Nonlinear Evolution Problems

HS 2017

Summary. The evolution of an initial state in nature or in geometry to an equilibrium state very often is described by a parabolic pde of second order, the prototype for which is the standard heat equation.

Starting from this example, we review issues of well-posedness, regularity, and a-priori bounds, in particular, maximum principles and energy estimates also for nonlinear variants of the heat equation arising in physics and geometry, including the Navier-Stokes equations, Yamabe flow, and the heat flow of harmonic maps.

Literature.

L. C. Evans: PDE, AMS (1998)

F. John: PDE, Springer (1971), 4th edition (1982)

O. Ladyzhenskaya et al.: Linear and quasilinear equations of parabolic type, AMS translations (1968)

Contents

1. The heat equation

For $\Omega \subset \mathbb{R}^n$ smoothly bounded, $u_0 : \Omega \to \mathbb{R}$ we consider solutions $u : \Omega \times [0,T[\to \mathbb{R}$ of the Cauchy problem for the heat equation

(1.1) $\quad u_t - \Delta u = 0 \quad \text{in } \Omega \times [0,T[$

with initial and boundary data

(1.2) $\quad u|_{\Omega \times \{0\} \cup \partial\Omega \times [0,T[} = u_0 .$

Here, $T \leq \infty$; if $T = \infty$ such u is a global solution.

Phys. motivation: $\frac{d}{dt}\left(\int_V u\,dx\right) = -\int_{\partial V} F \cdot \nu\, do$, hence $u_t = -\operatorname{div} F$, $F = -a\nabla u$: flux density.

Questions: i) Is the problem (1.1)-(1.2) "well-posed"?
Does there exist a unique smooth solution u for smooth data u_0?
Does the solution depend continuously on the data? Can one bound u in terms of u_0?
What can we say about the long-term behavior?

Similarly, for given $f : \Omega \times [0,T[\to \mathbb{R}$ we consider the inhomogeneous equation

(1.3) $\quad u_t - \Delta u = f \quad \text{in } \Omega \times [0,T[.$

1.1 Fundamental solution, Cauchy problem

On $\Omega = \mathbb{R}^n$, let

$$G(x,t) = \frac{1}{\sqrt{4\pi t}^{\,n}} e^{-\frac{|x|^2}{4t}}, \quad t > 0.$$

Compute

$$\nabla G(x,t) = -\frac{x}{2t} G,$$

$$\Delta G(x,t) = \left(-\frac{n}{2t} + \frac{|x|^2}{4t^2}\right) G,$$

$$\frac{\partial}{\partial t} G(x,t) = \left(-\frac{n}{2t} + \frac{|x|^2}{4t^2}\right) G = \Delta G(x,t).$$

Moreover,

$$\int_{\mathbb{R}^n} G(x,t)\,dx \overset{(y=\frac{x}{2\sqrt{t}})}{=} \frac{1}{\sqrt{\pi}^{\,n}} \underbrace{\int_{\mathbb{R}^n} e^{-|y|^2} dy}_{=\prod_{i=1}^n \int_{\mathbb{R}} e^{-y_i^2} dy_i = \sqrt{\pi}^{\,n}} = 1, \tag{1.4}$$

Theorem 1.1.1. For every $u_0 \in L^1(\mathbb{R}^n)$ there is a solution

$$u(x,t) = \int_{\mathbb{R}^n} G(x-y,t) u_0(y)\,dy = (G(\cdot,t) * u_0)(x)$$

to (1.1) – (1.2), and $u \in C^\infty(\underbrace{\mathbb{R}^n \times]0,\infty[}_{\text{even spatially analytic}})$ with

$$\|u(t)\|_{L^1} \leq \|u_0\|_{L^1},$$

$$u(t) \xrightarrow{(t \downarrow 0)} u_0 \text{ in } L^1(\mathbb{R}^n).$$

Proof. Since $G(\cdot,t)$ is smooth and decaying exponentially fast as $|x|\to\infty$, for any $u_0 \in L^1(\mathbb{R}^n)$ it is possible to differentiate under the integral and bound for any $t>0$:

$$\sup|\nabla^k u(x,t)| \leq \sup|\nabla^k G(\cdot,t)|\,\|u_0\|_{L^1}, \quad \forall k\in\mathbb{N}_0$$

In particular, $u_t - \Delta u = ((G_t - \Delta G)(\cdot,t) * u_0) = 0$

Moreover, $u(t)\in L^1(\mathbb{R}^n)$ with

$$\|u(t)\|_{L^1} \leq \|G(\cdot,t)\|_{L^1}\|u_0\|_{L^1} = \|u_0\|_{L^1}$$

by Fubini's theorem.

Finally, $u(x,t) \xrightarrow{(t\downarrow 0)} u_0(x)$ at any Lebesgue point (by a computation similar to (1.4)) and $(u(t))_{t>0}$ is equi-integrable (exercise). So $u(t) \xrightarrow{(t\downarrow 0)} u_0$ in $L^1(\mathbb{R}^n)$ by Vitali's thm. $\square$

<u>Remarks 1.1</u> 1.i) If $u_0 \geq 0$, $u_0 \not\equiv 0$ we have $u(t)>0$ on $\mathbb{R}^n$ for every $t>0$. Equation (1.1) thus has <u>infinite propagation speed</u>.

ii) If $u_0 \in C^0 \cap L^1(\mathbb{R}^n)$ we have pointwise convergence $u(x,t) \xrightarrow{(t\downarrow 0)} u_0(x)$ at every $x\in\mathbb{R}^n$. (Every $x\in\mathbb{R}^n$ is a Lebesgue point of u_0.)

iii) If $u_0 \in L^p(\mathbb{R}^n)$ for some $p \in [1,\infty]$, the solution $u(t) = G(\cdot,t) * u_0$ above is smooth with

$$\| u(t) \|_{L^p} \leq \| u_0 \|_{L^p}, \quad \forall t > 0,$$

by Fubini's theorem (exercise).

In particular, for $u_0 \in L^\infty(\mathbb{R}^n)$ we have

$$\| u(t) \|_{L^\infty} \leq \| u_0 \|_{L^\infty}, \quad \forall t > 0,$$

but $u(t) \not\to u_0$ in $L^\infty(\mathbb{R}^n)$, in general, e.g. if $u_0 \notin C^0$.

iv) For $u_0 \in L^1(\mathbb{R}^n)$ the solution $u(t) = G(\cdot,t) * u_0$ is the <u>unique</u> solution of (1.1)-(1.2) with $u(t) \in L^1(\mathbb{R}^n)$ for $t > 0$ satisfying

$$\| u(t) \|_{L^1} \leq \| u_0 \|_{L^1}, \quad \forall t > 0,$$

and such that $u(t) \to u_0$ in $L^1(\mathbb{R}^n)$ as $t \downarrow 0$ (exercise; hint: consider $u * \rho_\varepsilon$ and use Thm. 1.2.2).

However, even for $u_0 \equiv 0$ there are <u>infinitely</u> many distinct solutions of (1.1)-(1.2), as discovered by <u>Tychonoff</u> (see John, References).

<u>Example</u> 1.1.1 (Tychonoff; see John, p. 211 f.)

Solve (1.1) with Cauchy data <u>on</u> <u>the</u> t-<u>axis</u> given by

$$(1.5) \quad u = g(t), \quad u_x = 0 \quad \text{for } x = 0,$$

for $g = g(t)$ to be determined.

The <u>ansatz</u>

$$u(x,t) = \sum_{j=0}^{\infty} g_j(t)\, x^j$$

yields a solution to (1.1), (1.5) if

$$g_j' = (j+2)(j+1)\, g_{j+2}, \quad j \in \mathbb{N}_0,$$

with

$$g_0 = g, \quad g_1 = 0.$$

Thus $\quad g_{2j} = \dfrac{g^{(k)}}{(2k)!}, \quad g_{2j+1} = 0, \quad j \in \mathbb{N}_0.$

For

$$g(t) = \begin{cases} e^{-t^{-\alpha}}, & t > 0 \\ 0, & t \le 0 \end{cases}$$

with $\alpha > 1$ there is $C = C(\alpha) > 0$ such that

$$|g^{(k)}(t)| \le C \frac{k!}{(t/2)^k}\, e^{-t^{-\alpha}} \quad \text{(exercise)}$$

for all $0 < t \le 1$.

With $\frac{k!}{(2k)!} \leq \frac{1}{k!}$ we find

$$\sum_{k=0}^{\infty} \left| \frac{g^{(k)}(t)}{(2k)!} x^{2k} \right| \leq$$

$$\leq C \sum_{k=0}^{\infty} \frac{1}{k!} \frac{2^k x^{2k}}{t^k} e^{-t^{-\alpha}} \leq C \exp\left(\frac{2|x|^2}{t} - t^{-\alpha} \right)$$

and the series $u(x,t)$ converges smoothly locally uniformly for $0 < t \leq 1$ to a solution u of (1.1) with $u|_{t=0} = 0$.

Note that the bounds for the above function u grow faster than exponentially in $|x|$ for any $t > 0$.

Below we shall see that under the assumption

$$|u(x,t)| \leq M e^{a|x|^2}$$

for some $M, a > 0$ any solution u to (1.1), (1.2) agrees with the solution given by Thm. 1.1.1, see Thm. 1.2.2.

Moreover, by a result of Widder (cf. John, p. 222) the same is true when $u \geq 0$.

In order to avoid problems "at ∞" (or on the boundary), in the following we usually will study the equation (1.1) on the torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, that is, we consider only <u>periodic</u> solutions u of (1.1) with

$$u(x+z,t) = u(x,t), \quad \forall t \geq 0, z \in \mathbb{Z}^n.$$

Extending initial data u_0 periodically the representation formula

$$u(x,t) = \int_{\mathbb{R}^n} G(x-y,t)\, u_0(y)\, dy$$

then may still be used to obtain a solution u of (1.1), (1.2) with

$$u(x+z,t) = \int_{\mathbb{R}^n} G(x-(y-z),t)\, \overset{u_0(y-z)}{\overset{\shortparallel}{u_0(y)}}\, dy = u(x,t)$$

and satisfying

$$\|u(x,t)\|_{L^1(\mathbb{T}^n)} \leq \int_{\mathbb{T}^n} \left(\int_{\mathbb{R}^n} G(x-y,t)\, |u_0(y)|\, dy \right) dx$$

$$= \int_{\mathbb{T}^n} \sum_{z \in \mathbb{Z}^n} \int_{\mathbb{T}^n} G(x-y-z,t)\, |\overset{u_0(y)}{\overset{\shortparallel}{u_0(y+z)}}|\, dy\, dx$$

$$= \int_{\mathbb{R}^n} G(x,t)\, dx \int_{\mathbb{T}^n} |u_0|\, dy = \|u_0\|_{L^1(\mathbb{T}^n)}.$$

1.2 Maximum principle, uniqueness

We have the following fundamental result.

Theorem 1.2.1 Let $u \in C^2(\mathbb{T}^n \times]0,T]) \cap C^0(\mathbb{T}^n \times [0,T])$ be a smooth solution of (1.1) with data $u_0 = u(\cdot, 0) \in C^0(\mathbb{T}^n)$. Then there holds

$$m := \min_{\mathbb{T}^n} u_0 \leq u \leq \max_{\mathbb{T}^n} u_0 =: M;$$

in particular, $\max\limits_{\mathbb{T}^n \times [0,T]} u = \max\limits_{\mathbb{T}^n} u_0$.

Proof. It suffices to show $u \leq M$.
(Consider $v = -u$ to see that then also $u \geq m$.)
Moreover, we may assume that $M = 0$. (Else consider $v = u - M$.)
Suppose by contradiction that there is $x_0 \in \mathbb{T}^n$, $0 < t_0 \leq T$ such that $u(x_0, t_0) > 0$.

Then for sufficiently small $\lambda > 0$ there also holds $e^{-\lambda t_0} u(x_0, t_0) > 0$ and for

$$v(x,t) = e^{-\lambda t} u(x,t)$$

there exists $(y_0, s_0) \in \mathbb{T}^n \times]0,T]$ with

$$v(x_0, s_0) = \max_{\mathbb{T}^n \times [0,T]} v > 0.$$

Hence $\Delta v(x_0, s_0) \leq 0 \leq v_t(x_0, s_0)$, so that

$$0 \leq v_t - \Delta v = -\underbrace{\lambda v}_{>0} + e^{-\lambda t} \underbrace{(u_t - \Delta u)}_{=0} < 0 \text{ at } (y_0, s_0). \quad \square$$

Corollary 1.2.1 For any $u_0 \in C^\infty(\mathbb{T}^n)$ the solution $u(t) = G(\cdot, t) * u_0$ of (1.1), (1.2) is unique among all solutions $v \in C^\infty(\mathbb{T}^n \times [0, \infty[)$ of (1.1), (1.2).

Proof. For any such u and v the function $w = u - v$ smoothly solves (1.1) with $w|_{t=0} = 0$. By the maximum principle, $w \equiv 0$. □

Corollary 1.2.2. For any $u_0 \in L^1(\mathbb{T}^n)$ the solution $u(t) = G(\cdot, t) * u_0$ is unique among all solutions $v \in C^\infty(\mathbb{T}^n \times]0, \infty[)$ with

$$\| v(t) - u_0 \|_{L^1} \to 0 \quad (t \downarrow 0).$$

Proof. Shifting time by $t_0 > 0$, for any v, any $t > t_0 > 0$ by Cor. 1.2.1 there holds

$$v(t) = G(\cdot, t - t_0) * v(t_0);$$ similarly for u.

By Theorem 1.1.1 then, for any $t > 0$ we have

$$\| u(t) - v(t) \|_{L^1} \leq \| u(t_0) - v(t_0) \|_{L^1}$$

$$\leq \| u(t_0) - u_0 \|_{L^1} + \| v(t_0) - u_0 \|_{L^1} \xrightarrow{(t_0 \downarrow 0)} 0.$$ □

Remark 1.2.1, i) Write

$$u(t) = G(\cdot, t) * u_0 =: S(t) u_0$$

with an operator $S(t) \in \bigcap_{1 \le p \le \infty} L(L^p(\mathbb{T}^n)) \cap L(C^0(\mathbb{T}^n))$ by Thm. 1.1.1, satisfying $\|S(t)\|_{L(X)} \le 1$, $0 < t < \infty$, for $X = L^p(\mathbb{T}^n)$ or $X = C^0(\mathbb{T}^n)$.

ii) Then we can interpret the result of Corollary 1.2.2 as stating the semi-group property

$$S(t) = S(t-s) \circ S(s)$$

for any $0 < s < t$. (Proof: Use uniqueness.)

The idea of the proof of Thm. 1.2.1 may be applied to establish the uniqueness of the solution $u(t) = G(\cdot, t) * u_0$ to (1.1), (1.2) under mild growth assumptions also on $\mathbb{R}^n$ (or any other unbounded domain).

Theorem 1.2.2. Let $u \in C^0(\mathbb{R}^n \times [0,T])$ with $u_t, \nabla u, \nabla^2 u \in C^0(\mathbb{R}^n \times]0,T[)$ satisfy

$$u_t - \Delta u \leq 0 \quad \text{on } \mathbb{R}^n \times]0,T[, \quad u|_{t=0} = u_0,$$

and suppose there exist $M, a > 0$ such that

$$u(x,t) \leq M e^{a|x|^2} \quad \text{on } \mathbb{R}^n \times]0,T[.$$

Then $$u(x,t) \leq \sup_x [u_0(x)]^+ \quad \text{on } \mathbb{R}^n \times [0,T].$$

Proof. We may assume $8aT < 1$; else divide the interval $[0,T]$ into intervals of length $\tau < \frac{1}{8a}$ and iterate.

The function

$$K(x,t) = \frac{1}{\sqrt{4\pi(2T-t)}^{\,n}} e^{\frac{|x|^2}{4(2T-t)}}$$

satisfies

$$K_t(x,t) = \left(\frac{n}{2(2T-t)} + \frac{|x|^2}{4(2T-t)^2}\right) K(x,t) = \Delta K(x,t).$$

Hence for any $\mu > 0$ the function

$$u_\mu = u - \mu K$$

satisfies $(\partial_t - \Delta) u_\mu \leq 0$ on $\mathbb{R}^n \times]0,T[$.

Moreover, for any $0 < t \leq T$ there holds

$$u_\mu(x,t) \leq M e^{a|x|^2} - \frac{\mu}{\sqrt{8\pi T}^n} e^{\frac{|x|^2}{8T}}$$

$$\leq \sup [u_0]^+ \quad \text{for } |x| \geq R_0(T),$$

where we used that $a < \frac{1}{8T}$.

Suppose there exists $x_0 \in \mathbb{R}^n$, $0 < t_0 \leq T$ with

$$u(x_0, t_0) > \sup [u_0]^+ =: m_0 \geq 0.$$

Then for sufficiently small $\lambda, \mu > 0$ we also have $e^{-\lambda t_0} u_\mu(x_0, t_0) > m_0$, and there is (x_1, t_1) with $|x_1| \leq R_0$, $0 < t_1 \leq T$ such that

$$m_1 := e^{-\lambda t_1} u_\mu(x_1, t_1) = \max_{\mathbb{R}^n \times [0,T]} e^{-\lambda t} u_\mu(x,t) > m_0.$$

But then from computing

$$0 \leq -\Delta\left(e^{-\lambda t_1} u_\mu(x_1, t_1)\right) = -e^{-\lambda t_1} \Delta u_\mu(x_1, t_1)$$

$$0 \leq \partial_t \left(e^{-\lambda t} u_\mu(x_1, t)\right)\Big|_{t=t_1} = -\lambda m_1 + e^{-\lambda t_1} \partial_t u_\mu(x_1, t_1)$$

there results the contradiction

$$(\partial_t - \Delta) u_\mu(x_1, t_1) \geq \lambda m_1 > 0.$$

$\square$

Moreover, the strong maximum principle holds.

<u>Theorem</u> 1.2.3. If there is $x_0, t_0 > 0$ such that

$$u(x_0, t_0) = \max |u| =: M,$$

then $u \equiv M$.

<u>Proof</u>. If $u_0 \leq M \not\equiv u_0$ for all $t > 0$ we have

$$u(x,t) = \int_{\mathbb{R}^n} G(x-y,t)\, u_0(y)\, dy$$

$$< \int_{\mathbb{R}^n} G(x-y,t)\, dy \cdot M = M. \qquad \square$$

For $u \in C^0(\mathbb{T}^n)$, $0 < \alpha < 1$, let

$$[u]_{C^\alpha} = \sup_{\substack{x,y \in \mathbb{T}^n \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha}.$$

<u>Corollary</u> 1.2.3. There holds

$$[u(t)]_{C^\alpha} \leq [u_0]_{C^\alpha}.$$

Proof. For any $x_0, y_0 \in \mathbb{T}^n$, $x_0 \neq y_0$, let $h := y_0 - x_0$ and set

$$v(x,t) = u(x+h,t) - u(x,t),$$

satisfying (1.1) with

$$|v(x,0)| = |u_0(x+h) - u_0(x)| \leq |h|^\alpha [u_0]_{C^\alpha}.$$

By Theorem 1.2 we have

$$\|v(t)\|_{L^\infty} \leq \|v(0)\|_{L^\infty} \leq |h|^\alpha [u_0]_{C^\alpha};$$

in particular, for any $t_0 > 0$:

$$\frac{|u(y_0,t_0) - u(x_0,t_0)|}{|y_0 - x_0|^\alpha} \leq \frac{\|v(t_0)\|_{L^\infty}}{|h|^\alpha} \leq [u_0]_{C^\alpha}.$$

$\square$

Arguing similarly for any spatial derivative, we find

Corollary 1.2.4. For $u_0 \in C^{k,\alpha}(\mathbb{T}^n)$ there holds

$$\|u(t)\|_{C^{k,\alpha}(\mathbb{T}^n)} \leq \|u_0\|_{C^{k,\alpha}(\mathbb{T}^n)}.$$

1.3 Energy estimates, entropy

Let $u : \mathbb{T}^n \times [0, \infty[\to \mathbb{R}$ be a smooth solution of (1.1)-(1.2) for smooth data u_0.

