2. Harmonic maps and related problems

2.1 Harmonic maps

Let (M,g), (N,h) be smooth, closed Riemannian manifolds. For a C^1-map $u: M \to N$ let

$$e(u) = \frac{1}{2} g^{ij}(x)\, h_{\alpha\beta}(u)\, \partial_i u^\alpha \partial_j u^\beta,$$

with $(g^{ij}) = (g_{ij})^{-1}$, $h = (h_{\alpha\beta})$ in local coordinates on M and N, respectively, and let

$$E(u) = \int_M e(u)\, d\mu_g$$

be the Dirichlet energy.

Remark 2.1. The energy $E(u)$ is invariant under isometries. By Nash's embedding theorem, N may be isometrically embedded in some Euclidean $\mathbb{R}^n$. Thus, with no loss of generality we may assume that N is a smooth submanifold of some $\mathbb{R}^n$ and regard $u = (u^1, \dots, u^n)$ as a map into $N \hookrightarrow \mathbb{R}^n$.

For simplicity, moreover, in the following we often will assume that N is an oriented hypersurface of codimension 1 with a smooth unit normal vector field $\nu : N \to T^{\perp}N \subset \mathbb{R}^n$.

(In the general case, the normal bundle is locally spanned by k mutually orthogonal smooth unit vector fields $\nu_1, \dots, \nu_k$.)

For further simplification, we often will consider only the case when $M = T^m = \underbrace{S^1 \times \dots \times S^1}_{m\text{-fold}} = \mathbb{R}^m / \mathbb{Z}^m$, where we may regard any map $u : M \to N$ as a map $u : \mathbb{R}^m \to N$ which is 1-periodic in each variable $x^1, \dots, x^m$.

<u>Example</u> 2.2. Let $m = 1$. For a C^1 map $u : S^1 \to N \hookrightarrow \mathbb{R}^n$ the Dirichlet energy

$$E(u) = \frac{1}{2} \int_0^{2\pi} |\dot{u}|^2 d$$

is the "length energy" of the curve $u = u(t)$, whose critical points are geodesics with $|\dot{u}| = |\frac{du}{dt}| = \mathrm{const}$.

In general, a C^2-map $u: M \to N \hookrightarrow \mathbb{R}^n$ is called <u>harmonic</u> if it is a critical point of E. in the following sense.

For any $\delta > 0$ let

$$\mathcal{U}_\delta N = \bigcup_{p \in N} B_\delta(p) \subset \mathbb{R}^n$$

be the δ-neighborhood of N. Since N is compact, there exists $\delta > 0$ such that the nearest-neighbor projection

$$\pi_N : \mathcal{U}_\delta N \to N$$

is well-defined and smooth. For any $\varphi \in H^1 \cap L^\infty(M; \mathbb{R}^n)$ and sufficiently small $\varepsilon_0 > 0$ then the variation

$$u_\varepsilon = \pi_N(u + \varepsilon\varphi) : M \to N$$

is well-defined for $|\varepsilon| < \varepsilon_0$, and the map

$$\varepsilon \mapsto E(u_\varepsilon)$$

is differentiable at $\varepsilon = 0$. We then say:

<u>Def.</u> 2.3. A C^1-map $u: M \to N \hookrightarrow \mathbb{R}^n$ is <u>harmonic</u>, if

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(\pi_N(u+\varepsilon\varphi)) = 0 \text{ for any } \varphi \in H^1 \cap L^\infty(M, \mathbb{R}^n).$$

Euler equation. Differentiating, we then find the equation

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left(\frac{1}{2}\int_M |\nabla \pi_N(u+\varepsilon\varphi)|^2 d\mu_g\right)$$

$$= \int_M \nabla u \cdot \nabla \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \pi_N(u+\varepsilon\varphi)\, d\mu_g$$

$$= \int_M \nabla u \, \nabla\left(d\pi_N(u)\varphi\right) d\mu_g = -\int_M \Delta_g u \cdot d\pi_N(u)\varphi \, d\mu_g$$

$$= -\int_M d\pi_N(u)\Delta_g u \cdot \varphi \, d\mu_g .$$

Since this identity in particular must hold for any smooth variation field φ, by the fundamental lemma of the calculus of variations then u is harmonic iff there holds

$$- d\pi_N(u)\, \Delta_g u = 0,$$

that is, iff there holds

$$- \Delta_g u = \lambda \nu(u) \perp T_u N$$

with some scalar function $\lambda : M \to \mathbb{R}$, which may be interpreted as a Lagrange multiplier relative to the constraint N.

(Here we use our simplifying assumption that N is a hypersurface.)

The function λ may be determined by computing

$$\lambda = \lambda\, \nu(u)\cdot\nu(u) = -\Delta_g u \cdot \nu(u)$$

$$= -\underbrace{\operatorname{div}_g\big(\nabla u \cdot \nu(u)\big)}_{=0} + \big(d\nu(u)\, \nabla u \cdot \nabla u\big)_g ,$$

so that

$$\lambda\, \nu(u) = A(u)(\nabla u, \nabla u) \perp T_u N ,$$

with $A(p): T_pN \times T_pN \to T_p^{\perp}N$ the second fundamental form of N at any point $p \in N$.

<u>Example</u> 2.4. For $N = S^{n-1} \subset \mathbb{R}^n$ a C^2-map $u : T^m \to S^{n-1} \hookrightarrow \mathbb{R}^n$ is harmonic iff there holds

$$-\Delta u = u\,|\nabla u|^2 \quad \text{on } T^m.$$

Indeed, with $\nu(p) = p$ for $p \in S^{n-1} \subset \mathbb{R}^n$ there holds

$$-\Delta u = \lambda\,\nu(u) = \lambda u, \text{ with } \lambda = -\Delta u \cdot u = -\underbrace{\Delta \tfrac{|u|^2}{2}}_{=0} + |\nabla u|^2.$$

2.2 Weakly harmonic maps.

The energy E is well-defined for u in the space
$$H^1(M;N) = \{u \in H^1(M;\mathbb{R}^n);\ u(x) \in N \text{ a.e.}\}.$$

Def. 2.5. A map $u \in H^1(M;N)$ is weakly harmonic iff there holds

$$(2.1)\qquad \int_M \nabla u\, \nabla(d\pi_N(u)\varphi)\, d\mu_g = \int_M \big(\nabla u \nabla\varphi - A(u)(\nabla u, \nabla u)\varphi\big)\, d\mu_g = 0$$

for any $\varphi \in H^1 \cap L^\infty(M;\mathbb{R}^n)$.

The Campanato theory from FA II gives the following result (see also Schoen-Yau (1997)).

Theorem 2.6. Suppose $u \in H^1(M;N)$ is weakly harmonic, and suppose that $u \in C^\alpha$ for some $\alpha > 0$. Then $u \in C^{2,\alpha}$ for any $\alpha > 0$.

Proof. For simplicity, let $M = T^m = \mathbb{R}^m/\mathbb{Z}^m$,

$u: \mathbb{R}^m \to N \hookrightarrow \mathbb{R}^n$ periodic.

