4. Regularity of harmonic maps

4.1 Hélein's theorem

Finally, we now have all tools at our disposal to prove the following result.

<u>Theorem</u> 4.1 (Hélein (1991)). Let M be a closed surface, $N \subset \mathbb{R}^n$ a closed submanifold, and suppose that $u \in H^1(M; N)$ is weakly harmonic. Then u is smooth.

We present the proof of this result given by Rivière - Struwe (2008), building on Hélein's and Rivière's ideas sketched in Chapter 2; see in particular Rivière (2007).

For simplicity we again suppose that $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and identify the map u with a periodic function $u \in H^1_{loc}(\mathbb{R}^2; N)$. Moreover, we assume that $N \subset \mathbb{R}^n$ is a smooth hypersurface, oriented by a smooth unit normal vector field $\nu : N \to T^\perp N$.

With $w := \nu \circ u \in H^1_{loc}(\mathbb{R}^2; S^{n-1})$ then as in (2.17) we have

$$(4.1) \quad -\Delta u^i = \left(w^i \partial_\alpha w^j - w^j \partial_\alpha w^i\right) \partial_\alpha u^j =: \Omega^{ij}_\alpha \partial_\alpha u^i,$$

where

$$\Omega = \left(\Omega^{ij}\right)_{1 \le i,j \le n} \in L^2_{loc}\left(\mathbb{R}^2; T^*\mathbb{R}^2 \otimes so(n)\right)$$

with

$$\Omega^{ij} = \Omega^{ij}_\alpha dx^\alpha = -\Omega^{ji}$$

given by $\quad \Omega^{ij}_\alpha = w^i \partial_\alpha w^j - w^j \partial_\alpha w^i.$

Let $x_0 \in \mathbb{R}^2$ and let $R > 0$ such that

$$\|\Omega\|^2_{L^2(B_R(x_0))} < \varepsilon,$$

where $\varepsilon > 0$ is as in Thm. 3.8. We then show that u is Hölder continuous in $B_{R/2}(x_0)$ and hence smooth by Thm. 2.6. After translating the origin and scaling, we may assume that $x_0 = 0$, $R = 1$, and we set $B = B_1(0) \subset \mathbb{R}^2$.

We then let $P \in H^1(B; SO(n))$ with $P|_{\partial B} = id$, $\xi \in H^1_0(B; so(n))$ and $\omega \in L^2(B; T^*\mathbb{R}^2 \otimes so(n))$ with $\delta\omega = 0$, $d\omega = 0$ in B as in Thm. 3.8 so that

$$A := P^{-1}dP + P^{-1}\Omega P = * d\xi + \omega \text{ in } B$$

with

$$\|dP\|^2_{L^2} + \|d\xi\|^2_{L^2} + \|\omega\|^2_{L^2} \leq C\|\Omega\|^2_{L^2} \leq C\varepsilon.$$

Identifying $A = A_\alpha dx^\alpha$ with the vector field $A = (A_\alpha)_{1\leq\alpha\leq 2}$, etc., we have

$$A = P^{-1}\nabla P + P^{-1}\Omega P = \nabla^\perp\xi + \omega \text{ in } B.$$

From (4.1) we then obtain the equation

$$-\operatorname{div}(P^{-1}\nabla u) = -\nabla P^{-1}\nabla u + P^{-1}\Omega\nabla u$$

$$(4.2) \qquad = (P^{-1}\nabla P + P^{-1}\Omega P)P^{-1}\nabla u = AP^{-1}\nabla u$$

$$= \nabla^\perp\xi\, P^{-1}\nabla u + \omega P^{-1}\nabla u \quad \text{in } B,$$

exhibiting a determinant structure, perturbed by the factor P^{-1}, on the right hand side.

Observe that the system (4.2) in contrast to (4.1) no longer is diagonal; the rotation P^{-1} couples the components of u in the principal part.

Moreover, it seems that by allowing the partial derivatives $\partial_\alpha u$ to be acted on by an unconstrained rotation P^{-1} of the whole ambient space $\mathbb{R}^n$ we are giving up much of the geometric information from the target constraint $\partial_\alpha u(x) \in T_{u(x)}N$, $x \in M$. This apparent defect of our approach, however, disappears completely if we interpret the rotation P^{-1} not as acting on the components of ∇u but rather on the orthonormal frame of the ambient space in which we express these components, Fix $\eta \in C_c^\infty(B)$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{1/2}(0)$. Multiplying (4.2) by η we obtain the equation

$$(4.3)\quad -\operatorname{div}\left(P^{-1}\nabla(u\eta)\right) = \nabla^\perp \xi\, P^{-1}\nabla(u\eta) + \omega P^{-1}\nabla(u\eta) - e,$$

with "error" term

$$e = \operatorname{div}\left(P^{-1}u\nabla\eta\right) + P^{-1}\nabla u \cdot \nabla\eta$$

$$(4.4)\qquad + \nabla^\perp\xi\, P^{-1}u\nabla\eta + \omega P^{-1}u\nabla\eta \in L^2(B).$$

Set $v = u\eta \in H_0^1(B;\mathbb{R}^n)$, and for $B_R(x_0) \subset B$ let

$$P^{-1}dv = df + \delta g + h \quad \text{in } B_R(x_0)$$

be the Hodge decomposition of $P^{-1}dv$ on $B_R(x_0)$, where $f \in H_0^1(B_R(x_0))$, $g = *\gamma$ with $\gamma \in H_0^1(B_R(x_0))$ and where $h \in L^2(B_R(x_0), \cancel{T^*}\mathbb{R}^2)$ is harmonic, as in Thm. 3.26, satisfying in view of (4.3)

$$-\Delta f = \delta d f = -\operatorname{div}\left(P^{-1}\nabla v\right)$$

$$(4.5)\qquad = \nabla^\perp\xi\, P^{-1}\nabla v + \omega P^{-1}\nabla v - e$$

in $B_R(x_0)$, with

$$f = 0 \text{ on } \partial B_R(x_0),$$

and with γ solving

$$-\Delta\gamma = -*d\delta g = -*(dP^{-1}\wedge dv)$$

$$(4.6) \qquad = -\nabla^\perp P^{-1}\cdot\nabla v \quad \text{in } B_R(x_0),$$

$$\gamma = 0 \quad \text{on } \partial B_R(x_0).$$

Fix conjugate exponents $1 < p < 2 < q < \infty$ with $\frac{1}{p}+\frac{1}{q}=1$. By Cor. 3.29 then we have

$$\|df\|_{L^p} \leq C \sup_{\substack{\varphi\in W_0^{1,q}(B_R(x_0))\\ \|\nabla\varphi\|_{L^q}\leq 1}} (df, d\varphi)_{L^2}.$$