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth convex function, and define

$$F(u) = \int_{\mathbb{T}^n} f(u)\, dx .$$

Theorem 1.3.1 There holds

$$\frac{d}{dt} F(u(t)) = -\int_{\mathbb{T}^n} f''(u)\, |\nabla u|^2 dx \leq 0 .$$

Proof: Simply compute

$$\frac{d}{dt}\left(\int_{\mathbb{T}^n} f(u(t))\,dx \right) = \int_{\mathbb{T}^n} f'(u)\, u_t \, dx$$

$$\overset{(1.1)}{=} \int_{\mathbb{T}^n} f'(u)\, \Delta u \, dx = -\int_{\mathbb{T}^n} f''(u)\, |\nabla u|^2 dx . \qquad \square$$

Remark 1.3.1. If $u_0 \geq 0$, $u > 0$ the same result holds for any convex function $f : \mathbb{R}^+ \to \mathbb{R}$.

__Examples 1.3.1.__ i) Let $p > 1$ and set

$$f(u) = |u|^p$$

with $f'(u) = p\, u\, |u|^{p-2}$, $f''(u) = p(p-1)|u|^{p-2} \geq 0$.

Theorem 1.3.1 gives a sharpened version of the bound

$$\| u(t) \|_{L^p} \leq \| u_0 \|_{L^p}, \quad \forall t > 0$$

from Theorem 1.1.1.

ii) Let $u > 0$, $f(u) = u \log u$ so that

$$F(u) = \int_{\mathbb{T}^n} u \log u \, dx$$

is the __entropy__ of u. With

$$f'(u) = \log u + 1, \quad f''(u) = \frac{1}{u}$$

we find

$$\frac{d}{dt} F(u) = - \int_{\mathbb{T}^n} \frac{|\nabla u|^2}{u} dx = - \int_{\mathbb{T}^n} 4 |\nabla \sqrt{u}|^2 dx.$$

We can also apply Theorem 1.6 to derivatives of u. In particular, we find the <u>energy inequality</u>:

<u>Theorem</u> 1.3.2. For any sol. u of (1.1), (1.2) there holds

$$\| \nabla u(t) \|_{L^2} \leq \| \nabla u_0 \|_{L^2}$$

with

$$\frac{d}{dt}\left(\int_{\mathbb{T}^n} |\nabla u(t)|^2 dx \right) = -2 \int_{\mathbb{T}^n} |u_t|^2 dx$$

<u>Proof</u>. Compute

$$\frac{d}{dt}\left(\int_{\mathbb{T}^n} |\nabla u| \, dx \right) = 2 \int_{\mathbb{T}^n} \nabla u \, \nabla \underbrace{u_t}_{= \Delta u} \, dx = -2 \int_{\mathbb{T}^n} |\Delta u|^2 dx ,$$

and use (1.1).

$\square$

Remark 1.3.2

i) By means of Thm. 1.3.2 we can regard (1.1) as the gradient flow for the Dirichlet energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 \, dx$$

with respect to the standard inner product $g(\varphi, \psi) = \int_{\mathbb{T}^n} \varphi \psi \, dx$ on $L^2(\mathbb{T}^n)$.

Abstract setting. Let (M, g) be a Riemannian manifold, $E \in C^1(M)$.

For any $u \in M$ let $\nabla E(u) \in T_u M$ with

$$g_u(\nabla E(u), v) = dE(u) \cdot v$$

$$= \langle dE(u), v \rangle_{T_u^* M \times T_u M}, \quad \forall v \in T_u M.$$

A C^1-curve $u = u(t)$ evolves by gradient flow if u satisfies

$$\frac{du}{dt} = - \nabla E(u);$$

that is, if for all $t > 0$ there holds

$$g_u\left(\frac{du}{dt}, v\right) = - dE(u) \cdot v, \quad \forall v \in T_u M;$$

in particular

$$g_u\left(\frac{du}{dt}, \frac{du}{dt}\right) = - dE(u) \cdot \frac{du}{dt} = - \frac{d}{dt} E(u).$$

ii) There also is a "nonlinear" way of viewing the heat equation as a gradient flow.

Ref. Felix Otto: "The geometry of dissipative evolution equations: the porous medium equation", Comm. PDE 26 (2001), 101-174.

Let

$$\mathcal{M} = \{ 0 < \rho \in L^1(\mathbb{R}^n);\ \|\rho\|_{L^1} = 1 \}$$

be the space of (positive) probability measures, formally with tangent space

$$T_\rho \mathcal{M} = \{ s \in L^1(\mathbb{R}^n);\ \int_{\mathbb{R}^n} s\, dx = 0 \} =: L^1_0(\mathbb{R}^n).$$

Recall that for any $u_0 = \rho_0 \in \mathcal{M}$ the Cauchy problem (1.1), (1.2) has a unique smooth solution $u(t) = \rho(t)$, $t > 0$, where

$$\|\rho(t)\|_{L^1} \leq \|\rho_0\|_{L^1}, \quad \forall t > 0;$$

moreover, $\rho(t) > 0$ by the maximum principle, and in view of

$$\int_{\mathbb{R}^n} \rho(t)\, dx = \underbrace{\int_0^t \Big(\int_{\mathbb{R}^n} \Delta\rho\, dx \Big) dt}_{=0} + \underbrace{\int_{\mathbb{R}^n} \rho_0\, dx}_{=1} = 1$$

we have $\rho(t) \in \mathcal{M}$ for all $t > 0$.

As inner product on $T_\rho \mathcal{M}$

for any $\rho \in \mathcal{M}$ now define

$$g_\rho(s_1, s_2) = \int_{\mathbb{R}^n} \rho \nabla p_1 \cdot \nabla p_2 \, dx ,$$

where for $s = s_1, s_2 \in T_\rho \mathcal{M} = L^1_0(\mathbb{R}^n)$ we let p solve the equation

$$- \operatorname{div}(\rho \nabla p) = s \quad \text{in } \mathbb{R}^n$$

so that

$$g_\rho(s_1, s_2) = \int_{\mathbb{R}^n} s_1 p_2 \, dx .$$

We then can check

$$g_\rho\left(\frac{d\rho}{dt}, s\right) + dE(\rho) \cdot s$$

$$= \int_{\mathbb{R}^n} \left(\frac{d\rho}{dt} p - (\ln \rho + 1) \operatorname{div}(\rho \nabla p) \right) dx$$

$$= \int_{\mathbb{R}^n} \Big(\rho_t - \underbrace{\operatorname{div}\left(\rho \left(\frac{\nabla \rho}{\rho}\right)\right)}_{= \Delta \rho} \Big) p \, dx = 0$$

and find

$$\frac{d}{dt} E(\rho) = - g_\rho\left(\frac{d\rho}{dt}, \frac{d\rho}{dt}\right) .$$

Finally, we can characterize the metric

$$g_\rho(s,s) = \inf\left\{\int_{\mathbb{R}^n} \rho |u|^2 dx;\ u \text{ vector field with } s + \operatorname{div}(\rho u) = 0\right\}.^{1)}$$

Indeed, a minimizer u solves

$$\int_{\mathbb{R}^n} \rho u v \, dx = 0, \ \forall v: \operatorname{div}(\rho v) = 0.$$

Setting $w = \rho v$, we find

$$\int_{\mathbb{R}^n} u w \, dx = 0, \ \forall w: \operatorname{div} w = 0,$$

and[1)] there exists p such that $u = \nabla p$; hence

$$-\operatorname{div}(\rho \nabla p) = -\operatorname{div}(\rho u) = s,$$

and

$$\int_{\mathbb{R}^n} \rho |u|^2 dx = \int_{\mathbb{R}^n} \rho |\nabla p|^2 dx,$$

as claimed.

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1) by Weyl lemma; on $\mathbb{T}^n$ instead of $\mathbb{R}^n$ use Hodge decomposition.

Energy methods also give the following remarkable result. (Note that the Cauchy problem (1.1), (1.2) is not well-posed in backwards time!)

Backwards uniqueness. The following result is taken from Evans, p. 64 f.

Theorem 1.3.3. Suppose u, v smoothly solve (1.1) on $\mathbb{T}^n \times [0, T]$ with

$$u(T) = v(T).$$

Then $u \equiv v$.

Proof (Evans, p. 64 f.): Let $w = u - v$ and set

$$e(t) = \int_{\mathbb{T}^n} w^2(t)\, dx, \quad 0 \le t \le T$$

with

$$\dot{e} = \frac{de}{dt} = -2\int_{\mathbb{T}^n} |\nabla w|^2 dx, \quad 0 < t < T,$$

$$\ddot{e} = -4\int_{\mathbb{T}^n} \nabla w \nabla w_t\, dx = -4\int_{\mathbb{T}^n} \nabla w \nabla \Delta w\, dx$$

$$= 4\int_{\mathbb{T}^n} |\Delta w|^2 dX.$$

Integrating by parts we have

$$\dot{e} = -2\int_{\mathbb{T}^n} |\nabla w|^2 dx = 2\int_{\mathbb{T}^n} w \,\Delta w \, dx .$$

Then by Hölder's inequality

$$(\dot{e})^2 \leq 4 \int_{\mathbb{T}^n} |w|^2 dx \int_{\mathbb{T}^n} |\Delta w|^2 dx = e \cdot \ddot{e} .$$

Suppose by contradiction that $e > 0$ on $I = [t_1, t_2[\subset [0,T]$, $e(t_2) = 0$.

Then $f(t) = \log e(t)$ is well-defined in I with

$$\ddot{f} = \left(\frac{\dot{e}}{e}\right)^{\cdot} = \frac{\ddot{e}}{e} - \left(\frac{\dot{e}}{e}\right)^2 \geq 0 .$$

Hence f is convex in I, and for any $0 < \alpha < 1$, $t_1 < t < t_2$ there holds

$$f((1-\alpha)t_1 + \alpha t) \leq (1-\alpha) f(t_1) + \alpha f(t).$$

Exponentiating, we find

$$e((1-\alpha)t_1 + \alpha t) \leq \left(e(t_1)\right)^{1-\alpha} \cdot \left(e(t)\right)^{\alpha} \underset{(t \uparrow t_2)}{\longrightarrow} 0 .$$

Hence $e \equiv 0$ in $]t_1, t_2]$, contrary to assumption.

$\square$

1.4 Inhomogeneous equations

Consider now the equation (1.3), that is,

$$(1.3) \quad u_t - \Delta u = f \quad \text{in } \mathbb{T}^n \times]0,T[$$

with initial data $u|_{t=0} = u_0$. By linearity, we may assume that

$$u|_{t=0} = 0.$$

The superposition principle immediately yields a solution for (1.3) with $u|_{t=0} = 0$.

Theorem 1.4.1: For any $f \in L^1(\mathbb{T}^n \times [0,T])$ the function

$$u(x,t) = \int_0^t \int_{\mathbb{T}^n} G(x-y, t-s) f(y,s)\,dy\,ds$$

is the unique solution to (1,3) with $\sup_{0<t<T} \|u(t)\|_{L^1} \leq \|f\|_{L^1}$ and such that

$$\|u(t)\|_{L^1} \to 0 \quad (t \downarrow 0).$$

Proof. For any other such solution v the function $w = u - v$ is a smooth sol. of (1.1) with $\|w(t)\|_{L^1} \to 0$ $(t\downarrow 0)$. By Cor. 1.2.2. then $w \equiv 0$. □

<u>Hölder</u> <u>estimates</u>. Given f, from our experience with elliptic regularity theory we expect the solution u to (1.3) to "gain" two spatial <u>and</u> one time derivative over the regularity of f.

So far, we have seen that the solution $u(t) = S(t)u_0$ to the Cauchy problem (1.1), (1.2) preserves the Hölder bounds of the data u_0 by the maximum principle; see Cor. 1.2.3.

By Thm. 1.4.1 the solution u of (1.3) may be represented as

$$u(t) = \int_0^t S(t-s)f(s)\,ds\,;$$

hence for any $0 < \alpha \leq 1$ by Cor. 1.2.3 there holds

$$[u(t)]_{C^\alpha} \leq \int_0^t [f(s)]_{C^\alpha}\,ds\,.$$

But is there any regularity with respect to t? – Consider first again the Cauchy problem (1.1), (1.2).

Theorem 1.4.2. Let $0 < \alpha \leq 1$. For $u_0 \in C^\alpha(\mathbb{T}^n)$ let $u(t) = S(t)u_0$ be the unique smooth solution to (1.1), (1.2) given by Thm. 1.1.1. Then for any $z_1 = (x_1, t_1)$, $z_2 = (x_2, t_2)$ we have

$$|u(z_1) - u(z_2)| \leq C[u_0]_\alpha \left(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2} \right).$$

Proof. By Cor. 1.2.3 we may assume $x_1 = x_2 = x_0$. Fix $R = |t_1 - t_2|^{2}$. For $0 \leq \varphi \in C_c^\infty(B_R(x_0))$ with

$$\int_{B_R(x_0)} \varphi \, dx = 1, \quad \int_{B_R(x_0)} |\nabla^k \varphi| \leq C R^{-k}, \quad k = 1, 2,$$

let

$$\bar{u}_\varphi(t) = \int_{B_R(x_0)} u(t) \varphi \, dx.$$

Then

$$|\bar{u}_\varphi(t) - u(x_0, t)| \leq \int_{B_R(x_0)} |u(t) - u(x_0, t)| \varphi \, dx$$

$$\leq \sup_{|x - x_0| < R} |u(x,t) - u(x_0, t)| \leq [u(t)]_{C^\alpha} R^\alpha$$

$$\leq [u_0]_{C^\alpha} R^\alpha \quad \text{for all } t > 0$$

and

$$\bar{u}_\varphi(t_2) - \bar{u}_\varphi(t_1) = \int_{t_1}^{t_2}\int_{B_R(x_0)} u_t(x,t)\,\varphi(x)\,dx$$

$$= \int_{t_1}^{t_2}\int_{B_R(x_0)} \Delta\big(u(x,t) - u(x_0,t)\big)\,\varphi(x)\,dx$$

$$\leq \int_{t_1}^{t_2}\int_{B_R(x_0)} |u(x,t) - u(x_0,t)|\,|\Delta\varphi(x)|\,dx$$

$$\leq [u_0]_{C^\alpha} R^\alpha \underbrace{|t_2 - t_1|}_{=R^2} \underbrace{\int \max|\Delta\varphi|\,dx}_{\leq CR^{-2}}$$

$$\leq C\,[u_0]_{C^\alpha}\,|t_1 - t_2|^{\alpha/2}. \qquad \square$$

Remark. Applied to the solution u of (1.3), Thm. 1.4.2 likewise yields Hölder regularity of u in time (even if we only require $f(t) \in C^\alpha(\mathbb{T}^n)$ for every t, with $[f(t)]_{C^\alpha} \in L^1_{loc}([0,\infty[)$). However, in order to see the full gain in regularity we have to use more sophisticated tools, following the Campanato approach to the elliptic Schauder estimates.

<u>Function spaces</u>. Because of the different scaling properties of a solution u to (1.1) in the space and time variables, function spaces adapted to the analysis of parabolic pde in general will be defined by norms that distinguish space and time.

<u>Scaling property</u>. Let u be a smooth solution of (1.1) on $\mathbb{R}^n \times]0,\infty[$. Then for any $R > 0$ also the function

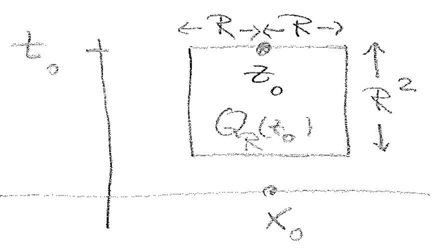
$$u_R(x,t) = u(Rx, R^2 t)$$

is a solution of (1.1).

Thus, in the analysis of (1.1) balls usually will be replaced by <u>parabolic cylinders</u>

$$Q_R(z_0) = \{ z = (x,t) ;\ |x - x_0| < R,\ 0 < t_0 - t < R^2 \},$$

centered at a point $z_0 = (x_0, t_0)$.



For any sufficiently smooth space-time domain $Q \subset \mathbb{R}^n \times \mathbb{R}$, $0 < \alpha \leq 1$ thus define

$$C^{\alpha,\alpha/2}(Q) = \Big\{ u \in C^0(Q) ;\ [u]_{C^{\alpha,\alpha/2}} = \sup_{z_1 \neq z_2} \frac{|u(z_1) - u(z_2)|}{d^\alpha(z_1, z_2)} \simeq \sup_{z_0, r} \Big(R^{-\alpha} \operatorname*{osc}_{Q \cap Q_r(z_0)} (u) \Big) < \infty \Big\},$$

where for $z_i = (x_i, t_i)$, $i = 1,2$, we let

$$d(z_1, z_2) = |x_1 - x_2| + |t_1 - t_2|^{1/2}$$

the <u>parabolic distance</u> between z_1 and z_2, and similarly for $Q \subset \mathbb{T}^n \times \mathbb{R}$.

In particular, we consider domains

$$Q = \Omega^T = \Omega \times]0, T[$$

for a sufficiently smooth $\Omega \subset \mathbb{R}^n$ or $\Omega = \mathbb{T}^n$.
For Ω of type A as in Campanato's work, $Q = \Omega^T$ we also let

$$\mathcal{L}_d^{p,\lambda}(Q) = \Big\{ u \in L^p(Q) ;\ \underbrace{\sup_{\substack{0 < r < r_0 \\ z_0 \in Q}} \Big(r^{-\lambda} \int_{Q_r(z_0) \cap Q} |u - \bar{u}_{r,z_0}|^p \, dz \Big)}_{=: [u]^p_{\mathcal{L}_d^{p,\lambda}}} < \infty \Big\},$$

where $r_0 = \min\{1, \sqrt{T}, \operatorname{diam} \Omega\}$ as in the time-independent case, $1 \leq p < \infty$, $\lambda > 0$, $\bar{u}_{r,z_0} = \int_{Q_r(z_0) \cap Q} u \, dz \Big/ \int_{Q_r(z_0) \cap Q} dz$.

As in the time-independent case we have the following result

<u>Theorem</u> 1.4.3 (Campanato (1963))
For any $0 < \alpha \leq 1$, any domain Ω of type A for $Q = \Omega^T$ there holds

$$C^{\alpha;\alpha/2}(Q) \cong \mathcal{L}_d^{p,\lambda}(Q), \quad \alpha = \frac{\lambda-(n+2)}{p}.$$

<u>Proof</u>. See Rufner, Thm. 4.6.1. □

<u>Goal</u>: Prove $u_t, \nabla^2 u \in C^{\alpha,\alpha/2}$ when u solves (1.3) for $f \in C^{\alpha,\alpha/2}$.
We start by deriving bounds for sol's u of (1.1) in Campanato spaces.

