3. Tools from harmonic analysis and gauge theory

3.1 Vector bundles and connections

Let (M, g) be a smooth Riemannian manifold of dimension $m \in \mathbb{N}$. Often we will simply take $M = B^m = B_1(0; \mathbb{R}^m)$. A good reference is Jost (2017).

Definition 3.1:
A real (or complex) vector bundle of rank k is a smooth manifold V with a smooth projection $\pi : V \to M$ such that for a cover of M by simply connected charts (U_α) there exist local trivializations $s_\alpha : U_\alpha \times \mathbb{R}^k \to V$ with $\pi \circ s_\alpha = id_{U_\alpha}$, and such that each s_α is a diffeomorphism onto its range.
Moreover, for each pair α, β with $U_\alpha \cap U_\beta \neq \emptyset$ the transition maps $s_{\alpha\beta} = s_\alpha^{-1} \circ s_\beta$ on $U_\alpha \cap U_\beta \times \mathbb{R}^k$ are given by

(3.1) $\quad s_{\alpha\beta}(x, v) = (x, g_{\alpha\beta} v), \ (x, v) \in U_\alpha \cap U_\beta \times \mathbb{R}^k$

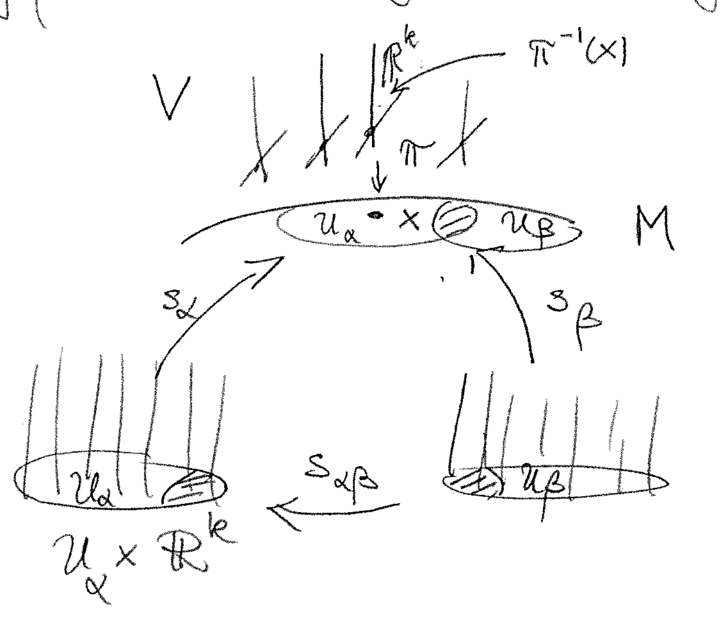
with a local gauge transformation

$g_{\alpha\beta}: \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow G \subset Gl(k)$, where G (e.g. $G = SO(k)$ or $SU(k)$) is the compact <u>structure group</u> of the bundle, and such that the co-cycle condition

(3.2) $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = id$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$

holds.

Conversely, given $(\mathcal{U}_\alpha)$ and smooth maps $g_{\alpha\beta}: \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$ satisfying (3.2), there exists a vector bundle $V \xrightarrow{\pi} M$ with <u>fibre</u> $\pi^{-1}(x) = \mathbb{R}^k$ for each $x \in M$ and local trivializations $s_\alpha: \mathcal{U}_\alpha \times \mathbb{R}^k \to V$ having $(g_{\alpha\beta})$ as local gauge transformations.



Standard examples are the tangent bundle or the co-tangent bundle

$$TM = \{(x, v);\ v \in T_x M\},$$

or

$$T^*M = \{(x, \lambda);\ \lambda \in T_x^* M\}$$

and their sections

$$\Gamma(TM) = \{ v : M \to TM;\ x \mapsto (x, v(x))$$
$$\text{with } v(x) \in T_x M \text{ for any } x \in M\},$$

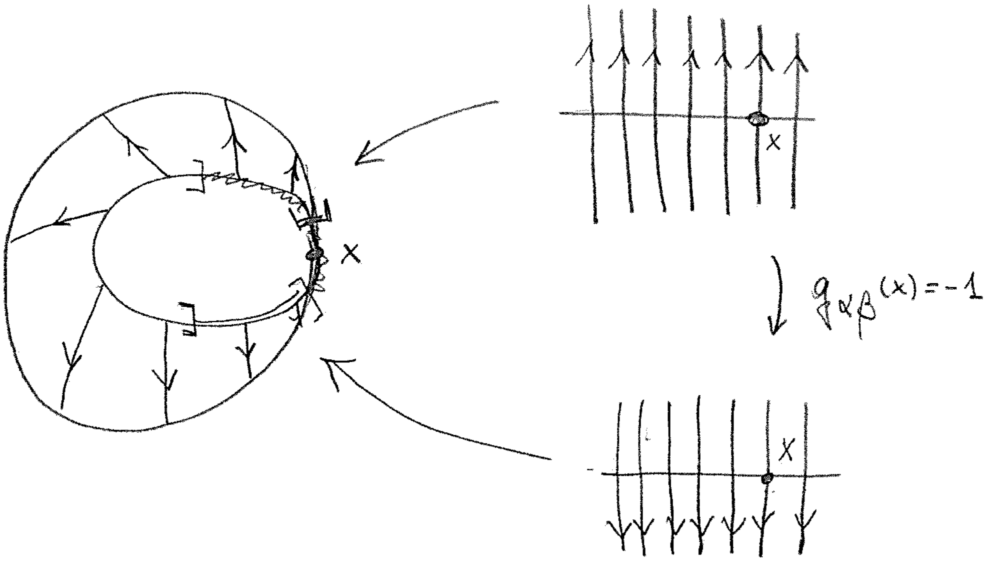
$$\Gamma(T^*M) = \Gamma^1(M),$$

and similarly for tensor-fields and multi-linear forms over M.

A (smooth) <u>section</u> in a vector bundle $V \xrightarrow{\pi} M$ is a smooth map $s : M \to V$ with $\pi \circ s = id$. We let

$$\Gamma(V) = \{ s \in C^\infty(M; V);\ \pi \circ s = id \}.$$

<u>Example</u> 3.2. i) The Möbius band can be realized as a real line bundle $V \xrightarrow{\pi} M$ over $M = S^1 = \mathbb{R}/\mathbb{Z}$ with fiber $\pi^{-1}(x) = \mathbb{R}$ for $x \in M$ and structure group $G = \{1, -1\}$.



If $s \in \Gamma(V)$ is given by $u : S^1 \to \mathbb{R}$ with

$$s(x) = (x, u(x)), \quad x \in S^1,$$

the twist condition implies

$$\forall x \in S^1 : \; s(x + 2\pi) = (x + 2\pi, u(x + 2\pi)) = (x, -u(x)).$$

Thus, any $s \in \Gamma(V)$ must intersect the 0-section in the sense $\exists\, x \in S^1 : u(x) = 0$.

ii) (De Giorgi / Modica-Mortola / Fröhlich-Struwe).

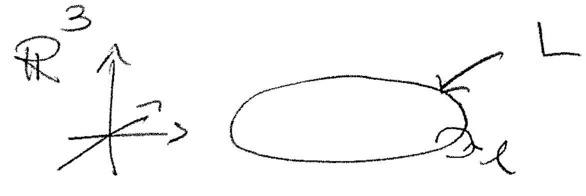
Let $L \subset \mathbb{R}^3$ be a smoothly embedded closed curve ("loop") in $\mathbb{R}^3$, and let $M = \mathbb{R}^3 \setminus L$, V the line bundle over $M = \mathbb{R}^3 \setminus L$ such that for any sufficiently small $\rho > 0$ and any $x \in L$ the bundle $V|_\ell$ restricted to the curve

$$\ell = \{x + \rho(\cos\phi\, a(x) + \sin\phi\, b(x));\ \phi \in \mathbb{R}\}$$

in the tubular neighborhood

$$\mathcal{U}_\rho(L) = \{x + \xi_1 a(x) + \xi_2 b(x);\ x \in L, |\xi| \leq \rho\}$$

of L with $a, b : L \to \mathbb{R}^3$ such that with the unit tangent vector $t(x) \in T_x L$ the triple $(t(x), a(x), b(x))$ is an orthonormal frame for $T_x \mathbb{R}^3$, smoothly depending on $x \in L$, is a Möbius bundle.



Then any section $s \in \Gamma(V)$, when restricted to such l by Ex. i) must intersect the 0-section.

Letting s be represented by $s(x, u(x))$, then the set

$$S = u^{-1}(\{0\}) \subset \mathbb{R}^3$$

satisfies $\partial S = L$. For given $L \subset \mathbb{R}^3$ we can thus hope to find an embedded minimal surface $\Sigma \subset \mathbb{R}^3$ spanning L in the sense that $\partial \Sigma = L$ as a suitable limit of

$$\Sigma_\varepsilon = u_\varepsilon^{-1}(\{0\}), \quad \varepsilon > 0,$$

where for $\varepsilon > 0$ the section $s_\varepsilon(x) = (x, u_\varepsilon(x))$ minimizes the energy

$$E_\varepsilon(u) = \int_{\mathbb{R}^3 \setminus L} \left(\varepsilon |\nabla u|^2 + \frac{1}{\varepsilon}(1-u^2)^2 \right) dx .$$

For loops $L \subset \partial \Omega$ on the boundary of a convex body this problem was studied by Modica-Mortola; see also Pacard or Guaraco, Let $V \xrightarrow{\pi} M$ be a vector bundle with structure group $G = SO(k)$ (or $G = SU(k)$).

A connection D on V is a map $D : \Gamma(V) \to \Omega^1(\mathrm{Ad}\, V)$, mapping a section s to a vector-valued 1-form and such that

$$D(fs)v = (df \cdot v)s + f\,Ds\,v$$

for any $s \in \Gamma(V)$, any $f \in C^\infty(M)$, and any smooth vector field $v \in \Gamma(TM)$.

Remark 3.3. For connections D_1, D_2 on V there holds

$$(D_1 - D_2)(fs) = f(D_1 - D_2)s;$$

thus, D_1 and D_2 differ by a zero-order differential operator given by a smooth section $A \in \Gamma(T^*M \otimes \mathrm{Ad}\, V)$, locally given by a smooth map $A_\alpha : U_\alpha \to T^*M \otimes \mathfrak{g}$, where $\mathfrak{g} = T_{id} G$ $(= so(k))$ is the Lie algebra of G.

Here, we also require the connection D to be <u>metric</u>; that is, such that

$$d\left(\langle p, q\rangle_{\mathbb{R}^k}\right) = \langle Dp, q\rangle_{\mathbb{R}^k} + \langle p, Dq\rangle_{\mathbb{R}^k}$$

for any $p, q \in \Gamma(V)$. In particular, letting $p = e_i$, $q = e_j$ and representing $D = d + A$ in a local chart, we have

$$0 = d\left(\langle e_i, e_j\rangle_{\mathbb{R}^k}\right) = \langle A e_i, e_j\rangle_{\mathbb{R}^k} + \langle e_i, A e_j\rangle_{\mathbb{R}^k}$$

$$= \langle (A + A^t) e_i, e_j\rangle, \quad 1 \le i, j \le k,$$

and $A \in \Omega^1(\mathrm{Ad}\, V) = \Gamma(T^*M \otimes \mathrm{Ad}\, V)$

$= C^\infty(M; T^*M \times \mathfrak{g})$ is anti-symmetric.