Given $x_0 \in \mathbb{R}^m$, $R > 0$ we split $u = v + w$ on $B_R(x_0)$, where $\Delta v = 0$ and $w \in H^1_0(B_R(x_0))$, By assumption $u \in C^\alpha$ for some $\alpha > 0$ and the maximum principle we have

$$\sup_{x,y \in B_R(x_0)} |v(x) - v(y)| \leq \sup_{\substack{x \in \partial B_R(x_0) \\ y \in B_R(x_0)}} |v(x) - v(y)|$$

$$\leq \sup_{x,y \in \partial B_R(x_0)} |v(x) - v(y)| \leq \sup_{x,y \in B_R(x_0)} |u(x) - u(y)|$$

$$\leq C R^\alpha, \quad C = [u]_{C^\alpha},$$

and then also

$$\sup_{x,y \in B_R(x_0)} |w(x) - w(y)|$$

$$\leq 2 \sup_{x,y \in B_R(x_0)} |u(x) - u(y)| \leq C R^\alpha. \tag{2.2}$$

The Campanato estimates [FA II, Satz 10.2.2] then may be used ~~to~~ bound the components of ∇u on balls $B_r(x_0)$, $0 < r < R \leq 1$

Repeatedly using the splitting $u = v + w$, we have

$$\int_{B_r(x_0)} |\nabla u|^2 dx \leq 2\int_{B_r(x_0)} |\nabla v|^2 dx + 2\int_{B_r(x_0)} |\nabla w|^2 dx$$

$$\leq C\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |\nabla v|^2 dx + 2\int_{B_r(x_0)} |\nabla w|^2 dx$$

$$\leq C\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |\nabla u|^2 dx + C\int_{B_R(x_0)} |\nabla w|^2 dx ,$$

and by (2.1), (2.2) we can bound

$$\int_{B_R(x_0)} |\nabla w|^2 dx = \int_{B_R(x_0)} A(u)(\nabla u, \nabla u)\, w\, dx$$

(2.3) $$\leq C \int_{B_R(x_0)} |\nabla u|^2 dx \cdot \sup_{B_R(x_0)} |w| \leq C R^\alpha .$$

Thus, for the non-decreasing function

$$\Phi(r) = \int_{B_r(x_0)} |\nabla u|^2 dx , \quad 0 < r \leq 1,$$

there holds

$$(2.4) \quad \phi(r) \leq C\left(\frac{r}{R}\right)^n \phi(R) + C R^\alpha, \quad 0 < r < R \leq 1.$$

Campanato's "useful lemma" [FA II, Lemma 10.3.2] then gives the bound

$$\phi(r) \leq C r^\alpha, \quad 0 < r \leq 1,$$

with a constant

$$C = C(n, N) \|u\|_{C^\alpha} > 0.$$

Inserting this bound in (2.3), we can improve (2.4) to obtain

$$\phi(r) \leq C\left(\frac{r}{R}\right)^n \phi(R) + C R^{2\alpha},$$

and after finitely many iterations, for any $\beta < n$ we find the bound

$$(2.5) \quad \phi(r) \leq C r^\beta.$$

Campanato's estimates for any $0<r<R\leq 1$ also give the bound

$$\int_{B_r(x_0)} |\nabla u - \overline{(\nabla u)}_r|^2 dx$$

$$\leq 2\int_{B_r(x_0)} |\nabla v - \overline{(\nabla v)}_r|^2 dx + 2\int_{B_r(x_0)} |\nabla w - \overline{(\nabla w)}_r|^2 dx$$

$$\leq C\left(\frac{r}{R}\right)^{n+2}\int_{B_R(x_0)} |\nabla v - \overline{(\nabla v)}_R|^2 dx + C\int_{B_r(x_0)} |\nabla w|^2 dx$$

$$\leq C\left(\frac{r}{R}\right)^{n+2}\int_{B_R(x_0)} |\nabla u - \overline{(\nabla u)}_R|^2 dx + C\int_{B_R(x_0)} |\nabla w|^2 dx,$$

where now (2.3) is replaced by

$$\int_{B_R(x_0)} |\nabla w|^2 dx \leq C\int_{B_R(x_0)} |\nabla u|^2 dx \sup_{B_R(x_0)} |w| \leq CR^{\alpha+\beta}$$

for any $\beta < n$, in view of (2.5).

Thus, for the function

$$\psi(r) = \int_{B_r(x_0)} |\nabla u - \overline{(\nabla u)}_r|^2 dx$$

There holds

$$\psi(r) \leq C\left(\frac{r}{R}\right)^{n+2} \psi(R) + C R^{\alpha+\beta}, \quad 0<r<R\leq 1.$$

Moreover, since

$$\psi(r) \leq \int_{B_r(x_0)} |\nabla u - (\overline{\nabla u})_R|^2 dx \leq \psi(R),$$

$\phi : [0,1] \to \mathbb{R}$ is non-decreasing.

Campanato's "useful lemma" [FA II, Lemma 10,3,2] then gives the bound

$$\psi(r) \leq C r^{\alpha+\beta}, \quad 0<r\leq 1,$$

for some $C>0$, uniformly for any $x_0 \in \mathbb{R}^m$.
Choosing $\beta < n$ such that $\alpha+\beta = n+2\delta > n$, then $\nabla u \in \mathcal{L}^{2,n+2\delta} \hookrightarrow C^\delta$ by Campanato's embedding theorem [FA II, ~~Satz~~ 8.6.5], and

$$-\Delta u = A(u)(\nabla u, \nabla u) \in C^\delta.$$

From the Schauder theory [FA II, Satz 10.5.1] it then follows that $u \in C^{2,\delta} \hookrightarrow \bigcap_{0<\beta<1} C^{1,\beta}$.
Iteration then yields $u \in C^{2,\beta}$ for any $\beta < 1$.

$\square$

Thus, for weakly harmonic maps again there is a regularity "gap" similar to the problem that De Giorgi / Nash were facing. Only now it is not always possible to obtain Hölder regularity.

<u>Example</u> 2.7. The "equator map"

$u : B_1^m(0) = B_1(0; \mathbb{R}^m) \to S^m \hookrightarrow \mathbb{R}^{m+1}$, given by $\forall x \neq 0 : u(x) = \frac{x}{|x|} \in \partial B_1^m(0) = S^{m-1} \subset S^m$,

is weakly harmonic when $m \geq 3$, and even locally minimizes $E(v)$ among maps $v \in H^1(B_1(0; \mathbb{R}^m), S^m)$ with $u - v \in H_0^1(B_1^m(0); S^m)$ when $m \geq 7$, see Jäger-Kaul (1979), Baldes (1984). (In fact, for $m \geq 7$ the map u is absolutely E-minimizing.)

<u>Proof</u>: Compute

$$\partial_j\left(\frac{x^i}{|x|}\right) = \frac{1}{|x|}\left(\delta^i_j - \frac{x^i x^j}{|x|^2}\right), \quad 1 \leq i, j \leq m;$$

so

$$|\nabla(\tfrac{x}{|x|})|^2 = \sum_{i,j} |\partial_j(\tfrac{x^i}{|x|})|^2 = \frac{1}{|x|^2} \sum_{i,j} \left(\delta^i_j - \frac{x^i x^j}{|x|^2}\right)^2$$

$$= \frac{1}{|x|^2} \sum_i \left(\left(1 - \frac{|x^i|^2}{|x|^2}\right)^2 + \frac{|x^i|^2(|x|^2 - |x^i|^2)}{|x|^4}\right) = \frac{m-1}{|x|^2},$$

and

$$-\Delta\left(\frac{x}{|x|}\right)^i = -\partial_j\left(\frac{1}{|x|}\left(\delta^i_j - \frac{x^i x^j}{|x|^2}\right)\right)$$

$$= \underbrace{\frac{x^j}{|x|^3}\left(\delta^i_j - \frac{x^i x^j}{|x|^2}\right)}_{=0} + \frac{m-1}{|x|^3} x^i$$

$$= \frac{n-1}{|x|^2} \cdot \frac{x^i}{|x|} = \left|\nabla\left(\frac{x}{|x|}\right)\right|^2 \left(\frac{x}{|x|}\right)^i .$$