<u>Lemma</u> 1.4.1 (Poincaré-type estimate)
Let u smoothly solve (1.1) on $Q = Q_{2R} = Q_{2R}(0)$.
Then

$$\int_{Q_R} |u - \bar{u}_R|^2 dz \leq C R^2 \int_{Q_{2R}} |\nabla u|^2 dz, \quad \bar{u}_R = \bar{u}_{R,0}.$$

<u>Proof</u>. We may assume $R = 1$. Let $\varphi = \varphi(x) \in C_c^\infty(B_2(0))$ with $\varphi = 1$ on $B_1(0)$, $0 \leq \varphi \leq 1$,

$\bar{u}(t) = \fint_{B_1(0)} u(t)\,dx$. Then we have

$$\bar{u} = \bar{u}_1 = \int_{-1}^{0} \bar{u}(t)\,dt;$$

and for any $-1 \le t_1 \le t_2 \le 0$

$$|\bar{u}(t_1) - \bar{u}(t_2)|^2 = \Big(\fint_{B_1(0)} (u(x,t_1) - u(x,t_2))\,dx \Big)^2$$

$$\le C \int_{B_2(0)} |u(t_1) - u(t_2)|^2 \varphi^2\,dx ,$$

But for any $x \in B_2(0)$ we have

$$u(x,t_2) - u(x,t_1) = \int_{t_1}^{t_2} u_t(x,t)\,dt = \int_{t_1}^{t_2} \Delta u(x,t)\,dt ,$$

so

$$\int_{B_2(0)} |u(t_1) - u(t_2)|^2 \varphi^2\,dx = \int_{B_2(0)} \int_{t_1}^{t_2} (-\Delta u(t))(u(t_1) - u(t_2))\varphi^2\,dt\,dx$$

$$\overset{\begin{pmatrix}\text{Fubini}\\ \text{part. int.}\end{pmatrix}}{=} \int_{t_1}^{t_2} \int_{B_2(0)} \nabla u(t)\, \nabla\big((u(t_1) - u(t_2))\varphi^2\big)\,dx\,dt$$

$$\le \underbrace{\int_{t_1}^{t_2} \|\nabla u(t)\|_{L^2(B_2(0))}\,dt}_{\le \|\nabla u\|_{L^2(Q_2)}} \cdot \Big(\|\nabla u(t_1)\|_{L^2(B_2(0))} + \|\nabla u(t_2)\|_{L^2(B_2(0))} + \|\nabla\varphi\|_{L^\infty} \|(u(t_1) - u(t_2))\varphi\|_{L^2(B_2(0))} \Big)$$

$$\le C \|\nabla u\|^2_{L^2(Q_2)} + \|\nabla u(t_1)\|^2_{L^2(B_2(0))} + \|\nabla u(t_2)\|^2_{L^2(B_2(0))} + \frac{1}{2} \int_{B_2(0)} |u(t_1) - u(t_2)|^2 \varphi^2\,dx .$$

Hence

$$\int_{-1}^{0}\int_{-1}^{0} |\bar{u}(t_1) - \bar{u}(t_2)|^2\,dt_1\,dt_2 \le C \|\nabla u\|^2_{L^2(Q_2)} .$$

Thus also

$$\int_{-1}^{0} |\bar{u} - \bar{u}(t_2)|^2 \, dt_2 = \int_{-1}^{0} \Big| \int_{-1}^{0} (\bar{u}(t_1) - \bar{u}(t_2)) \, dt_1 \Big|^2 dt_2$$

$$\leq \int_{-1}^{0}\int_{-1}^{0} |\bar{u}(t_1) - \bar{u}(t_2)|^2 \, dt_1 \, dt_2 \leq C \, \|\nabla u\|^2_{L^2(Q_2)}.$$

Finally, by Poincaré's inequality

$$\int_{Q_1} |u - \bar{u}|^2 \, dz = \int_{-1}^{0} \int_{B_1(0)} |u - \bar{u}|^2 \, dx \, dt$$

$$\leq 2 \int_{-1}^{0} \int_{B_1(0)} \left(|u - \bar{u}(t)|^2 + |\bar{u} - \bar{u}(t)|^2 \right) dx \, dt$$

$$\leq C \int_{-1}^{0} \|\nabla u(t)\|^2_{L^2(Q_1)} \, dt + C \int_{-1}^{0} |\bar{u} - \bar{u}(t)|^2 \, dt$$

$$\leq C \, \|\nabla u\|^2_{L^2(Q_2)},$$

and the claim follows. $\square$

Complementing Lemma 1.4.1 we also have

<u>Lemma 1.4.2</u> (Caccioppoli inequality)

Let u smoothly solve (1.1) on $Q = Q_{2R}$.

Then

$$\int_{Q_R} |\nabla u|^2 dz \leq C R^{-2} \int_{Q_{2R}} |u - \bar{u}_{2R}|^2 dz .$$

<u>Proof.</u> (NLoG $R = 1$.) Let $\varphi \in C^\infty(\mathbb{R}^n \times]-\infty, 0])$ with $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on Q_1, $\varphi = 0$ on $\mathbb{R}^n \times]-\infty, 0[\setminus Q_2$, $\bar{u} = \bar{u}_2$.

Multiply (1.1) with $(u - \bar{u})\varphi^2$ and integrate by parts to obtain

$$0 = \int_{Q_2} \left(u_t (u - \bar{u}) \varphi^2 + \nabla u \, \nabla\left((u - \bar{u})\varphi^2\right) \right) dz$$

$$= \int_{Q_2} \frac{d}{dt}\left(\frac{|u - \bar{u}|^2}{2} \right) \varphi^2 dz + \int_{Q_2} |\nabla u|^2 \varphi^2 dz$$

$$+ \underbrace{2 \int_{Q_2} \nabla u \, (u - \bar{u}) \, \varphi \, \nabla \varphi \, dz}_{\leq \left(\int_{Q_2} |\nabla u|^2 \varphi^2 dz \right)^{1/2} \cdot \left(\int_{Q_2} |u - \bar{u}|^2 |\nabla \varphi|^2 dz \right)^{1/2}}$$

$$\geq \frac{1}{2} \int_{B_2 \times \{0\}} (u - \bar{u})^2 \varphi^2 dx + \frac{1}{2} \int_{Q_2} |\nabla u|^2 \varphi^2 dx$$

$$- C \int_{Q_2} |u - \bar{u}|^2 \left(|\nabla \varphi|^2 + \left|\frac{\partial \varphi}{\partial t}\right| \right) dz$$

The claim follows.

Finally, there holds

Lemma 1.4.3.

Let u smoothly solve (1.1) on $Q = Q_{2R}$. Then
$$\max_{Q/2} |u| \le C \fint_{Q_{2R}} |u| \, dz .$$

Proof. NloG $R = 1$. Let φ as in the proof of Lemma 1.4.2 and set $v = u\varphi$. Then
$$v_t - \Delta v = \overbrace{(\cancel{u_t - \Delta u})}^{=0}\varphi + u(\varphi_t + \Delta\varphi) - 2\,\mathrm{div}(\nabla\varphi\, u) =: f$$
on $\mathbb{R}^n \times [-4,0]$ with $v = 0$ at $t = -4$. By Theorem 1.3.1 then
$$v(x,t) = \int_{-4}^{0}\int_{\mathbb{R}^n} G(x-y, t-s)\, f(y,s)\, dy\, ds .$$
$$= \int_{-4}^{0}\int_{\mathbb{R}^n} [G(x-y, t-s)\, u(\varphi_t + \Delta\varphi)$$
$$+ 2\nabla G(x-y, t-s) u \nabla\varphi]\, dy\, ds .$$

But $\mathrm{supp}(\varphi_t, \Delta\varphi, \nabla\varphi) \subset Q_2 \setminus Q_1$; and for $z_0 = (x_0, t_0) \in Q_{1/2}$ there holds
$$G(z_0 - z),\ |\nabla G(z_0 - z)| \le C$$
uniformly for $z \in Q_2 \setminus Q_1$. The claim follows. $\square$

We conclude the fundamental estimates.

Proposition 1.4.1. (Campanato)
For a smooth solution u of (1.1) on Q_R and any $0<r<R$ there holds

i) $$\int_{Q_r} |u|^2 dz \le C\left(\frac{r}{R}\right)^{n+2} \int_{Q_R} |u|^2 dz ,$$

ii) $$\int_{Q_r} |u-\bar{u}_r|^2 dz \le C\left(\frac{r}{R}\right)^{n+4} \int_{Q_R} |u-\bar{u}_R|^2 dz .$$

NLoG $R=1$

Proof. i) For $0<r<1/4$ by Lemma 1.4.3 we have
$$\int_{Q_r} |u|^2 dz \le C r^{n+2} \max_{Q_{1/4}} |u|^2 \le C r^{n+2} \int_{Q} |u|^2 dz .$$

ii) For $0<r<1/4$ by i) and Lemmas 1.4.1-2 we have
$$\int_{Q_r} |u-\bar{u}_r|^2 dz \le C r^2 \int_{Q_{2r}} |\nabla u|^2 dz \le C r^{n+4} \int_{Q_{1/2}} |\nabla u|^2 dz$$
$$\le C r^{n+4} \int_{Q_1} |u-\bar{u}_1|^2 dz .$$
□

Perturbation theory. To proceed, on any $Q=Q_R$ we need to be able to split a solution u to (1.3) into a solution of (1.1) and a remainder w solving (1.3) with $w=0$ on the parabolic boundary.

With energy methods we can easily solve the initial-boundary value problem on a (sufficiently smooth) domain $\Omega \subset\subset \mathbb{R}^n$.

For any $T > 0$ let $Q = \Omega \times]0,T[$ with "parabolic boundary" $\Gamma = \Omega \times \{0\} \cup \partial\Omega \times [0,T]$.

Given $u_0, f \in C^\infty(\overline{Q})$, consider the initial-boundary value problem

$$u_t - \Delta u = f \quad \text{in } Q\,,$$
$$u = u_0 \quad \text{on } \Gamma\,.$$

Letting $v = u - u_0$, $g = f - \left(\frac{\partial u_0}{\partial t} - \Delta u_0\right)$, we may assume that $u_0 \equiv 0$:

<u>Theorem</u> 1.4.4. For any $f \in L^2(Q)$ there exists a unique solution $u \in V_0^{2,1}(Q)$ of (1.3), where

$$V_0^{2,1}(Q) = \{u \in L^2(Q);\ u_t, \nabla^2 u \in L^2(Q),\ u(t) \in H_0^1(\Omega) \text{ for a.e. } t,\ \operatorname*{ess\,sup}_{0<t<T} \|\nabla u(t)\|_{L^2} < \infty\}, \text{ and}$$

with

$$u|_{t=0} = 0 \text{ in the sense of traces}\},$$

and

$$\|u\|_{V_0^{2,1}} = \|u_t\|_{L^2(Q)} + \|\nabla^2 u\|_{L^2(Q)} + \operatorname*{ess\,sup}_{0<t<T} \|\nabla u(t)\|_{L^2(\Omega)} \leq C\|f\|_{L^2(Q)};$$

Proof: Existence. Let $(\varphi_i)_{i\in\mathbb{N}}$ be an L^2-orthonormal basis of eigenfunctions of $A = -\Delta$ in $L^2(\Omega)$ satisfying

$$-\Delta \varphi_i = \lambda_i \varphi_i \quad \text{in } \Omega$$
$$\varphi_i = 0 \quad \text{on } \partial\Omega$$

with eigenvalues $0 < \lambda_1 < \lambda_2 \leq \dots$

Given $j \in \mathbb{N}$ define an approximate solution $u^{(j)}$ to (1.) by letting

$$u^{(j)}(t) = \sum_{i=1}^{j} \alpha_i^{(j)}(t)\varphi_i\,,$$

where

$$\frac{d\alpha_i^{(j)}}{dt} + \lambda_i \alpha_i^{(j)} = \int_\Omega f(t)\varphi_i\,dx, \quad 1 \leq i \leq j$$
$$\alpha_i^{(j)}(0) = 0\,.$$

Then for $0 < t < T$ there holds

$$\int_\Omega \left(u_t^{(j)} - \Delta u^{(j)} - f\right)\varphi_i\,dx = 0, \quad 1 \leq i \leq j.$$

Using that $-\Delta u^{(j)} = \sum_{i=1}^{j} \alpha_i^{(j)} \lambda_i \varphi_i$ we conclude

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u^{(j)}\|_{L^2}^2\right) + \|\Delta u^{(j)}\|_{L^2}^2 = \int_\Omega f\left(-\Delta u^{(j)}\right)dx$$
$$\leq \|f\|_{L^2}\|\Delta u^{(j)}\|_{L^2} \leq \frac{1}{2}\|f\|_{L^2}^2 + \frac{1}{2}\|\Delta u^{(j)}\|_{L^2}^2, \quad 0 < t < T,$$

It follows that $\nabla u^{(j)} \in L^\infty_t L^2_x$, $\Delta u^{(j)} \in L^2$

with
$$\|\nabla u^{(j)}\|^2_{L^{\infty,2}_{t,x}} \leq \|f\|^2_{L^2},$$

$$\|\Delta u^{(j)}\|^2_{L^2} \leq \|f\|^2_{L^2}.$$

By elliptic regularity theory therefore also $u^{(j)}(t) \in H^2(\Omega)$ for all $0<t<T$ with

$$\|\nabla^2 u^{(j)}\|^2_{L^2} \leq C \int_0^T \|\Delta u^{(j)}(t)\|^2_{L^2} dt \leq C \|f\|^2_{L^2}.$$

Hence $(u^{(j)})_{j\in\mathbb{N}}$ is bounded in $V_0^{2,1}(Q)$.

A subsequence $u^{(j)} \xrightarrow{w} u$ in $V_0^{2,1}(Q)$,

where u solves (1.3) for every $i \in \mathbb{N}$, and

$$\|\nabla^2 u\|^2_{L^2} \leq \liminf_{j\to\infty} \|\nabla^2 u^{(j)}\|^2_{L^2} \leq \|f\|^2_{L^2};$$

in particular

$$\|u_t\|_{L^2} \leq \|\Delta u\|_{L^2} + \|f\|_{L^2} \leq C\|f\|_{L^2}.$$

Moreover, by the Poincaré inequality

$$\operatorname{ess\,sup}_{0<t<T} \left(\|u(t)\|^2_{L^2} + \|\nabla u(t)\|^2_{L^2}\right) \leq C \|\nabla u\|^2_{L^{\infty,2}_{t,x}}$$
$$\leq C\|f\|^2_{L^2}.$$

<u>Uniqueness.</u> Let $u, v \in \overset{\circ}{V}{}^{2,1}(Q)$ with $u|_{t=0} = 0 = v|_{t=0}$ solve (1.3) on Q.

Then $w = u - v \in \overset{\circ}{V}{}^{2,1}(Q)$ solves (1.1) with $w|_{t=0} = 0$.

Testing (1.1) with w is admissible, and we obtain

$$0 = \int_0^t \int_\Omega (w_t - \Delta w)\, w \, dx\, dt = \int_0^t \int_\Omega \left(\frac{d}{dt}\left(\frac{|w|^2}{2}\right) + |\nabla w|^2 \right) dx\, dt$$

$$= \frac{1}{2}\left(\|w(t)\|^2_{L^2(\Omega)} - \underbrace{\|w(0)\|^2_{L^2(\Omega)}}_{=0} \right) + \int_0^t \int_\Omega |\nabla w|^2 \, dx\, dt, \quad 0 < t < T;$$

so $w \equiv 0$, as claimed. $\square$

Consider now the inhomogeneous equation

(1.6) $\quad u_t - \Delta u = -\operatorname{div} f$

for some $f = (f^i)_{1\le i\le n} \in C^\infty(\overline{Q}_R)$.

<u>Lemma</u> 1.4.4.

Let u smoothly solve (1.6) on Q_R.

Then for any $0<r<R$ there holds

$$\int_{Q_r} |\nabla u - \overline{\nabla u}_r|^2 dz \le C\left(\frac{r}{R}\right)^{n+4} \int_{Q_R} |\nabla u - \overline{\nabla u}_R|^2 dz$$

$$+ C\int_{Q_R} |f - f_R|^2 dz .$$

<u>Proof</u>. NLoG $R=1$. Split $u = v + w$ on Q_R,

where w solves

$$w_t - \Delta w = \operatorname{div} f \quad \text{in } Q_R$$

$$w\,|_{\Gamma_R} = 0,$$

with $\quad \Gamma_R = B_R \times \{-R^2\} \cup \partial B_R \times]-R^2, 0]$

the parabolic boundary of Q_R, and where

v solves (1.1) in Q_R.

By Theorem 1.4.4 there is a unique solution $w \in V_0^{2,1}(Q_R)$, and we have

$$\|\nabla w\|^2_{L^2(Q_R)} \leq \frac{1}{2}\|w(0)\|^2_{L^2(B_R)} + \|\nabla w\|^2_{L^2(Q_R)}$$

$$= \int_{Q_R} (w_t - \Delta w) w \, dz = \int_{Q_R} \operatorname{div}(f - f_R) w \, dz$$

$$= -\int_{Q_R} (f - f_R) \cdot \nabla w \, dz \leq \|f - f_R\|_{L^2(Q_R)} \|\nabla w\|_{L^2(Q_R)}$$

so that

$$(1.7) \quad \|\nabla w\|^2_{L^2(Q_R)} \leq \|f - f_R\|^2_{L^2(Q_R)} .$$

Estimating

$$\|\nabla u - \nabla v\|_{L^2} \leq \|\nabla w\|_{L^2}$$

on Q_r, Q_R, respectively, our claim follows from Proposition 1.4.1.

$\square$

Theorem 1.4.5. For any $u_0 \in C^{2,\alpha}(\mathbb{T}^n)$, any $f \in C^{\alpha,\alpha/2}(\mathbb{T}^n \times]0,T[)$ for some $0 < \alpha < 1$ there exists a unique solution u to (1.2), (1.3) with u_t, $\nabla^2 u \in C^{\alpha,\alpha/2}$, satisfying

$$[u_t]_{C^{\alpha,\alpha/2}} + [\nabla^2 u]_{C^{\alpha,\alpha/2}} \leq C\left([u_0]_{C^{2,\alpha}} + [f]_{C^{\alpha,\alpha/2}}\right).$$

Proof. By Theorem 1.4.2 we may assume $u_0 = 0$.
i) Suppose $f \in C^\infty(\mathbb{T}^n \times [0,T])$, and let u as in Thm. 1.4.1.
Let $v = \frac{\partial}{\partial x_i} u$ be any spatial derivative of u solving (1.6) with $f^i = f$, $f^j = 0$ $(j \neq i)$.
By Lemma 1.4.4, for any $z_0 = (0,0)$, any $0 < r < R_0$:
for

$$\Phi(r) = \int_{Q_r} |\nabla v - \overline{\nabla v}_r|^2 dz$$

we have

$$\Phi(r) \leq C\left(\frac{r}{R}\right)^{n+4} \Phi(R) + C[f]_{C^{\alpha,\alpha/2}} R^{n+2+2\alpha},$$

for all $0 < r < R \leq R_0$.
Giaquinta's "useful lemma" (see FA II, Lemma 10.3.2) then implies $\nabla v \in \mathcal{L}_d^{2,n+2+2\alpha} \cong C^{\alpha,\alpha/2}$.
Hence, $\nabla^2 u \in C^{\alpha,\alpha/2}$, $u_t = \Delta u + f \in C^{\alpha,\alpha/2}$ with the desired bound.

ii) Given $f \in C^{\alpha,\alpha/2}(\mathbb{T}^n \times [0,T])$ we can extend f to $f \in C^{\alpha,\alpha/2}(\mathbb{T}^n \times \mathbb{R})$. Let $\rho \in C_c^\infty(B_1(0))$ be a mollifier, $\rho_\varepsilon(x,t) = \varepsilon^{-n-2}\rho\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)$, $\varepsilon > 0$.

Then
$$f_\varepsilon = f * \rho_\varepsilon \in C^\infty \cap C^{\alpha,\alpha/2}(\mathbb{T}^n \times [0,T])$$

with
$$[f_\varepsilon]_{C^{\alpha,\alpha/2}} = \sup_{\substack{h \neq 0 \\ 0 \leq a \leq n}} \frac{\|(f - \tau_h^a f) * \rho_\varepsilon\|_{L^\infty}}{|h|^\alpha} \leq [f]_{C^{\alpha,\alpha/2}}$$

for any $\varepsilon > 0$, where
$$\tau_h^i f(x,t) = f(x + he_i, t), \quad 1 \leq i \leq n,$$
$$\tau_h^0 f(x,t) = f(x, t + h^2).$$

By i) the solution u_ε to (1.3) with $u_\varepsilon|_{t=0} = 0$ satisfies

$$\left[\frac{\partial u_\varepsilon}{\partial t}, \nabla^2 u_\varepsilon\right]_{C^{\alpha,\alpha/2}} \leq C [f_\varepsilon]_{C^{\alpha,\alpha/2}}$$
$$\leq C [f]_{C^{\alpha,\alpha/2}}.$$

By Arzelà-Ascoli, $u_\varepsilon \to u$, $\frac{\partial u_\varepsilon}{\partial t} \to u_t$, $\nabla^2 u_\varepsilon \to \nabla^2 u$ uniformly on $\mathbb{T}^n \times [0,T]$ as $\varepsilon \downarrow 0$ suitably, where u solves (1.3) with

$$[u_t]_{C^{\alpha,\alpha/2}} + [\nabla^2 u]_{C^{\alpha,\alpha/2}} \leq C [f]_{C^{\alpha,\alpha/2}}.$$

Uniqueness follows from Thm. 1.2.1 (max. princ.). $\square$

1.5 Variable coefficients

For a fixed (positively definite) matrix $a^{(0)} = (a^{(0)}_{ij})_{1\le i,j\le n}$

now consider also the equation

(1.7) $\quad u_t + A^{(0)}u = 0$ on $\mathbb{T}^n \times]0,T[$,

where

$$A^{(0)}u = -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a^{(0)}_{ij}\frac{\partial u}{\partial x_j}\right).$$

We may assume $a^{(0)}_{ij} = a^{(0)}_{ji}$, $1 \le i,j \le n$. Then there exists a rotation $R \in SO(n)$ such that

$$R\, a^{(0)} R^t = \operatorname{diag}(d_1^2, \dots, d_n^2), \; d_i > 0.$$

Letting $D = \operatorname{diag}(d_1, \dots, d_n)$, $S = D^{-1}R$, the function v with

$$v(y,t) = u(S^{-1}y, t)$$

then satisfies (1.1) on $\mathbb{R}^n \times]0,T[$, and $v(t)$ is periodic with fundamental domain $S(]0,1[^n)$. Hence all the results of sections 1.2–1.4 also hold for (1.7).