Given a background connection $\mathcal{D}_0$, in view of Rem. 3.3 the space $\mathcal{A}$ of connections is an affine space

$$\mathcal{A} = \{\mathcal{D} = \mathcal{D}_0 + A;\ A \in \Gamma(T^*M \otimes \mathrm{Ad}\, V)\}.$$

In any local trivialization $U_\alpha \times \mathbb{R}^k$ of V we may choose the exterior differential d as background connection. Observing how connections transform under gauge transformations, we can then extend this to all of V.

A global gauge transformation $\sigma : V \to V$ is a map such that

$$\forall x \in M,\ v \in \pi^{-1}(x) :\ \sigma(x, v) = (x, g v)$$

with a smooth map $g : M \to G$. Let

$$\mathcal{D} = \mathrm{Aut}(V)$$

denote the space of gauge transformations σ as above.

For $\mathcal{D} = \mathcal{D}_0 + A \in \mathcal{A}$, $\sigma \in \mathcal{D}$ let

$$\sigma^* \mathcal{D} = \sigma^{-1} \circ \mathcal{D} \circ \sigma$$

be the pull-back connection with

$$\begin{aligned}(\sigma^* \mathcal{D})\, s &= \sigma^{-1} \mathcal{D}(\sigma s) = \sigma^{-1}(\mathcal{D}_0 + A)(\sigma s) \\ &= \left(\sigma^{-1} d\sigma + \sigma^{-1} A \sigma\right) s + \mathcal{D}_0 s, \\ &= \left(\mathcal{D}_0 + \tilde{A}\right) s,\end{aligned}$$

where

$$\tilde{A} = \sigma^{-1} d\sigma + \sigma^{-1} A \sigma \tag{3.3}$$

acts on a section $s \in \Gamma(V)$ with $s(x) = (x, v(x))$ via

$$(\tilde{A} s)(x) = \left(x, \left(g^{-1} dg + g^{-1} A g\right) v(x)\right), \quad x \in M.$$

Thus, with the help of the local gauge transformations $g_{\alpha\beta}$ one can extend the trivial connection d on $U_\alpha \times \mathbb{R}^k$ to $U_\beta \times \mathbb{R}^k$, and so on, to obtain a base connection on V, which is well-defined thanks to the co-cycle condition (3.2).

Given a connection $\mathcal{D}$ on V, the <u>curvature</u> of $\mathcal{D}$ is defined to be

$$F = F(\mathcal{D}) = \mathcal{D} \circ \mathcal{D}.$$

<u>Remark</u> 3.4. i) For any $f \in C^\infty(M)$, any $s \in \Gamma(V)$ there holds

$$\begin{aligned} F(fs) &= D(df\, s + f\, Ds) \\ &= \underbrace{d^2 f\, s}_{=0} \underbrace{- df \wedge Ds + df \wedge Ds}_{=0} + f\, F(s) \\ &= f\, F(s). \end{aligned}$$

Thus $F \in \Omega^2(\mathrm{Ad}\, V) = \Gamma(T^*M \otimes T^*M \otimes \mathrm{Ad}\, V)$ is locally given by a $\mathfrak{g}$-valued 2-form

$$F = F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta.$$

ii) For $D = D_0 + A$ we have

$$\begin{aligned} F(D_0 + A) &= (D_0 + A) \circ (D_0 + A) = \\ &= F(D_0) + \underbrace{D_0 \circ A + A \circ D_0}_{=(D_0 A)} + A \circ A, \end{aligned}$$

since $(D_0 \circ A + A \circ D_0)\, s = (D_0 A) s \underbrace{- A \wedge D_0 s + A \wedge D_0 s}_{=0}$

for any sections s.

iii) Locally, if $A = A_\alpha dx^\alpha$, then

$$A \circ A = A \wedge A = A_\alpha A_\beta \, dx^\alpha \wedge dx^\beta$$
$$= \frac{1}{2} \sum_{\alpha,\beta} (A_\alpha A_\beta - A_\beta A_\alpha) \, dx^\alpha \wedge dx^\beta$$
$$= \sum_{\alpha<\beta} [A_\alpha, A_\beta] \, dx^\alpha \wedge dx^\beta = \frac{1}{2}[A, A],$$

where we either highlight the exterior product or the Lie bracket. Thus, for $D = d + A$ we have

$$F(D) = \underbrace{F(d)}_{= d^2 = 0} + dA + \frac{1}{2}[A, A] ;$$

that is, $F = \sum_{\alpha,\beta} F_{\alpha\beta} \, dx^\alpha \wedge dx^\beta$ with

$$(3.4) \quad F_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]\right).$$

iv) For any $D \in \mathcal{A}$, any $\sigma \in \mathcal{D}$ there holds

$$F(\sigma^* D) = \sigma^{-1} \circ D \circ \sigma \circ \sigma^{-1} \circ D \circ \sigma = \sigma^{-1} F(D) \cdot \sigma .$$

Thus, if $G = SO(k)$ we have, in particular,

$$|F(\sigma^* D)| = |F(D)| .$$

With the help of the inner product

$$A \cdot B = tr(AB^t), \quad A, B \in so(k)$$

we can then define the Yang-Mills energy

$$YM(D) = \frac{1}{2}\int_M |F|^2 d\mu,$$

where the energy density is locally given by

$$|F|^2 = \sum_{\alpha<\beta} |F_{\alpha\beta}|^2 = \sum_{\alpha<\beta} F_{\alpha\beta} \cdot F_{\alpha\beta}.$$

By Rem. 3.4. iv), the energy density is gauge-invariant.
If $M = B^m = B_1(0; \mathbb{R}^m)$, for $D = d + A$ we then have

$$YM(D) = YM(A) = \frac{1}{2}\int_B |dA + \frac{1}{2}[A,A]|^2 dx,$$

similar to the Dirichlet energy.

Def. 3.5. A connection D is a Yang-Mills connection if D is critical for the Yang-Mills energy in the sense that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} YM(D+\varepsilon a) = 0, \quad \forall a \in \Omega^1(\mathrm{Ad}\, V).$$

If $M = B^m$, $D = d + A$ is Yang-Mills iff there holds

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} YM(D+\varepsilon a)$$

$$= \int_B F(D) \cdot \underbrace{\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F(D+\varepsilon a)}_{= da + a\wedge A + A \wedge a} \, dx \quad .$$

Observing that

$$a \wedge A + A \wedge a = \frac{1}{2}\sum_{\alpha,\beta}\left([a_\alpha, A_\beta] + [A_\alpha, a_\beta]\right) dx^\alpha \wedge dx^\beta$$

$$= \sum_{\alpha,\beta} [a_\alpha, A_\beta]\, dx^\alpha \wedge dx^\beta = [a, A] \,,$$

we see that

$$0 = \sum_{\alpha,\beta} \int_B F_{\alpha\beta} \left(\partial_\alpha a_\beta + [a_\alpha, A_\beta]\right) dx$$

$$= -\sum_{\alpha,\beta}\int_B \Big(\partial_\alpha F_{\alpha\beta} \cdot a_\beta \underbrace{- \underbrace{F_{\alpha\beta}}_{= F_{\beta\alpha}} \cdot [a_\alpha, A_\beta]}_{= F_{\alpha\beta}\cdot [a_\beta, A_\alpha]}\Big) dx .$$

Compute, using the identity

$$tr(ABC) = tr(BCA)$$

for $k\times k$-matrices A, B, C, the last term

$$F_{\alpha\beta}\cdot[a_\beta, A_\alpha] = tr\left(F_{\alpha\beta}(a_\beta A_\alpha)^t\right) - tr\left(F_{\alpha\beta}(A_\alpha a_\beta)^t\right)$$

$$= tr\left(F_{\alpha\beta} A_\alpha^t a_\beta^t\right) - tr\left(F_{\alpha\beta} a_\beta^t A_\alpha^t\right)$$

$$= - tr\left(F_{\alpha\beta} A_\alpha a_\beta^t\right) + tr\left(F_{\alpha\beta} a_\beta^t A_\alpha\right)$$

$$= tr\left(\left(A_\alpha F_{\alpha\beta} - F_{\alpha\beta} A_\alpha\right) a_\beta^t\right) = [A_\alpha, F_{\alpha\beta}]\cdot a_\beta$$

to see that we have as a condition for criticality of $D = d + A$ that

$$0 = \sum_{\alpha,\beta} \int_B \left(\partial_\alpha F_{\alpha\beta} + [A_\alpha, F_{\alpha\beta}]\right)\cdot a_\beta \, dx$$

for all $a \in C^\infty(B, T^*M \otimes \mathrm{Ad}\, V)$.

Thus, $D = d + A$ is Yang-Mills iff

$$(D^* F)_\beta = \partial_\alpha F_{\alpha\beta} + [A_\alpha, F_{\alpha\beta}] = 0 .$$

With (3.) this Yang-Mills equation

takes the form

$$\partial_\alpha^2 A_\beta - \partial_\alpha \partial_\beta A_\alpha + \partial_\alpha\big([A_\alpha, A_\beta]\big)$$

$$+ [A_\alpha, \partial_\alpha A_\beta - \partial_\beta A_\alpha] + [A_\alpha, [A_\alpha, A_\beta]] = 0.$$

The <u>Coulomb</u> gauge

$$\partial_\alpha A_\alpha = 0$$

turns this equation into a diagonal elliptic system

$$\Delta A_\beta + 2[A_\alpha, \partial_\alpha A_\beta] - [A_\alpha, \partial_\beta A_\alpha]$$

$$+ [A_\alpha, [A_\alpha, A_\beta]] = 0.$$

<u>Question</u> 3.6. i) Can any connection $D = d + A$ on $\mathcal{B}$ be transformed into Coulomb gauge?

ii) What does this mean for our purposes; that is how can we exploit this in order to show regularity of weakly harmonic maps?

The first question is answered by a theorem of Karen Uhlenbeck from 1982. For this we need the space

$$\mathcal{A}_1^p = \{ D = d + A ;\ A \in W^{1,p}(B; \mathbb{R}^m \times \mathfrak{g}) \},$$

where we identify $T_x^* B^m \cong \mathbb{R}^m$ for $x \in B^m$ and with $\mathfrak{g} = so(k)$. Moreover, we let

$$\mathcal{D}_2^p = W^{2,p}(B; G)$$

be the space of local gauge transformations of class $W^{2,p}$, where $p \geq m/2$. Then we have the following result.

<u>Theorem</u> 3.7 (Uhlenbeck (1982)). Let $m \geq 2$, $p \geq m/2$ and let $D = d + \Omega \in \mathcal{A}_1^p$. There exists a number $\kappa = \kappa(m) > 0$ and a constant $c = c(m) > 0$ such that if

$$\|F\|_{L^{m/2}}^{m/2} = \|d\Omega + [\Omega, \Omega]\|_{L^{m/2}}^{m/2} \leq \kappa(m),$$

then D is gauge equivalent via a $\sigma \in \mathcal{D}_2^p$ to a connection $d + A$, where

$$i)\ \partial_\alpha A_\alpha = 0\ ,\quad ii)\ \|A\|_{W^{1,p}} \leq c(m) \|F\|_{L^p}.$$

To see the relevance of Thm. 3.7, recall from (3.3) that for $\sigma \in \mathcal{D}_2^p$ acting via $\sigma(x,v) = (x, gv)$, $(x,v) \in B^m \times \mathbb{R}^k$, we have that

$$A = \sigma^{-1} d\sigma + \sigma^{-1}\Omega\sigma$$

acts on fibers via

$$A = g^{-1}dg + g^{-1}\Omega g, \quad g \in W^{2,p}(B, SO(k)).$$

If A is in Coulomb gauge with $\partial_\alpha A_\alpha = 0$ then in $m=2$ dimensions the Hodge decomposition of A reads

$$A = \nabla^\perp a + h,$$

for some function a and some harmonic h.