But for any $\varphi \in W_0^{1,q}(B_R(x_0))$ with $\|\nabla\varphi\|_{L^q}\leq 1$ by Sobolev's embedding $W_0^{1,q}(B_R(x_0)) \hookrightarrow C^{1-2/q}(B_R(x_0))$ we can bound

$$\|\varphi\|_{L^\infty} \leq C\|\nabla\varphi\|_{L^q} R^{1-2/q} \leq C R^{\frac{2-p}{p}},$$

$$(4.7) \qquad \|d\varphi\|_{L^2} \leq C\|d\varphi\|_{L^q} R^{1-2/q} \leq C R^{\frac{2-p}{p}}.$$

Thus, for any such φ we can

estimate the terms

$$(df, d\varphi)_{L^2} = -\int_{B_R(x_0)} \Delta f \cdot \varphi \, dx$$

$$= \int_{B_R(x_0)} \nabla^\perp \xi \, P^{-1} \nabla v \cdot \varphi \, dx$$

$$+ \int_{B_R(x_0)} \omega P^{-1} \nabla v \cdot \varphi \, dx - \int_{B_R(x_0)} e \varphi \, dx$$

$$=: I + II + III$$

resulting from (4.5), as follows. From (3.12), since $\varphi \in H_0^1(B_R(x_0))$, $v \in H^1 \cap L^\infty(B)$ we have

$$|I| = \left| \int_{B_R(x_0)} \nabla^\perp \xi \, P^{-1} \nabla v \cdot \varphi \, dx \right|$$

$$= \left| \int_{B_R(x_0)} \nabla^\perp \xi \, \nabla (P^{-1} \varphi) \, v \, dx \right|$$

$$\leq C \| \nabla^\perp \xi \, \nabla (P^{-1}\varphi) \|_{\mathcal{H}^1} [v]_{BMO}$$

$$\leq C \| d\xi \|_{L^2} \| d(P^{-1}\varphi) \|_{L^2} [v]_{BMO}$$

$$\leq C \sqrt{\varepsilon} \left(\| dP \|_{L^2} \| \varphi \|_{L^\infty} + \| d\varphi \|_{L^2} \right) [v]_{BMO}$$

$$\leq C \sqrt{\varepsilon} \, R^{\frac{2-p}{p}} [v]_{BMO}$$

while we can simply bound

$$|II| = \left| \int_{B_R(x_0)} \omega P^{-1} \nabla v \cdot \varphi \, dx \right|$$

$$\leq C \|\omega\|_{L^\infty(\operatorname{supp} \eta)} \|\nabla v\|_{L^1} \|\varphi\|_{L^\infty}$$

$$\leq C R^{2-2/q} \|\nabla v\|_{L^2} \leq C R^{\frac{2-p}{p}+1}$$

and

$$|III| = \left| \int_{B_R(x_0)} e \varphi \, dx \right| \leq \|e\|_{L^1} \|\varphi\|_{L^\infty}$$

$$\leq C R \|e\|_{L^{2'}} R^{1-2/q} \leq C R^{\frac{2-p}{p}+1}$$

Similarly, we have

$$\|\delta g\|_{L^p} = \|d\gamma\|_{L^2} \leq C \sup_{\substack{\psi \in W_0^{1,q}(B_R(x_0)) \\ \|\nabla \psi\|_{L^q} \leq 1}} (d\gamma, d\psi),$$

where for any $\psi \in W_0^{1,q}(B_R(x_0))$ with

$\|\nabla\psi\|_{L^q} \leq 1$ in view of (4.6) and (4.7) there holds

$$|(d\gamma, d\psi)_{L^2}| = \Big|\int_{B_R(x_0)} \Delta\gamma\, \psi\, dx\Big|$$

$$= \Big|\int_{B_R(x_0)} \nabla^\perp P^{-1}\nabla v \cdot \psi\, dx\Big|$$

$$= \Big|\int_{B_R(x_0)} \nabla^\perp P^{-1}\nabla\psi \cdot v\, dx\Big|$$

$$\leq C\|\nabla^\perp P^{-1}\nabla\psi\|_{\mathcal{H}^1}[v]_{BMO}$$

$$\leq C\|\nabla P\|_{L^2}\|\nabla\psi\|_{L^2}[v]_{BMO}$$

$$\leq C\sqrt{\varepsilon}\, R^{\frac{2-p}{p}}[v]_{BMO}.$$

Thus, by Campanato's estimates, for $0 < r < R$, letting

$$\phi(r) = \phi(x_0, r) = \int_{B_r(x_0)} |\nabla v|^p dx,$$

we find the bound

$$\phi(r) = \int_{B_r(x_0)} |\nabla v|^p dx = \int_{B_r(x_0)} |P^{-1}\nabla v|^p dx$$

$$\leq C \int_{B_r(x_0)} \left(|df|^p + |\delta g|^p + |h|^p \right) dx$$

(4.8)

$$\leq C\left(\frac{r}{R}\right)^2 \int_{B_R(x_0)} |h|^p dx + C\int_{B_R(x_0)} \left(|df|^p + |\delta g|^p\right) dx$$

$$\leq C\left(\frac{r}{R}\right)^2 \int_{B_R(x_0)} |\nabla v|^p dx + C\int_{B_R(x_0)} \left(|df|^p + |\delta g|^p\right) dx$$

$$\leq C\left(\frac{r}{R}\right)^2 \phi(R) + C R^{2-p} \varepsilon^{\frac{p}{2}} [v]^p_{BMO}$$

$$+ C R^{2-p+p}.$$

Scaling with r^{p-m}, we set

$$\Phi(r) = \Phi(x_0, r) = r^{p-2}\phi(r).$$

Also letting

$$\Psi(R) = \sup_{x_0 \in B,\, 0<r<R} \Phi(x_0, r),$$

we first can bound

$$(4.9)\quad \fint_{B_r(x_0)} |v - \bar{v}_{r,x_0}|^p dx \leq C r^{p-2} \int_{B_r(x_0)} |\nabla v|^p dx \leq C\Psi(r)$$

for all $x_0 \in B$, $0 < r < R$, and then

$$\sup_{x_0 \in B} [v]_{BMO(B_R(x_0))} \leq C\,\Psi(R).$$

With this, from (4.8) we obtain

$$\Phi(r) \leq C\left(\frac{r}{R}\right)^p \Phi(R) + C\left(\frac{r}{R}\right)^{p-2} \varepsilon^{p/2}\, \Psi(R)$$

$$+ C\left(\frac{r}{R}\right)^{p-2} R^p$$

$$\leq C_1\left(\frac{r}{R}\right)^p \left(1 + \left(\frac{r}{R}\right)^{-2} \varepsilon^{p/2}\right) \Psi(R) + C\left(\frac{r}{R}\right)^{p-2} R.$$