In particular, given any $f \in L^1(\mathbb{T}^n \times]0,T[)$ the unique solution extend periodically

$$v(x,t) = \int_{\mathbb{R}^n} G(x-y, t-s)\, f(S^{-1}y, s)\, dy$$

of the equation (1.3) with right hand side $f \circ S^{-1}$ yields a solution $u = v \circ S$ of the equation

(1.8) $\quad u_t + A^{(0)} u = f \quad$ in $\mathbb{T}^n \times]0,T[$

with $u_{|t=0} = 0$. Moreover Theorems 1.4.2 and 1.4.5 also hold for (1.7), (1.8), resp.

By using the estimates of Thm. 1.4.5 we can then also obtain existence and Schauder estimates for solutions to the problem

(1.9) $\quad u_t + Au = f \quad$ in $\mathbb{T}^n \times]0,T[$

for an arbitrary operator

$$Au = -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)$$

with uniformly elliptic coefficients $a_{ij} = a_{ji}$ satisfying

$$\forall \xi \in \mathbb{R}^n: \lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(x,t)\xi^i\xi^j \leq \Lambda|\xi|^2$$

for some uniform constants $0 < \lambda \leq \Lambda$.

Theorem 1.5.1 (Schauder estimates)
Let $a = (a_{ij})$ with $a_{ij}, \nabla a_{ij} \in C^{\alpha, \alpha/2}$ be symmetric and uniformly parabolic, and let u with $u_t, \nabla^2 u \in C^{\alpha, \alpha/2}$ be a solution α!
of (1.9) for some $f \in C^{\alpha, \alpha/2}$. Then with a constant $C = C(n, T, \lambda, \Lambda, \alpha, [a_{ij}]_{C^{\alpha,\alpha/2}} + [\nabla a_{ij}]_{C^{\alpha,\alpha/2}})$ there holds

$$\|u\|_{C^{2+\alpha, 1+\alpha/2}(\mathbb{T}^n \times]0,T[)} := [u_t]_{C^{\alpha,\alpha/2}} + [\nabla^2 u]_{C^{\alpha,\alpha/2}} + \|u\|_{C^{2,1}}$$

$$\leq C\left(\|u_0\|_{C^{2,\alpha}} + \|f\|_{C^{\alpha,\alpha/2}}\right),$$

with

$$\|u\|_{C^{2,1}} = \sum_{k=0}^{2} \|\nabla^k u\|_{L^\infty} + \|u_t\|_{L^\infty}.$$

Proof. Fix $z_0 = (x_0, t_0) \in \mathbb{T}^n \times]0,T[$, $R > 0$ such that

$$\sup_{z \in Q_{2R}(z_0)} |a_{ij}(z) - a_{ij}(z_0)| < \delta$$

for some $\delta > 0$ to be determined.

Fix $\varphi \in C^\infty(Q_2)$ with $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in Q_1 and $\varphi \equiv 0$ in $\mathbb{R}^n \times]-\infty, 0[\setminus Q_2$ as in the proof of Lemma 1.4.2, and let

$$\varphi_{R, z_0}(x,t) = \varphi\left(\frac{x - x_0}{R}, \frac{t - t_0}{R^2}\right).$$

The function $v = u\varphi_{R,z_0}$ then solves the equation on $Q_{2R}(z_0) \cap \mathbb{T}^n \times]0,T[$:

$$v_t + Av = v_t + A^{(0)}v + (A - A^{(0)})v$$

$$= (u_t + Au)\varphi + u(\varphi_t + A\varphi) - 2\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j}\dots$$

$$= f\varphi + u(\varphi_t + A\varphi) - 2a(\nabla u, \nabla\varphi)\dots$$

with $A^{(0)}v = \sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}(z_0)\frac{\partial v}{\partial x_j}\right)$, and with

$v|_{t=0} = u_0\varphi$, $\quad \varphi = \varphi_{R,z_0}$ for brevity.

By Thm. 1.4.5 then $[u_t]_{C^{\alpha,\alpha/2}(Q_R(z_0))} + [\nabla^2 u]_{C^{\alpha,\alpha/2}(Q_R(z_0))}$

$$\leq [v_t]_{C^{\alpha,\alpha/2}} + [\nabla^2 v]_{C^{\alpha,\alpha/2}} \leq C\,[u_0\varphi]_{C^\alpha} + I,$$

where (essentially)

$$I = C\Big([(A - A^{(0)})v]_{C^{\alpha,\alpha/2}} + [f\varphi]_{C^{\alpha,\alpha/2}}$$

$$+ [u(\varphi_t + A\varphi)]_{C^{\alpha,\alpha/2}} + [a(\nabla u, \nabla\varphi)]_{C^{\alpha,\alpha/2}}\Big)$$

$$+\dots$$

$$\leq \underbrace{C\delta}^{\leq 1/2}[\nabla^2 v]_{C^{\alpha,\alpha/2}} + C\|f\|_{C^{\alpha,\alpha/2}}$$

$$+ C\|u\|_{C^{2,1}}.$$

Note: $[f\varphi]_{C^{\alpha,\alpha/2}} \leq [f]_{C^{\alpha,\alpha/2}}\underbrace{\|\varphi\|_{L^\infty}}_{=1} + \|f\|_{L^\infty}[\varphi]_{C^{\alpha,\alpha/2}} \leq C\|f\|_{C^{\alpha,\alpha/2}}$,

etc.

Moreover, we estimate $[\nabla u]_{C^{\alpha,\alpha/2}}$, as follows.

For any $z_1=(x_1,t_1)$, $z_2=(x_2,t_2)$ with

$$0<|x_1-x_2|+|t_1-t_2|^{1/2}=h<1$$

we have

$$|\nabla u(z_1)-\nabla u(z_2)|\leq |\nabla u(x_1,t_1)-\nabla u(x_1,t_2)|$$

$$+\|\nabla^2 u\|_{L^\infty}\underbrace{h}_{\leq h^\alpha}.$$

Next, fixing $\rho\in C_c^\infty(B_1(0))$ with $0\leq\rho$, $\int_{\mathbb{R}^n}\rho\,dx=1$, and letting $\rho_h(x)=h^{-n}\rho\left(\frac{x-x_1}{h}\right)\in C_c^\infty(B_h(x_1))$, we have

$$\left|\nabla u(x_1,t_1)-\int_{B_h(x_1)}\nabla u(x,t_1)\rho_h\,dx\right|\leq\|\nabla^2 u\|_{L^\infty}h^\alpha.$$

Finally,

$$\int_{B_h(x_1)}\Big(\nabla u(x,t_1)-\nabla u(x,t_2)\Big)\rho_h\,dx$$

$$=\int_{B_h(x_1)}\underbrace{\Big(u(x,t_2)-u(x,t_1)\Big)}_{\leq|t_2-t_1|\,\|u_t\|_{L^\infty}}\nabla\rho_h\,dx\leq C\|u_t\|_{L^\infty}\underbrace{|t_2-t_1|\,h^{-1}}_{\leq h\leq h^\alpha}.$$

So

$$[\nabla u]_{C^{\alpha,\alpha/2}}=\sup_{d(z_1,z_2)=h}\frac{|\nabla u(z_1)-\nabla u(z_2)|}{h^\alpha}\leq C\|u\|_{C^{2,1}}.$$

Covering $\mathbb{T}^n \times]0,T]$ with finitely many cylinders $Q_R(z_i)$ as above, and summing, we obtain

$$\|u\|_{C^{2+\alpha,1+\alpha/2}} = [u_t]_{C^{\alpha,\alpha/2}} + [\nabla^2 u]_{C^{\alpha,\alpha/2}} + \|u\|_{C^{2,1}}$$

$$\le C\left(\|u_0\|_{C^{2,\alpha}} + \|f\|_{C^{\alpha,\alpha/2}}\right) + C\|u\|_{C^{2,1}}. \tag{1.10}$$

The last term may be absorbed on the left. Indeed, observe that we have

$$0 = \int_{\mathbb{T}^n} (u_t + Au - f)(-\Delta u)\,dx\,dt$$

$$= \frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2} + \int_{\mathbb{T}^n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_k}\frac{\partial^2 u}{\partial x_j \partial x_k}\,dx$$

$$+ \int_{\mathbb{T}^n}\left(\frac{\partial a_{ij}}{\partial x_k}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_k} + f\,\Delta u\right)dx$$

$$\ge \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2(\mathbb{T}^n)} + \frac{\lambda}{2}\|\nabla^2 u\|^2_{L^2(\mathbb{T}^n)} - C\|f\|^2_{L^2(\mathbb{T}^n)} - C\|\nabla u\|^2_{L^2(\mathbb{T}^n)}$$

for every $t \in]0,T[$; hence, upon integrating in time, we find

$$\|\nabla^2 u\|_{L^2(\mathbb{T}^n \times]0,T[)} + \sup_t \|\nabla u(t)\|_{L^2(\mathbb{T}^n)} \le$$

$$\le C\|f\|_{L^2(\mathbb{T}^n \times]0,T[)} + C\|\nabla u_0\|_{L^2(\mathbb{T}^n)} \tag{1.11}$$

and then also

$$\|u_t\|_{L^2(\mathbb{T}^n\times]0,T[)} = \|f - Au\|_{L^2(\mathbb{T}^n\times]0,T[)}$$

$$(1.12) \qquad \leq \|f\|_{L^2(\mathbb{T}^n\times]0,T[)} + C\,\||\nabla u| + |\nabla^2 u|\|_{L^2(\mathbb{T}^n\times]0,T[)}$$

$$\leq C\|f\|_{L^2(\mathbb{T}^n\times]0,T[)} + C\|\nabla u_0\|_{L^2(\mathbb{T}^n)}.$$

Since the space

$$C^{2+\alpha,1+\alpha/2}(\mathbb{T}^n\times]0,T[) = \{w \in C^{2,1};\ w_t, \nabla^2 w \in C^{\alpha,\alpha/2}\}$$

by Arzelà-Ascoli embeds compactly into $C^{2,1}(\mathbb{T}^n\times[0,T]) \hookrightarrow V^{2,1}(\mathbb{T}^n\times]0,T[)$, where

$$V^{2,1}(\mathbb{T}^n\times]0,T[) = \{w \in L^2;\ w_t, \nabla^2 w \in L^2\},$$

by Ehrling's lemma for any $\varepsilon > 0$ there is $C = C(\varepsilon)$ such that (FA II; Lemma 10.)

$$\|u\|_{C^{2,1}} \leq \varepsilon\|u\|_{C^{2+\alpha,1+\alpha/2}} + C\|u\|_{V^{2,1}}.$$

From (1.10)–(1.12) we obtain the claim.

$\square$

Theorem 1.5.2 (Existence). Let $a = (a_{ij})$ with $\nabla a_{ij} \in C^{\alpha, \alpha/2}$ be symmetric and uniformly elliptic in $\mathbb{T}^n \times]0,T[$. Then for any $u_0 \in C^{2,\alpha}(\mathbb{T}^n)$, any $f \in C^{\alpha,\alpha/2}(\mathbb{T}^n \times]0,T[)$ there exists a unique solution $u \in C^{2+\alpha, 1+\alpha/2}(\mathbb{T}^n \times]0,T[)$ of (1.9) with $u|_{t=0} = u_0$, and

$$\|u\|_{C^{2+\alpha,1+\alpha/2}} \leq C\left(\|u_0\|_{C^{2,\alpha}(\mathbb{T}^n)} + \|f\|_{C^{\alpha,\alpha/2}(\mathbb{T}^n \times]0,T[)}\right).$$

Proof. We use the "method of continuity", similar to the proof of FA II, Theorem 10.5.2.

For $0 \leq s \leq 1$ let

$$A^{(s)} u = -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}^{(s)} \frac{\partial u}{\partial x_j}\right), \quad a_{ij}^{(s)} = (1-s)\delta_{ij} + s\, a_{ij},$$

and define

$$S = \{ s \in [0,1];\ \text{our claim holds for } a^{(s)} \}.$$

By Theorem 1.4.5 we have $0 \in S$; so $S \neq \emptyset$.

We proceed to show that S is both relatively open and closed in $[0,1]$, which then implies the Theorem.

Claim 1. S is closed.

Proof. Let $(s_k)_{k\in\mathbb{N}} \subset S$ with $s_k \to s_0$ $(k\to\infty)$. For each s_k let $u_k \in C^{2+\alpha,\, 1+\alpha/2}(\mathbb{T}^n \times]0,T[)$ solve

$$u_{k,t} + A^{(s_k)} u_k = f \quad \text{in } \mathbb{T}^n \times]0,T[,$$
$$u_k|_{t=0} = u_0 .$$

By Theorem 1.5.1 we have the uniform estimate

$$(1.13) \quad \|u_k\|_{C^{2+\alpha,1+\alpha/2}} \leq C\left(\|u_0\|_{C^{2,\alpha}} + \|f\|_{C^{\alpha,\alpha/2}}\right),$$

with $C = C(\ldots,(a_{ij}))$ independent of $k\in\mathbb{N}$.

By Arzelà-Ascoli a subsequence $u_k \to u$ in $C^{2,1}$, where

$$u_t + A^{(s_0)} u = \lim_{k\to\infty} \left(u_{k,t} + A^{(s_k)} u_k\right) = f,$$

and $u \in C^{2+\alpha,\, 1+\alpha/2}(\mathbb{T}^n \times]0,T[)$ again satisfies (1.13). Thus, $s_0 \in S$, and S is closed. $\square$

Claim 2. S is open.

Proof. Let $s_0 \in S$. Set $X := C^{2+\alpha, 1+\alpha/2}(\mathbb{T}^n \times]0,T[)$,

and $R = \|u^{(s_0)}\|_{C^{2+\alpha, 1+\alpha/2}(\mathbb{T}^n \times]0,T[)} + 1$

$$\leq C\left(\|u_0\|_{C^{2,\alpha}(\mathbb{T}^n)} + \|f\|_{C^{\alpha,\alpha/2}(\mathbb{T}^n \times]0,T[)}\right) + 1,$$

with $C = C(\dots, (a_{ij})) > 0$ as in Thm. 1.5.1.

For any $v \in \overline{B_R(0,X)}$, any $s \in [0,1]$ then let

$$\Phi(v) := \tilde{v} \in X$$

be the solution of the problem

$$\tilde{v}_t + A^{(s_0)}\tilde{v} = (A^{(s_0)} - A^{(s)})v + f \quad \text{in } \mathbb{T}^n \times]0,T[,$$
$$\tilde{v}_{|t=0} = u_0,$$

whose existence is guaranteed by our assumption that $s_0 \in S$. By linearity we have

$$\tilde{v} = u^{(s_0)} + w,$$

where

$$w_t + A^{(s_0)}w = (A^{(s_0)} - A^{(s)})v,$$
$$w_{|t=0} = 0,$$

with

$$\|w\|_X \leq C\|(A^{(s_0)} - A^{(s)})v\|_{C^{\alpha,\alpha/2}}$$

$$\leq C|s_0 - s|\,\|v\|_X \leq CR|s_0 - s| \leq 1,$$

if $|s-s_0|<\delta$ for sufficiently small $\delta>0$.

Thus, for such $s\in[0,1]$ we have

$$\Phi : \overline{B_R(0;X)} \to \overline{B_R(0;X)}.$$

Moreover, given $v_1, v_2 \in B_R(0;X)$ with corresponding solutions $w_1, w_2 \in X$ we have

$$\partial_t(w_1-w_2) + A^{(s_0)}(w_1-w_2) = \left(A^{(s_0)}-A^{(s)}\right)(v_1-v_2),$$
$$(w_1-w_2)|_{t=0} = 0,$$

with

$$\|w_1-w_2\|_X \leq C|s_0-s|\,\|v_1-v_2\|_X \leq \tfrac{1}{2}\|v_1-v_2\|_X,$$

if $|s-s_0|<\delta$. For any such $s\in[0,1]$ Banach's fixed point thm. then yields $v\in\overline{B_R(0;X)}$ with $\Phi(v)=v$, where $v\in X$ solves

$$v_t + A^{(s)}v = f \quad \text{in } \mathbb{T}^n\times]0,T[,$$
$$v|_{t=0} = u_0.$$

With Thm. 1.5.1 it follows that $B_\delta(s_0)\cap[0,1]\subset S$, and S is open, as claimed. $\square$

<u>Remark 1.5.1.</u> A priori the constants $C > 0$ in Thm. 1.5.1 and 1.5.2 may depend on the choice of final time $T > 0$.

However, for any given (a_{ij}) and $T > 0$ as in these theorems, the constants $C > 0$ may be chosen uniformly for all $T_0 \leq T$.

Indeed, given $T_0 \leq T$, we may extend any $f \in C^{\alpha, \alpha/2}(\mathbb{T}^n \times]0, T_0[)$ periodically in $t \in \mathbb{R}$ to obtain $f \in C^{\alpha, \alpha/2}(\mathbb{T}^n \times]0, T[)$ with the same Hölder bound as the original f. Solving (1.2), (1.3) for $u \in C^{2+\alpha, 1+\alpha/2}(\mathbb{T}^n \times]0, T[)$ and restricting u to $0 \leq t \leq T_0$, we obtain the solution of (1.2), (1.3) on $\mathbb{T}^n \times]0, T_0[$, with the desired bound.

1.6 Nonlinear equations

Consider the equation

$$(1.14)\qquad \begin{aligned} Lu &:= u_t - \sum_{i,j} \frac{\partial}{\partial x_i}\Big(a_{ij}(x,t,u)\frac{\partial u}{\partial x_j}\Big) \\ &= f(x,t,u,\nabla u) \quad \text{in } \mathbb{T}^n \times]0,T[, \end{aligned}$$

where, for any $R>0$, with constants $0<\lambda=\lambda(R)\le\Lambda=\Lambda(R)$ for any $|u|<R$ there holds

$$\forall \xi\in\mathbb{R}^n:\ \lambda|\xi|^2\le\sum_{i,j}a_{ij}(x,t,u)\xi^i\xi^j\le\Lambda|\xi|^2,$$

uniformly on $\mathbb{T}^n\times]0,T[$, and where the functions

$$a_{ij}(x,t,u)=a_{ji}(x,t,u),\quad f(x,t,u,p)$$

are smooth in $u\in\mathbb{R}$ and $p\in\mathbb{R}^n$ with uniform bounds

$$|\nabla^k_{u,p}f(x,t,u,p)|+|\nabla^k_u a_{ij}(x,t,u)|\le C=C(R)$$

for $0\le k\le 2$ and with uniform Hölder bounds

$$\|\nabla^k_u\nabla^l_x a_{ij}\|_{C^{\alpha,\alpha/2}}+\|\nabla^k_{u,p}f\|_{C^{\alpha,\alpha/2}}\le C=C(R)$$

for some $0<\alpha<1$, uniformly for $|u|,|p|<R$, for all $0\le k\le 2$, $0\le l\le 1$.