Moreover, for any $\varphi \in H^1_0 \cap L^\infty(B^m_1(0), \mathbb{R}^{m+1})$ there holds $\pi_N(u+\varepsilon\varphi) = \frac{u+\varepsilon\varphi}{|u+\varepsilon\varphi|}$, $|\varepsilon| < \varepsilon_0 < \frac{1}{\|\varphi\|_{L^\infty}}$ and

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} E\left(\pi_N(u+\varepsilon\varphi)\right) = \frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0}\left(\frac{1}{2}\int_{B^m_1(0)} \left|\nabla\left(\frac{u+\varepsilon\varphi}{|u+\varepsilon\varphi|}\right)\right|^2 dx\right)$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{B^m_1(0)} \nabla\left(\frac{u+\varepsilon\varphi}{|u+\varepsilon\varphi|}\right) \cdot \nabla\left(\frac{1}{|u+\varepsilon\varphi|}\left(\varphi - \frac{\varphi\cdot(u+\varepsilon\varphi)(u+\varepsilon\varphi)}{|u+\varepsilon\varphi|^2}\right)\right) dx$$

$$= \int_{B^m_1(0)} \left(|\nabla(\varphi - (\varphi\cdot u)u)|^2 + \nabla u \, \nabla n_\varphi\right) dx ,$$

where n_φ is the normal part of

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left(\frac{1}{|u+\varepsilon\varphi|}\left(\varphi - \frac{\varphi\cdot(u+\varepsilon\varphi)(u+\varepsilon\varphi)}{|u+\varepsilon\varphi|^2}\right)\right)$$

$$= -\varphi(\varphi\cdot u) - |\varphi|^2 u - (\varphi\cdot u)\varphi + 3(\varphi\cdot u)^2 u .$$

That is,

$$n_\varphi = \left((\varphi\cdot u)^2 - |\varphi|^2 \right) u ,$$

and in view of $\nabla u \cdot u = 0$ we have

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} E\left(\pi_N(u+\varepsilon\varphi)\right)$$

$$= \int_{B_1^m(0)} \left(|\nabla(\varphi - (\varphi\cdot u)u)|^2 - |\nabla u|^2 \left(|\varphi|^2 - (\varphi\cdot u)^2 \right) \right) dx$$

$$= \int_{B_1^m(0)} \left(|\nabla v_\varphi|^2 - \frac{m-1}{|x|^2} |v_\varphi|^2 \right) dx ,$$

with $v_\varphi = \varphi - (\varphi\cdot u) u \in T_u S^m$.

But by Hardy's inequality, since we also have $v_\varphi \in H_0^1(B_1^m(0))$ we can estimate

$$\int_{B_1^m(0)} \frac{|v_\varphi|^2}{|x|^2} dx \leq \frac{4}{(m-2)^2} \int_{B_1^m(0)} |\nabla v_\varphi|^2 dx ,$$

and

$$\frac{4(m-1)}{(m-2)^2} < 1 \text{ if } m \geq 7 .$$

This proves strict stability of u, as claimed.

$\square$

2.3 Harmonic maps with "small" range

A uniform smallness condition for u suffices to show Hölder continuity (and then smoothness, by Thm. 2.6) of any weakly harmonic map $u \in H^1(M; N)$.

More generally, this is true for any weak solution $u \in H^1 \cap L^\infty(M; \mathbb{R}^n)$ of any quasilinear elliptic system

(2.6) $\quad -\Delta u = f(x, u, \nabla u)$

with smooth $f: M \times \mathbb{R}^n \times \mathbb{R}^{m \cdot n} \to \mathbb{R}^n$ satisfying

(2.7) $\quad |f(x, u, p)| \leq a|p|^2 + b$

with constants $a, b \in \mathbb{R}$.

<u>Theorem</u> 2.8. Let $u \in H^1 \cap L^\infty(M; \mathbb{R}^n)$ weakly solve (2.6), (2.7) with

(2.8) $\quad a \cdot \sup_{x,y \in M} |u(x) - u(y)| < 1.$

Then $u \in C^\alpha$ for some $\alpha > 0$.

The proof is easiest in the case when $m=2$. Moreover, for simplicity we again carry out the details only for the torus $M=T^m$ and when $b=0$.

<u>Proof of Thm. 2.8 for $m=2$:</u> Fix any point $x_0 \in \mathbb{R}^2$, and let $R>0$. With a cut-off function $\eta \in C_c^\infty(B_R(x_0))$ satisfying $0 \le \eta \le 1$, $\eta = 1$ on $B_{R/2}(x_0)$, and $|\nabla\eta| \le 4/R$, and with $\tilde{u}_R = \fint_{B_R \setminus B_{R/2}(x_0)} u(x)\,dx$ then let

$$\varphi = (u-\tilde{u}_R)\,\eta^2 \in H_0^1 \cap L^\infty(B_R(x_0))$$

to obtain

$$0 = \int_{B_R(x_0)} \big(\nabla u \,\nabla\varphi - f(x,u,\nabla u)\,\varphi\big)\,dx$$

$$\ge \int_{B_R(x_0)} |\nabla u|^2 \eta^2\,dx + 2\int_{B_R(x_0)} \nabla u\,(u-\tilde{u}_R)\nabla\eta\cdot\eta\,dx$$
$$- a \sup_{B_R(x_0)} |u(x)-\tilde{u}_R| \int_{B_R(x_0)} |\nabla u|^2\eta^2\,dx$$

$$\ge (1-\varepsilon-\delta)\int_{B_R(x_0)} |\nabla u|^2\eta^2\,dx + \frac{C}{\varepsilon}\int_{B_R(x_0)} |u-\tilde{u}_R|^2|\nabla\eta|^2\,dx,$$

where (2.8) implies

$$S := a \cdot \sup_{B_R(x_0)} |u - \tilde{u}_R| \leq a \cdot \sup_{x \in B_R(x_0)} \fint_{B_R \setminus B_{R/2}(x_0)} |u(x) - u(y)|\, dy < 1 .$$

For $\varepsilon = \frac{1-S}{2} > 0$ there results the bound

$$\int_{B_R(x_0)} |\nabla u|^2 \eta^2 \, dx \leq C \int_{B_R(x_0)} |u - \tilde{u}_R|^2 |\nabla \eta|^2 dx$$

$$(2.9) \qquad \leq C R^{-2} \int_{B_R \setminus B_{R/2}(x_0)} |u - \tilde{u}_R|^2 dx$$

("Caccioppoli inequality").
The Poincaré inequality allows to estimate

$$(2.10) \quad \int_{B_R \setminus B_{R/2}(x_0)} |u - \tilde{u}_R|^2 dx \leq C R^2 \int_{B_R \setminus B_{R/2}(x_0)} |\nabla u|^2 dx .$$ [1)]

---

1) Proof of (2.10): Scale $R = 1$, $x_0 = 0$. Suppose by contradiction there exists $(u_k) \subset H^1(B_1 \setminus B_{1/2}(0))$ with

$$1 = \int_{B_1 \setminus B_{1/2}(0)} |u_k|^2 dx \geq k \int_{B_1 \setminus B_{1/2}(0)} |\nabla u_k|^2 dx , \quad \fint_{B_1 \setminus B_{1/2}(0)} u_k \, dx = 0 , \quad k \in \mathbb{N} .$$