Fix $\frac{r}{R} = a$ such that $2C_1 a^{\frac{p-1}{2}} \leq 1$. We may assume that $\varepsilon^{p/2} \leq a^2$. Thus we find the bound

$$\Phi(x_0, aR) \leq C_1 a^p \left(1 + a^{-2}\varepsilon^{p/2}\right)\Psi(R) + C a^{p-2} R$$

$$\leq a^{\frac{p+1}{2}}\, \Psi(R) + CR$$

$$\leq a^{\frac{p+1}{2}}\, \Psi(R_0) + CR_0$$

for any $R \leq R_0 \leq 1$. Taking the supremum with respect to $x_0 \in B$ and $R \leq R_0$ on the left, we then arrive at the estimate

$$\Psi(aR) \leq a^{\frac{p+1}{2}}\Psi(R) + CR, \quad 0 < R \leq 1,$$

again writing R instead of R_0.

For any $0 < r < a$ let $k \in \mathbb{N}$ such that $a^{k+1} \leq r \leq a^k$. Then by definition $\Psi(r) \leq \Psi(a^k)$, and by iteration we obtain the bound

$$\Psi(r) \leq \Psi(a^k) \leq a^{\frac{p+1}{2}}\Psi(a^{k-1}) + Ca^{k-1}$$

$$\leq \dots \leq a^{k\frac{p+1}{2}}\Psi(1) + Ca^{k-1}\sum_{j=0}^{k-1} a^{\frac{p-1}{2}\cdot j}$$

$$\leq Ca^k \leq Cr.$$

It then follows from (4.9) that

$$v \in \mathcal{L}^{p,2+1}(B) \hookrightarrow C^{\frac{3-p}{p}}(B) \hookrightarrow C^{1/p}(B),$$

proving our claim.

4.2 Higher dimensions

Concerning the regularity of weakly harmonic maps in dimensions $m \geq 3$ we first have a negative result.

Theorem 4.2 (Rivière (1995)) Let $B = B^3 = B_1(0; \mathbb{R}^3)$, $u_0 \in C^\infty(\partial B; S^2)$ non-constant. Then there exists a weakly harmonic map $u \in H^1(B; S^2)$ satisfying $u = u_0$ on ∂B in the sense of traces and with $Sing(u) = \overline{B}$.

Here, $Sing(u) = \overline{B \setminus Reg(u)}$, where

$$Reg(u) = \{ x_0 \in B;\ \exists\, r > 0 : u|_{B_r(x_0)} \in C^\infty(B_r(x_0)) \}$$

is the set of regular points of u. Note that, by definition, $Reg(u)$ is open.

Rivière's proof builds on the notion of relaxed energy, introduced by Brezis-Coron-Lieb (1986), and their notion of dipole, as well as a construction of axially symmetric dipoles by Hardt-Lin-Poon (1992).

Rivière's construction reminds of
similar constructions of "strange" or
"unphysical" weak solutions for a
number of other problems described
by pde's that often may be characterized
by a "super-critical" scaling behavior.
A prominent example is the Navier-Stokes
system or the Euler equations of fluid
dynamics in 3 space dimensions. After
Scheffer's (1985) pioneering work, recently
a plethora of "unphysical" weak solutions
for these equations was constructed by
De Lellis - Szekelyhidi (2009), using tools
of "convex integration" previously developed
by Nash (1954) and Kuiper (1955) in the construction
of (non-geometric) C^1 solutions to the
isometric embedding problem; see also Gromov (1986).
Dacorogna-Marcellini (2008) construct Lipschitz
local isometric immersions using Baire's theorem.

In view of Thm. 4.2, even partial regularity results can only be expected to hold for classes of weakly harmonic maps satisfying additional requirements.

Def. 4.3. Let $N \subset \mathbb{R}^n$ be a smooth, closed submanifold, $B = B^m = B_1(0; \mathbb{R}^m)$, $m \geq 3$.

A map $u \in H^1(B; N)$ is locally minimizing, if for any $x_0 \in B$ there is $r > 0$ such that

$$\int_{B_r(x_0)} |\nabla u|^2 dx \leq \int_{B_r(x_0)} |\nabla v|^2 dx$$

for any $v \in H^1(B, N)$ such that $u - v \in H^1_0(B_r(x_0); N)$.

For locally minimizing (and hence weakly harmonic) maps the following result holds.

Theorem 4.4 (Schoen-Uhlenbeck (1982)). Let $u \in H^1(B, N)$ be locally minimizing, where $B = B^m$, $m \geq 3$. Then $\mathrm{Sing}(u)$ is discrete when $m = 3$, and for $m \geq 4$ the set $\mathrm{Sing}(u)$ has locally finite $(m-3)$-dimensional Hausdorff measure.

Remark 4.5: Recalling Ex. 2.7 (and its sharpening by Baldes (1984)), we see that Thm. 4.4 is best possible.

A less stringent assumption than local minimality is the following.

Def. 4.6. Let $\Omega \subset\subset \mathbb{R}^m$ be open. A weakly harmonic map $u \in H^1(\Omega; N)$ is stationary if there holds

$$(4.10) \quad \forall \zeta \in C_c^\infty(\Omega; \mathbb{R}^m): \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u \circ (id + \varepsilon\zeta)) = 0;$$

that is, if u is a critical point of E both with respect to "variations of the map" in the sense that

$$\forall \varphi \in C_c^\infty(\Omega; \mathbb{R}^n): \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(\pi_N(u+\varepsilon\varphi)) = 0$$

and with respect to "variations of the independent variables".