If $u \in H^1(B^2; N)$ with $N \subset \mathbb{R}^n$ a smooth hypersurface is weakly harmonic with

$$-\Delta u^i = \underbrace{(\omega^i \partial_j \omega^k - \omega^k \partial_j \omega^i)}_{=:\Omega^{ik}} \partial_j u^k$$

upon letting $P = g^{-1} \in W^{2,p}(B, SO(n))$ we find the following identity

$$
\begin{aligned}
-\operatorname{div}(P\nabla u) &= P(-\Delta u) - \nabla P \cdot \nabla u \\
&= P\Omega P^{-1}(P\nabla u) - \nabla P P^{-1}(P\nabla u) \\
&= A P\nabla u = \nabla^{\perp} a P\nabla u + h P\nabla u,
\end{aligned}
$$

again exhibiting the desired determinant structure (up to a harmless, bounded factor P) on the right hand side.

3.2 Existence of local Coulomb gauges

Instead of Thm. 3.7, requiring bounds on curvature, we will use the following result, due to Rivière-Struwe (2008), requiring only L^2-bounds on the connection 1-form in $m=2$ space dimensions.

Let $m=2$, $B=B_1(0;\mathbb{R}^2)$, $N\subset\mathbb{R}^n$ closed submanifold.

<u>Theorem</u> 3.8 (Rivière-Struwe (2008)) There are constants $\varepsilon=\varepsilon(n)>0$, $C>0$ with the following property. For any $\Omega=(\Omega^{ij}_\alpha dx^\alpha)\in L^2(B;T^*\mathbb{R}^2\otimes so(n))$ with

$$(3.5)\qquad \int_B |\Omega|^2 dx < \varepsilon$$

there exist $P\in H^1(B;SO(n))$, $\xi\in H^1_0(B;so(n))$ and a harmonic $h\in L^2(B;T^*\mathbb{R}^2\otimes so(n))$ such that

$$(3.6)\qquad P^{-1}dP+P^{-1}\Omega P=*d\xi+h,$$

where $*d\xi=\partial_2\xi\,dx^1-\partial_1\xi\,dx^2$, and satisfying

$$(3.7)\qquad \|dP\|^2_{L^2}+\|d\xi\|^2_{L^2}\leq C\|\Omega\|^2_{L^2}\leq C\varepsilon,$$

and then also $\|h\|^2_{L^2}\leq C\|\Omega\|^2_{L^2}\leq C\varepsilon$.

Moreover, $P=id$ on ∂B.

The proof can be achieved via the following lemma.

Lemma 3.9. The assertion of Thm. 3.8 holds true if in addition to (3.5) there holds $\Omega \in L^{2,\alpha}(B)$ for some $\alpha > 0$, in which case there exist P, ξ, and h as in Thm. 3.8 satisfying (3.6), (3.7) and, in addition, $dP, d\xi \in L^{2,\alpha}(B)$ with

(3.8) $\quad \|dP\|_{L^{2,\alpha}} + \|d\xi\|_{L^{2,\alpha}} \leq C \|\Omega\|_{L^{2,\alpha}}$.

Proof of Thm. 3.8: Extend Ω to $\Omega \in L^2(B_2(0))$ with

$$\int_{B_2(0)} |\Omega|^2 dx < C\varepsilon .$$

Let $0 \leq \rho \in C_c^\infty(B_1(0))$ with $\int_{\mathbb{R}^2} \rho\, dx = 1$ and set $\rho_\delta(x) = \delta^{-2}\rho(\frac{x}{\delta}) \in C_c^\infty(B_\delta(0))$ to obtain a standard mollifying family $(\rho_\delta)_{\delta>0}$. For any sufficiently small $\delta > 0$ then $\Omega_\delta = \Omega * \rho_\delta \in C^\infty(B, T^*\mathbb{R}^2 \otimes so(n))$ satisfies (3.5), and $\Omega_\delta \in L^{2,\alpha}(B)$ for $0 < \alpha < 2$.

By Lemma 3.9 there exist P_δ, ξ_δ of class H^1 with dP_δ, $d\xi_\delta \in L^{2,\alpha}(B)$ satisfying (3.7) and harmonic $h_\delta \in L^{2,\alpha}(B)$ such that

$$(3.9)\quad P_\delta^{-1} dP_\delta + P_\delta^{-1} \Omega_\delta P_\delta = *d\xi_\delta + h_\delta \quad \text{in } B.$$

In view of (3.7) for suitable $\delta = \delta_k \downarrow 0$ we have

$$P_\delta \xrightarrow{w} P \text{ in } H^1(B, SO(n)) \text{ and strongly in } L^2(B),$$

$$\xi_\delta \xrightarrow{w} \xi \text{ in } H^1(B, so(n)),$$

which allows to pass to the limit $\delta = \delta_k \downarrow 0$ in (3.9) to conclude (3.6). Moreover, (3.7) follows by weak lower semi-continuity of the L^2-norm. □

Thus, it is enough to prove Lemma 3.9.

Proof of Lemma 3.9. The proof is modelled on the proof of Thm. 3.7 in Uhlenbeck's 1982 paper.

For given $\alpha > 0$ and sufficiently small $\varepsilon > 0$ and sufficiently large $C > 0$ to be defined let

$$\mathcal{U}^\alpha_{\varepsilon,C} = \{ \Omega \in L^{2,\alpha}(B, T^*\mathbb{R}^2 \otimes so(n)); \ \|\Omega\|^2_{L^2} \le \varepsilon$$
$$\text{and there exist } P, \xi \in H^1, h \in L^2 \text{ with } (3.6)-(3.8)\}$$

$$\subset \mathcal{V}^\alpha_\varepsilon := \{\Omega \in L^{2,\alpha}(B, T^*\mathbb{R}^2 \otimes so(n)); \ \|\Omega\|^2_{L^2} \le \varepsilon\}.$$

Note that $0 \in \mathcal{U}^\alpha_{\varepsilon,C}$; hence $\mathcal{U}^\alpha_{\varepsilon,C} \neq \emptyset$.
Moreover, the set $\mathcal{V}^\alpha_\varepsilon$ is star-shaped with respect to $\Omega_0 = 0 \in \mathcal{V}^\alpha_\varepsilon$ and hence is path-connected.

<u>Claim 1.</u> $\mathcal{U}^\alpha_{\varepsilon,C} \subset \mathcal{V}^\alpha_\varepsilon$ is closed.

<u>Proof</u>: Let $(\Omega_k)_{k\in\mathbb{N}} \subset \mathcal{U}^\alpha_{\varepsilon,C}$ with $\Omega_k \xrightarrow{(k\to\infty)} \Omega \in \mathcal{V}^\alpha_\varepsilon$ in $L^{2,\alpha}(B; T^*\mathbb{R}^2 \otimes so(n))$, and for each $k \in \mathbb{N}$ let P_k, ξ_k, h_k satisfy (3.6), (3.7), (3.8) for Ω_k.
By (3.7) we may assume $P_k \xrightarrow{w} P$, $\xi_k \xrightarrow{w} \xi$ in H^1 and strongly in L^2 and a.e., and $h_k \xrightarrow{w} h$ in L^2. Then (3.6) holds, and (3.7), (3.8) again hold by weak lower semi-continuity of the L^2-norm. $\square$

Lemma 3.9 then is a consequence of the following claim.

<u>Claim 2</u>: $\mathcal{U}^{\alpha}_{\varepsilon,C}$ is open.

We give the proof of this claim in a sequence of Lemmas. For convenience we identify $T^*_x\mathbb{R}^2 \cong \mathbb{R}^2$.

<u>Lemma</u> 3.10. Suppose $A \in \mathcal{U}^{\alpha}_{\varepsilon,C}$ satisfies $\partial_\alpha A_\alpha = 0$. Then there exists $\delta > 0$ such that for any $\lambda \in L^{2,\alpha}(B; T^*\mathbb{R}^2 \otimes so(n))$ with $\|\lambda\|_{L^{2,\alpha}} < \delta$ the equation

(3.10) $\operatorname{div}\left(P^{-1}\nabla P + P^{-1}(A+\lambda)P\right) = 0$

has a solution $P = P(\lambda) \in \mathcal{D}^{2,\alpha}_{1,0}$, smoothly depending on $\lambda \in L^{2,\alpha}$.

Here

$$\mathcal{D}^{2,\alpha}_{1,0} = \{P \in H^1(B; SO(n));\ dP \in L^{2,\alpha},\ P = id \text{ on } \partial B\}.$$

Proof: Consider the exponential map

$$T_{id}\mathcal{D}_1^{2,\alpha} = \{ U \in H_0^1(B; so(n)) ;\ \nabla U \in L^{2,\alpha}(B) \}$$

$$\ni U \mapsto e^U \in \mathcal{D}_1^{2,\alpha} .$$

Note that by Campanato's theorem and Poincaré's inequality

$$\int_{B_r(x_0)} |U - \bar{U}_{x_0,r}|^2 dx \leq C r^2 \int_{B_r(x_0)} |\nabla U|^2 dx$$

$$\leq C [\nabla U]_{L^{2,\alpha}} r^{2+\alpha}$$

there holds $T_{id}\mathcal{D}_1^{2,\alpha} \hookrightarrow C^{\alpha/2}$ and

$$\| e^U - id \|_{L^\infty} \leq C [\nabla U]_{L^{2,\alpha}}$$

for $[\nabla U]_{L^{2,\alpha}} \leq \rho$ and sufficiently small $\rho > 0$.

Writing $P = e^U$ for $[\nabla U]_{L^{2,\alpha}} \leq \rho$, we then may regard the map

$$\Phi : (U, \lambda) \mapsto div\left(e^{-U} \nabla e^U + e^{-U}(A+\lambda) e^U \right)$$

from $T_{id}\mathcal{D}_1^{2,\alpha} \times L^{2,\alpha}$ to the space

$$L_{-1}^{2,\alpha} = \{ div\, f ;\ f \in L^{2,\alpha}(B, \mathbb{R}^2 \otimes so(n)) \} .$$

Clearly, the map Φ is smooth.
Moreover, by the following Lemma 3.11 the linearization $H = \partial_u \Phi$ at $(u, \lambda) = (0,0)$ is an isomorphism. The claim then follows by the implicit function theorem. $\square$

<u>Lemma</u> 3.11. There is $\varepsilon > 0$ such that the linearized operator

$$L = \partial_u|_{(u,\lambda)=(0,0)} \Phi : T_{id}\mathcal{D}^{2,\alpha}_1 \to L^{2,\alpha}_{-1}$$

given by

$$LV = \Delta V + div([A,V]) = \Delta V + [A; \nabla V]$$

is an isomorphism, if $A = \nabla^\perp \xi \in L^{2,\alpha}$ with $\xi \in H^1_0(B; so(n))$ satisfies $\|A\|_{L^2} < \varepsilon$.

Postponing the proof of Lemma 3.11 to the next section, we proceed with the proof of Lemma 3.9.

Proof of Lemma 3.9 (cont.) Let $\Omega \in \mathcal{U}^{\alpha}_{\varepsilon, C}$ such that for suitable $P, \xi \in H^1$ with $P|_{\partial B} = id$, $\xi|_{\partial B} = 0$ and $dP, d\xi \in L^{2,\alpha}$ conditions (3.6) - (3.8) hold.