Then (u_k) is bounded in $H^1(B_1 \setminus B_{1/2}(0))$ and a subsequence $u_k \xrightarrow{(k\to\infty)} u$ in L^2; moreover, $\nabla u_k \xrightarrow{(k\to\infty)} 0 = \nabla u$ in L^2.
Since $B_1 \setminus B_{1/2}(0)$ is connected, $u \equiv \fint_{B_1 \setminus B_{1/2}(0)} u \, dx = 0$. ↯

Letting

$$\phi(R) = \int_{B_R(x_0)} |\nabla u|^2 dx ,$$

thus from (2.9) we obtain

$$\phi(R/2) \leq \int_{B_R(x_0)} |\nabla u|^2 \eta^2 dx$$

$$\leq C_1 \int_{B_R \setminus B_{R/2}(x_0)} |\nabla u|^2 dx = C_1\big(\phi(R) - \phi(R/2)\big).$$

"Filling the hole" on the right by adding $C_1 \phi(R/2)$, and dividing by $1+C_1$, we find

$$\phi(R/2) \leq \frac{C_1}{1+C_1} \phi(R)$$

with $\theta := \dfrac{C_1}{1+C_1} < 1$.

Iteration yields the bound

$$\phi\left(2^{-k}R\right) \leq \theta\, \phi\left(2^{1-k}R\right) \leq \ldots \leq \theta^k \phi(R), \quad k \in \mathbb{N}.$$

Given $0 < r < R$ let $k \in \mathbb{N}$ such that

$$2^{-k-1} < \frac{r}{R} \leq 2^{-k}.$$

With $\alpha > 0$ such that $\theta = 2^{-\alpha}$ then there holds

$$\phi(r) \leq \phi(2^{-k}R) \leq 2^{-k\cdot\alpha}\phi(R)$$

$$\leq 2^{\alpha}\left(\frac{r}{R}\right)^{\alpha}\phi(R), \quad 0 < r < R \leq 1,$$

uniformly in $x_0 \in \mathbb{R}^2$; in particular, letting $R = 1$ we find the uniform bound

$$\phi(r) \leq C r^{\alpha}, \quad 0 < r \leq 1,$$

where $C = 2^{\alpha}\phi(1) > 0$.

Poincaré's inequality [FA II, Satz 8,6.6]

$$\int_{B_r(x_0)} |u - \bar{u}_r|^2 dx \leq C r^2 \int_{B_r(x_0)} |\nabla u|^2 dx$$

with

$$\bar{u}_r = \fint_{B_r(x_0)} u(x)\, dx$$

then gives the estimate

$$\int_{B_r(x_0)} |u - \bar{u}_r|^2 dx \leq C r^2 \phi(r) \leq C r^{2+\alpha}, \quad 0 < r \leq 1,$$

uniformly in $x_0 \in \mathbb{R}^2$, and $u \in \mathcal{L}^{2,2+\alpha}(\mathbb{R}^2) \hookrightarrow C^{\alpha/2}(\mathbb{R}^2)$ by Campanato's thm. [FA II, Satz 8,6.5] $\square$

If $m \geq 3$ we can argue similarly for a suitably weighted Dirichlet integral.

Let

$$G(x,y) = \frac{c(m)}{|x-y|^{m-2}}$$

be the Green's function for the Laplacian, satisfying

$$-\Delta G(\cdot, x_0) = \delta_{\{x = x_0\}} \quad \text{on } \mathbb{R}^m .$$

Mollifying with $(\rho_\varepsilon)_{\varepsilon > 0}$, where $\rho \in C_c^\infty(B_1(0))$ with $\int_{\mathbb{R}^m} \rho(x)\, dx = 1$ and where

$$\rho_\varepsilon(x) = \varepsilon^{-m} \rho\left(\frac{x}{\varepsilon}\right) \in C_c^\infty(B_\varepsilon(0)), \quad \varepsilon > 0,$$

we obtain

$$G^\varepsilon(x,y) = \int_{\mathbb{R}^m} G(x-z, y)\, \rho_\varepsilon(z)\, dz$$

with

$$-\Delta G^\varepsilon(x, x_0) = \rho_\varepsilon(x - x_0), \quad x \in \mathbb{R}^m,$$

for any fixed $x_0 \in \mathbb{R}^m$.

Proof of Thm 2.8 for $n \geq 3$. For any $x_0 \in \mathbb{R}^n$, any $R > 0$, any $\eta \in C_c^\infty(B_R(x_0))$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_{R/2}(x_0)$, and $|\nabla \eta| \leq 4/R$, again letting

$$\tilde{u}_R = \fint_{B_R \setminus B_{R/2}(x_0)} u(x)\,dx$$

we test (2.6) with

$$\varphi^\varepsilon = (u - \tilde{u}_R)\, G^\varepsilon(\cdot, x_0)\, \eta^2 \in H_0^1 \cap L^\infty(B_R(x_0))$$

to obtain (with $G^\varepsilon = G^\varepsilon(\cdot, x_0)$)

$$0 = \int_{B_R(x_0)} \left(\nabla u \nabla \varphi^\varepsilon - f(x, u, \nabla u)\varphi^\varepsilon\right) dx$$

$$\geq \int_{B_R(x_0)} |\nabla u|^2 G^\varepsilon \eta^2\, dx + I + II$$

$$- a \sup_{B_R(x_0)} |u - \tilde{u}_R| \int_{B_R(x_0)} |\nabla u|^2 G^\varepsilon \eta^2\, dx$$

$$\geq (1 - s) \int_{B_R(x_0)} |\nabla u|^2 G^\varepsilon \eta^2\, dx + I + II,$$

where

$$s = a \sup_{B_R(x_0)} |u - \tilde{u}_R| \leq a \sup_{x,y \in B_R(x_0)} |u(x) - u(y)| < 1$$

as before, and with

$$I = 2 \int_{B_R(x_0)} \nabla u \,(u - \tilde{u}_R)\, G^\varepsilon \eta \,\nabla \eta \, dx$$

as well as

$$II = \int_{B_R(x_0)} \nabla u \,(u - \tilde{u}_R)\, \nabla G^\varepsilon \eta^2 \, dx$$

$$= \int_{B_R(x_0)} \nabla \left(\frac{|u - \tilde{u}_R|^2}{2} \eta^2 \right) \nabla G^\varepsilon \, dx$$

$$- \int_{B_R(x_0)} |u - \tilde{u}_R|^2 \eta \,\nabla \eta \,\nabla G^\varepsilon \, dx$$

$$\geq \int_{B_R(x_0)} \frac{|u - \tilde{u}_R|^2}{2} \eta^2 \rho_\varepsilon(\cdot - x_0) \, dx$$

$$- \int_{B_R(x_0)} |u - \tilde{u}_R|^2 \eta \,\nabla \eta \,\nabla G^\varepsilon \, dx .$$