Remark 4.7. i) Any smooth harmonic map u is stationary. (Note: $u \circ (id+\varepsilon\zeta) = u + \varepsilon\zeta\cdot\nabla u + O(\varepsilon^2)$.)

ii) Any minimizing harmonic map is stationary.

Lemma 4.8. Condition (4.10) is equivalent to the condition

(4.11) $\forall \zeta \in C_c^\infty(\Omega;\mathbb{R}^m): \int_\Omega \left(|\nabla u|^2 \operatorname{div}\zeta - 2\,\partial_\alpha u^j \partial_\beta u^j \partial_\beta \zeta^\alpha\right)dx = 0.$

Proof: Compute for any $\zeta \in C_c^\infty(\Omega;\mathbb{R}^m)$, $|\varepsilon| << 1$:

$$E(u\circ(id+\varepsilon\zeta)) =$$

$$= \int_\Omega (\partial_\alpha u^j)\circ(id+\varepsilon\zeta)(\delta_\beta^\alpha + \varepsilon\partial_\beta\zeta^\alpha) \cdot (\partial_\gamma u^j)\circ(id+\varepsilon\zeta)(\delta_\beta^\gamma+\varepsilon\partial_\beta\zeta^\gamma)\,dx$$

$$\overset{(y=(id+\varepsilon\zeta)(x))}{=} \int_\Omega \partial_\alpha u^j \partial_\gamma u^j\left((\delta_\beta^\alpha+\varepsilon\partial_\beta\zeta^\alpha)(\delta_\beta^\gamma+\varepsilon\partial_\beta\zeta^\gamma)\circ(id+\varepsilon\zeta)^{-1}\right) \cdot \det\left(d(id+\varepsilon\zeta)^{-1}\right)dy.$$

Hence, with $d(id+\varepsilon\zeta)^{-1} = \mathbb{1} - \varepsilon d\zeta + O(\varepsilon^2)$,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u\circ(id+\varepsilon\zeta))$$

$$= \int_\Omega \left(\partial_\alpha u^j \partial_\beta u^j \partial_\beta\zeta^\alpha + \partial_\beta u^j \partial_\gamma u^j \partial_\beta \zeta^\gamma - |\nabla u|^2 \operatorname{tr}(d\zeta)\right)dx$$

$$= \int_\Omega \left(2\,\partial_\alpha u^j \partial_\beta u^j \partial_\beta\zeta^\alpha - |\nabla u|^2 \partial_\alpha\zeta^\alpha\right)dx.$$

Note that this computation only requires $u \in H^1(\Omega;N)$. $\square$

Stationarity implies "monotonicity".

Proposition 4.9 (Monotonicity formula, Price (1983))
Suppose $u \in H^1(\Omega; N)$ is a stationary weakly harmonic map, and suppose $B_{R_0}(0) \subset \Omega$. Then there holds

$$(4.12) \quad \forall\, 0 < r < R \leq R_0: \quad r^{2-m} \int_{B_r(0)} |\nabla u|^2 dx \leq R^{2-m} \int_{B_R(0)} |\nabla u|^2 dx .$$

Proof: For $0 < r < r+h < R_0$ let

$$\zeta(x) = \eta(|x|)\, x ,$$

where $\eta = 1$ on $[0, r]$, $\eta = 0$ on $[r+h, \infty[$ and where $\eta' \leq 0$ so that $0 \leq \eta \leq 1$. Then

$$\partial_\alpha \zeta^\alpha = \eta'(|x|)|x| + m\eta ,$$

$$\partial_\beta \zeta^\alpha = \eta'(|x|) \frac{x^\beta}{|x|} x^\alpha + \eta\, \delta^\alpha_\beta$$

so that (4.11) yields the equation

$$0 = \int_\Omega \Big(2\eta'(|x|) \frac{|x \cdot \nabla u|^2}{|x|} + (2-m)\eta |\nabla u|^2 - \eta'(|x|)|x|\,|\nabla u|^2 \Big) dx$$

$$\leq \int_\Omega \Big((2-m)\eta |\nabla u|^2 - \eta'(|x|)|x|\,|\nabla u|^2 \Big) dx .$$

Passing to the limit $h \downarrow 0$ we have

$$-\int_{\Omega} \eta'(|x|)\,|x|\,|\nabla u|^2 dx \longrightarrow r \int_{\partial B_r(0)} |\nabla u|^2 do.$$

Hence we obtain

$$0 \leq (2-m)\int_{B_r(0)} |\nabla u|^2 dx + r\int_{\partial B_r(0)} |\nabla u|^2 do$$

$$= r^{m-1}\frac{d}{dr}\left(r^{2-m}\int_{B_r(0)} |\nabla u|^2 dx\right),$$

and the claim follows. $\square$

From monotonicity of the scaled Dirichlet integral partial regularity can be deduced.

<u>Theorem</u> 4.10 (Evans (1991), Bethuel (1993), Rivière - Struwe (2008)) Suppose $u \in H^1(B; N)$ is a stationary weakly harmonic map on $B = B^m$, $m \geq 3$. Then $Sing(u)$ has vanishing $(m-2)$-dimensional Hausdorff measure.

We sketch the proof: Note that (4.12) implies that $\nabla u \in L^{2,m-2}(B)$; moreover, if for $B_R(x_0) \subset B$ we have

$$R^{2-m} \int_{B_R(x_0)} |\nabla u|^2 dx < \varepsilon$$

it follows that the Morrey norm

$$\| \nabla u \|^2_{L^{2,m-2}(B_{R/2}(x_0))} < C\varepsilon$$

with $C = C(m)$, and thus

$$\| \Omega \|^2_{L^{2,m-2}(B_{R/2}(x_0))} < C\varepsilon,$$

where Ω is as in (4.1). We then have the following result similar to Thm. 3.8.

<u>Theorem</u> 4.11 (Rivière-Struwe (2008)). There is $\varepsilon(m) > 0$ such that for $\Omega \in L^{2,m-2}(B; T^*\mathbb{R}^m \times so(n))$ with

$$\| \Omega \|^2_{L^{2,m-2}} < \varepsilon(m)$$

there exist $P \in H^1(B, SO(n))$, $\xi \in H^1_0(B; \Lambda^{m-2} T^*\mathbb{R}^m \otimes so(n))$ and a harmonic ω such that $dP, d\xi, \omega \in L^{2,m-2}$ with

$$\| dP \|^2_{L^{2,m-2}} + \| d\xi \|^2_{L^{2,m-2}} + \| \omega \|^2_{L^{2,m-2}} \leq C \| \Omega \|^2_{L^{2,m-2}}$$

so that $A = P^{-1} dP + P^{-1} \Omega P = * d\xi + \omega$ in B.