Theorem 1.6.1. For any a_{ij}, f as above, for any $u_0 \in C^{2,\alpha}(\Pi^n)$ there exists $0 < T_0 \leq T$ and a unique solution $u \in C^{2+\alpha,\, 1+\alpha/2}(\Pi^n \times]0,T_0])$ of (1.14) with $u|_{t=0} = u_0$.

Proof: For suitably small $0 < T_0 \leq T$ to be determined let $X := C^{2+\alpha,\, 1+\alpha/2}(\Pi^n \times]0,T_0])$ and set $M := \overline{B_1(0;X)}$. Also let

$$R = \|u_0\|_{C^1} + 2 > 0.$$

We seek to obtain a solution $u \in X$ of (1.14) of the form $u = u^{(0)} + v$, where $v \in M$ and where $u^{(0)} \in X$ solves

$$L_0 u^{(0)} := u^{(0)}_t - \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(\cdot, u_0) \frac{\partial u^{(0)}}{\partial x_j}\right) = f(\cdot, u_0, \nabla u_0)$$

$$u^{(0)}\big|_{t=0} = u_0.$$

By choice of $R > 0$, in view of Thm. 1.5.1-2 there exists $u^{(0)}$ as above, and

$$\|u^{(0)}\|_X \leq C(R)\underbrace{\left(\|f(\cdot, u_0, \nabla u_0)\|_{C^{\alpha,\alpha/2}} + \|u_0\|_{C^{2,\alpha}}\right)}_{\leq C\left(\|u_0\|_{C^{1,\alpha}} + \sup_{|u|,|p| \leq R} \|f(\cdot,u,p)\|_{C^{\alpha,\alpha/2}}\right)}.$$

In particular, for sufficiently small $0 < T_0 \le T$ we have

$$\|u^{(0)} - u_0\|_{C^1} \le T_0^{\alpha/2} \, \|u^{(0)} - u_0\|_{C^{2+\alpha,\, 1+\alpha/2}} \le 1.$$

Solving (1.14) then is equivalent to solving

$$L_0 v = v_t - \sum_{i,j} \frac{\partial}{\partial x_i}\Big(a_{ij}(\cdot, u_0)\, \frac{\partial v}{\partial x_j}\Big) = F_0(\cdot, v, \nabla v, \nabla^2 v),$$

where, for any v, writing $u = u^{(0)} + v$ for brevity,

$$F_0(\cdot, v, \nabla v, \nabla^2 v) \Big(\overset{\text{motivation}}{=} L_0 u - L_0 u^{(0)}$$

$$= Lu - (L - L_0) u - L_0 u^{(0)} = \Big)$$

$$:= f(\cdot, u, \nabla u) - f(\cdot, u_0, \nabla u_0)$$

$$+ \sum_{i,j} \frac{\partial}{\partial x_i}\Big(\big(a_{ij}(\cdot, u) - a_{ij}(\cdot, u_0)\big) \frac{\partial u}{\partial x_i}\Big),$$

and with data

$$v|_{t=0} = 0.$$

[Observe that since $u^{(0)} = u_0$ at $t = 0$ we then also have

$$v_t = F_0(\cdot, v, \nabla v, \nabla^2 v) = 0 \quad \text{at } t = 0.$$

Hence we may impose the condition

$$v \in M_0 = \{ w \in M;\ w = 0 \text{ at } t = 0 \}.$$

Also writing $u^{(0)} - u_0 =: v^{(0)}$ for brevity, we have

$$I := f(\cdot, u^{(0)} + v, \nabla(u^{(0)} + v)) - f(\cdot, u_0, \nabla u_0)$$

$$= f(\cdot, u^{(0)} + v, \nabla(u^{(0)} + v)) - f(\cdot, u^{(0)} + v, \nabla u_0)$$

$$+ f(\cdot, u^{(0)} + v, \nabla u_0) - f(\cdot, u_0, \nabla u_0)$$

$$= \int_0^1 f_p(\cdot, u^{(0)} + v, \nabla(u_0 + s(v^{(0)} + v))) \cdot \nabla(v^{(0)} + v)\, ds$$

$$+ \int_0^1 f_u(\cdot, u_0 + s(v^{(0)} + v), \nabla u_0)(v^{(0)} + v)\, ds$$

and then can estimate

$$\|I\|_{C^{\alpha, \alpha/2}} \leq C(R)\left(\|\nabla(v^{(0)} + v)\|_{C^{\alpha, \alpha/2}} + \|v^{(0)} + v\|_{C^{\alpha, \alpha/2}}\right)$$

$$+ C(R)\|v^{(0)} + v\|_{C^1}\left(1 + \sup_{\substack{0 \leq s \leq 1 \\ 0 \leq k \leq 1}} \|\nabla^k(u_0 + s(v^{(0)} + v)\|_{C^{\alpha, \alpha/2}}\right)$$

$$\leq C(R) \sum_{k=0}^{1} \|\nabla^k(v^{(0)} + v)\|_{C^{\alpha, \alpha/2}}$$

for any $v \in M_0$, observing that

$$\sup_{0 \leq s \leq 1} \|u_0 + s(v^{(0)} + v)\|_{C^1} \leq \|u_0\|_{C^1} + \underbrace{\|v^{(0)}\|_{C^1}}_{\leq 1} + 1 \leq R.$$

Similarly, for $u = u^{(0)} + v = u_0 + (v^{(0)} + v)$ we write

$$II_i := \left(a_{ij}(\cdot, u) - a_{ij}(\cdot, u_0)\right) \frac{\partial u}{\partial x_j}$$

$$= \int_0^1 a_{ij,u}\left(\cdot, u_0 + s(v^{(0)} + v)\right)(v^{(0)} + v)\, ds \cdot \frac{\partial (u^{(0)} + v)}{\partial x_j}$$

to bound

$$\left\| \frac{\partial II_i}{\partial x_i} \right\|_{C^{\alpha, \alpha/2}} \leq C(R) \| v^{(0)} + v \|_{L^\infty} \left\| \frac{\partial^2 (u^{(0)} + v)}{\partial x_i \partial x_j} \right\|_{C^{\alpha, \alpha/2}}$$

$$+ C(R)\left(1 + \| v^{(0)} + v \|_{C^{2,1}}\right) \sum_{k=0}^{1} \| \nabla^k (v^{(0)} + v) \|_{C^{\alpha, \alpha/2}}$$

In view of $v^{(0)}|_{t=0} = 0 = v|_{t=0}$, by lemma 1.6.1 below

$$\| \nabla^k (v^{(0)} + v) \|_{C^{\alpha, \alpha/2}} \leq C\, T_0^{\frac{1-\alpha}{2}} \| v^{(0)} + v \|_X, \quad k = 0, 1.$$

Thus, we find

$$\| F_0(\cdot, v, \nabla v, \nabla^2 v) \|_{C^{\alpha, \alpha/2}} \leq$$

$$\leq C(R)\, T_0^{\alpha/2} \| v^{(0)} + v \|_X + C(R) \| v^{(0)} + v \|_{C^{2,1}}$$

for any $v \in M_0$.

For $v \in M_0$ now let $w = \Phi(v) \in X$ be the solution to the Cauchy problem

$$L_0 w = F_0(\cdot, v, \nabla v, \nabla^2 v) \quad \text{in } \mathbb{T}^n \times]0, T_0],$$
$$w|_{t=0} = 0$$

guaranteed by Thm. 1.5.2, satisfying

$$\|w\|_X \leq C(R) \|F_0(\cdot, v, \nabla v, \nabla^2 v)\|_{C^{\alpha, \alpha/2}}$$
$$\leq C(R) T_0^{\alpha/2} \|v^{(0)} + v\|_X \leq 1$$

for sufficiently small $0 < T_0 \leq T$.

Hence $\Phi : M_0 \to M_0$. Moreover, for any $v_1, v_2 \in M_0$ we have

$$L_0(w_1 - w_2) = F_0(\cdot, v_1, \nabla v_1, \nabla^2 v_1) - F_0(\cdot, v_2, \nabla v_2, \nabla^2 v_2)$$
$$= f(\cdot, u^{(0)} + v_1, \nabla(u^{(0)} + v_1)) - f(\cdot, u^{(0)} + v_2, \nabla(u^{(0)} + v_2))$$
$$+ \sum_{i,j} \frac{\partial}{\partial x_i}\Big((a_{ij}(\cdot, u^{(0)} + v_1) - a_{ij}(\cdot, u_0)) \frac{\partial (u^{(0)} + v_1)}{\partial x_j} \Big)$$
$$- \sum_{i,j} \frac{\partial}{\partial x_i}\Big((a_{ij}(\cdot, u^{(0)} + v_2) - a_{ij}(\cdot, u_0)) \frac{\partial (u^{(0)} + v_2)}{\partial x_j} \Big).$$

Set

$$\beta_{ij}(v) := a_{ij}(\cdot, u^{(0)} + v) - a_{ij}(\cdot, u_0)$$

for brevity.

Expanding as before, we find

$$I := f(\cdot, u^{(0)} + v_1, \nabla(u^{(0)} + v_1)) - f(\cdot, u^{(0)} + v_2, \nabla(u^{(0)} + v_2))$$

$$= \int_0^1 f_u(\cdot, u^{(0)} + v_2 + s(v_1 - v_2), \nabla(u^{(0)} + v_1))(v_1 - v_2)\, ds$$

$$+ \int_0^1 f_p(\cdot, u^{(0)} + v_2, \nabla(u^{(0)} + v_2 + s(v_1 - v_2))\, \nabla(v_1 - v_2)\, ds$$

with

$$\| I \|_{C^{\alpha, \alpha/2}} \leq C(R) \sum_{k=0}^{1} \| \nabla^k (v_1 - v_2) \|_{C^{\alpha, \alpha/2}}$$

and

$$II_i := \sum_j \left(\beta_{ij}(v_1) \frac{\partial(u^{(0)} + v_1)}{\partial x_j} - \beta_{ij}(v_2) \frac{\partial(u^{(0)} + v_2)}{\partial x_j} \right)$$

$$= \sum_j \int_0^1 \beta_{ij,u}(v_2 + s(v_1 - v_2))(v_1 - v_2)\, ds \; \frac{\partial(u^{(0)} + v_1)}{\partial x_j}$$

$$+ \sum_j \beta_{ij}(v_2) \frac{\partial(v_1 - v_2)}{\partial x_j}$$

with

$$\left\| \sum_i \frac{\partial II_i}{\partial x_i} \right\|_{C^{\alpha, \alpha/2}} \leq C(R) \| v_1 - v_2 \|_{L^\infty} \| u^{(0)} + v_1 \|_X$$

$$+ C(R) \left(\sum_{k=0}^{1} \| \nabla^k (v_1 - v_2) \|_{C^{\alpha, \alpha/2}} + \| v_1 - v_2 \|_{C^2} \right) + \| \beta_{ij}(v_2) \|_{L^\infty} \| v_1 - v_2 \|_X$$

to conclude from Lemma 1.6.1

$$\| w_1 - w_2 \|_X \leq C(R) \| v_1 - v_2 \|_{C^2} + \left(C T_0^{\frac{1-\alpha}{2}} + \| \beta_{ij}(v_2) \|_{L^\infty} \right) \| v_1 - v_2 \|_X .$$

Observing that

$$\| v_1 - v_2 \|_{C^2} \leq T_0^{\alpha/2} \| v_1 - v_2 \|_X ,$$

$$\| \beta_{ij}(v_2) \|_{L^\infty} \leq C(R) \| u^{(0)} + v_2 - u_0 \|_{L^\infty}$$

$$\leq C(R) \| v^{(0)} + v_2 \|_{L^\infty} \leq C(R) T^{\alpha/2} ,$$

for sufficiently small $0 < T_0 < T$ then we find

$$\| w_1 - w_2 \|_X \leq \frac{1}{2} \| v_1 - v_2 \|_X ,$$

and Banach's fixed point theorem yields a unique $v \in M_0$ with $v = \Phi(v)$, inducing a solution $u = u^{(0)} + v \in X$ of (1.14) on $\mathbb{T}^n \times]0, T_0]$.

Uniqueness. Let u_1, u_2 be solutions to (1.14) of class $C^{2+\alpha, 1+\alpha/2}$ with initial data

$$u_1|_{t=0} = u_2|_{t=0} = u_0 \in C^{2,\alpha},$$

defined for $0 < t \leq T_i$, $i = 1, 2$. Let $T = \min\{T_1, T_2\}$, $0 < T_0 \leq T$ such that

$$\|u_1 - u_0\|_{C^1}, \ \|u_2 - u_0\|_{C^1} \leq 1 \quad \text{on } [0, T_0].$$

Writing $u_i = u^{(0)} + v_i$ as above, then

$$L_0 v_i = F_0(\cdot, v_i, \nabla v_i, \nabla^2 v_i), \ i = 1, 2,$$

and, with $X = C^{2+\alpha, 1+\alpha/2}(\mathbb{T}^n \times]0, T_0])$,

$$\|v_1 - v_2\|_X \leq C(R)\Big(\|v_1 - v_2\|_{C^2} + \sum_{k=0}^{1} \|\nabla^k(v_1 - v_2)\|_{C^{\alpha, \alpha/2}}\Big)$$

$$+ C(R)\|v^{(0)} + v_2\|_{L^\infty} \|v_1 - v_2\|_X$$

$$\leq C_1(R) T_0^{\alpha/2} \|v_1 - v_2\|_X$$

$$+ C_2(R)\|v_2\|_{L^\infty} \|v_1 - v_2\|_X .$$

as before. Since $v_2(t) \to 0$ uniformly as $t \downarrow 0$, upon reducing $T_0 > 0$, if necessary, we can achieve that

$$C_1(R) T_0^{\alpha/2} + C_2(R)\|v_2\|_{L^\infty} < 1,$$

and it follows that $v_1 = v_2$, hence $u_1 = u_2$.

Alternatively, we may use energy methods.

Expanding

$$\partial_t(u_1-u_2) = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(\cdot,u_1)\frac{\partial u_1}{\partial x_j} - a_{ij}(\cdot,u_2)\frac{\partial u_2}{\partial x_j}\right)$$

$$+ f(\cdot,u_1,\nabla u_1) - f(\cdot,u_2,\nabla u_2)$$

$$= \sum_{i,j} \frac{\partial}{\partial x_i}\left(\int_0^1 a_{ij,u}(\cdot,u_2+s(u_1-u_2))(u_1-u_2)\,ds\cdot\frac{\partial u_1}{\partial x_j}\right)$$

$$+ \frac{\partial}{\partial x_i}\left(a_{ij}(\cdot,u_2)\frac{\partial(u_1-u_2)}{\partial x_j}\right)$$

$$+ \int_0^1 f_u(\cdot,u_2+s(u_1-u_2),\nabla u_1)(u_1-u_2)\,ds$$

$$+ \int_0^1 f_p(\cdot,u_2,\nabla(u_2+s(u_1-u_2)))\,\nabla(u_1-u_2)\,ds$$

in view of the bounds

$$\sup_{0\le s\le 1}\|u_2+s(u_1-u_2)\|_{C^1} \le \|u_0\|_{C^1} + \max_i \|u_i-u_0\|_{C^1} \le R$$

we can estimate

$$L_2(u_1-u_2) := (u_1-u_2)_t - \sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}(\cdot,u_2)\frac{\partial(u_1-u_2)}{\partial x_j}\right) =: g$$

with

$$|g| \le C(R)\left(1+\|u_1\|_{C^2}\right)\left(|u_1-u_2| + |\nabla(u_1-u_2)|\right).$$

Multiplying with $u_1 - u_2$ and integrating by parts we find

$$\frac{1}{2}\frac{d}{dt}\left(\|u_1-u_2\|^2_{L^2(\mathbb{T}^n)}\right) + \lambda \|\nabla(u_1-u_2)\|^2_{L^2(\mathbb{T}^n)}$$

$$\leq C\left(\|u_1-u_2\|_{L^2} + \|\nabla(u_1-u_2)\|_{L^2}\right)\|u_1-u_2\|_{L^2}$$

$$\leq C_1\|u_1-u_2\|^2_{L^2} + \frac{\lambda}{2}\|\nabla(u_1-u_2)\|^2_{L^2}.$$

Absorbing the gradient term on the left, we obtain

$$\frac{d}{dt}\left(e^{-2C_1 t}\|(u_1-u_2)(t)\|^2_{L^2}\right) \leq 0.$$

so that

$$\|(u_1-u_2)(t)\|^2_{L^2} \leq e^{2C_1 t}\|(u_1-u_2)(0)\|^2_{L^2} = 0$$

for all $t > 0$. $\square$

We have used:

Lemma 1.6.1. For $v \in X$ with $v|_{t=0} = 0$ there holds
$$\sum_{k=0}^{1} \|\nabla^k v\|_{C^{\alpha,\alpha/2}} \le C\, T_0^{\min\{\alpha, 1-\alpha\}/2} \|v\|_X .$$

Proof. Consider $k=1$. (The argument for $k=0$ is similar and somewhat simpler.)

Let $0 \le \rho \in C_c^\infty(B_1(0))$ with $\int_{B_1(0)} \rho \, dx = 1$, and for any $h>0$ let
$$\rho_h(x) = h^{-n}\rho\left(\tfrac{x}{h}\right), \quad v_h(t) = v(t) * \rho_h$$
as in the proof of Thm. 1.4.2.

Given $0 < h < T_0^{1/2}$, for any $(x,t),(y,s) \in \mathbb{T}^n \times]0,T_0]$ with
$$|x-y| + |t-s|^{1/2} = h$$
we then estimate
$$|\nabla v(x,t) - \nabla v(y,s)|$$
$$\le 2\|\nabla v - \nabla v_h\|_{L^\infty} + |\nabla v_h(x,t) - \nabla v_h(y,s)|$$
where
$$\|\nabla v - \nabla v_h\|_{L^\infty} \le h\|\nabla^2 v\|_{L^\infty} \le h\|v\|_X .$$
Moreover, for any $0<t<T_0$ we have
$$|\nabla v_h(x,t) - \nabla v_h(y,t)| \le |x-y|\,\|\nabla^2 v(t)\|_{L^\infty} \le h\|v\|_X ,$$
and for any $y \in \mathbb{T}^n$, any $0 \le t_1 < t_2 \le T_0$ there holds
$$|\nabla v_h(y,t_2) - \nabla v_h(y,t_1)| = \left|\int_{t_1}^{t_2} \overset{\curvearrowright}{\nabla v_t(y-z,t)\rho_h}(z)\, dz\, dt\right|$$
$$\le |t_2-t_1|\,\|v_t\|_{L^\infty} \int_{B_h(0)} |\nabla\rho_h|\, dx \le \frac{|t_2-t_1|}{h}\|v\|_X .$$

Thus we find

$$|\nabla v(x,t) - \nabla v(y,s)| \leqslant C h \|v\|_X$$

and in view of $h \leqslant T_0^{1/2}$ there holds

$$\frac{|\nabla v(x,t) - \nabla v(y,s)|}{h^\alpha} \leqslant C h^{1-\alpha} \|v\|_X \leqslant C T_0^{\frac{1-\alpha}{2}} \|v\|_X.$$

If $T_0^{1/2} \leqslant h \leqslant 1$, for any $(x,t) \in \mathbb{T}^n \times]0, T_0]$ with $h_0 := T_0^{1/2}$ we can bound

$$|\nabla v(x,t)| \leqslant \|\nabla v - \nabla v_{h_0}\|_{L^\infty} + |\nabla v_{h_0}(x,t)|$$

$$\leqslant h_0 \|v\|_X + |\nabla v_{h_0}(x,t) - \underbrace{\nabla v_{h_0}(x,0)}_{=0}|$$

$$\leqslant \left(h_0 + \frac{t}{h_0}\right) \|v\|_X \leqslant 2 T_0^{1/2} \|v\|_X$$

to conclude

$$\frac{|\nabla v(x,t)|}{h^\alpha} \leqslant 2\, T_0^{\frac{1-\alpha}{2}} \|v\|_X ,$$

uniformly in $(x,t) \in \mathbb{T}^n \times]0, T_0]$.