Given $\omega \in L^{2,\alpha}(B; \mathbb{R}^2 \otimes so(n))$ with $\|\omega\|_{L^{2,\alpha}} < \delta$ for sufficiently small $\delta > 0$ (to be determined), let

$$\lambda = P^{-1}\nabla P + P^{-1}(\Omega + \omega)P - A = P^{-1}\omega P$$

where

$$A = \nabla^{\perp}\xi \cong * d\xi$$

with

$$\operatorname{div} A = \partial_\alpha A_\alpha = 0 .$$

Note that we can bound

$$\|\lambda\|_{L^{2,\alpha}} \leq \|\omega\|_{L^{2,\alpha}} < \delta .$$

Thus, for sufficiently small $\delta > 0$ given by Lemma 3.10 we can find $Q \in H^1(B; SO(n))$ with $dQ \in L^{2,\alpha}$ and $Q|_{\partial B} = id$ such that

$$\operatorname{div}\left(Q^{-1}\nabla Q + Q^{-1}(A + \lambda)Q\right) = 0 .$$

Letting $R = P \circ Q$, then we have

$$R^{-1}\nabla R + R^{-1}(\Omega+\omega)R$$

$$= Q^{-1}\circ P^{-1}\nabla P \circ Q + Q^{-1}\nabla Q$$

$$+ Q^{-1}\circ P^{-1}(\Omega+\omega)P\circ Q$$

$$= Q^{-1}\nabla Q + Q^{-1}\left(P^{-1}\nabla P + P^{-1}(\Omega+\omega)P\right)Q$$

$$= Q^{-1}\nabla Q + Q^{-1}(A+\lambda)Q$$

is divergence-free, and by Hodge decomposition there exists $\eta \in H^1_0(B)$ and harmonic $k \in L^2$ with

$$R^{-1}\nabla R + R^{-1}(\Omega+\omega)R = \nabla^\perp\eta + k,$$

showing existence of R, η, k with (3.6).

The proof of the bounds (3.7), (3.8) is achieved via the following Lemma 3.12, whose proof again will be given in the next section.

Lemma 3.12. There exist $\varepsilon_0 > 0$, $C > 0$ such that when $0 < \varepsilon < \varepsilon_0/C$ for any $\Omega \in \mathcal{V}^\alpha_\varepsilon$ for which there exist P, ξ, and h as above with (3.6) and satisfying

$$\|dP\|_{L^2} + \|d\xi\|_{L^2} < \varepsilon_0$$

there hold the bounds (3.7) and (3.8).

Proof of Lemma 3.9 (completed): By Lemma 3.10 for any $\Omega \in \mathcal{U}^\alpha_{\varepsilon,C}$ with associated $P_0 = P_\Omega$, $\xi_0 = \xi_\Omega$, $h_0 = h_\Omega$ satisfying (3.6), where $\varepsilon > 0$, $C > 0$ with $C\varepsilon < \varepsilon_0$ as in Lemma 3.12, there exists $\delta > 0$ such that for $\omega \in L^{2,\alpha}(B; \mathbb{R}^2 \otimes so(4))$ with

$$\|\omega\|_{L^{2,\alpha}} < \delta$$

there exist P, ξ, h such that (3.6) holds for $\Omega + \omega$, and $dP \to dP_0$, $d\xi \to d\xi_0$ in $L^{2,\alpha}$ as $\delta \to 0$.

For sufficiently small $\delta > 0$ then we have

$$\|dP\|_{L^2} + \|d\xi\|_{L^2} < \varepsilon_0,$$

and $\Omega + \omega \in \mathcal{U}^\alpha_{\varepsilon,C}$ by Lemma 3.12 for any ω with $\|\omega\|_{L^{2,\alpha}} < \delta$. Hence $\mathcal{U}^\alpha_{\varepsilon,C}$ is (rel.) open in $\mathcal{V}^\alpha_\varepsilon$.

$\square$

3. The Hardy space $\mathcal{H}^1$ and BMO

There are several equivalent definitions of $\mathcal{H}^1$, the simplest one being the maximal characterization.

Let $0 \le \varphi \in C_c^\infty(B_1(0))$ with $\int_{\mathbb{R}^n} \varphi \, dx = 1$, and for $t > 0$ let

$$\varphi_t(x) = t^{-n} \varphi\left(\frac{x}{t}\right) \in C_c^\infty(B_t(0))$$

be a standard mollifier or "approximation of the identity".

For $f \in L^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ then let

$$M_\varphi f(x) = \sup_{t>0} |(f * \varphi_t)(x)|$$

denote the maximal function of f based on φ.

Def. 3.13. The Hardy space $\mathcal{H}^1$ is

$$\mathcal{H}^1 = \{ f \in L^1(\mathbb{R}^n) ; \; M_\varphi f \in L^1(\mathbb{R}^n) \}$$

with norm

$$\|f\|_{\mathcal{H}^1} = \|f\|_{L^1} + \|M_\varphi f\|_{L^1}, \; f \in \mathcal{H}^1.$$

Remark 3.14. For a function $f \in L^1(\mathbb{R}^n)$ to belong to $\mathcal{H}^1$, necessarily the moment condition

$$\int_{\mathbb{R}^n} f(x)\,dx = 0$$

must hold.

Proof: For any $x \in \mathbb{R}^n$, any $t > 0$ write

$$(f * \varphi_t)(x) = \int f(y)\,\varphi_t(x-y)\,dy$$

$$= t^{-n} \int_{\mathbb{R}^n} f(y)\left(\varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x}{t}\right)\right)dy + t^{-n} \int_{\mathbb{R}^n} f(y)\,\varphi\left(\frac{x}{t}\right)dy$$

$$= \mathrm{I} + \mathrm{II}$$

with

$$\mathrm{I} \leq t^{-n} \int_{B_R(0)} |f(y)| \sup|\nabla\varphi| \cdot \left|\frac{y}{t}\right| dy$$

$$+ 2 \sup|\varphi|\, t^{-n} \int_{\mathbb{R}^n \setminus B_R(0)} |f(y)|\,dy$$

$$\leq C R \|f\|_{L^1} t^{-(n+1)} + o(1)\, t^{-n},$$

where $o(1) \to 0$ as $R \to \infty$, and with

$$\mathrm{II} = t^{-n}\varphi\left(\frac{x}{t}\right)\int_{\mathbb{R}^n} f(y)\,dy .$$

We may assume (possibly after a translation) that $\varphi(0) \geq c_0 > 0$ in $B_\rho(0)$ for some $\rho > 0$, $c_0 > 0$. Let $c_1 = \left|\int_{\mathbb{R}^n} f(y)\,dy\right|$. Then if $c_1 \neq 0$, for $|x| \geq 1$ and $t = |x|/\rho$ we have

$$|\mathrm{II}| \geq t^{-n} c_0 c_1 = \frac{c_0 c_1 \rho^n}{|x|^n}$$

whereas

$$|\mathrm{I}| \cdot |x|^n \leq CR\|f\|_{L^1}\rho^{n+1}/|x| + o(1)$$

$$\leq c_0 c_1 \rho^n / 2$$

for sufficiently large $R \geq 1$ and sufficiently large $|x| \geq R$. Thus

$$\int_{\mathbb{R}^n} M_\varphi f(x)\,dx \geq \frac{c_0 c_1 \rho^n}{2} \int_{\mathbb{R}^n \setminus B_R(0)} \frac{dx}{|x|^n} = \infty .$$

Example 3.15. i) Let $n=2$, $u \in H^1(B)$, $v \in H^1_0(B)$, where $B = B_1(0;\mathbb{R}^2)$. Then we have

$$\partial_1 u\, \partial_2 v - \partial_2 u\, \partial_1 v = \det(du, dv) \in \mathcal{H}^1,$$

where we extend $v \equiv 0$, $\det(du, dv) = 0$ on $\mathbb{R}^2 \setminus B$, and

$$\|\det(du, dv)\|_{\mathcal{H}^1} \leq C \|du\|_{L^2} \|dv\|_{L^2}.$$

Proof: Let $(\varphi_t)_{t>0}$ as above. For any $t>0$, any $x \in \mathbb{R}^2$, choosing $c = \fint_{B_t(x)} u\, dy$ we can bound

$$\Big| \int_{\mathbb{R}^2} \varphi_t(x-y) \det(du, dv)(y)\, dy \Big|$$

$$= \Big| \int_{\mathbb{R}^2} \varphi_t \Big(\partial_1 \big((u-c)\partial_2 v\big) - \partial_2 \big((u-c)\partial_1 v\big)\Big) dy \Big|$$

$$= \Big| \int_{\mathbb{R}^2} \Big(\partial_2 \varphi_t\, \partial_1 v\, (u-c) - \partial_1 \varphi_t \big(\partial_2 v\, (u-c)\big)\Big) dy \Big|$$

$$\leq C \frac{\sup|\nabla\varphi|}{t} \fint_{B_t(x)} |u-c|\,|\nabla v|\, dy$$

$$\leq C \frac{\sup|\nabla\varphi|}{t} \Big(\fint_{B_t(x)} |u-c|^p dy\Big)^{1/p} \Big(\fint_{B_t(x)} |\nabla v|^q dy\Big)^{1/q}$$

by Hölder's inequality, with suitable

exponents $1<q<2<p<q^* = \frac{2q}{2-q}$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then by Sobolev's embedding we have $W^{1,q} \hookrightarrow L^p$, and the Poincaré-Sobolev inequality allows to bound

$$\left(\fint_{B_t(x)} |u-c|^p dx\right)^{1/p} \leq Ct\left(\fint_{B_t(x)} |\nabla u|^q dx\right)^{1/q}$$

so that we can estimate

$$\begin{aligned} &|(\varphi_t * \det(du, dv))(x)| \\ &\leq C \sup|\nabla\varphi| \left(\fint_{B_t(x)} |\nabla u|^q dy\right)^{1/q}\left(\fint_{B_t(x)} |\nabla v|^q dy\right)^{1/q} \\ &\leq C \sup|\nabla\varphi| \left(M(|\nabla u|^q)(x)\right)^{1/q}\left(M(|\nabla v|^q)(x)\right)^{1/q}, \end{aligned} \tag{3.11}$$

where

$$Mf(x) = \sup_{r>0}\left(\fint_{B_r(x)} |f(y)|dy\right)$$

is the Hardy-Littlewood maximal function.

Since $|\nabla u|^q, |\nabla v|^q \in L^{2/q}(\mathbb{R}^2)$ with $2/q > 1$, by the Hardy-Littlewood maximal theorem (see for instance Stein: "Harmonic analysis", Thm. 1, p. 13) $M(|\nabla u|^q)$, $M(|\nabla v|^q) \in L^{2/q}(\mathbb{R}^2)$ with

$$\| M(|\nabla u|^q)\|_{L^{2/q}}^{1/q} = \|(M(|\nabla u|^q))^{1/q}\|_{L^2} \leq C \|\nabla u\|_{L^2},$$

etc.