Remark that the Morrey norm estimates for $m \geq 3$ can be obtained exactly as in dimension $m = 2$.

Moreover, by the Poincaré inequality

$$[u]_{BMO} \leq C \|\nabla u\|_{L^{p, m-p}}, \quad 1 < p < m.$$

Thus, the proof of Thm. 4.1 also may be carried over to the case $m \geq 3$ easily; see Rivière - Struwe (2008). In particular, we have

$$\{x_0 \in B; \exists R > 0 : R^{2-m} \int_{B_R(x_0)} |\nabla u|^2 dx < \varepsilon\} \subset Reg(u),$$

and thus

$$x_0 \in Sing(u) \Rightarrow \forall r > 0 : r^{2-m} \int_{B_r(x_0)} |\nabla u|^2 dx \geq \varepsilon.$$

For any $\delta > 0$ then the family $(B_\delta(x_0))_{x_0 \in Sing(u)}$ by Vitali's covering lemma has a subfamily of disjoint balls $B_\delta(x_i^\delta)$, $1 \leq i \leq I_\delta$, such that

$$Sing(u) \subset \bigcup_{i=1}^{I_\delta} B_{5\delta}(x_i^\delta)$$

and

$$\mathcal{H}^{m-2}(Sing(u)) \leq \limsup_{\delta \downarrow 0} C \sum_{i=1}^{I_\delta} (5\delta)^{m-2}$$

$$\leq \limsup_{\delta \downarrow 0} \frac{C}{\varepsilon} \sum_{i=1}^{I_\delta} \int_{B_\delta(x_i^\delta)} |\nabla u|^2 dx = \limsup_{\delta \downarrow 0} \frac{C}{\varepsilon} \int_{\bigcup_{i=1}^{I_\delta} B_\delta(x_i^\delta)} |\nabla u|^2 dx = 0.$$

5. Appendix

For completeness, we present the proof of the Calderòn-Zygmund estimate in Thm. 3.27. Recall the statement:

Thm 5.1 (Calderòn-Zygmund)
Let $\Omega \subset\subset \mathbb{R}^m$ be smoothly bounded, $A = (A^i)_{1 \le i \le m} \in L^q(\Omega; \mathbb{R}^m)$ for some $1 < q < \infty$. Then there is a unique solution $u \in W_0^{1,q}(\Omega)$ to the Dirichlet problem

$$(5.1) \quad \begin{aligned} -\Delta u &= -\operatorname{div} A = -\partial_i A^i \text{ in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and

$$\|\nabla u\|_{L^q} \le C \|A\|_{L^q}.$$

The proof we present in contrast to the classical proofs does not use harmonic analysis. For $q > 2$ we follow the presentation of Mingione (2019), based on T. Iwaniec (1983, 1992). The case $1 < q < 2$ then is obtained by duality.

Let $q > 2$, and let $A \in L^q(\Omega; \mathbb{R}^m)$ be given. Since $L^q(\Omega) \hookrightarrow L^2(\Omega)$ by Hölder's inequality, there is a unique weak solution $u \in H_0^1(\Omega)$ of (5.1), and we have

$$(5.2) \quad \int_\Omega |\nabla u|^2 dx \leq \int_\Omega |A|^2 dx \leq C\left(\int_\Omega |A|^q dx\right)^{2/q}.$$

Let $B_{R_0}(x_0) \subset \Omega$. After a translation we may assume $x_0 = 0$. Set $B_r := B_r(0)$, $r > 0$.

Fix $0 < R \leq R_0$, and let $R/2 \leq r_1 < r_2 \leq R$. For any $\lambda > 0$, any $r \in [R/2, R]$ set

$$E_\lambda^r := \{x \in B_r ; |\nabla u(x)|^2 > \lambda\},$$

and set

$$E_\lambda := E_\lambda^R .$$

For any ball $B \subset B_R$ let

$$\Psi(B) = \fint_B \left(|\nabla u|^2 + M|A|^2\right) dx,$$

with $M \geq 1$ to be determined. Note that by (5.2) the function Ψ is well-defined.

Since $A, \nabla u \in L^2$, by Lebesgue's theorem we have

$$(5.3) \quad \lim_{r \downarrow 0} \Psi(B_r(x_0)) > \lambda \quad \text{for a.e. } x_0 \in E_\lambda .$$

Moreover, for any $x_0 \in B_{r_1}$, any $r > 0$ with

$$\frac{r_2 - r_1}{20} \leq r \leq r_2 - r_1$$

we have $B_r(x_0) \subset B_{r_2}$ and, with $|G| = \mathcal{L}^m(G)$, $G \subset \mathbb{R}^m$,

$$\Psi(B_r(x_0)) \leq \frac{|B_{r_2}|}{|B_r|} \Psi(B_{r_2})$$

$$(5.4) \quad \leq \frac{20^m r_2^m}{(r_2 - r_1)^m} \fint_{B_{r_2}} (|\nabla u|^2 + M|A|^2)\, dx =: \lambda_0 .$$

Henceforth we consider only $\lambda > \lambda_0$.

<u>Def</u>. 5.2. For any $x_0 \in B_{r_1}$, any $\lambda > \lambda_0$ define the <u>exit time</u> $r_{x_0} < \frac{r_2 - r_1}{20}$ to be

$$r_{x_0} = \inf \{ \rho > 0 ;\ \forall r \in]\rho, r_2 - r_1[: \Psi(B_r(x_0)) < \lambda \} .$$

Note that by (5.3), (5.4) the number r_{x_0} is well-defined and $r_{x_0} > 0$ for a.e. $x_0 \in E_\lambda^{r_1}$, any $\lambda > \lambda_0$.

Fix some $\lambda > \lambda_0$.