Moreover, we can bound

$$\|\nabla^k v(t)\|_{L^\infty} \leqslant T^{\alpha/2} \left\| \frac{\nabla^k (v(t) - v(0))}{t^{\alpha/2}} \right\|_{L^\infty} \leqslant T^{\alpha/2} \|v\|_X$$

for $k = 0, 1$. $\square$

2. The heat flow for harmonic maps

2.1 Harmonic maps

Let N be a smooth closed (compact, without boundary) Riemannian manifold, isometrically embedded in $\mathbb{R}^n$ for suitable $n \in \mathbb{N}$.

For smooth $u: M = \mathbb{T}^m \to N \hookrightarrow \mathbb{R}^n$ define the energy

$$E(u) = \frac{1}{2} \int_{\mathbb{T}^m} |\nabla u|^2 dx.$$

Definition 2.1. A map $u \in C^1(M; N)$ is harmonic if u is a critical point of E, i.e. if

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(\pi_N(u + \varepsilon\varphi)) = 0$$

for any $\varphi \in C^1(M; \mathbb{R}^n)$.

Here, $\pi_N : U_\delta(N) \subset \mathbb{R}^n \to N$ is the smooth nearest-neighbor projection, defined in the δ-neighborhood

$$U_\delta(N) = \bigcup_{p \in N} B_\delta(p)$$

of N for sufficiently small $\delta > 0$.

<u>Examples</u> 2.1. i) For $N = \mathbb{R}$, the energy is the familiar Dirichlet integral, and critical points of E are harmonic functions.

ii) If $m = 1$ the energy $E(u)$ is the familiar energy of the curve $u : S^1 = T^1 \to N$. Critical points are geodesics.

1<u>st</u> <u>variation</u>. Let $u \in C^2(M; N)$, $\varphi \in C^1(M, \mathbb{R}^n)$.

Compute

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E\left(\pi_N(u+\varepsilon\varphi)\right) = \int_M \nabla u \cdot \nabla\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \pi_N(u+\varepsilon\varphi)\right) dx$$

$$(2.1) \qquad = -\int_M \Delta u \cdot d\pi_N(u) \cdot \varphi \, dx = -\int_M d\pi_N(u) \Delta u \cdot \varphi \, dx.$$

We conclude

Theorem 2.1.1 A map $u \in C^2(M,N)$ is harmonic iff

$$\Delta u \perp T_u N$$

or, equivalently, if

$$-\Delta u = A(u)(\nabla u, \nabla u),$$

where $A(p): T_p N \times T_p N \longrightarrow T_p^{\perp} N$ is the second fundamental form of $N \hookrightarrow \mathbb{R}^n$ at any point $p \in N$.

Note: If $\nu_1, \dots, \nu_\ell$ are smooth orthonormal vector fields spanning the normal space $T_p^{\perp} N = (T_p N)^{\perp} \subset \mathbb{R}^n$ of N at any point $p \in N$ near p_0, then

$$A(p_0)(X,Y) = \sum_{i=1}^{\ell} \nu_i(p_0) \langle d\nu_i(p_0) X, Y \rangle_{\mathbb{R}^n}$$

for any $X, Y \in T_{p_0} N$.

Proof of Thm 2.1. By (2.1) and the fundamental lemma in the calculus of variations (see FA I, Satz 3,4.3), if $u \in C^2(M;N)$ is harmonic

$$d\pi_N(u)\,\Delta u \equiv 0\,;$$

i.e. $\Delta u(x) \in \ker d\pi_N(u(x)) = T^{\perp}_{u(x)} N$ for every $x \in M$, and conversely.

For harmonic $u \in C^2(M;N)$, given any $x_0 \in M$ choose a smooth orthonormal frame $\nu_1, \dots, \nu_\ell$ for $T^{\perp}N$ near $p_0 = u(x_0)$.

Then for x near x_0 we have

$$\Delta u = \sum_{i=1}^{\ell} \nu_i(u)\,\lambda_i$$

with coefficients

$$\lambda_i = \lambda_i(x) = \langle \nu_i(u), \Delta u \rangle_{\mathbb{R}^n}$$

$$= \sum_{j=1}^{m} \frac{\partial}{\partial x_j} \underbrace{\langle \nu_i(u), \frac{\partial u}{\partial x_j} \rangle_{\mathbb{R}^n}}_{=0} - \sum_{j=1}^{m} \langle d\nu_i(u) \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_j} \rangle_{\mathbb{R}^n}.$$

Hence

$$-\Delta u = \sum_{i,j} \nu_i(u) \langle d\nu_i(u) \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_j} \rangle_{\mathbb{R}^n} = A(u)(\nabla u, \nabla u).$$

$\square$

Example 2.2. The sphere $N = S^k \subset \mathbb{R}^{k+1}$.

By Thm. 2.1 a map $u \in C^2(M; S^k)$ is harmonic iff

$$-\Delta u = \lambda u \perp T_u S^k \subset \mathbb{R}^{k+1},$$

where

$$\lambda = \lambda |u|^2 = -\Delta u \cdot u = -\operatorname{div}(\underbrace{\nabla u \cdot u}_{=0}) + |\nabla u|^2.$$

Hence, u is harmonic iff there holds

$$-\Delta u = u |\nabla u|^2.$$

Bochner inequality. Differentiating the equation

$$(2.2) \quad -\Delta u = A(u)(\nabla u, \nabla u)$$

for a harmonic map $u \in C^2(M; N)$ and multiplying with $\frac{\partial u}{\partial x_i}$, upon summing over $1 \le i \le m$ we obtain

$$-\Delta\left(\frac{|\nabla u|^2}{2}\right) + |\nabla^2 u|^2 = -\sum_{i,j} \frac{\partial}{\partial x_j}\left(\frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial u}{\partial x_i}\right) + |\nabla^2 u|^2$$

$$(2.3) \quad = \sum_i \left\langle \frac{\partial}{\partial x_i}\big(A(u)(\nabla u, \nabla u)\big), \frac{\partial u}{\partial x_i} \right\rangle_{\mathbb{R}^n} = \operatorname{div}\big(\underbrace{\langle A(u)(\nabla u, \nabla u), \nabla u\rangle}_{=0}\big) - A(u)(\nabla u, \nabla u) \cdot \Delta u$$

$$= |A(u)(\nabla u, \nabla u)|^2 \le C |\nabla u|^4.$$

Consequence. In particular, we may conclude the following threshold result.

Theorem 2.1.2. For any $m \geq 2$, any smooth, closed $N \hookrightarrow \mathbb{R}^n$ there is $\varepsilon_0 = \varepsilon_0(m, N) > 0$ such that for any non-constant harmonic map $u \in C^2(M; N)$ there holds

$$\int_M |\nabla u|^m dx \geq \varepsilon_0 .$$

Proof: i) $m > 2$. In this case we use Sobolev's embedding

$$\|v\|_{L^{2^*}} \leq C\left(\|\nabla v\|_{L^2} + \left|\int_M v\,dx\right|\right),$$

where

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{m} .$$

Integrating, from (2.3) for $v = \nabla u$ we obtain

$$\|\nabla u\|^2_{L^{2^*}} \leq C\|\nabla^2 u\|^2_{L^2} \leq C\int_{\mathbb{T}^m} \underbrace{|\nabla u|^2}_{\in L^{m/2}} \underbrace{|\nabla u|^2}_{\in L^{\frac{m}{m-2}}} dx$$

$$\leq C\|\nabla u\|^2_{L^m} \|\nabla u\|^2_{L^{2^*}}$$

and the claim follows.

ii) $m=2$. In this case we use the interpolation inequality

$$\|v\|_{L^4}^4 \leq C\|v\|_{L^2}^2 \|v\|_{H^1}^2$$

$$\leq C\|v\|_{L^2}^2 \left(\|\nabla v\|_{L^2}^2 + \left|\int_{\mathbb{T}^m} v \, dx\right|^2 \right)$$

of Gagliardo-Nirenberg / Ladyzhenskaya for $v = \nabla u$ to bound

$$\|\nabla^2 u\|_{L^2}^2 \leq C\|\nabla u\|_{L^4}^4 \leq C\|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^2$$

and conclude, as before. $\square$

<u>Remark 2.1.1</u> Working intrinsically, for a target N with non-positive sectional curvature one has the improved Bochner-type bound

$$-\Delta\left(\frac{|\nabla u|^2}{2}\right) + |\nabla^N \nabla u|^2 \leq 0,$$

where ∇^N is the pull-back covariant derivative of $\nabla u \in u^* TN$. In particular, $|\nabla u|^2$ will be subharmonic for a harmonic map $u: \mathbb{T}^m \to N$ and $|\nabla u| \equiv \text{const.}$ (e.g. constant-speed geodesic).

Homotopy problem. Given a smooth map $u_0 : M \to N$, is there a harmonic map homotopic to it?

Eells - Sampson (1964) gave an affirmative answer if the sectional curvature $K_N \leq 0$.

Sacks - Uhlenbeck (1981) showed the same result in dimension $m = 2$ (for an arbitrary closed surface M), provided $\pi_2(N) = 0$.

Here we want (among other things) to give a proof of the Sacks-Uhlenbeck result via the harmonic map heat flow introduced by Eells - Sampson (1964) in their work.
(The original proof of Sacks-Uhlenbeck uses minimizers u_α of the α-energy

$$E_\alpha(u) = \int_M (1 + |\nabla u|^2)^\alpha dx, \quad \alpha > 1,$$

in the homotopy class of u_0 and an analysis of the behavior of u_α as $\alpha \downarrow 1$.)

Remark 2.1.1: The direct approach via minimizing the energy E in the homotopy class

$$\mathcal{E}_0 = \{ u \in H^1 \cap C^0(M;N);\ u \text{ is homotopic to } u_0 \}$$

fails in general because $\mathcal{E}_0$ is not weakly closed in $H^1(M;\mathbb{R}^n)$.

For instance the maps $u_k = \pi^{-1} \circ \delta_k \circ \pi : S^2 \to S^2$ obtained via stereographic projection $\pi : S^2 \to \mathbb{R}^2$ and dilation

$$\delta_k : \mathbb{R}^2 \ni x \mapsto kx \in \mathbb{R}^2$$

all are homotopic to the identity and satisfy (by conformality)

$$E(u_k) = \text{area}(S^2) = 4\pi;$$

but $u_k \xrightarrow{w} u_\infty \equiv \text{const}$ in $H^1(S^2; S^2)$ as $k \to \infty$.

2.2 Harmonic map heat flow

Given a smooth map $u_0 : \mathbb{T}^m \to N \hookrightarrow \mathbb{R}^n$ we consider the Cauchy problem for the equation

$$(2.4)\quad u_t - \Delta u = A(u)(\nabla u, \nabla u) \quad \text{in } \mathbb{T}^m \times]0,T],$$

with initial data $u|_{t=0} = u_0$, introduced by Eells-Sampson (1964).

Note: Equation (2.4) is a system of equations for the components $u = (u^1, \dots, u^n) : \mathbb{T}^m \times]0,T] \to N \hookrightarrow \mathbb{R}^n$ with diagonal main part $u^i_t - \Delta u^i$, $1 \le i \le n$.

Theorem 1.6.1 may be generalized to this case and for data $u_0 \in C^{2,\alpha}(\mathbb{T}^m; N)$ yields $T_0 > 0$ and a solution $u \in C^{2+\alpha, 1+\alpha/2}(\mathbb{T}^m \times]0,T_0]; \mathbb{R}^n)$ of (2.4) for $0 < t \le T_0$ for suff. small $T_0 > 0$, where we replace $A(u)$ by $A(\pi_N(u))$. Since $u_0(x) \in N$, in particular, $u(x,t) \in \mathcal{U}_\delta(N)$ for any given $\delta > 0$ and sufficiently small $0 < t \le T_0$, uniformly in $x \in \mathbb{T}^m$, and $A(u) = A(\pi_N(u))$ is well-defined for all such (x,t), independently of whether $u(x,t) \in N$ or not.

We then have the following result.

Lemma 2.2.1. The local solution u of (2.4) guaranteed by Thm. 1.6.1 for any $u_0 \in C^{2,\alpha}(\mathbb{T}^m, N)$ satisfies $u(x,t) \in N$ for any $(x,t) \in \mathbb{T}^m \times [0, T_0]$.

Proof. For simplicity of exposition, we assume that $N \hookrightarrow \mathbb{R}^n$ is a hypersurface, oriented by a smooth normal $\nu : N \to S^2$. With $\pi_N : \mathcal{U}_\delta(N) \to N$ as above, then we have

$$u - \pi_N(u) = \nu(u) \langle \nu(u), u - \pi_N(u) \rangle_{\mathbb{R}^n},$$
$$A(u)(\nabla u, \nabla u) = \nu(u) \langle \nabla u, d\nu(u) \nabla u \rangle_{\mathbb{R}^n},$$

so

$$\frac{1}{2} \frac{d}{dt} \left(|u - \pi_N(u)|^2 \right) = \langle u_t - d\pi_N(u) \cdot u_t, u - \pi_N(u) \rangle_{\mathbb{R}^n}$$

$$= \langle u_t, \nu(u) \rangle_{\mathbb{R}^n} \langle \nu(u), u - \pi_N(u) \rangle_{\mathbb{R}^n}$$

$$= \langle \Delta u + A(u)(\nabla u, \nabla u), \nu(u) \rangle_{\mathbb{R}^n} \langle \nu(u), u - \pi_N(u) \rangle_{\mathbb{R}^n}$$

$$= \Big[\operatorname{div}\left(\langle \nabla u, \nu(u) \rangle_{\mathbb{R}^n} \right) \underbrace{- \langle \nabla u, d\nu(u) \cdot \nabla u \rangle_{\mathbb{R}^n} + \langle A(u)(\ldots), \nu(u) \rangle_{\mathbb{R}^n} \Big]}_{=0}$$
$$\cdot \langle \nu(u), u - \pi_N(u) \rangle_{\mathbb{R}^n}$$

$$= \operatorname{div}\left(\langle \nabla (u - \pi_N(u)), \nu(u) \rangle_{\mathbb{R}^n} \right) \langle \nu(u), u - \pi_N(u) \rangle_{\mathbb{R}^n},$$

But using that $\langle u-\pi_N(u), \nabla\nu(u)\rangle_{\mathbb{R}^n} = 0$ we have

$$\operatorname{div}\left(\langle \nabla(u-\pi_N(u)), \nu(u)\rangle_{\mathbb{R}^n}\right)\langle \nu(u), u-\pi_N(u)\rangle_{\mathbb{R}^n}$$

$$= \operatorname{div}\left(\nabla\langle u-\pi_N(u), \nu(u)\rangle_{\mathbb{R}^n}\right)\langle \nu(u), u-\pi_N(u)\rangle_{\mathbb{R}^n}$$

$$= \operatorname{div}\left(\nabla\left(\frac{\langle u-\pi_N(u), \nu(u)\rangle^2_{\mathbb{R}^n}}{2}\right)\right) - |\nabla(\langle u-\pi_N(u), \nu(u)\rangle_{\mathbb{R}^n})|^2$$

we then arrive at the identity

$$\left(\frac{d}{dt} - \Delta\right)\left(\frac{|u-\pi_N(u)|^2}{2}\right) + |\nabla\langle u-\pi_N(u), \nu(u)\rangle_{\mathbb{R}^n}|^2 = 0 .$$

Integrating over $\mathbb{T}^m \times]0,t]$, we conclude

$$\|(u-\pi_N(u))(t)\|^2_{L^2} \leqslant \|(u-\pi_N(u))(0)\|^2_{L^2} = 0$$

for all $t > 0$. Thus, $u = \pi_N(u) \in N$. $\square$

We conclude:

<u>Theorem</u> 2.2.1. For any $u_0 \in C^{2,\alpha}(\mathbb{T}^m; N)$, $0<\alpha<1$, there exists a unique solution $u \in C^{2+\alpha, 1+\alpha/2}(\mathbb{T}^m \times [0,T_0]; N)$ of (2.4) for some $T_0 = T_0(u_0) > 0$.

Moreover, a solution $u \in C^{2+\alpha, 1+\alpha/2}$ of (2.4) with smooth data $u_0 \in C^\infty(M; N)$ is smooth.

<u>Theorem</u> 2.2.2. Suppose $u \in C^{2+\alpha, 1+\alpha/2}(\Pi^m \times [0,T]; N)$ solves (2.4). Then $u \in C^\infty(\Pi^m \times]0,T]; N)$.

<u>Proof</u>. For a solution $u \in C^{2+\alpha, 1+\alpha/2}(\Pi^m \times]0,T]; N)$ of (2.4) any spatial derivative $v = \frac{\partial u}{\partial x_i}$, $1 \le i \le m$, solves an equation

$$v_t - \Delta v = \frac{\partial}{\partial x_i}\left(A(u)(\nabla u, \nabla u)\right) =: f_i$$

with

$$f_i = dA(u) \cdot \frac{\partial u}{\partial x_i}(\nabla u, \nabla u) + 2A(u)\left(\nabla \frac{\partial u}{\partial x_i}, \nabla u\right)$$
$$\in C^{\alpha, \alpha/2}(\Pi^m \times]0,T]).$$

By Theorem 1.4.5 then $v \in C^{2+\alpha, 1+\alpha/2}(\Pi^m \times]0,T])$ and we may differentiate

$$f_i = dA(u) \cdot v(\nabla u, \nabla u) + 2A(u)(\nabla v, \nabla u)$$

further to obtain

$$\frac{\partial f_i}{\partial x_j} \in C^{\alpha, \alpha/2}, \quad \frac{\partial v}{\partial x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j} \in C^{2+\alpha, 1+\alpha/2}.$$

It now follows from (2.4) that $w = u_t$ with $\frac{\partial}{\partial x_i} u_t = v_t \in C^{\alpha, \alpha/2}$ satisfies

$$w_t - \Delta w = dA(u)\, u_t(\nabla u, \nabla u) + 2A(u)(\nabla u_{t'}, \nabla u) \in C^{\alpha, \alpha/2}.$$

Hence $v \in C^{2+\alpha, 1+\alpha/2}$, and $u \in C^{4+\alpha, 2+\alpha/2}$. Iterating, we obtain the claim. $\square$

<u>Remark 2.2.1.</u> It suffices to assume $u_0 \in C^{2+\alpha}(\Pi^m, N)$ to obtain smoothness of u for $t > 0$.

Energy inequality. The harmonic map heat flow may be regarded as the (negative) L^2-gradient flow for the energy.

In fact, for a smooth solution u of (2.4) on $[0,T]$ in view of $\langle u_t, A(u)(\nabla u, \nabla u)\rangle_{\mathbb{R}^n} \equiv 0$ there holds

$$\frac{d}{dt} E(u(t)) = \int_M \nabla u \cdot \nabla u_t \, dx$$

$$(2.5) \qquad = -\int_M \Delta u \cdot u_t \, dx = -\int_M |u_t|^2 dx .$$

Hence we find the energy bound

$$(2.6) \qquad E(u(T)) + \int_0^T \int_M |u_t|^2 dx\, dt \le E(u_0).$$

In particular, for a "global" solution of (2.4) for all $t > 0$ we have

$$\int_0^\infty \int_M |u_t|^2 dx\, dt \le E(u_0) < \infty$$

and we may expect convergence $u(t) \to u_\infty$ as $t \to \infty$ suitably, where $u_\infty : M \to N$ is harmonic.

Bochner formula. Similar to (2.3), upon differentiating in x_i and multiplying with $\frac{\partial u}{\partial x_i}$, from (2.4) we obtain

$$\left(\partial_t - \Delta\right)\left(\frac{|\nabla u|^2}{2}\right) + |\nabla^2 u|^2 =$$

$$= \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(\underbrace{\left\langle A(u)(\nabla u, \nabla u), \frac{\partial u}{\partial x_i}\right\rangle_{\mathbb{R}^n}}_{=0}\right)$$

(2.7) $$- A(u)(\nabla u, \nabla u) \cdot \Delta u$$

$$= |A(u)(\nabla u, \nabla u)|^2 \leq C(N)|\nabla u|^4,$$

where again we have used orthogonality $\langle u_t, A(u)(\nabla u, \nabla u)\rangle_{\mathbb{R}^n} \equiv 0$.