Observing that from (3.11) we have

$$|M_\varphi \det(du, dv)(x)|$$
$$\leq C \left(M(|\nabla u|^q)(x)\right)^{1/q} \left(M(|\nabla v|^q)(x)\right)^{1/q}$$

we then obtain

$$\int_{\mathbb{R}^2} |M_\varphi \det(du, dv)(x)|\, dx$$
$$\leq C \; \|(M(|\nabla u|^q))^{1/q}\|_{L^2} \; \|(M(|\nabla v|^q))^{1/q}\|_{L^2}$$
$$\leq C \; \|\nabla u\|_{L^2} \; \|\nabla v\|_{L^2}.$$

Moreover, clearly

$$\|\det(du, dv)\|_{L^1} \leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2},$$

proving the claim. $\square$

ii) Let $n=2$, $P \in H^1(B; SO(n))$ with $P = id$ on ∂B, $h \in L^2(B; \mathbb{R}^2 \otimes so(n))$ harmonic with $\operatorname{div} h = 0$. Then

$$\operatorname{div}(P \cdot h) = \operatorname{div}((P - id) \cdot h) \in \mathcal{H}^1$$

with

$$\| \operatorname{div}((P-id) \cdot h) \|_{\mathcal{H}^1} \leq C \| dP \|_{L^2} \| h \|_{L^2} ,$$

where we extend $P \equiv id$ on $\mathbb{R}^2 \setminus B$.

<u>Proof</u>: Similar to i), with $(\varphi_t)_{t>0}$ as above for any $x \in \mathbb{R}^2$, any $t > 0$ with $c = \fint_{B_t(x)} P \, dy$ if $B_t(x) \subset B$ we estimate

$$\left| \int_{\mathbb{R}^2} \varphi_t(x-y) \operatorname{div}((P-id) h)(y) \, dy \right|$$

$$= \left| \int_{\mathbb{R}^2} \varphi_t(x-y) \operatorname{div}((P-c) h)(y) \, dy \right|$$

$$= \left| \int_{\mathbb{R}^2} \nabla \varphi_t(x-y) . ((P-c) h)(y) \, dy \right|$$

$$\leq C \frac{\sup |\nabla \varphi|}{t} \fint_{B_t(x)} |P-c| |h| \, dy$$

$$\leq C \frac{\sup |\nabla \varphi|}{t} \left(\fint_{B_t(x)} |P-c|^p dy \right)^{\frac{1}{p}} \left(\fint_{B_t(x)} |h|^q dx \right)^{\frac{1}{q}}$$

with conjugate exponents $1 < q < 2 < p < q^* = \frac{2q}{2-q}$ as in the proof of i) to conclude the bound

$$|\varphi_t * \operatorname{div}((P - id) h)|$$

$$\leq C \sup |\nabla \varphi| \left(M(|\nabla P|^q)(x) \right)^{\frac{1}{q}} \left(M(|h|^q)(x) \right)^{\frac{1}{q}}$$

as before.

On the other hand, if $1 - |x| < t$ we have $B_t(\hat{x}) \setminus B \subset B_{2t}(x)$, where $\hat{x} = x/|x| \in \partial B$, and we can estimate

$$\left| \int_{\mathbb{R}^2} \varphi_t(x-y) \operatorname{div}((P - id) h)(y)\, dy \right|$$

$$= \left| \int_{\mathbb{R}^2} \nabla \varphi_t(x-y) ((P - id) h)(y)\, dy \right|$$

$$\leq C \frac{\sup |\nabla \varphi|}{t} \left(\fint_{B_{2t}(x) \cap B} |P - id|^p dy \right)^{\frac{1}{p}} \left(\fint_{B_{2t}(x) \cap B} |h|^q dy \right)^{\frac{1}{q}}$$

$$\leq C \sup |\nabla \varphi| \left(\fint_{B_{2t}(x) \cap B} |\nabla P|^q dy \right)^{\frac{1}{q}} \left(\fint_{B_{2t}(x) \cap B} |h|^q dy \right)^{\frac{1}{q}}$$

$$\leq C \sup |\nabla \varphi| \left(M(|\nabla P|^q)(x) \right)^{\frac{1}{q}} \left(M(|h|^q)(x) \right)^{\frac{1}{q}}$$

with the help of Poincaré's inequality (like Satz 8.6.9, FA II)

$$\left(\fint_{B_{2t}(x) \cap B} |u|^p dy \right)^{\frac{1}{p}} \leq C t \left(\fint_{B_{2t}(x) \cap B} |\nabla u|^q dy \right)^{\frac{1}{q}}$$

for $u \in W_0^{1,q}(B)$. The claim follows as in i). $\square$

Example 3.15.i) is a particular case of a theorem by Coifman - Lions - Meyer - Semmes (1993), prompted by an observation of Stefan Müller (1990).

Def. 3.16. A function $a \in L^1(\mathbb{R}^n)$ is an $\mathcal{H}^1$-atom (associated to a ball $B \subset \mathbb{R}^n$) if there hold

i) $\operatorname{supp}(a) \subset B$;

ii) $|a(x)| \leq \frac{1}{|B|}$ for a.e. x, where $|B| = \mathcal{L}^n(B)$;

iii) $\int_{\mathbb{R}^n} a(x)\,dx = 0$.

Remark 3.17. Any $\mathcal{H}^1$-atom a (associated to $B = B_r(x_0)$) is an element of $\mathcal{H}^1$.

Proof: With φ as above, for any $t > 0$, any $x \in \mathbb{R}^n$ with $|x| \geq 1 + r$ we can estimate

$$|(a * \varphi_t)(x)| = t^{-n} \int_B a(y) \left(\varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x}{t}\right) \right) dy$$

$$\leq \sup_{|y| \leq r} |(\nabla \varphi)(\tfrac{x-y}{t})|\, t^{-(n+1)} \leq C|x|^{-(n+1)}$$

Since $\nabla\varphi\left(\frac{x-y}{t}\right)=0$ for $\left|\frac{x-y}{t}\right|\geq 1$ allows to assume $t\geq |x-y|\geq |x|-r$.

Thus, $M_\varphi a\in L^1(\mathbb{R}^n)$. Since $a\in L^1$ with $\|a\|_{L^1}\leq 1$, therefore $a\in\mathcal{H}^1$. $\square$

The usefulness of the concept of $\mathcal{H}^1$-atom is two-fold. On the one hand the following decomposition result holds.

<u>Theorem</u> 3.18. Let $f\in\mathcal{H}^1$. Then there sequences $(a_k)_{k\in\mathbb{N}}$, $(\lambda_k)_{k\in\mathbb{N}}$ of $\mathcal{H}^1$-atoms a_k and numbers λ_k with $\sum_{k\in\mathbb{N}}|\lambda_k|<\infty$ such that

$$f=\sum_k \lambda_k a_k$$

and

$$\sum_k |\lambda_k|\leq C\|f\|_{\mathcal{H}^1}.$$

See Stein: "Harmonic analysis", Thm. 2, p. 107.

On the other hand, the following characterization of BMO as the dual of $\mathcal{H}^1$ can be easily established with the help of Thm. 3.18.

<u>Theorem</u> 3.19 (Fefferman - Stein (1972)).
There holds $(\mathcal{H}^1)^* \cong BMO(\mathbb{R}^n)$; in particular, for any $f \in \mathcal{H}^1 \hookrightarrow L^1(\mathbb{R}^n)$ and any $g \in L^\infty(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n)$ we can estimate

$$(3.12) \quad \int_{\mathbb{R}^n} f g \, dx \le C \|f\|_{\mathcal{H}^1} \|g\|_{BMO} .$$

<u>Proof</u> <u>of</u> (3.12): Let $f \in \mathcal{H}^1$, and let $f = \sum_k \lambda_k a_k$ with atoms a_k, $\lambda_k \in \mathbb{R}$, $k \in \mathbb{N}$, satisfying $\|a_k\|_{L^1} \le 1$, $\sum_k |\lambda_k| < \infty$.

Since the sum $\sum_k \lambda_k a_k$ thus converges in L^1, for any given $g \in L^\infty$ there holds

$$\int_{\mathbb{R}^n} f g \, dx = \sum_k \lambda_k \int_{\mathbb{R}^n} a_k g \, dx .$$

Moreover, since $\int_{\mathbb{R}^n} a_k(x)\,dx = 0$, for each $k \in \mathbb{N}$ there holds

$$\left|\int_{\mathbb{R}^n} a_k g \,dx\right| = \left|\int_{\mathbb{R}^n} a_k (g - \bar{g}_{B_k})\,dx\right|$$

$$\leq \fint_{B_k} |g - \bar{g}_{B_k}|\,dx \leq [g]_{BMO},$$

where B_k is the ball associated with a_k.

Hence

$$\left|\int_{\mathbb{R}^n} f g \,dx\right| \leq \sum_k |\lambda_k| \, [g]_{BMO}$$

$$\leq C \|f\|_{\mathcal{H}^1} \|g\|_{BMO},$$

as claimed. $\square$

Thm. 3.18, in particular, the estimate (3.12), now allows to complete the proofs of Lemmas 3.10 and 3.12.

Proof of Lemma 3.11: Let $A = \nabla^{\perp}\xi \in L^{2,\alpha}(B, \mathbb{R}^2 \otimes so(n))$ with $\xi \in H^1_0(B, so(n))$ satisfying $d\xi \in L^{2,\alpha}$ and

$$\|d\xi\|_{L^2} \leq C\|A\|_{L^2} \leq C\varepsilon.$$

Claim 1: For sufficiently small $\varepsilon > 0$ the bilinear form

$$a(\varphi, \psi) = \int_B \left(\nabla\varphi \cdot \nabla\psi + [A; \nabla\varphi]\psi\right) dx$$

satisfies

$$|a(\varphi, \psi)| \leq 2\|\nabla\varphi\|_{L^2}\|\nabla\psi\|_{L^2}$$

and

$$a(\varphi, \varphi) \geq \frac{1}{2}\|\nabla\varphi\|^2_{L^2}$$

for any $\varphi, \psi \in H^1_0(B; so(n))$.

Proof: It suffices to bound

$$\left|\int_B [A; \nabla\varphi]\psi \, dx\right| = \left|\int_B \nabla^{\perp}\xi \cdot \nabla\varphi \, \psi \, dx\right|$$

$$\leq C\|\nabla^{\perp}\xi \cdot \nabla\varphi\|_{\mathcal{H}^1}\|\psi\|_{BMO}$$

$$\leq C\|d\xi\|_{L^2}\|\nabla\varphi\|_{L^2}\|\psi\|_{BMO}$$

$$\leq C\varepsilon^{1/2}\|\nabla\varphi\|_{L^2}\|\psi\|_{BMO}$$

by Example 3.15 and (3.12). Finally, extending $\psi \equiv 0$ outside $\mathcal{B}$, for any cube $Q \subset Q_0 =]-1,1[^2$ we can bound

$$\fint_Q |\psi - \bar{\psi}_{r,x_0}|^2 dx \leq C \fint_{B_r(x_0)} |\psi - \bar{\psi}_{r,x_0}|^2 dx$$

$$\leq C \int_{B_r(x_0)} |\nabla\psi|^2 dx \leq C \|\nabla\psi\|_{L^2},$$

where $B_r(x_0)$ is the smallest ball containing Q and where we let $\bar{\psi}_{r,x_0} = \fint_{B_r(x_0)} \psi \, dx$ and use Poincaré's inequality. □

In view of Claim 1 the Lax-Milgram theorem (see Functional Analysis I, Satz 4.3.3) is applicable to the bilinear form $a(\cdot,\cdot)$. Given $f \in L^{2,\alpha}$, the map

$$H^1_0 \ni \psi \longmapsto \int_{\mathcal{B}} \operatorname{div} f \, \psi \, dx = -\int_{\mathcal{B}} f \cdot \nabla\psi \, dx$$

defines a bounded linear functional $l \in H^{-1}(\mathcal{B})$.