Using Young's inequality to bound

$$|I| \leq \frac{1-s}{2} \int_{B_R(x_0)} |\nabla u|^2 G^\varepsilon \eta^2 \, dx + C \int_{B_R(x_0)} |u - \tilde{u}_R|^2 G^\varepsilon |\nabla \eta|^2 \, dx$$

we then arrive at the inequality

$$\left(\frac{|u-\tilde{u}_R|^2\eta^2}{2}*\rho_\varepsilon\right)(x_0)+\frac{1-s}{2}\int_{B_R(x_0)}|\nabla u|^2G^\varepsilon\eta^2\,dx$$

$$\leq C\int_{B_R}|u-\tilde{u}_R|^2\left(G^\varepsilon|\nabla\eta|^2+|\nabla\eta||\nabla G^\varepsilon|\right)dx.$$

Since $G^\varepsilon \xrightarrow{(\varepsilon\downarrow 0)} G = G(\cdot, x_0)$ smoothly away from $x=x_0$, by Fatou's lemma we may pass to the limit $\varepsilon\downarrow 0$ to obtain

$$\int_{B_R(x_0)}|\nabla u|^2G\eta^2\,dx \leq C\int_{B_R(x_0)}|u-\tilde{u}_R|^2\left(G|\nabla\eta|^2+|\nabla\eta||\nabla G|\right)dx$$

$$\leq CR^{-m}\int_{B_R\setminus B_{R/2}(x_0)}|u-\tilde{u}_R|^2dx\,,$$

where we used that $G+R|\nabla G|\leq CR^{2-m}$ in $B_R\setminus B_{R/2}(x_0)$. Poincaré's inequality (2.10) and the estimate $G\geq c(m)(R/2)^{2-m}$ on $B_R\setminus B_{R/2}(x_0)$ the yield the bound

$$R^{-m}\int_{B_R\setminus B_{R/2}(x_0)}|u-\tilde{u}_R|^2dx\leq C\int_{B_R\setminus B_{R/2}(x_0)}|\nabla u|^2G\,dx\,,$$

and for the function

$$\phi(R) = \int_{B_R(x_0)} |\nabla u|^2 G \, dx$$

we obtain

$$\phi(R/2) \leq \int_{B_R(x_0)} |\nabla u|^2 G \eta^2 dx \leq C$$

$$\leq C_1 \int_{B_R \setminus B_{R/2}(x_0)} |\nabla u|^2 G \, dx = C_1 \big(\phi(R) - \phi(R/2)\big).$$

As in the case $m=2$ we then may "fill the hole" and iterate to find

$$\phi(r) \leq C r^{\alpha}, \quad 0 < r < 1,$$

for some $\alpha > 0$, uniformly in $x_0 \in \mathbb{R}^m$.
With Poincaré's inequality we then obtain

$$\int_{B_r(x_0)} |u - \bar{u}_r|^2 dx \leq C r^2 \int_{B_r(x_0)} |\nabla u|^2 dx$$

$$\leq C r^m \int_{B_r(x_0)} |\nabla u|^2 G \, dx = C r^m \phi(r) \leq C r^{m+\alpha},$$

uniformly in $x_0 \in \mathbb{R}^m$, $0 < r \leq 1$. Thus,

$u \in \mathcal{L}^{2,m+\alpha}(\mathbb{R}^m) \hookrightarrow C^{\alpha/2}(\mathbb{R}^m)$ by Campanato's theorem.

□

Remark 2.9. i) The method of "filling the hole" was introduced by Hildebrandt-Widman (1975).

ii) Thm. 2.8 was obtained by Hildebrandt-Widman (1975).

iii) Condition (2.8) can be relaxed to the condition

$$a \|u\|_{L^\infty} < 1 \quad \text{(Wiegner (1976))},$$

which is optimal, as Example 2.7 shows; see also Hildebrandt-Kaul-Widman (1977).

iv) Also Thm. 2.6 holds for weak solutions of system (2.6), (2.7).

2.4 H-surfaces

For equations of type (2.6), (2.7) with a special "determinant" structure, our Thm. 2.8 can be further improved.

We demonstrate this in the case of "H-surfaces" $u \in H^1(B;\mathbb{R}^3)$ weakly solving the equation

(2.11) $\quad -\Delta u = 2u_x \wedge u_y \quad$ in $B = B_1(0) \subset \mathbb{R}^2$,

where

$$u_x \wedge u_y = \begin{vmatrix} e_1 & u_x^1 & u_y^1 \\ e_2 & u_x^2 & u_y^2 \\ e_3 & u_x^3 & u_y^3 \end{vmatrix} = \begin{pmatrix} u_x^2 u_y^3 - u_y^2 u_x^3 \\ u_x^3 u_y^1 - u_y^3 u_x^1 \\ u_x^1 u_y^2 - u_y^1 u_x^2 \end{pmatrix}$$

with $(x,y) = z \in B$.

<u>Remark 2.10.</u> If $u \in C^2(B;\mathbb{R}^3)$ is conformal with

$$(|u_x|^2 - |u_y|^2) - 2i\, u_x \cdot u_y = 0 \quad \text{in } B,$$

then u parametrizes a surface $S = u(B) \subset \mathbb{R}^3$ (away from branch-points where $\nabla u = 0$), whose mean curvature H is given by

$$-\Delta u = 2H u_x \wedge u_y .$$

Theorem 2.11 (Wente (1980)). Let $u \in H^1(B; \mathbb{R}^3)$ weakly solve (2.11). Then u is smooth.

In order to illustrate the method without too many technicalities we first prove the following result.

Proposition 2.12 (Wente (1980)) If $u \in C^2(\mathbb{R}^2; \mathbb{R}^3)$ solves (2.11) and has compact support, then there holds

$$\sup_{z \in \mathbb{R}^2} |u(z)| \leq \frac{1}{2\pi} \|\nabla u\|_{L^2}^2 .$$

Proof: Identifying $\mathbb{R}^2 \cong \mathbb{C}$ let $z = x+iy = re^{i\phi}$ be the representation of a generic point $z \in \mathbb{R}^2$.

Claim 1: We have the identity

$$u_x \wedge u_y = \frac{1}{r} u_r \wedge u_\phi .$$

Proof: Use $x = r\cos\phi$, $y = r\sin\phi$ to compute

$$\frac{1}{r} u_r \wedge u_\phi = (u_x \cos\phi + u_y \sin\phi) \wedge (-u_x \sin\phi + u_y \cos\phi)$$
$$= u_x \wedge u_y (\cos^2\phi + \sin^2\phi) = u_x \wedge u_y . \quad \square$$

Recalling that for any $z_0 \in \mathbb{R}^2$

$$G_{z_0}(z) = \frac{1}{2\pi} \log \frac{1}{|z - z_0|}$$

is the Green's function for the Laplacian on $\mathbb{R}^2$ with

$$-\Delta G_{z_0}(z) = \delta_{\{z = z_0\}} \, ,$$

we have

$$u(z_0) = -\int_{\mathbb{R}^2} \Delta u \, G_{z_0} \, dz$$

$$= -2 \int_{\mathbb{R}^2} u_x \wedge u_y \, G_{z_0} \, dz$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} u_r \wedge u_\phi \, g \, dr \, d\phi \, ,$$

where
$$g(r) = \log r \, , \quad r > 0 \, ,$$

in polar coordinates r, ϕ around z_0; that is, with $z = z_0 + re^{i\phi}$.