Rem. 2.2.2. If $K_N \leq 0$, working intrinsically, Eells-Sampson (1964) find the improved Bochner-type estimate

$$\left(\partial_t - \Delta\right)\left(\frac{|\nabla u|^2}{2}\right) \leq \left(\partial_t - \Delta\right)\left(\frac{|\nabla u|^2}{2}\right) + |\nabla^N \nabla u|^2 \leq 0,$$

yielding the a-priori bound

$$\| \nabla u \|_{L^\infty} \leq \| \nabla u_0 \|_{L^\infty},$$

from which higher-order bounds and smooth sub-convergence $u(t) \to u_\infty$ follow, where u_∞ is harmonic.

2.3 The harmonic map heat flow on surfaces

If $m = 2$, we have the following result, which holds for any closed surface M.
(For simplicity, the proof will only be given in the case when $M = T^2$.)

Ref.: M. S. "Variational Methods", 4th edition, Chapter III, 6.

Theorem 2.3.1 (S. CMH 60 (1985)). For any $u_0 \in H^1(M; N)$ there exists a global weak solution $u : M \times [0, \infty[\to N$ of (2.4) and satisfying $u|_{t=0} = u_0$ in the trace sense, which is smooth away from finitely many points $(\bar{x}_k, \bar{t}_k)$, $1 \le k \le K$, and satisfies the energy inequality

$$E(u(t)) \le E(u(s)) \text{ for all } 0 \le s \le t < \infty,$$

and

$$\int_0^\infty \int_M |u_t|^2 \, dx \, dt \le E(u_0).$$

The solution is unique in this class.

At any singularity $(\bar{x}, \bar{t})$, a non-constant "harmonic sphere" $\bar{u}: S^2 \to N$ separates in the sense that for suitable $x_i \to \bar{x}$, $t_i \uparrow \bar{t}$, $R_i \downarrow 0$ as $i \to \infty$ we have

$$u_i(x) = u(x_i + R_i x, t_i) \to \tilde{u}(x) \text{ in } H^2_{loc}(\mathbb{R}^2; N),$$

where $\tilde{u}: \mathbb{R}^2 \to N$ is a non-constant, smooth harmonic map with energy $E(\tilde{u}) \leq E(u_0)$ whose conformal lift $\bar{u} = \tilde{u} \circ \pi : S^2 \setminus \{p^*\} \to N$ may be extended to a non-constant, smooth harmonic map $\bar{u}: S^2 \to N$. Here, $\pi: S^2 \setminus \{p^*\} \to \mathbb{R}^2$ is the conformal stereographic projection from $p^* \in S^2$.

Finally, as $t = t_i \to \infty$ suitably, we have

$$u(t_i) \xrightarrow{w} u_\infty \text{ in } H^1(M; N),$$

where $u_\infty : M \to N$ is smooth and harmonic and where the convergence is smooth away from finitely many points $\tilde{x}_\ell$, $1 \leq \ell \leq L$, where non-constant harmonic spheres separate in the sense defined above.

Remarks 2.3.1. i) Uniqueness. Originally uniqueness was established in the class of partially regular solutions, as constructed in Thm. 2.3.1. Freire (1994/96) showed uniqueness of the solution constructed above among solutions of class H^1 satisfying the energy inequality.

Topping (2002) gave examples of non-uniqueness via "backwards bubbling", by constructing flows with upward jumps of the energy.

Rupflin (2008) showed uniqueness of the solution above among solutions u of class H^1 with $t \mapsto E(u(t))$ of locally finite total variation and with upward jumps of the energy strictly less than

$$\varepsilon_0 = \inf\{E(u);\ u: S^2 \to N \text{ is non-constant, smooth and harmonic}\} > 0.$$

ii) <u>Energy identity</u>. At a singular time $\bar{t} > 0$ by Thm. 2.3.1 we have

$$\lim_{t \uparrow \bar{t}} E(u(t)) \geq E(u(\bar{t})) + \sum_{\{(\bar{x}_k, \bar{t}_k);\ \bar{t}_k = \bar{t}\}} E(\bar{u}_k).$$

In fact, equality holds (Qing (1995), Ding-Tian (1995)).
("No energy loss in necks.")
In particular, it follows that

$$K + L \leq E(u_0)/\varepsilon_0 < \infty.$$

iii) <u>Regularity</u>. Examples of Chang-Ding-Ye (1992) show that the flow (2.4) may blow up in finite time. Thus Thm. 2.3.1 is best possible.

The following local regularity result is a core element in the proof of Thm. 2.3.1.

Proposition 2.3.1. There exists $\varepsilon_1 = \varepsilon_1(N) > 0$ with the following property: Any smooth solution $u : B_R(0) \times [0, T[$ with

(2.8) $\sup_{0<t<T} \|\nabla u(t)\|^2_{L^2} \leq \varepsilon_1$, $T \leq CR^2$,

may be smoothly extended to $B_R(0) \times [0, T]$. With smooth bounds on u_0, u satisfies smooth estimates on $B_r(0) \times [0, T]$ for any $0 < r < R$, and on $B_r(0) \times]\tau, T]$ for any $0 < r < R$, $0 < \tau < T$ in the general case.

Fix a cut-off function $\varphi \in C_c^\infty(B_1(0))$ with $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_{1/2}(0)$, and for $R > 0$ let

$$\varphi_R(x) = \varphi\left(\frac{x}{R}\right) \in C_c^\infty(B_R(0)).$$

Lemma 2.3.1. For any $v \in H^1(\mathbb{R}^2)$, any $R > 0$ there holds

$$\int_{\mathbb{R}^2} |v|^4 \varphi_R^2 \, dx \leq C \int_{B_R(0)} |v|^2 dx \int_{B_R(0)} \left(|\nabla v|^2 \varphi_R^2 + R^{-2}|v|^2\right) dx$$

Proof: By density of $C_c^\infty(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$ we may assume that $v \in C_c^\infty(\mathbb{R}^2)$. For any $x = (x^1, x^2)$ write

$$(|v|^2\varphi)(x) = \int_{-\infty}^{x_0^1} \frac{\partial}{\partial x_1}(|v|^2\varphi)(s, x^2)\, ds$$

$$\leq \int_{-\infty}^{\infty} \left(2|\nabla v|\varphi + |v||\nabla\varphi|\right)|v|(s, x^2)\, ds,$$

where we write φ instead of φ_R for brevity, and similarly with respect to x^2. Thus by Fubini

$$\int_{\mathbb{R}^2} |v|^4\varphi^2 dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (|v|^4\varphi^2)(x^1, x^2)\, dx^1 dx^2$$

$$\leq \left(\int_{\mathbb{R}^2} \left(2|\nabla v|\varphi + |v||\nabla\varphi|\right)|v|\, dx\right)^2$$

$$\leq C \int_{B_R(0)} |v|^2 dx \int_{\mathbb{R}^2} \left(|\nabla v|^2\varphi^2 + |\nabla\varphi|^2|v|^2\right) dx. \qquad \square$$

Lemma 2.3.2. Let u as in Prop. 2.3.1 satisfy (2.8). Then, if $\varepsilon_1 = \varepsilon_1(N) > 0$ is sufficiently small there holds

$$\int_0^T \int_{B_R(0)} (|\nabla^2 u|^2 + |\nabla u|^4) \varphi_R^2 \, dx \, dt \leq C\left(1 + \frac{T}{R^2}\right) \varepsilon_1 ,$$

where $C = C(N)$.

Proof. Multiply the terms in the Bochner inequality (2.7) with φ_R^2 and integrate by parts to obtain

$$\frac{1}{2} \frac{d}{dt}\left(\int_{B_R(0)} |\nabla u|^2 \varphi_R^2 \, dx \right) + \int_{B_R(0)} |\nabla^2 u|^2 \varphi_R^2 \, dx$$

$$= \int_{B_R(0)} |A(u)(\nabla u, \nabla u)|^2 \varphi_R^2 \, dx + \int_{B_R(0)} |\nabla u|^2 \Delta(\varphi_R^2) \, dx$$

$$\leq C \int_{B_R(0)} |\nabla u|^4 \varphi_R^2 \, dx + C R^{-2} \varepsilon_1$$

$$\leq C_1 \varepsilon_1 \int_{B_R(0)} |\nabla^2 u|^2 \varphi_R^2 \, dx + C \varepsilon_1 \int_{B_R(0)} |\nabla u|^2 |\nabla \varphi|^2 \, dx + \frac{C}{R^2} \varepsilon_1 .$$

by Lemma 2.3.1. For $\varepsilon_1 > 0$ with $C_1 \varepsilon_1 \leq \frac{1}{2}$ we can absorb the first term on the right in the left hand side. Integrating in $0 < t < T$ we obtain the claim. □

Proof. Differentiating and multiplying the equation (2.9) with $\nabla u\, \varphi_R^2$, upon integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{B_R(0)} |\nabla u|^2 \varphi_R^2\, dx\Big) + \int_{B_R(0)} |\nabla^2 u|^2 \varphi_R^2\, dx\, dt$$

$$= -\int_{B_R(0)} 2\nabla^2 u \cdot \nabla u\, \nabla\varphi_R\, \varphi_R\, dx - \int_{B_R(0)} f\, \mathrm{div}(\nabla u\, \varphi_R^2)\, dx$$

$$\leq \frac{1}{2}\int_{B_R(0)} |\nabla^2 u|^2 \varphi_R^2\, dx + C\int_{B_R(0)} \big(|\nabla u|^2 |\nabla\varphi_R|^2 + |f|^2 \varphi_R^2\big)\, dx.$$

Absorbing and integrating, for any $0 < t_1 < T$ we find

$$\int_{B_R(0)} |\nabla u(t_1)|^2 \varphi_R^2\, dx + \int_0^{t_1}\int_{B_R(0)} |\nabla^2 u|^2 \varphi_R^2\, dx\, dt$$

$$\leq C\int_0^T\int_{B_R(0)} |f|^2 \varphi_R^2\, dx\, dt + CR^{-2}\int_0^T\int_{B_R(0)} |\nabla u|^2\, dx\, dt$$

$$+ \int_{B_R(0)} |\nabla u_0|^2 \varphi_R^2\, dx$$

$$\leq M_0 + CM_1,$$

as claimed. □

To proceed via H^k-regularity (instead of improving the exponent of integrability of the gradient, as in M.S. "Var. Methods"), we use induction based on the following lemma.

<u>Lemma</u> 2.3.3. Suppose v smoothly solves

(2.9) $\quad v_t - \Delta v = f$ in $B_R(0) \times [0,T[$

with
$$M_1 = \int_0^T \int_{B_R(0)} \left(|f|^2 \varphi_R^2 + R^{-2} |\nabla v|^2 \right) dx\,dt < \infty$$

and $v|_{t=0} = v_0$ satisfying
$$M_0 = \int_{B_R(0)} |\nabla v_0|^2 \varphi_R^2\, dx < \infty .$$

Then
$$M_2 := \sup_{0<t<T} \left(\int_{B_R(0)} |\nabla v|^2 \varphi_R^2\, dx \right) + \int_0^T \int_{B_R(0)} |\nabla^2 v|^2 \varphi_R^2\, dx\,dt$$
$$\leq M_0 + C M_1 .$$

We now use Lemma 2.3.3 to inductively bound $\nabla^3 u$, $\nabla^4 u$ in space-time L^2.

Lemma 2.3.4. Let u as in Prop. 2.3.1 with $u|_{t=0} = u_0$ satisfying

$$M_{20} = \int_{B_R(0)} |\nabla^2 u_0|^2 \varphi_{R/2}^2 \, dx < \infty$$

and satisfying (2.8) with sufficiently small $\varepsilon_1 = \varepsilon_1(N) > 0$. Then for $T/R^2 \leq C$ there holds

$$M_3 := \sup_{0<t<T} \left(\int_{B_R(0)} |\nabla^2 u|^2 \varphi_{R/2}^2 \, dx \right) + \int_0^T \int_{B_R(0)} |\nabla^3 u|^2 \varphi_{R/2}^2 \, dx$$

$$\leq C M_{20} + C\left(1 + \frac{T}{R^2}\right) \frac{\varepsilon_1}{R^2}.$$

Proof: Apply Lemma 2.3.3 with $v = \nabla u$ and $f = \nabla(A(u)(\nabla u, \nabla u))$.

Observe that we have

$$|\nabla(A(u)(\nabla u, \nabla u))| \leq C\left(|\nabla u|^3 + |\nabla^2 u||\nabla u|\right);$$

hence

$$|f|^2 \leq C\left(|\nabla u|^6 + |\nabla^2 u|^2 |\nabla u|^2\right)$$

$$\leq C\left(|\nabla u|^6 + |\nabla^2 u|^3\right)$$

by Young's inequality.

Recall from Lemma 2.3.2 that we have

$$\int_0^T \int_{B_{R/2}(0)} \left(|\nabla^2 u|^2 + |\nabla u|^4 \right) dx\, dt \leq C\left(1+\frac{T}{R^2}\right)\varepsilon_1 ,$$

provided $\varepsilon_1 = \varepsilon_1(N) > 0$ is sufficiently small.

By Hölder's inequality and standard Gagliardo-Nirenberg interpolation, applied to $v = \nabla^2 u \varphi$, then there holds

$$\int_0^T \int_{B_R(0)} |\nabla^2 u|^3 \varphi_{R/2}^2\, dx\, dt$$

$$\leq \left(\int_0^T \int_{B_{R/2}(0)} |\nabla^2 u|^2\, dx\, dt \right)^{1/2} \left(\int_0^T \int_{B_R(0)} |\nabla^2 u|^4 \varphi_{R/2}^4\, dx\, dt \right)^{1/2}$$

(2.9)
$$\leq \left(C\left(1+\frac{T}{R^2}\right)\varepsilon_1 \sup_{0<t<T} \left(\int_{B_{R/2}(0)} |\nabla^2 u|^2 \varphi_{R/2}^2\, dx \right) \int_0^T \int_{B_R(0)} \left(|\nabla^3 u|^2 \varphi_{R/2}^2 + |\nabla^2 u|^2 |\nabla \varphi_{R/2}|^2 \right) dx\, dt \right)^{1/2}$$

$$\leq \left(C\left(1+\frac{T}{R^2}\right)\varepsilon_1 \right)^{1/2} \left(M_3 + C\left(1+\frac{T}{R^2}\right)\frac{\varepsilon_1}{R^2} \right).$$

Moreover, integrating by parts, for any $0<t<T$ we have

$$\int_{B_R(0)} |\nabla u|^6 \varphi_{R/2}^2\, dx \leq C \int_{B_R(0)} \left(|u| |\nabla^2 u| |\nabla u|^4 \varphi_{R/2}^2 + |u| |\nabla u|^5 |\nabla \varphi_{R/2}| \varphi_{R/2} \right) dx$$

$$\leq \frac{1}{2} \int_{B_R(0)} |\nabla u|^6 \varphi_{R/2}^2\, dx + C(N) \int_{B_R(0)} \left(|\nabla^2 u|^3 \varphi_{R/2}^2 + |\nabla u|^4 |\nabla \varphi_{R/2}|^2 \right) dx ,$$

and then

$$\int_0^T\int_{B_R(0)} |\nabla u|^6 \varphi_{R/2}^2 \,dx\,dt \leq$$

$$\leq C \int_0^T\int_{B_R(0)} |\nabla^2 u|^3 \varphi_{R/2}^2 \,dx\,dt + \frac{C}{R^2}\int_0^T\int_{B_{R/2}} |\nabla u|^4 \,dx\,dt \tag{2.10}$$

$$\leq C\left(\left(1+\frac{T}{R^2}\right)\varepsilon_1\right)^{1/2} M_3 + C\left(1+\frac{T}{R^2}\right)\frac{\varepsilon_1}{R^2}.$$

Lemma 2.3.3 then yields

$$M_3 \leq M_{20} + C\left(\left(1+\frac{T}{R^2}\right)\varepsilon_1\right)^{1/2} M_3 + C\left(1+\frac{T}{R^2}\right)\frac{\varepsilon_1}{R^2},$$

and our claim follows for sufficiently small $\varepsilon_1 = \varepsilon_1(N) > 0$, provided $T/R^2 \leq C$. □

<u>Remark 2.3.2.</u> Note that (2.9) also give

$$\int_0^T\int_{B_{R/4}(0)} \left(|\nabla^2 u|^4 + |\nabla u|^6\right) dx\,dt \leq C M_3 + C\left(1+\frac{T}{R^2}\right)\frac{\varepsilon_1}{R^2}.$$

Lemma 2.3.5 Let u be as in Prop. 2.3.1 with $u_{/t=0} = u_0$ satisfying

$$M_{30} = \int_{B_R(0)} |\nabla^3 u_0|^2 \varphi^2_{R/4}\, dx < \infty ,$$

and satisfying (2.8) with sufficiently small $\varepsilon_1 = \varepsilon_1(N) > 0$. Then if $T/R^2 \leq C$ we have

$$M_4 := \sup_{0<t<T} \int_{B_R(0)} |\nabla^3 u(t)|^2 \varphi^2_{R/4}\, dx + \int_0^T \int_{B_R(0)} |\nabla^4 u|^2 \varphi^2_{R/4}\, dx\, dt$$

$$\leq C\left(M_{30} + \frac{M_{20}}{R^2}\right) + C\left(1 + \frac{T}{R^2}\right)\frac{\varepsilon_1}{R^4} .$$

Proof. Apply Lemma 2.3.3 with $v = \nabla^2 u$ and

$$f = \nabla^2 (A(u)(\nabla u, \nabla u)),$$

satisfying

$$|f|^2 \leq C\left(|\nabla u|^4 + |\nabla^2 u||\nabla u|^2 + |\nabla^3 u||\nabla u|\right)^2$$

$$\leq C\left(|\nabla u|^8 + |\nabla^2 u|^4 + |\nabla^3 u|^2 |\nabla u|^2\right).$$

Integrating by parts, we have

$$\int_{B_R(0)} |\nabla u|^8 \varphi^2_{R/4}\, dx \leq C \int_{B_R(0)} |u| \left(|\nabla^2 u||\nabla u|^6 \varphi^2_{R/4} + |\nabla u|^7 |\nabla \varphi_{R/4}| \varphi_{R/4}\right) dx$$

$$\leq \frac{1}{2} \int_{B_R(0)} |\nabla u|^8 \varphi^2_{R/4}\, dx + C(N) \int_{B_R(0)} \left(|\nabla^2 u|^4 \varphi^2_{R/4} + |\nabla u|^6 |\nabla \varphi_{R/4}|^2\right) dx ,$$

so that in view of Remark 2.3.2

$$\int_0^T \int_{B_R(0)} (|\nabla u|^8 + |\nabla^2 u|^4)\varphi^2_{R/4}\,dx\,dt$$

$$\leq C \int_0^T \int_{B_R(0)} \left(|\nabla^2 u|^4 \varphi^2_{R/4} + |\nabla u|^6 |\nabla \varphi_{R/4}|^2\right) dx\,dt$$

$$\leq C M_3 + C\left(1+\frac{T}{R^2}\right)\frac{\varepsilon_1}{R^4}.$$

Finally, we estimate

$$\int_0^T \int_{B_R(0)} |\nabla^3 u|^2 |\nabla u|^2 \varphi^2_{R/4}\,dx\,dt \leq$$

$$\leq \left(\int_0^T \int_{B_{R/4}(0)} |\nabla u|^4\,dx\,dt\right)^{1/2} \left(\int_0^T \int_{B_R(0)} |\nabla^3 u|^4 \varphi^4_{R/4}\,dx\,dt\right)^{1/2}$$

$$\leq \left(C\left(1+\frac{T}{R^2}\right)\varepsilon_1\right)^{1/2} \left(\sup_{0<t<T} \left(\int_{B_R(0)} |\nabla^3 u|^2 \varphi^2_{R/4}\,dx\right) \int_0^T \int_{B_R(0)} \left(|\nabla^4 u|^2 \varphi^2_{R/4} + |\nabla^3 u|^2 |\nabla \varphi_{R/4}|^2\right) dx\,dt\right)^{1/2}$$

$$\leq \left(C\left(1+\frac{T}{R^2}\right)\varepsilon_1\right)^{1/2} \left(M_4 + R^{-2} M_3\right).$$

Lemmas 2.3.3-4 then yield for sufficiently small $\varepsilon_1 = \varepsilon_1(N) > 0$

$$M_4 \leq 2M_{30} + C\left(1 + \left(\left(1+\frac{T}{R^2}\right)\varepsilon_1\right)^{1/2}\right)\frac{M_{20}}{R^2} + C\left(1+\frac{T}{R^2}\right)\frac{\varepsilon_1}{R^4}.$$

Our claim follows. $\square$

Estimating

$$|\nabla u_t|^2 \leq \left(|\nabla\Delta u| + |\nabla(A(u)(\nabla u, \nabla u))|\right)^2$$

$$\leq C\left(|\nabla^3 u| + |\nabla^2 u||\nabla u| + |\nabla u|^3\right)^2$$

$$\leq C\left(|\nabla^3 u|^2 + |\nabla^2 u|^3 + |\nabla u|^6\right)$$

via Young's inequality, and bounding

$$\int_{B_R(0)}\left(|\nabla u|^6 + |\nabla^2 u|^3\right)\varphi_{R/4}^2\,dx \leq C\int_{B_R(0)}\left(|\nabla^2 u|^3\varphi_{R/4}^2 + |\nabla u|^4|\nabla\varphi_{R/4}|^2\right)dx$$

$$\leq C M_3 M_4^{1/2} + C\frac{\varepsilon_1}{R^2}M_3 + C\frac{\varepsilon_1^2}{R^4} \leq C\frac{\varepsilon_1}{R^4}$$

at any time $0<t<T$ on account of lemmas 2.3.4-5 if $T/R^2 \leq C$ we have

$$(2.11)\quad M_5 := \sup_{0<t<T}\left(\|\nabla u_t\|^2_{L^2(B_{R/4}(0))} + \|\nabla u\|^2_{H^2(B_{R/4}(0))}\right) \leq C\frac{\varepsilon_1}{R^4}$$

<u>Lemma</u> 2.3.6. Let u smoothly solve (2.4) on $B_R(0)\times[0,T[$ with (2.11). Then

$$\nabla u \in C^{\alpha,\alpha/2}(B_{R/4}(0)\times]0,T[)$$

for any $0<\alpha<1$, and $[\nabla u]^2_{C^{\alpha,\alpha/2}} \leq C M_5$

Higher regularity estimates then hold as well.