By the Lax-Milgram theorem there exists a unique $\varphi \in H^1_0(B; so(n))$ such that

$$a(\varphi, \psi) = \ell_f(\psi) = -\int_B f \cdot \nabla \psi \, dx$$

for all $\psi \in H^1_0(B; so(n))$, and φ solves

$$(3.13) \quad \begin{aligned} L\varphi = \Delta\varphi + [A, \nabla\varphi] &= -\operatorname{div} f \text{ in } B, \\ \varphi &= 0 \text{ on } \partial B. \end{aligned}$$

with

$$\|\varphi\|^2_{H^1_0} \leq 2\, a(\varphi, \varphi) \leq 2\int_B |f|\, |\nabla\varphi|\, dx$$

$$\leq 2\|f\|_{L^2} \|\nabla\varphi\|_{L^2} = 2\|f\|_{L^2}\|\varphi\|_{H^1_0},$$

showing that $L : H^1_0(B; so(n)) \to H^{-1}(B; so(n))$ is surjective with bounded inverse. Here,

$$H^{-1}(B; so(n)) = \{ \operatorname{div} f;\ f \in L^2(B; \mathbb{R}^2 \otimes so(n)) \}.$$

<u>Claim 2</u>: For $f \in L^{2,\alpha}$ the unique solution $\varphi \in H^1_0(B; so(n))$ of (3.13) satisfies $d\varphi \in L^{2,\alpha}$ and

$$\|d\varphi\|_{L^{2,\alpha}} \leq C\|f\|_{L^{2,\alpha}}.$$

Proof: For any $B_R(x_0) \subset B_2(0)$ split $\varphi = v + w$ on $B_R(x_0) \cap B =: B_R^+(x_0)$, where $\Delta v = 0$ and $w \in H_0^1(B \cap B_R(x_0); so(n))$. By Campanato's estimates for v, there holds

$$\int_{B_r^+(x_0)} |\nabla\varphi|^2 dx \leq 2\int_{B_r^+(x_0)} |\nabla v|^2 dx + 2\int_{B_r^+(x_0)} |\nabla w|^2 dx$$

$$\leq C\left(\frac{r}{R}\right)^2 \int_{B_R^+(x_0)} |\nabla v|^2 dx + 2\int_{B_r^+(x_0)} |\nabla w|^2 dx$$

$$\leq C\left(\frac{r}{R}\right)^2 \int_{B_R^+(x_0)} |\nabla\varphi|^2 dx + C\int_{B_R^+(x_0)} |\nabla w|^2 dx .$$

Testing the equation

$$-\Delta w = [A; \nabla\varphi] + \operatorname{div} f \quad \text{in } B_R^+(x_0),$$
$$w = 0 \quad \text{on } \partial B_R^+(x_0)$$

with w, as in the proof of Claim 1 we obtain

$$\int_{B_R^+(x_0)} |\nabla w|^2 dx \leq \int_{B_R^+(x_0)} [A; \nabla\varphi]\, w\, dx - \int_{B_R^+(x_0)} f \cdot \nabla w\, dx$$

$$\leq C\,\varepsilon^{1/2} \|\nabla\varphi\|_{L^2(B_R^+(x_0))} \|\nabla w\|_{L^2(B_R^+(x_0))}$$

$$+ \|f\|_{L^{2,\alpha}} \cdot R^{\alpha/2} \|\nabla w\|_{L^2(B_R^+(x_0))};$$

so $$\|\nabla w\|^2_{L^2(B_R^+(x_0))} \leq C\varepsilon \|\nabla\varphi\|^2_{L^2(B_R^+(x_0))} + C\|f\|^2_{L^{2,\alpha}} R^\alpha .$$

For
$$\Phi(r) = \int_{B_r^+(x_0)} |\nabla\varphi|^2 dx \ , \quad 0 < r < R_0 (< 2) ,$$

thus there holds

$$\Phi(r) \leq C\left(\left(\frac{r}{R}\right)^2 + \varepsilon\right) \Phi(R) + C\|f\|^2_{L^{2,\alpha}} R^\alpha .$$

Campanato's "useful lemma" (see FA II, Lemma 10.3.2) then yields that for sufficiently small $\varepsilon > 0$ we have

$$\begin{aligned} \Phi(r) &\leq C r^\alpha \left(\Phi(1) + \|f\|^2_{L^{2,\alpha}}\right) \\ &\leq C r^\alpha \|f\|^2_{L^{2,\alpha}} , \quad 0 < r < R_0 , \end{aligned}$$

uniformly in $x_0 \in B$. That is, $d\varphi \in L^{2,\alpha}(B)$ and

$$\|d\varphi\|_{L^{2,\alpha}} \leq C \|f\|_{L^{2,\alpha}} ,$$

as claimed. □

Proof of Lemma 3.12: Suppose for $\Omega \in \mathcal{V}_\varepsilon^\alpha$ there exist $P \in H^1(B; SO(m))$, $\xi \in H^1_0(B; so(m))$ with $P = id$ on ∂B and such that dP, $d\xi$ satisfy

$$\|dP\|^2_{L^2} + \|d\xi\|^2_{L^2} < \varepsilon_0 , \tag{3.14}$$

so that (3.6) holds, that is, $\operatorname{div} A = 0$, where

$$A = P^{-1}\nabla P + P^{-1}\Omega P = \nabla^\perp \xi + h \quad \text{in } B.$$

Multiplying by P and taking the divergence, in view of

$$\operatorname{div} \nabla^\perp \xi = \operatorname{div} h = 0$$

we obtain the equation

$$\begin{aligned} \Delta P + \operatorname{div}(\Omega P) &= \operatorname{div}\left(P(\nabla^\perp \xi + h)\right) \\ &= \nabla P \cdot (\nabla^\perp \xi + h), \end{aligned} \tag{3.15}$$

exhibiting a determinant structure.

Moreover, letting $\nabla^\perp$ act on both sides of (3.6) and using

$$\nabla^\perp \nabla^\perp \xi = \nabla^\perp \begin{pmatrix} \partial_2 \xi \\ -\partial_1 \xi \end{pmatrix} = \partial_2^2 \xi + \partial_1^2 \xi = \Delta \xi ,$$

we obtain

$$
\begin{aligned}
\Delta \xi &= \nabla^{\perp}\left(P^{-1}\nabla P + P^{-1}\Omega P\right)\\
(3.16) \qquad &= \nabla^{\perp}P^{-1}\cdot\nabla P + \nabla^{\perp}\left(P^{-1}\Omega P\right),
\end{aligned}
$$

again with a determinant structure.

Testing (3.16) with $\xi \in H^1_0(B; so(m))$ yields

$$
\begin{aligned}
\|\nabla\xi\|^2_{L^2} &= \int_B \nabla^{\perp}\xi \left(P^{-1}\nabla P + P^{-1}\Omega P\right) dx\\
&\leq C\|\det(d\xi, dP)\|_{\mathcal{H}^1}\|P^{-1}\|_{BMO}\\
&\quad + C\|\nabla\xi\|_{L^2}\|\Omega\|_{L^2}.
\end{aligned}
$$

Thus, by Ex. 3.15 and bounding

$$
\|P^{-1}\|_{BMO} \leq C\|\nabla P^{-1}\|_{L^2} = C\|\nabla P\|_{L^2}
$$

we find the estimate

$$
\begin{aligned}
\|\nabla\xi\|^2_{L^2} &\leq C\|\nabla\xi\|_{L^2}\left(\|\nabla P\|^2_{L^2} + \|\Omega\|_{L^2}\right)\\
(3.17) \qquad &\leq \frac{1}{4}\|\nabla\xi\|^2_{L^2} + C\left(\|\nabla P\|^4_{L^2} + \|\Omega\|^2_{L^2}\right).
\end{aligned}
$$

Similarly, testing (3.15) with $P - id \in H_0^1 \cap L^\infty(B)$, with (3.12) we obtain

$$\|\nabla P\|_{L^2}^2 = -\int_B \left(\nabla P\, \Omega P + (P - id)\nabla P \cdot (\nabla^\perp \xi + h)\right) dx$$

$$\leq \frac{1}{2}\|\nabla P\|_{L^2}^2 + \frac{1}{2}\|\Omega\|_{L^2}^2$$

$$+ C\|\nabla P \cdot (\nabla^\perp \xi + h)\|_{\mathcal{H}^1}\|P - id\|_{BMO}.$$

Thus, estimating

$$\|P - id\|_{BMO} \leq C\|\nabla P\|_{L^2}$$

via Poincaré's inequality and using the result in Ex. 3.15 we find

$$\|\nabla P\|_{L^2}^2 \leq \|\Omega\|_{L^2}^2 + C\|\nabla \xi\|_{L^2}\|\nabla P\|_{L^2}^2.$$

Adding (3.17) and recalling our assumption (3.14) for sufficiently small $\varepsilon_0 > 0$ we then have

$$\|\nabla P\|_{L^2}^2 + \|\nabla \xi\|_{L^2}^2 \leq \frac{1}{2}\left(\|\nabla P\|_{L^2}^2 + \|\nabla \xi\|_{L^2}^2\right)$$

$$+ C\|\Omega\|_{L^2}^2,$$

and (3.7) follows.

To see (3.8), fix $B_R(x_0) \subset B_2$ and split $\xi = \eta + \zeta$, $P = Q + R$ on $B_R(x_0) \cap B^+ =: B_R^+(x_0)$, where

$$\Delta \eta = 0, \quad \Delta Q = 0 \quad \text{in } B_R^+(x_0)$$

and $\zeta, R \in H_0^1(B_R^+(x_0))$.

With Campanato's estimates for η and Q, we then obtain for any $0 < r < R$ the bound

$$\int_{B_r^+(x_0)} (|\nabla \xi|^2 + |\nabla P|^2)\, dx$$

$$\leq C\left(\frac{r}{R}\right)^2 \int_{B_R^+(x_0)} \left(|\nabla \xi|^2 + |\nabla P|^2\right) dx +$$

$$(3.18) \qquad + C \int_{B_R^+(x_0)} \left(|\nabla \zeta|^2 + |\nabla R|^2\right) dx,$$

similar to the proof of Claim 2 in the proof of Lemma 3.11 above.

