$.