Lemma 5.3. There exists an at most countable family of disjoint balls $\tilde{B}_i = B_{r_{x_i}}(x_i)$, $i \in \mathbb{N}$, and a null-set N with $|N| = 0$ such that

$$E_\lambda^{r_1} \subset N \cup \bigcup_{i \in \mathbb{N}} B_i,$$

where

$$B_i = 5\tilde{B}_i = B_{r_i}(x_i) \quad \text{with} \quad r_i = 5 r_{x_i}, \; i \in \mathbb{N},$$

and there holds $4B_i = B_{4r_i}(x_i) \subset B_{r_2}$ as well as

$$(5.5) \quad \fint_{\tilde{B}_i} \left(|\nabla u|^2 + M|A|^2\right) dx = \lambda, \quad \fint_{4B_i} \left(|\nabla u|^2 + M|A|^2\right) dx \leq \lambda, \; i \in \mathbb{N}.$$

Proof: By (5.3) there exists $N \subset E_\lambda^{r_1}$ with $|N| = 0$ such that the collection $\left(B_{r_{x_0}}(x_0)\right)_{x_0 \in E_\lambda^{r_1} \setminus N}$ is an open cover of $E_\lambda^{r_1} \setminus N$. By Vitali's covering lemma there then exist $\tilde{B}_i$, B_i, $i \in \mathbb{N}$, as claimed.

Since by definition we have

$$r_i = 5 r_{x_i} \leq \frac{r_2 - r_1}{4}, \quad 4r_i = 20\, r_{x_i} < r_2 - r_1$$

and since $x_i \in B_{r_1}$ we have $20\tilde{B}_i = 4B_i \subset B_{r_2}$, $i \in \mathbb{N}$.

Def. 5.2 then immediately gives (5.5). $\square$

For any $i \in \mathbb{N}$ let $v_i = u + w_i$ with $w_i \in H^1_0(4B_i)$ solve the boundary value problem

$$
\begin{aligned}
-\Delta v_i &= 0 \quad \text{in } 4B_i = B_{4r_i}(x_i), \\
(5.6) \qquad v_i &= u \quad \text{on } \partial B_{4r_i}(x_i).
\end{aligned}
$$

<u>Lemma</u> 5.4. There exists $C_1 > 0$ such that for any $i \in \mathbb{N}$ there holds

$$
(5.7) \quad \fint_{4B_i} |\nabla w_i|^2 dx \leq \fint_{4B_i} |A|^2 dx \leq \frac{C_1}{M} \lambda,
$$

as well as

$$
(5.8) \quad \sup_{B_i} |\nabla v_i|^2 \leq C \fint_{4B_i} |\nabla v_i|^2 dx \leq C_1 \lambda .
$$

<u>Proof</u>: Compute

$$
\fint_{4B_i} |\nabla w_i|^2 dx = \fint_{4B_i} |\nabla v_i - \nabla u|^2 dx = \fint_{4B_i} \Delta u \, (v_i - u)\, dx
$$

$$
= \fint_{4B_i} A \cdot \nabla \underbrace{(v_i - u)}_{= w_i} dx \leq \frac{1}{2} \fint_{4B_i} |\nabla w_i|^2 dx + \frac{1}{2} \fint_{4B_i} |A|^2 dx
$$

to obtain (5.7) with $C_1 = 1$, using (5.5), $i \in \mathbb{N}$.

By harmonicity, $v_i \in C^\infty(4B_i)$ for each $i \in \mathbb{N}$, and the mean value property of harmonic functions yields the bound

$$\sup_{B_i} |\nabla v_i| \leq C \fint_{4B_i} |\nabla v_i| dx \leq C \left(\fint_{4B_i} |\nabla v_i|^2 dx \right)^{1/2}.$$

Together with (5,7), (5.5), then we can bound

$$\sup_{B_i} |\nabla v_i|^2 \leq C \fint_{4B_i} |\nabla v_i|^2 dx$$

$$\leq C \fint_{4B_i} |\nabla u|^2 dx + C \fint_{4B_i} |\nabla w_i|^2 dx$$

$$\leq C \fint_{4B_i} \left(|\nabla u|^2 + M |A|^2 \right) dx \leq C \lambda,$$

as claimed. $\square$

<u>Lemma</u> 5.5. There exists $C_2 > 0$ such that for any $\lambda > 4C_1\lambda_0$ there holds

$$(5.9)\quad \int_{E_\lambda^{r_1}} |\nabla u|^2 dx \leq \frac{C_2}{M} \int_{E^{r_2}_{\lambda/16C_1}} |\nabla u|^2 dx + C_2 \int_{F^{r_2}_{\lambda/16C_1 M}} |A|^2 dx,$$

where $F_\lambda^r = \{x \in B_r;\ |A(x)|^2 > \lambda\}$.

<u>Proof</u>: For $i \in \mathbb{N}$, $\lambda > \lambda_0$ we use (5.8) to bound

$$2C_1\lambda\,|\{x \in B_i;\ |\nabla u(x)|^2 > 4C_1\lambda\}|$$

$$+ \frac{1}{2} \int_{\{x \in B_i;\ |\nabla u(x)|^2 > 4C_1\lambda} |\nabla u|^2 dx$$

$$\leq \int_{\{x \in B_i;\ |\nabla u|^2 > 4C_1\lambda\}} |\nabla u|^2 dx$$

$$\leq 2\int_{\{x \in B_i;\ |\nabla u(x)|^2 > 4C_1\lambda\}} |\nabla v_i|^2 dx + 2\int_{B_i} |\nabla w_i|^2 dx$$

$$\leq 2C_1\lambda\,|\{x \in B_i;\ |\nabla u(x)|^2 > 4C_1\lambda\}| + 2\int_{B_i} |\nabla w_i|^2 dx.$$

Thus, we obtain

$$\int_{\{x\in B_i;\ |\nabla u(x)|^2 > 4C_1\lambda\}} |\nabla u|^2 dx \leq 4 \int_{B_i} |\nabla w_i|^2 dx ,$$

and with (5.7) as well as (5.5) we conclude

$$\int_{\{x\in B_i;\ |\nabla u(x)|^2 > 4C_1\lambda\}} |\nabla u|^2 dx \leq 4|4B_i| \fint_{4B_i} |\nabla w_i|^2 dx$$

(5.10)
$$\leq 4 \cdot 20^m |\tilde{B}_i| \fint_{4B_i} |A|^2 dx \leq \frac{C}{M} |\tilde{B}_i| \lambda .$$