Moreover, similar to the proof of (3.7) we can bound

$$\int_{B_R^+(x_0)} |\nabla \xi|^2 dx = \int_{B_R^+(x_0)} \nabla^\perp \xi \left(P^{-1}\nabla P + P^{-1}\Omega P\right) dx$$

$$\leq C \, \| \det(d\xi, dP) \|_{\mathcal{H}^1} \, \| P^{-1} \|_{BMO}$$

$$+ C \Big(\int_{B_R^+(x_0)} |\nabla^\perp \xi|^2 dx \Big)^{\frac{1}{2}} \Big(\int_{B_R^+(x_0)} |\Omega|^2 dx \Big)^{\frac{1}{2}}$$

$$\leq C \, \| \nabla \xi \|_{L^2(B_R^+(x_0))} \, \| \nabla P \|^2_{L^2(B_R^+(x_0))}$$

$$+ \frac{1}{4} \| \nabla \xi \|^2_{L^2(B_R^+(x_0))} + C \| \Omega \|^2_{L^2(B_R^+(x_0))}$$

and, integrating by parts, also

$$\int_{B_R^+(x_0)} |\nabla R|^2 dx = - \int_{B_R^+(x_0)} \left(\nabla R \, \Omega P - R \nabla P (\nabla^\perp \xi + h) \right) dx$$

$$\leq \frac{1}{4} \| \nabla R \|^2_{L^2(B_R^+(x_0))} + C \| \Omega \|^2_{L^2(B_R^+(x_0))}$$

$$+ \int_{B_R^+(x_0)} \nabla R \left(\nabla^\perp \xi + h \right) P \, dx$$

Again we note that since $P \in H^1 \cap L^\infty(B)$ and since $R \in H^1_0(B^+_R(x_0))$ we may use (3.12) to estimate

$$\left| \int_{B^+_R(x_0)} \nabla R \,(\nabla^\perp \xi + h)\, P \,dx \right|$$

$$\leq \left(\| \det(dR, d\xi) \|_{\mathcal{H}^1} + \| \operatorname{div}(R \cdot h) \|_{\mathcal{H}^1} \right) [P]_{BMO}$$

$$\leq C \, \| \nabla R \|_{L^2(B^+_R(x_0))} \| |\nabla \xi| + |h| \|_{L^2(B^+_R(x_0))} \| \nabla P \|_{L^2} .$$

With the help of Young's inequality, and estimating $|h| \leq |\nabla P| + |\nabla \xi| + |\Omega|$, thus we obtain

$$\int_{B^+_R(x_0)} \left(|\nabla \xi|^2 + |\nabla R|^2 \right) dx$$

$$\leq C \| \Omega \|^2_{L^2(B^+_R(x_0))} +$$

$$+ C \, \| \nabla P \|^2_{L^2} \int_{B^+_R(x_0)} \left(|\nabla \xi|^2 + |\nabla P|^2 \right) dx .$$

Letting

$$\Phi(r) = \int_{B^+_r(x_0)} \left(|\nabla \xi|^2 + |\nabla P|^2 \right) dx , \quad 0 < r < R ,$$

from (3.18) and (3.14) then we deduce the estimate

$$\phi(r) \leq C\left(\left(\frac{r}{R}\right)^2+\varepsilon_0\right)\phi(R)+C\|\Omega\|^2_{L^{2,\alpha}} R^\alpha$$

for $0<r<R$, and Campanato's useful lemma (FA II, Lemma 10.3.2) again may be applied to give the bound

$$\phi(r) \leq C r^\alpha \left(\phi(1)+\|\Omega\|^2_{L^{2,\alpha}}\right)$$

$$\leq C r^\alpha \|\Omega\|^2_{L^{2,\alpha}}, \quad 0<r<R,$$

where we observe that (3.7) allows to bound

$$\phi(1) \leq C\|\Omega\|^2_{L^2} \leq C\|\Omega\|^2_{L^{2,\alpha}}.$$

This concludes the proof of (3.8). $\square$

3.4. Elements of Hodge theory

Let $\Lambda^p \mathbb{R}^m$ be the p-fold exterior product of $\mathbb{R}^n$, $0 \leq p \leq m$. With the canonical basis $(e_i)_{1\leq i\leq n}$ for $\mathbb{R}^m$, the collection

$$e_{i_1} \wedge \dots \wedge e_{i_p}, \quad 1 \leq i_1 < i_2 < \dots < i_p \leq m,$$

is a basis for $\Lambda^p \mathbb{R}^m$.

Similarly, we can define

$$\Lambda^p T_x^* \mathbb{R}^m = \mathrm{span}\{dx^{i_1} \wedge \dots \wedge dx^{i_p};\ 1 \leq i_1 < \dots < i_p \leq m\}$$

for any $x \in \mathbb{R}^m$.

Finally, for any smooth Riemannian m-dimensional manifold M, we define $\Lambda^p T_x^* M$ as above in terms of a local basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}$ for $T_x M$.

Then we also let

$$\Omega^p(M) = \Gamma(\Lambda^p T^* M)$$

be the set of smooth sections in the bundle $\Lambda^p T^* M \to M$ with fibres $\Lambda^p T_x^* M$, $x \in M$.

An element of $\Omega^p(M)$ is called a *p-form*.

A <u>scalar product</u> can be defined, as follows.

For $v = v_1 \wedge \ldots \wedge v_p$, $w = w_1 \wedge \ldots \wedge w_p \in \Lambda^p \mathbb{R}^m$ let

$$\langle v, w \rangle := \det (v_i \cdot w_j)_{1 \leq i, j \leq p},$$

where $v_i \cdot w_j = (v_i, w_j)_{\mathbb{R}^m}$. Then the above basis $(e_{i_1} \wedge \ldots \wedge e_{i_p})_{1 \leq i_1 < \ldots < i_p \leq m}$ for $\Lambda^p \mathbb{R}^m$ is orthonormal.

Similarly, with the metric on $T_x^* M$ given by $(g^{ij}(x)) = (g_{ij}(x))^{-1}$ we can define the scalar product on $\Lambda^p T_x^* M$.

An <u>orientation</u> on $\mathbb{R}^m$ can be defined by distinguishing the standard basis $e_1, \ldots, e_m$ as positive. Expressing any other basis in terms of $e_1, \ldots, e_m$, a basis $f_1, \ldots, f_m$ is positively oriented if $\det(f_1, \ldots, f_m) > 0$.

The Hodge $*$-operator is the linear map

$$*: \Lambda^p \mathbb{R}^m \longrightarrow \Lambda^{m-p} \mathbb{R}^m$$

with

$$*(e_{i_1} \wedge \ldots \wedge e_{i_p}) = e_{j_1} \wedge \ldots \wedge e_{j_{m-p}}$$

such that $e_{i_1}, \ldots, e_{i_p}, e_{j_1}, \ldots, e_{j_{m-p}}$ is a positively oriented basis of $\mathbb{R}^m$. In particular, we have

$$*(1) = e_1 \wedge \ldots \wedge e_m,$$

$$* e_1 \wedge \ldots \wedge e_m = 1,$$

and $** = (-1)^{p(m-p)} \cdot id : \Lambda^p \mathbb{R}^m \to \Lambda^p(\mathbb{R}^m)$.

Moreover, with the above scalar product we have

(3.19) $\langle v, w \rangle = *(w \wedge * v) = *(v \wedge * w), \quad v, w \in \Lambda^p \mathbb{R}^m.$

Proof: It suffices to check (3.19) for the basis vectors $v, w \in \{ e_{i_1} \wedge \ldots \wedge e_{i_p} ; 1 \le i_1 < \ldots < i_p \le m \}$. Then, if $v \neq w$ we have

$$\langle v, w \rangle = 0, \quad v \wedge * w = 0,$$

while for $v = w$ we have

$$v \wedge * w = e_1 \wedge \ldots \wedge e_m$$

and

$$*(v \wedge * w) = 1 = \langle v, w \rangle .$$

For an arbitrary positively oriented basis $v_1, \ldots, v_m \in \mathbb{R}^m$ there holds

$$v_1 \wedge \ldots \wedge v_m = \left(\det\left((v_i \cdot v_j)_{1 \le i,j \le m}\right)\right)^{1/2} e_1 \wedge \ldots \wedge e_m,$$

and thus

$$(3.20) \qquad *(v_1 \wedge \ldots \wedge v_m) = \sqrt{\det\left((v_i \cdot v_j)_{1 \le i,j \le m}\right)}.$$

Indeed, since $\dim(\Lambda^m \mathbb{R}^m) = 1$ we have

$$v_1 \wedge \ldots \wedge v_m = c\, e_1 \wedge \ldots \wedge e_m$$

with $0 < c = \sqrt{\langle v_1 \wedge \ldots \wedge v_m, v_1 \wedge \ldots \wedge v_m \rangle} = \sqrt{\det\left((v_i \cdot v_j)_{1 \le i,j \le m}\right)}$.

In particular, on any m-manifold M with metric $g = (g_{ij})$ we have the <u>volume form</u>

$$*(1) = \frac{1}{\sqrt{\det(g^{ij})}}\, dx^1 \wedge \ldots \wedge dx^m$$

$$= \sqrt{\det(g_{ij})}\, dx^1 \wedge \ldots \wedge dx^m.$$

Indeed, with $v_i = dx^i \in T_x^* M$, $1 \le i \le m$, and with the metric on $T_x^* M$ given by $(g^{ij}) = (g_{ij})^{-1}$, the result follows from (3.20).

For closed M we can then define the L^2 scalar product on $\Omega^p(M)$ by letting

$$(\alpha,\beta)_{L^2} = \int_M \langle\alpha,\beta\rangle * (1) = \int_M * \langle\alpha,\beta\rangle$$

$$= \int_M \alpha\wedge *\beta$$

by (3.19) and since $** = 1$ on $\Lambda^0 T^*M$.

The exterior differential $d : \Omega^p M \to \Omega^{p+1} M$ then has a formal adjoint $\delta : \Omega^{p+1}(M) \to \Omega^p(M)$ such that

$$(d\alpha,\beta)_{L^2} = (\alpha,\delta\beta)_{L^2} \text{ for all } \alpha\in\Omega^p(M),\ \beta\in\Omega^{p+1}(M).$$

Lemma 3.20: For $\delta : \Omega^{p+1}(M) \to \Omega^p(M)$ there holds

$$\delta = (-1)^{mp+1} * d *.$$

Proof: For $\alpha\in\Omega^p(M)$, $\beta\in\Omega^{p+1}(M)$ we have

$$d(\alpha\wedge *\beta) = d\alpha\wedge *\beta + (-1)^p \alpha\wedge d(*\beta)$$

$$= d\alpha\wedge *\beta + (-1)^{p+(m-p)p}\alpha\wedge **\overbrace{d(*\beta)}^{\in\Omega^{m-p}(M)}$$

$$= \pm * \left(\langle d\alpha,\beta\rangle + (-1)^{p(m-p+1)}\langle\alpha, *d*\beta\rangle\right).$$

Upon integration, by Stokes' theorem the left hand side gives zero. Since $p(p-1)$ is even, the claim follows. $\square$

Definition 3.21. i) The Hodge-Laplace-Beltrami operator on $\Omega^p(M)$ is

$$\Delta = \Delta^H = \delta d + d\delta : \Omega^p(M) \longrightarrow \Omega^p(M).$$

ii) A form $\alpha \in \Omega^p(M)$ is harmonic if

$$\Delta\alpha = 0.$$

Remark 3.22. The Laplace operator is symmetric; that is

$$\forall \alpha, \beta \in \Omega^p(M) : (\Delta\alpha, \beta)_{L^2} = (\alpha, \Delta\beta)_{L^2}.$$

This follows immediately from the definitions.

Lemma 3.23. For $\alpha \in \Omega^p(M)$ there holds

$$\Delta\alpha = 0 \iff d\alpha = 0 \text{ and } \delta\alpha = 0.$$

Proof. Suppose $\alpha \in \Omega^p(M)$ is harmonic with $\Delta\alpha = 0$. Then from

$$0 = (\Delta\alpha, \alpha)_{L^2} = (\delta d\alpha, \alpha)_{L^2} + (d\delta\alpha, \alpha)_{L^2}$$
$$= (d\alpha, d\alpha)_{L^2} + (\delta\alpha, \delta\alpha)_{L^2} = \|d\alpha\|^2_{L^2} + \|\delta\alpha\|^2_{L^2}$$

we obtain $d\alpha = 0$, $\delta\alpha = 0$.

The converse implication is immediate. $\square$

Corollary 3.24. Suppose M is closed, connected, $\alpha \in C^\infty(M) = \Omega^0(M)$ harmonic. Then $\alpha \equiv$ const.