Integrating by parts[1], we have

$$2\int_0^{2\pi}\int_0^{\infty} u_r \wedge u_\phi \, g \, dr \, d\phi$$

$$= -\int_0^{2\pi}\int_0^{\infty} u_{r\phi} \wedge u \, g \, dr \, d\phi$$

$$-\int_0^{2\pi}\int_0^{\infty} u \wedge u_{\phi r} \, g \, dr \, d\phi$$

$$-\int_0^{2\pi}\int_0^{\infty} u \wedge u_\phi \, g_r \, dr \, d\phi$$

$$= -\int_0^{2\pi}\int_0^{\infty} u \wedge u_\phi \, g_r \, dr \, d\phi$$

in view of cancellation

$$u \wedge u_{\phi r} + u_{r\phi} \wedge u = 0$$

by anti-symmetry of the exterior product. Finally, integrating by parts again and using that

$$g_r(r) = \frac{1}{r}, \quad r > 0,$$

---

1) Concerning boundary terms, see the discussion at the end of this section.

with

$$\bar{u}(r) = \int_0^{2\pi} u(x_0 + re^{i\phi})\, d\phi$$

we have that

$$-\int_0^{2\pi}\int_0^{\infty} u \wedge u_\phi \, g_r \, dr\, d\phi$$

$$= \int_0^{\infty}\int_0^{2\pi} u_\phi \wedge (u - \bar{u}(r)) \frac{1}{r}\, d\phi\, dr$$

is bounded by

$$\int_0^{\infty}\int_0^{2\pi} |\nabla u| \frac{|u-\bar{u}(r)|}{r} \, r\, d\phi\, dr$$

$$\leq \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{1}{2}\int_0^{\infty}\left(\int_0^{2\pi} |u - \bar{u}(r)|^2 d\phi\right)\frac{dr}{r}.$$

But by Poincaré's inequality

$$\int_0^{2\pi} |u - \bar{u}(r)|^2 d\phi \leq \int_0^{2\pi} |u_\phi|^2 d\phi$$

$$\leq r^2 \int_0^{2\pi} |\nabla u|^2 d\phi$$

for any $r > 0$, and

$$\int_0^{\infty}\int_0^{2\pi} |u - \bar{u}(r)|^2 d\phi \frac{dr}{r} \leq \|\nabla u\|_{L^2}^2 ,$$

as well. It follows that

$$|u(z_0)| \leq \frac{1}{2\pi} \left| \int_0^{2\pi} \int_0^{\infty} u_r u_\phi \, g_r \, dr \, d\phi \right|$$

$$\leq \frac{1}{2\pi} \| \nabla u \|^2_{L^2} .$$

Since $z_0 \in \mathbb{R}^2$ is arbitrary, the claim follows. $\square$

For the proof of Thm. 2.11 we again need to use a regularized Green's function.

Given $R > 0$ let

$$G(z) = \frac{1}{2\pi} \log\left(\frac{R}{r}\right), \quad 0 < r = |z| \leq R,$$

be the Green's function $G = G(\cdot, 0)$ on $B_R(0)$.

Set $\quad g_R(r) = \log\left(\frac{R}{r}\right), \quad 0 < r < R,$

and for $0 < \varepsilon < R$ set

$$g_R^{\varepsilon}(r) = \begin{cases} g_R(r), & \varepsilon < r < R, \\ g_R(\varepsilon) - \frac{r^2}{2\varepsilon^2} + \frac{1}{2}, & \text{else}, \end{cases}$$

to obtain a regularized Green's function in C^1, Proof of Thm 2.11. Fix $z_0 \in B = B_1(0) \subset \mathbb{R}^2$, and for $R > 0$ with $B_R(z_0) \subset B$ let $z = z_0 + re^{i\phi}$, $0 < r < R$, to define

$$G^\varepsilon_{z_0}(z) = \frac{1}{2\pi} g^\varepsilon_R(r).$$

Then testing (2.11) with $G^\varepsilon_{z_0} \in H^1_0 \cap L^\infty(B_R(z_0))$, on the one hand we obtain

$$-\int_B \Delta u \, G^\varepsilon_{z_0} \, dz = \int_{B_R(z_0)} \nabla u \, \nabla G^\varepsilon_{z_0} \, dz$$

$$(2.12) \quad = \int_{\partial B_R(z_0)} u \, \partial_r G^\varepsilon_{z_0} \, do - \int_{B_R(z_0)} u \, \Delta G^\varepsilon_{z_0} \, dz$$

$$= -\fint_{\partial B_R(z_0)} u \, do + \fint_{B_\varepsilon(z_0)} u(z) \, dz,$$

in view of

$$-\Delta G^\varepsilon_{z_0} = \frac{2}{2\pi \varepsilon^2} \chi_{B_\varepsilon(z_0)}.$$

On the other hand, with Claim 1 and integrating by parts as in the proof of Proposition 2.12, we have

$$2\int_{B} u_x \wedge u_y \, G_{z_0}^{\varepsilon} \, dz$$

$$= \frac{1}{\pi}\int_0^{2\pi}\int_0^{R} u_r \wedge u_\phi \, g_R^{\varepsilon} \, dr \, d\phi$$

$$= \frac{1}{2\pi}\int_0^{R}\int_0^{2\pi} u_\phi \wedge (u - \bar{u}(r)) \, \partial_r g_R^{\varepsilon} \, d\phi \, dr.$$

But

$$\partial_r g_R^{\varepsilon}(r) = \begin{cases} -\frac{1}{r}, & \varepsilon < r < R, \\ -\frac{r}{\varepsilon^2}, & 0 < r < \varepsilon \end{cases}$$

gives the bound

$$|\partial_r g_R^{\varepsilon}(r)| \leq \frac{1}{r}, \quad 0 < r < R,$$

and thus we can estimate

$$\left| 2\int_{B} u_x \wedge u_y \, G_{z_0}^{\varepsilon} \, dz \right| \leq \frac{1}{2\pi} \, \|\nabla u\|^2_{L^2(B_R(z_0))}$$

as in the proof of Prop. 2.12.

For any Lebesgue point z_0 of u, any $R>0$ with $B_R(z_0) \subset B$, upon passing to the limit $\varepsilon \downarrow 0$ in (2.12) we then find

$$\Big| u(z_0) - \fint_{\partial B_R(z_0)} u \, do \Big| \leq \frac{1}{2\pi} \|\nabla u\|^2_{L^2(B_R(z_0))} .$$

For $R/2 < S < R$ then we also have the bound

$$\Big| u(z_0) - \fint_{\partial B_S(z_0)} u \, do \Big| \leq \frac{1}{2\pi} \|\nabla u\|^2_{L^2(B_R(z_0))} .$$

Taking the average in S with respect to the measure $\frac{4S}{3R^2} dS$ on $[R/2, R]$, we obtain the estimate

$$\Big| u(z_0) - \fint_{B_R \setminus B_{R/2}(z_0)} u \, dz \Big|$$

$$(2.13) \qquad = \Big| \int_{R/2}^{R} \frac{4S}{3R^2} \Big(u(z_0) - \fint_{\partial B_S(z_0)} u \, do \Big) dS \Big|$$

$$\leq \frac{1}{2\pi} \|\nabla u\|^2_{L^2(B_R(z_0))} .$$

For any $z_0 \in B$ and sufficiently small $R > 0$ with $B_{2R}(z_0) \subset B$ then the condition

$$\underset{B_R(z_0)}{\operatorname{osc}} u = \underset{z_1, z_2 \in B_R(z_0)}{\operatorname{ess\,sup}} |u(z_1) - u(z_2)| < 1$$

is satisfied, which in view of the bound

$$|2 u_x \wedge u_y| \leq 2 |u_x| |u_y| \leq |\nabla u|^2$$

and Thm. 2.8 gives Hölder continuity and then smoothness of u in $B_R(z_0)$.