But by (5.5), for any $i \in \mathbb{N}$, any $\lambda > \lambda_0$ we have

$$|\tilde{B}_i| = \frac{1}{\lambda} \int_{\tilde{B}_i} \left(|\nabla u|^2 + M|A|^2 \right) dx$$

$$\leq \frac{2}{\lambda} \int_{\{x\in \tilde{B}_i;\ |\nabla u(x)|^2 > \lambda/4\}} |\nabla u|^2 dx + \frac{2}{\lambda} \int_{\{x\in \tilde{B}_i;\ M|A(x)|^2 > \lambda/4\}} M|A|^2 dx$$

$$+ \frac{1}{2} |\tilde{B}_i|$$

and then

$$|\tilde{B}_i| \leq \frac{4}{\lambda} \int_{\{x\in \tilde{B}_i;\ |\nabla u(x)|^2 > \lambda/4\}} |\nabla u|^2 dx + \frac{4}{\lambda} \int_{\{x\in \tilde{B}_i;\ M|A(x)|^2 > \lambda/4\}} M|A|^2 dx .$$

Together with (5.10) this yields

$$\int_{\{x\in B_i;\ |\nabla u(x)|^2 > 4C_1\lambda\}} |\nabla u|^2 dx$$

$$\leq \frac{C}{M} \int_{\{x\in \tilde{B}_i;\ |\nabla u(x)|^2 > \lambda/4\}} |\nabla u|^2 dx + C\int_{\{x\in \tilde{B}_i;\ M|A(x)|^2 > \lambda/4\}} |A|^2 dx$$

Renaming $4C_1\lambda =: \tilde{\lambda}$ and summing over $i\in\mathbb{N}$, using Lemma 5.3, the claim follows.

$\square$

For $t > 4C_1\lambda$ define

$$\left[|\nabla u|^2\right]_t := \min\{|\nabla u|^2, t\},$$

and let $\gamma = q/2 > 1$.

<u>Lemma 5.6</u>. There exist $M\geq 1$, $C_3 > 0$ such that for any $R/2 \leq r_1 < r_2 \leq R$ we have

$$\fint_{B_{r_1}} \left[|\nabla u|^2\right]_t^{\gamma-1} |\nabla u|^2 dx \leq \frac{1}{2} \fint_{B_{r_2}} \left[|\nabla u|^2\right]_t^{\gamma-1} |\nabla u|^2 dx$$

$$+ C_3 \fint_{B_R} |A|^q dx + \left(\frac{CR^m}{(r_2-r_1)^m} \fint_{B_R} \left(|\nabla u|^2 + M|A|^2\right) dx\right)^{\gamma}.$$

Proof: Multiplying (5,9) by $\lambda^{\gamma-2}$ and integrating over $4C_1\lambda_0 < \lambda < t$, we obtain $I \leq II + III$, where

$$I = \int_{4C_1\lambda_0}^{t} \lambda^{\gamma-2} \int_{E_\lambda^{r_1}} |\nabla u|^2 dx\, d\lambda = \int_0^t \dots d\lambda - \int_0^{4C_1\lambda_0} \dots d\lambda$$

with

$$\int_0^t \lambda^{\gamma-2} \int_{E_\lambda^{r_1}} |\nabla u|^2 dx\, d\lambda = \int_{B_{r_1}} \int_0^{[|\nabla u(x)|^2]_t} \lambda^{\gamma-2} |\nabla u|^2 d\lambda\, dx$$

$$= \frac{1}{\gamma-1} \int_{B_{r_1}} [|\nabla u|^2]_t^{\gamma-1} |\nabla u|^2 dx$$

by Fubini's theorem, and with

$$\int_0^{4C_1\lambda_0} \lambda^{\gamma-2} \int_{E_\lambda^{r_1}} |\nabla u|^2 dx\, d\lambda$$

$$\leq \int_0^{4C_1\lambda_0} \lambda^{\gamma-2} d\lambda \cdot \int_{B_{r_2}} |\nabla u|^2 dx$$

$$\leq \frac{(4C_1\lambda_0)^{\gamma-1}}{\gamma-1} \cdot \lambda_0 \frac{|B_{r_2}| (r_2-r_1)^m}{20^m r_2^m}$$

$$\leq C \lambda_0^{\gamma} |B_{r_2}| = C\left(\frac{r_2^m}{(r_2-r_1)^m} \fint_{B_{r_2}} (|\nabla u|^2 + |A|^2) dx\right)^{\gamma} |B_{r_2}|$$

by (5,4).

Similarly, we can bound

$$\underline{\mathrm{II}} \leq \frac{C_2}{M}\int_0^t \lambda^{\gamma-2} \int_{F^{r_2}_{\lambda/16C_1}} |\nabla u|^2 dx\, d\lambda = \frac{C_2}{M}\int_{B_{r_2}} \int_0^{\min\{16C_1|\nabla u(x)|^2, t\}} \lambda^{\gamma-2}|\nabla u|^2 d\lambda\, dx$$

$$\leq \frac{(16C_1)^{\gamma-1}C_2}{(\gamma-1)M}\int_{B_{r_2}} [|\nabla u|^2]^{\gamma-1}_{t/16C_1} |\nabla u|^2 dx$$

$$\leq \frac{1}{2}\int_{B_{r_2}} [|\nabla u|^2]^{\gamma-1}_t |\nabla u|^2 dx,$$

if $M \geq 1$ is sufficiently large, where we also use that

$$[|\nabla u|^2]_{t/16C_1} \leq [|\nabla u|^2]_t .$$

Finally, we also have

$$\underline{\mathrm{III}} = \int_{4C_1\lambda_0}^t \lambda^{\gamma-2} \int_{F^{r_2}_{\lambda/16C_1M}} |A|^2 dx\, d\lambda$$

$$\leq \int_0^\infty \lambda^{\gamma-2} \int_{F^{r_2}_{\lambda/16C_1M}} |A|^2 dx\, d\lambda$$

$$= \int_{B_{r_2}} \int_0^{16C_1M|A(x)|^2} \lambda^{\gamma-2} d\lambda\, |A(x)|^2 dx$$

$$= \frac{(16C_1M)^{\gamma-1}}{\gamma-1}\int_{B_{r_2}} |A|^{2\gamma} dx = CM^{\gamma-1}\int_{B_{r_2}} |A|^q dx.$$

□

We then may apply the following "simple, but fundamental" lemma of Giaquinta - Giusti (1982)