Of special interest for us is the case when $M = B = B_1(0; \mathbb{R}^2)$, and when $A \in L^2(B; T^*M)$ $=: L^2\Omega^1(M)$.

Example 3.25. i) Let $m = 2$, $A = A_1 dx^1 + A_2 dx^2 \in L^2(\Omega^1(B))$. Then

$$*A = A_1 * dx^1 + A_2 * dx^2 = A_1 dx^2 - A_2 dx^1$$

and

$$d(*A) = \partial_1 A_1 \, dx^1 \wedge dx^2 - \partial_2 A_2 \, dx^2 \wedge dx^1$$
$$= (\partial_1 A_1 + \partial_2 A_2) \, dx^1 \wedge dx^2$$

so that

$$\delta A = - *(d * A) = - \partial_\alpha A_\alpha .$$

Thus, A is in Coulomb gauge if $\delta A = 0$.

ii) The Laplace operator in $m = 2$ dimensions thus acts, as follows.

For $f \in C^\infty(B) = \Omega^0(B)$ we have $\delta f = 0$; thus by i) there holds

$$\Delta^H f = \delta d f = \delta(\partial_1 f \, dx^1 + \partial_2 f \, dx^2) = -(\partial_1^2 + \partial_2^2) f = -\Delta_{\mathbb{R}^2} f .$$

Similarly, for $g = \gamma\, dx^1 \wedge dx^2 \in \Omega^2(B)$ we have $dg = 0$ and with $*g = \gamma$ we find

$$\Delta^H g = d\delta g = d * d\gamma = -* \delta d\gamma = -* \Delta_{\mathbb{R}^2}\gamma .$$

Finally, for $\varphi = \varphi_1 dx^1 + \varphi_2 dx^2$ we compute

$$\delta\varphi = -(\partial_1\varphi_1 + \partial_2\varphi_2), \quad d\varphi = (\partial_1\varphi_2 - \partial_2\varphi_1)\, dx^1 \wedge dx^2$$

and thus

$$\Delta^H\varphi = d\delta\varphi + \delta d\varphi = d\delta\varphi - * d(\partial_1\varphi_2 - \partial_2\varphi_1)$$

$$= -(\partial_1^2\varphi_1 + \partial_1\partial_2\varphi_2)\, dx^1 - (\partial_1\partial_2\varphi_1 + \partial_2^2\varphi_2)\, dx^2$$

$$- * \left[(\partial_1^2\varphi_2 - \partial_1\partial_2\varphi_1)\, dx^1 + (\partial_1\partial_2\varphi_2 - \partial_2^2\varphi_1)\, dx^2 \right]$$

$$= -(\partial_1^2 + \partial_2^2)\varphi_1\, dx^1 - (\partial_1^2 + \partial_2^2)\varphi_2\, dx^2$$

$$= -\Delta_{\mathbb{R}^2}\varphi_1\, dx^1 - \Delta_{\mathbb{R}^2}\varphi_2\, dx^2 .$$

A similar result holds for $m \geq 3$ and any $p \leq m$.

With the help of the Laplace operator, any $A \in L^2\Omega^1(B)$ now may be split into an L^2-orthogonal sum of closed, co-closed and harmonic forms.

Theorem 3.26. For any $A \in L^2(\Omega^1(B))$ there exist $f \in H_0^1(B)$, $g \in H^1(\Omega^2(B))$ and $h \in L^2(\Omega^1(B))$ such that

(3.21) $A = df + \delta g + h$ in B,

where h is harmonic with $dh = \delta h = 0$ and $*g|_{\partial B} = 0$. Moreover, there holds

$$\|A\|_{L^2}^2 = \|df\|_{L^2}^2 + \|\delta g\|_{L^2}^2 + \|h\|_{L^2}^2$$

and the Hodge decomposition (3.21) is L^2-orthogonal.

Proof: Let $f \in H_0^1(B)$, $g \in H^1(B; \Lambda^2 T^*\mathbb{R}^2)$ solve

$$-\Delta f = \delta d f = \delta A \text{ in } B, \quad f = 0 \text{ on } \partial B,$$
$$-\Delta g = d\delta g = dA \text{ in } B, \quad *g = 0 \text{ on } \partial B,$$

and set

$$A - df - \delta g =: h \in L^2(\Omega^1(B)).$$

Then

$$dh = dA - d\delta g = 0, \quad \delta h = \delta A - \delta d f = 0,$$

and h is harmonic.

Moreover, with $\gamma := *g \in H_0^1(\mathcal{B})$ there holds

$$(df, \delta g)_{L^2} = \int_{\mathcal{B}} * \langle df, \delta g \rangle$$

$$= \int_{\mathcal{B}} df \wedge * \delta g = -\int_{\mathcal{B}} df \wedge d\gamma$$

$$= -\int_{\mathcal{B}} d(f\, d\gamma) = 0$$

by Stokes' theorem, since $f \in H_0^1(\mathcal{B})$.

Similarly, we have

$$(df, h)_{L^2} = \int_{\mathcal{B}} * \langle df, h \rangle$$

$$= \int_{\mathcal{B}} df \wedge * h = \int_{\mathcal{B}} d(\underbrace{f * h}_{= *fh}) = 0,$$

since $d * h = * \delta h = 0$. Finally, we also note

$$(\delta g, h)_{L^2} = (h, \delta g)_{L^2} = \int_{\mathcal{B}} * \langle h, \delta g \rangle$$

$$= -\int_{\mathcal{B}} h \wedge d\gamma = \int_{\mathcal{B}} d(h\gamma) = 0$$

since $\gamma \in H_0^1(\mathcal{B})$ and $dh = 0$.

The claim follows. $\square$

The decomposition (3.21) also is bounded for $A \in L^q\Omega^1(B)$ for any $1 < q < \infty$. This is due to the Calderòn-Zygmund inequality.

Theorem 3.27. Let $\Omega \subset\subset \mathbb{R}^m$ be smoothly bounded domain, $A = (A^i)_{1 \le i \le m} \in L^q(\Omega; \mathbb{R}^m)$. Then there is a unique solution $u \in W_0^{1,q}(\Omega)$ of the Dirichlet problem

$$-\Delta u = -\operatorname{div} A = -\partial_i A^i \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

and

$$\|\nabla u\|_{L^q} \le C \|A\|_{L^q}.$$

Sketch of proof[1]: Let G be the Green's function for Ω, solving

$$-\Delta_y G(x,y) = \delta_{\{y=x\}} \quad \text{in } \Omega$$
$$G(x,\cdot) = 0 \quad \text{on } \partial\Omega$$

for every $x \in \Omega$. Thus, if $m = 2$,

$$G(x,y) = \frac{1}{2\pi} \log\left(\frac{1}{|x-y|}\right) + H(x,y),$$

---

[1] An L^2-based proof is given in the Appendix.

where $H : \Omega \times \Omega \to \mathbb{R}$ is the Robin's function with

$$-\Delta_y H(x,y) = 0 \quad \text{in } \Omega,$$

$$H(x,y) = \frac{1}{2\pi} \log |x-y|, \quad y \in \partial\Omega$$

for every $x \in \Omega$, and similarly for $m \geq 2$.

Then we have the representation

$$u(x) = \int_\Omega G(x,y)\left(-\partial_i A^i(y)\right) dy$$

$$= \int_\Omega \nabla_y G(x,\cdot) \cdot A \, dy \quad , \quad x \in \Omega,$$

and

$$\partial_i u(x) = \int_\Omega \frac{\partial^2}{\partial x^i \partial y^j} G(x,y) \cdot A^j(y) \, dy \, .$$

$$= c(m) \int_\Omega \frac{(x-y)^i (x-y)^j}{|x-y|^{m+2}} A^i(y) \, dy$$

$$+ R^i(x)$$

with

$$|R^i(x)| \leq C \int_B |\nabla^2 H(x,y)| |A(y)| \, dy \, .$$

The result then follows from harmonic analysis. $\square$

Corollary 3.28. Let $A \in L^q \Omega^1(\mathcal{B})$ for some $1 < q < \infty$, with $\mathcal{B} = B_1(0; \mathbb{R}^2)$. Then there exist $f \in W_0^{1,q}(\mathcal{B})$, $g = *\gamma$ with $\gamma \in W_0^{1,q}(\mathcal{B})$ and $h \in L^q \Omega^1(\mathcal{B})$ with $dh = \delta h = 0$ and such that (3.21) holds, and, in addition, with

$$\|df\|_{L^q} + \|\delta g\|_{L^q} + \|h\|_{L^q} \leq C \|A\|_{L^q}.$$

Proof: Letting $f \in W_0^{1,q}(\mathcal{B})$, $\gamma \in W_0^{1,q}(\mathcal{B})$ solve

$$-\Delta f = \delta A \text{ in } \mathcal{B}, \quad -\Delta \gamma = * dA \text{ in } \mathcal{B}$$

with $f = \gamma = 0$ on $\partial\mathcal{B}$, and setting $g = *\gamma$

$$h = A - df - \delta g$$

as in the proof of Thm. 3.26 we obtain that h is harmonic with $dh = 0$, $\delta h = 0$, and Thm. 3.27 gives the bounds

$$\|df\|_{L^q} \leq C \|A\|_{L^q}, \quad \|\delta g\|_{L^q} = \|d\gamma\|_{L^q} \leq C \|A\|_{L^q};$$

hence also

$$\|h\|_{L^q} \leq \|A\|_{L^q} + \|df\|_{L^q} + \|\delta g\|_{L^q} \leq C \|A\|_{L^q}.$$

$\square$

A particularly useful consequence of Thm. 3.27 is the following result.

Corollary 3.29. Let $1<p<2$. Then there holds

$$\forall u \in W_0^{1,p}(B): \ \|\nabla u\|_{L^p} \leq C \sup_{\substack{v \in W_0^{1,q}(B) \\ \|\nabla v\|_{L^q} \leq 1}} (\nabla u, \nabla v)_{L^2},$$

where $\frac{1}{p}+\frac{1}{q}=1$, $C=C(p)$.

Proof: From FA I, Lemma 4.4.1, we know that for any given $u \neq 0$ there is

$$A = \nabla u |\nabla u|^{p-2} / \|\nabla u\|_{L^p}^{p-1} \in L^q(B;\mathbb{R}^2) \cong L^q(B;T^*\mathbb{R}^2)$$

with

$$\|\nabla u\|_{L^p} = (\nabla u, A)_{L^2}, \quad \|A\|_{L^q} = 1.$$

Let

$$A = df + \delta g + h \in L^2(B;T^*\mathbb{R}^2)$$

be the Hodge decomposition of $A \in L^q(B) \hookrightarrow L^2(B)$ according to Theorem 3.26. Then

$$(\nabla u, A)_{L^2} = (du, df)_{L^2} + (du, \delta g)_{L^2} + (du, h)_{L^2}$$

$$= (du, df)_{L^2}.$$

But by Thm. 3.27, from

$$-\Delta f = \delta A = -\operatorname{div} A \text{ in } B$$
$$f = 0 \text{ on } \partial B$$

we obtain

$$\|\nabla f\|_{L^q} \leq C \|A\|_{L^q} \leq C$$

and hence

$$\|\nabla u\|_{L^p} = (\nabla u, A)_{L^2} = (\nabla u, \nabla f)_{L^2}$$

$$\leq C \sup_{\substack{v \in W_0^{1,q}(B) \\ \|\nabla v\|_{L^q} \leq 1}} (\nabla u, \nabla v)_{L^2},$$

as claimed. $\square$

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