Indeed, for any $z_1, z_2 \in B_R(z_0)$ we can estimate

$$\Big| \fint_{B_R \setminus B_{R/2}(z_1)} u \, dz - \fint_{B_R \setminus B_{R/2}(z_2)} u \, dz \Big| = \Big| \fint_{B_R \setminus B_{R/2}(0)} \big(u(z_1 + z) - u(z_2 + z)\big) \, dz \Big|$$

$$= \Big| \fint_{B_R \setminus B_{R/2}(0)} \int_0^1 \frac{d}{dt} u\big(z_2 + t(z_1 - z_2) + z\big) \, dt \, dz \Big|$$

$$\leq |z_1 - z_2| \sup_{0 \leq t \leq 1} \fint_{B_R \setminus B_{R/2}(0)} |\nabla u(z_2 + t(z_1 - z_2) + z| \, dz$$

$$\leq R \sup_{0 \leq t \leq 1} \Big(\fint_{B_R \setminus B_{R/2}(0)} |\nabla u(z_2 + t(z_1 - z_2) + z)|^2 \, dz \Big)^{1/2}$$

and the later can be bounded by $C \|\nabla u\|_{L^2(B_{2R}(z_0))}$. Also bounding

$$\|\nabla u\|^2_{L^2(B_R(z_1))} + \|\nabla u\|^2_{L^2(B_R(z_2))} \leq 2\|\nabla u\|^2_{L^2(B_{2R}(z_0))},$$

from (2.13) we thus obtain the estimate

$$|u(z_1) - u(z_2)| \leq \frac{1}{\pi}\|\nabla u\|^2_{L^2(B_{2R}(z_0))} + C\|\nabla u\|_{L^2(B_{2R}(z_0))},$$

$$\leq 1/2 < 1$$

for sufficiently small $R > 0$, uniformly in $z_1, z_2 \in B_R(z_0)$. $\square$

Addendum: Boundary terms at $r=0$ in the proof of Prop. 2.12 can be dealt with, as follows.

i) For smooth u we have $u_\phi = 0$ at $r=0$; thus

$$\int_0^{2\pi}\int_0^\infty u_r \wedge u_\phi \, g \, dr \, d\phi$$

$$= -\int_0^{2\pi}\int_0^\infty \left(u \wedge u_{\phi r} \, g + u \wedge u_\phi \, g_r \right) dr \, d\phi .$$

For $u \in H^1$, by density of smooth functions in H^1 again no boundary term appears.

ii) Write

$$(u_x \wedge u_y)^1 = u^2_x u^3_y - u^2_y u^3_x$$

$$= \partial_x (u^2 u^3_y) - \partial_y (u^2 u^3_x)$$

and then

$$\int_{\mathbb{R}^2} (u_x \wedge u_y)^1 g \, dz = \int_{\mathbb{R}^2} \left(\partial_x (u^2 u^3_y) - \partial_y (u^2 u^3_x) \right) g \, dz$$

$$= \int_{\mathbb{R}^2} u^2 \left(u^3_x g_y - u^3_y g_x \right) dz$$

$$= \int_0^{2\pi}\int_0^\infty u^2 \left(u^3_r g_\phi - u^3_\phi g_r \right) dr \, d\phi \quad \text{(by Claim 1)}$$

$$= -\int_0^{2\pi}\int_0^\infty u^2 u^3_\phi g_r \, dr \, d\phi = \int_0^{2\pi}\int_0^\infty u^2_\phi \left(u^3 - \bar{u}^3(r) \right) g_r \, dr \, d\phi .$$

2.5 Hélein's result

For weakly harmonic maps from a surface to the sphere, Hélein discovered a hidden determinant structure, allowing him to obtain the following result.

Theorem 2.13 (Hélein (1990)) Let M be a closed surface, $u \in H^1(M, S^2)$ weakly harmonic. Then u is smooth.

Proof: For simplicity, we only consider the case when $M = T^2 = \mathbb{R}^2/T^2$, and when $u \in H^1_{loc}(\mathbb{R}^2; S^2)$ is a periodic weak solution of the equation

$$-\Delta u = u|\nabla u|^2 \quad \text{on } \mathbb{R}^2.$$

Following Hélein, we write this in the form

$$(2.14) \quad -\Delta u^i = u^i \partial_j u^k \partial_j u^k$$

for each of the components u^i, $1 \leq i \leq 3$, of u. Observing that

$$u^k \partial_j u^k = \partial_j \left(\frac{|u|^2}{2}\right) = 0, \quad 1 \leq j \leq 2,$$

then we have

$$(2.15) \quad -\Delta u^i = \left(u^i \partial_j u^k - u^k \partial_j u^i\right) \partial_j u^k,$$

where

$$v_j^{ik} := u^i \partial_j u^k - u^k \partial_j u^i \in L^2$$

in view of (2.14) is divergence-free with

$$\partial_x v_1^{ik} + \partial_y v_2^{ik} = u^i \Delta u^k - u^k \Delta u^i$$
$$= u^i u^k |\nabla u|^2 - u^k u^i |\nabla u|^2 = 0.$$

Let $w^{ik} \in H^1(T^2)$ weakly solve the equation

$$-\Delta w^{ik} = \partial_y v_1^{ik} - \partial_x v_2^{ik} \quad \text{on } T^2.$$

Then for each $1 \leq i, k \leq 3$ the vector field $h = h^{ij} \in L^2(T^2; \mathbb{R}^2)$ with components $h = (h_j)_{1 \leq j \leq 2}$

given by $h_1 = \partial_x w - v_2$, $h_2 = \partial_y w + v_1$, $v = v^{ik}$, satisfies the conditions

$$\operatorname{div} h = \partial_x h_1 + \partial_y h_2 = \Delta w - \partial_x v_2 + \partial_y v_1 = 0,$$
$$\operatorname{curl} h = -\partial_y h_1 + \partial_x h_2 = \partial_x v_1 + \partial_y v_2 = 0$$

and hence is constant in view of

$$\int_{T^2} \left(|\operatorname{div} h|^2 + |\operatorname{curl} h|^2 \right) dz$$
$$= \int_{T^2} \left(|\partial_x h_1|^2 + |\partial_y h_2|^2 + |\partial_y h_1|^2 + |\partial_x h_2|^2 \right) dz$$
$$+ 2 \int_{T^2} \left(\partial_x h_1 \cdot \partial_y h_2 - \partial_y h_1 \, \partial_x h_2 \right) dz$$
$$= \int_{T^2} |\nabla h|^2 dz .$$

It follows that (2.14) may be written

$$-\Delta u^i = v_1^{ik} \partial_x u^k + v_2^{ik} \partial_y u^k$$
$$= \partial_x w^{ik} \partial_y u^k - \partial_y w^{ik} \partial_x u^k$$
$$+ h_2^{ik} \partial_x u^k - h_1^{ik} \partial_y u^k,$$

where the leading term on the right

once again has a determinant structure. In fact, similar to Claim 1 in the proof of Prop. 2.12, if we let $z = re^{i\phi}$ the following holds.