<u>Lemma</u> 5.7. Let $0 \leq T_0 < T_1 < \infty$, and let $f : [T_0, T_1] \to \mathbb{R}$ with $0 \leq f(t) \leq C < \infty$ for some $C > 0$ and all $t \in [T_0, T_1]$.
Suppose that with constants $A, B, \alpha \geq 0$ and $0 < \theta < 1$ for any $T_0 \leq t < T \leq T_1$ there holds

$$f(t) \leq \theta f(T) + \frac{A}{(T-t)^\alpha} + B .$$

Then there exists $C = C(\alpha, \theta) > 0$ such that

$$\forall\, T_0 \leq t < T \leq T_1 : \; f(t) \leq C \left(\frac{A}{(T-t)^\alpha} + B \right) .$$

<u>Proof</u>: Given $T_0 \leq t < T \leq T_1$, let $(t_k)_{k \in \mathbb{N}_0}$ be given by $t_0 = t$, $t_{k+1} = t_k + (1-\tau)\tau^k (T-t)$, $k \in \mathbb{N}_0$, for $0 < \tau < 1$ to be determined. By iteration,

$$f(t_0) \leq \theta^k f(t_k) + \left(\frac{A}{(1-\tau)^\alpha (T-t)^\alpha} + B \right) \sum_{j=0}^{k-1} \theta^j \tau^{-\alpha j}.$$

Fixing $0<\tau<1$ such that $\theta \tau^{-\alpha} < 1$, and letting $k \to \infty$ we obtain the claim with $C = (1-\tau)^{-\alpha} (1 - \theta \tau^{-\alpha})^{-1}$. □

Proof of Thm. 5.1: For given $q > 2$ we now apply Lemma 5.7 to the function

$$\phi(r) := \int_{B_r} [|\nabla u|^2]_t^{\gamma-1} |\nabla u|^2 dx, \quad R/2 \leq r \leq R.$$

By Lemma 5.6 we have

$$\phi(r_1) \leq \frac{1}{2}\phi(r_2) + \frac{K}{(r_2 - r_1)^{m\gamma}} + L$$

with

$$K = \left(C R^m \int_{B_R} (|\nabla u|^2 + M|A|^2) dx\right)^\gamma$$

and with

$$L = C \int_{B_R} |A|^q dx.$$

Thus, the hypotheses of Lemma 5.7 are satisfied with $\theta = 1/2$, $\alpha = m\gamma$, and with some constant $C = C(\alpha, \theta) > 0$ there holds

$$\phi(R/2) \leq C \frac{K}{(R/2)^{m\gamma}} + CL.$$

Passing to the limit $t \to \infty$ we then obtain $\nabla u \in L^q(B_{R/2})$, and

$$(5.11) \quad \int_{B_{R/2}} |\nabla u|^q dx \leq C\left(\int_{B_R} (|\nabla u|^2 + M|A|^2) dx\right)^\gamma + C \int_{B_R} |A|^q dx,$$

We can argue similarly for a ball $B_R(x_0)$ with $x_0 \in \partial\Omega$, using the techniques developed in the proof of boundary regularity for elliptic equations (FA II,

Finitely many balls $B_{R/2}(x_i)$, $x_i \in \overline{\Omega}$, $1 \leq i \leq I$, cover $\overline{\Omega}$. Multiplying (5.11) with R^m and summing over $1 \leq i \leq I$, we obtain

$$\int_\Omega |\nabla u|^q\,dx \leq C R^m \sum_{i=1}^{I} \left(\fint_{B_R(x_i)} \left(|\nabla u|^2 + M|A|^2\right) dx \right)^\gamma + C \int_\Omega |A|^q\,dx$$

$$\leq C(R) \left(\int_\Omega \left(|\nabla u|^2 + M|A|^2\right) dx \right)^\gamma + C\|A\|_{L^q}^q$$

$$\leq C(R,M) \left(\int_\Omega |A|^2\,dx \right)^\gamma + C\|A\|_{L^q}^q$$

$$\leq C\|A\|_{L^q}^q\,,$$

as claimed. Note that we used (5.2), that is, the bound $\|\nabla u\|_{L^2} \leq C\|A\|_{L^2}$ in this estimate.

Let $1<p<2$ with dual exponent $q=\frac{p}{p-1}>2$.

Given $u \in W_0^{1,p}(\Omega)$ with

$$-\Delta u = -\operatorname{div} f \text{ in } \Omega, \quad u=0 \text{ on } \partial\Omega,$$

we let $v \in W_0^{1,q}(\Omega)$ solve

$$-\Delta v = -\operatorname{div} g \text{ in } \Omega, \quad v=0 \text{ on } \partial\Omega,$$

where

$$g = \nabla u \, |\nabla u|^{p-2} \in L^q(\Omega) \hookrightarrow L^2(\Omega).$$

By Thm. 5.1 for the case $q>2$ we then have $v \in W_0^{1,q}(\Omega)$ with

$$\|\nabla v\|_{L^q} \leq C \|g\|_{L^q} = C \|\nabla u\|_{L^p}^{p-1}.$$

It follows that

$$\begin{aligned}
\|\nabla u\|_{L^p}^p &= (\nabla u, g)_{L^2} = (u, -\operatorname{div} g)_{L^2} \\
&= (u, -\Delta v)_{L^2} = (-\Delta u, v)_{L^2} \\
&= (-\operatorname{div} f, v)_{L^2} = (f, \nabla v)_{L^2} \\
&\leq \|f\|_{L^p} \|\nabla v\|_{L^q} \leq C \|f\|_{L^p} \|\nabla u\|_{L^p}^{p-1};
\end{aligned}$$

hence

$$\|\nabla u\|_{L^p} \leq C \|f\|_{L^p}. \tag{5.12}$$

Given $f \in L^p(\Omega)$, there exists $(f_k)_{k\in\mathbb{N}} \subset L^2(\Omega)$ with $f_k \overset{(k\to\infty)}{\longrightarrow} f$ in $L^p(\Omega)$.

The associated solutions $u_k \in H^1_0(\Omega)$ of

$$-\Delta u_k = -\operatorname{div} f_k \text{ in } \Omega, \quad u_k = 0 \text{ on } \partial\Omega,$$

by (5.12) are uniformly bounded in $W^{1,p}_0(\Omega)$. Thus, as $k\to\infty$ suitably we have $u_k \overset{w}{\to} u$ in $W^{1,p}_0(\Omega)$, where u solves

$$-\Delta u = -\operatorname{div} f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

and by (5.12) again $u \in W^{1,p}_0(\Omega)$ is the unique solution of this equation. $\square$

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