Claim 1: We have

$$\partial_x w^{ik} \partial_y u^k - \partial_y w^{ik} \partial_x u^k$$

$$= \frac{1}{r}\left(w_r^{ik} u_\phi^k - w_\phi^{ik} u_r^k\right).$$

Proof. With $x = r\cos\phi$, $y = r\sin\phi$ we compute

$$\frac{w_r^{ik} u_\phi^k - w_\phi^{ik} u_r^k}{r}$$

$$= \left(w_x^{ik}\cos\phi + w_y^{ik}\sin\phi\right)\cdot\left(-u_x^k \sin\phi + u_y^k\cos\phi\right)$$

$$-\left(-w_x^{ik}\sin\phi + w_y^{ik}\cos\phi\right)\left(u_x^k\cos\phi + u_y^k\sin\phi\right)$$

$$= w_x^{ik} u_y^k\left(\cos^2\phi + \sin^2\phi\right) - w_y^{ik} u_x^k\left(\sin^2\phi + \cos^2\phi\right)$$

$$= w_x^{ik} u_y^k - w_y^{ik} u_x^k.$$

□

For any $z_0 \in \mathbb{R}^2$, any $R > 0$, letting $G_R^\varepsilon \in H^2 \cap H_0^1 \cap L^\infty(B_R(z_0))$ as in the proof of Thm. 2.11 then we have

$$-\int_{B_R(z_0)} \Delta u^i \, G_R^\varepsilon \, dz = \fint_{B_\varepsilon(z_0)} u \, dz - \fint_{\partial B_R(z_0)} u \, do ,$$

and on the other hand

$$I := \int_{B_R(z_0)} \left(\omega_x^{ik} u_y^k - \omega_y^{ik} u_x^k\right) G_R^\varepsilon \, dz$$

$$= \int_0^{2\pi}\int_0^R \left(\omega_r^{ik} u_\phi^k - \omega_\phi^{ik} u_r^k\right) G_R^\varepsilon \, dr \, d\phi$$

$$= -\frac{1}{2\pi} \int_0^{2\pi}\int_0^R \omega^{ik} u_\phi^k \, \partial_r g_R^\varepsilon \, dr \, d\phi$$

$$= \frac{1}{2\pi} \int_0^{2\pi}\int_0^R \omega_\phi^{ik} \left(u^k - \bar{u}^k(r)\right) \partial_r g_R^\varepsilon \, dr \, d\phi .$$

Estimating as before, we find

$$|I| \leq \frac{1}{2\pi} \int_0^{2\pi}\int_0^R \left|\frac{1}{r}\omega_\phi^{ik}\right| \frac{|u^k - \bar{u}^k(r)|}{r} \, r \, dr \, d\phi$$

$$\leq \frac{1}{2\pi} \|\nabla \omega^{ik}\|_{L^2(B_R(z_0))} \|\nabla u^k\|_{L^2(B_R(z_0))} .$$

Since $\log\left(\frac{R}{r}\right) \in L^2(B_R(z_0))$, moreover, we can bound

$$\underline{II} := \int_{B_R(z_0)} \left(h_2^{ik} u_x^k - h_1^{ik} u_y^k\right) \sigma_R^\varepsilon \, dz$$

by Hölder's inequality to obtain

$$|\underline{II}| \leq C \|\nabla u\|_{L^2(B_R(z_0))},$$

uniformly in $\varepsilon > 0$.

Letting $\varepsilon \downarrow 0$, for any Lebesgue point $z_0 \in \mathbb{R}^2$ of u we then find the estimate

$$\left| u(z_0) - \int_{\partial B_R(z_0)} u \, do \right| = \lim_{\varepsilon \downarrow 0} \left| \int_{B_\varepsilon(z_0)} u \, dz - \int_{\partial B_R(z_0)} u \, do \right|$$

$$\leq |\underline{I}| + \underline{II} \leq \frac{1}{2\pi} \|\nabla u\|_{L^2(B_R(z_0))} \left(\|\nabla w\|_{L^2(B_R(z_0))} + C \right),$$

and after averaging in $R/2 < S < R$ again there results the bound

$$\left| u(z_0) - \int_{B_R \setminus B_{R/2}(z_0)} u \, dz \right| \leq C \|\nabla u\|_{L^2(B_R(z_0))}.$$

The proof may now be completed exactly as the proof of Prop. 2.12. $\square$

For general closed target manifolds $N \subset \mathbb{R}^n$ Hélein's idea leads to the following equivalent form of the harmonic map equation

(2.16) $\quad -\Delta u = A(u)(\nabla u, \nabla u) \perp T_u N$

from Section 2.1. For ease of exposition, we will assume that N is a hypersurface in $\mathbb{R}^n$, oriented by a smooth unit normal vector field $\nu : N \to T^\perp N$.

Let $w = \nu \circ u$ be the composite function $w \in H^1(M; T^\perp N)$. Recalling that (2.16) is equivalent to the equation

$$-\Delta u = \lambda \nu \circ u = \lambda w$$

with

$$\lambda = \lambda |w|^2 = -\operatorname{div}(\nabla u, w) + \nabla u \nabla w,$$

observing again that $w \cdot \partial_j u = 0$, we find

$$(2.17) \quad -\Delta u^i = w^i \partial_j w^k \partial_j u^k$$
$$= (w^i \partial_j w^k - w^k \partial_j w^i) \partial_j u^k$$

as an equivalent form of (2.16), analogous to (2.15). In general, however, we cannot expect the vector fields

$$v^{ik} := \left(w^i \partial_j w^k - w^k \partial_j w^i\right)_{1 \le j \le 2}$$

to be divergence-free.

As we shall see, following Hélein and Rivière, in spite of this apparent failure of the method, it is possible to once again produce the desired structure after a suitable transformation; in fact, as observed by Rivière–Struwe, this is possible via gauge theory in the spirit of Uhlenbeck. For this, only the anti-symmetry $v^{ik} = -v^{ki}$ is needed.

References

[1] Baldes, Alfred: *Stability and uniqueness properties of the equator map from a ball into an ellipsoid*, Math. Z. 185 (1984), no. 4, 505-516.

[2] Hélein, Frédéric: *Régularité des applications faiblement harmoniques entre une surface et une sphère*, C. R. Acad. Sci. Paris Sér. I Math. 311 (1990), no. 9, 519-524.

[3] Hélein, Frédéric: *Harmonic maps, conservation laws and moving frames*, Translated from the 1996 French original. With a foreword by James Eells. Second edition. Cambridge Tracts in Mathematics, 150. Cambridge University Press, Cambridge, 2002.

[4] Hildebrandt, Stefan; Kaul, Helmut; Widman, Kjell-Ove: *An existence theorem for harmonic mappings of Riemannian manifolds*, Acta Math. 138 (1977), no. 1-2, 1-16.

[5] Hildebrandt, Stefan; Widman, Kjell-Ove: *Some regularity results for quasilinear elliptic systems of second order*, Math. Z. 142 (1975), 67-86.

[6] Jäger, Willi; Kaul, Helmut: *Uniqueness and stability of harmonic maps and their Jacobi fields*, Manuscripta Math. 28 (1979), no. 1-3, 269-291.

[7] Schoen, R.; Yau, S. T.: *Lectures on harmonic maps*, Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997.

[8] Wente, Henry C.: *Large solutions to the volume constrained Plateau problem*, Arch. Rational Mech. Anal. 75 (1980/81), no. 1, 59?77.

[9] Wiegner, Michael: *Ein optimaler Regularitätssatz für schwache Lösungen gewisser elliptischer Systeme*, Math. Z. 147 (1976), no. 1, 21-28.

[10] Widman, K.-O.: *The singularity of the Green function for non-uniformly elliptic partial differential equations with discontinuous coefficients*, Uppsala University, 1970.