Proof. By Sobolev's embedding, for any $0 < \alpha < 1$ we have

$$\|\nabla u\|_{C^\alpha} \leq C \, \|\nabla u\|_{H^2} .$$

Given $0 < h < \min\{1, R/2\}$, for any pair $(x,t), (y,s) \in B_R(0) \times]0,T[$ with

$$|x-y| + |t-s|^{1/2} = h$$

we estimate

$$|\nabla u(x,t) - \nabla u(y,t)| \leq h^\alpha [\nabla u(t)]_{C^\alpha},$$

$$|\nabla u(y,t) - \overline{\nabla u}_h(y,t)| \leq C h^\alpha [\nabla u(t)]_{C^\alpha},$$

where

$$\overline{\nabla u}_h(y,t) = \int_{\mathbb{R}^2} \nabla u(x,t) \rho_h(y-x) \, dx$$

with

$$\rho_h(x) = \frac{1}{h^2} \rho\left(\frac{x}{h}\right)$$

for some smooth cut-off function $0 \leq \rho \in C_c^\infty(B_1(0))$ with $\int_{\mathbb{R}^2} \rho \, dx = 1$.

Note that

$$\int_{\mathbb{R}^2} \rho_h^2 \, dx = \frac{1}{h^2} \int_{\mathbb{R}^2} \rho^2 \, dx = \frac{c}{h^2} .$$

Moreover, for $|t_2-t_1|<h^2$ we have

$$
\begin{aligned}
|\overline{\nabla u_h}(y,t_2) - \overline{\nabla u_h}(y,t_1)|
&= \Big|\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \nabla u_t(x,t)\,\rho_h(y-x)\,dx\,dt\Big| \\
&\leqslant |t_2-t_1| \sup_t \|\nabla u_t\|_{L^2}\,\|\rho_h\|_{L^2} \\
&\leqslant \frac{C|t_2-t_1|}{h}\sup_t \|\nabla u_t\|_{L^2} \leqslant Ch\sup_t\|\nabla u_t\|_{L^2}.
\end{aligned}
$$

It follows that

$$|\nabla u(x,t) - \nabla u(y,s)| \leqslant C h^{\alpha} \sup_t\big(\|\nabla u_t\|_{L^2} + \|u\|_{H^3}\big),$$

and $\nabla u \in C^{\alpha,\alpha/2}(B_{R/4}(0)\times]0,T])$.

Truncating, $v = u\varphi_{R/4}$ then solves

$$
\begin{aligned}
(\partial_t-\Delta)v &= (\partial_t-\Delta)u\cdot\varphi_{R/4} - 2\nabla u\nabla\varphi_{R/4} - u\Delta\varphi_{R/2} \\
&= A(u)(\nabla u,\nabla u)\,\varphi_{R/4} - 2\nabla u\nabla\varphi_{R/4} - u\Delta\varphi_{R/2} \\
&\in C^{\alpha,\alpha/2}
\end{aligned}
$$

and $v\in C^{2+\alpha,1+\alpha/2}$ by Theorem 1.4.5.

Since $u\equiv v$ on $B_{R/8}(0)\times[0,T]$, smooth estimates for u follow by Theorem 2.2.2. $\square$

Proof of Prop. 2.3.1.

i) Let $0 < \tau < T$, $0 < r < R$ and let

$z_0 = (x_0, t_0) \in B_r(0) \times [\tau, T]$, $R_0 = \min\{R - |x_0|, \frac{\sqrt{\tau}}{2}\}$.

Shifting

$$u_{x_0}(x,t) = u(x - x_0, t)$$

we may assume that $x_0 = 0$ and u smoothly solves (2.4) on $B_{R_0}(0) \times [t_0 - 4R_0^2, t_0[$ with (2.8).

By Lemma 2.3.2 we may choose

$$t_1 \in [t_0 - 4R_0^2, t_0 - 3R_0^2]$$

such that

$$R_0^2 \overbrace{\int_{B_{R_0/2}(0)} |\nabla^2 u(t_1)|\, dx}^{\approx M_{20}} \leq \int_{t_0-4R_0^2}^{t_0-3R_0^2} \int_{B_{R_0/2}(0)} |\nabla^2 u|^2 dx\, dt \leq C\varepsilon_1 ,$$

and Lemma 2.3.4 yields

$$t_2 \in [t_0 - 3R_0^2, t_0 - 2R_0^2] \subset [t_1, t_0 - 2R_0^2]$$

such that

$$R_0^2 \overbrace{\int_{B_{R_0/4}(0)} |\nabla^3 u(t_2)|^2 dx}^{\approx M_{30}} \leq \int_{t_0-3R_0^2}^{t_0-2R_0^2} \overbrace{\int_{B_{R_0/4}(0)} |\nabla^3 u|^2 dx\, dt}^{\leq M_3} \leq C \frac{\varepsilon_1}{R_0^2} .$$

Lemma 2.3.5-6 then show that

$$[\nabla u]^2_{C^{\alpha,\alpha/2}(B_{R_0/8}(0) \times [t_0 - R_0^2, t_0[)} \leq M_5 \leq C\frac{\varepsilon_1}{R_0^4}$$

and higher regularity estimates up to time t_0 follow.

ii) Letting $R_0 = R - |x_0|$, for $t_0 \leq R_0^2$ with M_{20}, M_{30} as in Lemmas 2.3.4–5 we find the bounds

$$M_3 \leq C M_{20} + C \frac{\varepsilon_1}{R_0^2},$$

$$M_4 \leq C\left(M_{30} + R_0^{-2} M_{20}\right) + C \frac{\varepsilon_1}{R_0^4}$$

and the corresponding bound for (2.11), giving smooth bounds on $B_{R_0/8}(0) \times [0, t_0[$ by Lemma 2.3.6. □

<u>Corollary</u> 2.3.1. For any $u_0 \in C^{2,\alpha}(M; N)$ there exists a unique smooth solution u of (2.4) with $u|_{t=0} = u_0$, defined on a maximal time interval $[0, T[$, where either $T = \infty$ or T is characterized by concentration of Dirichlet energy in the sense

$$\exists x_0 \in M \ \forall R > 0: \limsup_{t \uparrow T} \left(\int_{B_R(x_0)} |\nabla u(t)|^2 dx \right) \geq \varepsilon_1,$$

where $\varepsilon_1 = \varepsilon_1(N) > 0$ is as determined in Prop. 2.3.1.

Def. 2.3.1. A point $x_0 \in M$ is a <u>blow-up point</u> of a smooth sol. u to (2.4) on $[0, T[$, if

$$\forall R > 0: \limsup_{t \uparrow T} \left(\int_{B_R(x_0)} |\nabla u(t)|^2 dx \right) \geq \varepsilon_1.$$

<u>Local</u> <u>energy</u> <u>inequality</u>. For a smooth solution u of (2.4) energy may be controlled locally. Testing (2.4) with u_t, in view of orthogonality $\langle u_t, A(u)(\nabla u, \nabla u)\rangle_{\mathbb{R}^n} = 0$ we have the identity

$$(2.12) \quad |u_t|^2 - \operatorname{div}\langle \nabla u, u_t\rangle_{\mathbb{R}^n} + \frac{d}{dt}\left(\frac{|\nabla u|^2}{2}\right) = 0.$$

From this the next result follows easily.

<u>Lemma</u> 2.3.7. Let u smoothly solve (2.4) on $B_R(0) \times [0,T]$ with energy $E(u(t)) \le E_0$. Then there holds

$$\int_{B_{R/2}(0)} |\nabla u(T)|^2 dx \le \int_{B_R(0)} |\nabla u(0)|^2 dx + C\frac{T}{R^2}E_0.$$

<u>Proof</u>: Multiply (2.12) with φ_R^2 and integrate to obtain

$$\frac{1}{2}\int_{B_R(0)} |\nabla u(t)|^2 \varphi_R^2 dx\Big|_{t=0}^{T} + \int_0^T\int_{B_R(0)} |u_t|^2 \varphi_R^2 dx\,dt$$

$$\le 2\int_0^T\int_{B_R(0)} |\nabla u|\,|u_t|\,|\nabla\varphi_R|\,\varphi_R\, dx\,dt$$

$$\le \frac{1}{2}\int_0^T\int_{B_R(0)} |u_t|^2\varphi_R^2 dx\,dt + \underbrace{2\int_0^T\int_{B_R(0)} |\nabla u|^2 |\nabla\varphi_R|^2 dx\,dt}_{\le C\frac{T}{R^2}E_0}.$$

The claim follows. □

Corollary 2.3.2. Let u smoothly solve (2.4) on $[0,T[$, where $T<\infty$ is maximal. Then there exist finitely many blow-up points $\bar{x}_i$, $1\le i\le i_0$, at time T; in fact, $i_0 \le CE(u_0)$.

Proof. Suppose $\bar{x}_i$, $1\le i\le i_0$, are blow-up points. Choose

$$R<\frac{1}{2}\min_{i\neq j}|\bar{x}_i-\bar{x}_j|.$$

By Lemma 2.3.7 at time $t_0=T-\delta R^2$ for each $i\in\{1,\dots,i_0\}$ we have

$$\int_{B_R(\bar{x}_i)}|\nabla u(t_0)|^2dx \ge \limsup_{t\uparrow T}\Big(\int_{B_{R/2}(\bar{x}_i)}|\nabla u(t)|^2dx\Big)-C\frac{T-t_0}{R^2}E(u_0)$$

$$\ge \varepsilon_1 - C\delta E(u_0)\ge \frac{\varepsilon_1}{2},$$

if $0<\delta$ is sufficiently small.

Since the balls $B_R(\bar{x}_i)$ are disjoint, we conclude

$$\frac{i_0\varepsilon_1}{2}\le \sum_{i=1}^{i_0}\int_{B_R(\bar{x}_i)}|\nabla u(t_0)|^2dx=\int_{\bigcup_{i=1}^{i_0}B_R(\bar{x}_i)}|\nabla u(t_0)|^2dx$$

$$\le 2E(u(t_0))\le 2E(u_0),$$

and our claim follows. $\square$

Corollary 2.3.3. For any

$$u_0 \in H^1(M;N) = \{u \in H^1(M;\mathbb{R}^n);\ u(x) \in N \text{ for almost every } x \in M\}$$

there exists a unique maximal solution u of (2.4) on $[0,T[$ which is smooth for $t>0$, satisfies the energy identity on $[0,T[$, and which attains the initial data in H^1.

Density of smooth maps. We can approximate any $u_0 \in H^1(M;N)$ by smooth maps from M to N.

Lemma 2.3.8 (Courant (1950), Sacks-Uhlenbeck (1981))
For any $u_0 \in H^1(M;N)$ there exists $(u_k)_{k\in\mathbb{N}} \subset C^\infty(M;N)$ with $u_k \to u_0$ in $H^1(M;\mathbb{R}^n)$.

Proof. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be standard mollifiers defined by letting

$$\rho_\varepsilon(x) = \varepsilon^{-2}\rho\left(\frac{x}{\varepsilon}\right)$$

for some $0 \le \rho \in C_c^\infty(B_1(0))$ with $\int_{B_1(0)} \rho\, dx = 1$, and let $\delta>0$ such that $\pi_N : U_\delta(N) \to N$ is smooth and well-defined.

Claim 1. For $v \in H^1(B_r(0))$ there holds

$$\int_{B_r(0)} \int_{B_r(0)} |u(x) - u(y)|^2 \, dx\, dy \leq C r^4 \|\nabla u\|_{L^2}^2 .$$

Proof. We may assume $v \in C^1(\overline{B_r(0)})$.

For any $x, y \in B_r(0)$ write

$$|u(x) - u(y)| = \left| \int_0^1 \nabla u(y + t(x-y)) \cdot (x-y)\, dt \right|$$

$$\leq 2r \left(\int_0^1 |\nabla u(y + t(x-y))|^2 dt \right)^{1/2} .$$

Then, by symmetry in x and y and Fubini,

$$\int_{B_r(0)} \int_{B_r(0)} |u(x) - u(y)|^2 dx\, dy$$

$$\leq 4r^2 \int_{B_r(0)} \int_{B_r(0)} \int_0^1 |\nabla u(y + t(x-y))|^2 dt\, dx\, dy$$

$$\leq 8r^2 \int_{B_r(0)} \int_{B_r(0)} \int_0^{1/2} |\nabla u(\underbrace{(1-t)y + tx}_{=:\, z \,\in\, B_r(0)})|^2 dt\, dx\, dy$$

$$\leq C r^4 \int_{B_r(0)} |\nabla u(z)|^2 dz .$$

$\square$

<u>Claim</u> 2. There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ there holds

$$u_\varepsilon = u_0 * \rho_\varepsilon \in C^\infty(M; \mathcal{U}_\delta(N)).$$

<u>Proof</u>. Compute for any $x \in M$

$$\operatorname{dist}(u_\varepsilon(x), N) \leq \inf_{y \in M} |u_\varepsilon(x) - u_0(y)|$$

$$\leq \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} (u_0(z) - u_0(y)) \rho_\varepsilon(x-z)\, dz \right| \rho_\varepsilon(x-y)\, dy$$

$$\leq \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u_0(z) - u_0(y)|^2 \rho_\varepsilon(x-z) \rho_\varepsilon(x-y)\, dz\, dy \right)^{1/2}$$

$$\leq \frac{C}{\varepsilon^2} \left(\int_{B_\varepsilon(x)} \int_{B_\varepsilon(x)} |u_0(z) - u_0(y)|^2\, dy\, dz \right)^{1/2}$$

$$\leq C_1 \left(\int_{B_\varepsilon(x)} |\nabla u_0|^2 dz \right)^{1/2}.$$

By absolute continuity of Lebesgue's integral there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ we have

$$C_1^2 \int_{B_\varepsilon(x)} |\nabla u_0|^2 dz < \delta^2, \text{ uniformly in } x.$$

$\square$

For $0 < \varepsilon_k < \varepsilon_0$ with $\varepsilon_k \downarrow 0$ $(k \to \infty)$ then

$$u_k := \pi_N \circ u_{\varepsilon_k} \in C^\infty(M, N)$$

is well-defined with

$$\| u_k - u_{\varepsilon_k} \|_{L^\infty} \leq \sup_x \operatorname{dist}(u_{\varepsilon_k}(x), N)$$

$$\leq C_1 \sup_{x_0} \Big(\int_{B_{\varepsilon_k}(x_0)} |\nabla u_0|^2 dx \Big)^{1/2} \to 0 \ (k \to \infty)$$

and satisfies

$$\| \nabla u_k \|_{L^2} = \| d\pi_N(u_{\varepsilon_k}) \nabla u_{\varepsilon_k} \|_{L^2} \leq \| \nabla u_{\varepsilon_k} \|_{L^2} \xrightarrow{(k \to \infty)} \| \nabla u_0 \|_{L^2}.$$

Thus, $u_k \xrightarrow{w} u_0$ in H^1 and, by strict convexity of the L^2-norm, in view of

$$\limsup_{k \to \infty} \| \nabla u_k \|_{L^2} \leq \| \nabla u_0 \|_{L^2}$$

we have $u_k \to u_0$ strongly in H^1 as $k \to \infty$. $\square$

<u>Proof</u> of Cor. 2.3.3: Given $u_0 \in H^1(M;N)$, let $(u_{0k}) \subset C^\infty(M;N)$ with $u_{0k} \to u_0$ in H^1 and for each $k \in \mathbb{N}$ let u_k solve (2.4) with initial data $u_k|_{t=0} = u_{0k}$, as given by Cor. 2.3.1, on a maximal time interval $[0, T_k[$.

<u>Claim</u> 1. There is $T > 0$ with $T_k \geq T$ for all k.

<u>Proof</u>. There is $R > 0$ such that

$$\sup_{x_0} \left(\int_{B_R(x_0)} |\nabla u_0|^2 dx \right) \leq \frac{\varepsilon_1}{3} .$$

Since $u_{0k} \to u_0$ in H^1 then for sufficiently large $k_0 \in \mathbb{N}$ we have

$$\sup_{k \geq k_0} \sup_{x_0} \left(\int_{B_R(x_0)} |\nabla u_{0k}|^2 dx \right) \leq \frac{\varepsilon_1}{2} ,$$

and Lemma 2.3.7 yields $\delta > 0$ such that for $0 < t \leq T = \delta R^2$ we have

$$(2.13) \quad \sup_{0<t<T} \sup_{x_0} \left(\int_{B_{R/2}(x_0)} |\nabla u_k(t)|^2 dx \right) \leq \varepsilon_1 ,$$

uniformly in $k \geq k_0$. The claim then follows from Prop. 2.3.1. $\square$

Proposition 2.3.1 and (2.13) now also yield smooth estimates for u_k on any time interval $[\tau, T]$, $\tau > 0$, uniformly in k.

A subsequence $u_k \to u$ then smoothly for $t > 0$ where u solves (2.4).

Moreover, the energy identity (2.6) and H^1-convergence $u_{0k} \to u_0$ give the uniform bound

$$\int_0^T \int_M |u_{kt}|^2 \, dx\, dt \leq E(u_{0k}) - E(u_k(T)) \xrightarrow{(k \to \infty)} E(u_0) - E(u(T))$$

and by compactness of the trace operator $H^1(M \times [0,T]) \ni v \mapsto v|_{t=0} \in L^2(M)$ we have

$$\lim_{t \downarrow 0} u(t) = \lim_{k \to \infty} \lim_{t \downarrow 0} u_k(t) = u_0 \in L^2(M)$$

in the sense of traces. But then, and since by (2.6) again, we have

$$\forall t > 0 : E(u(t)) = \lim_{k \to \infty} E(u_k(t)) \leq \lim_{k \to \infty} E(u_{0k}) = E(u_0),$$

we also have $u(t) \to u_0$ strongly in H^1 as $t \downarrow 0$.

Lemma 2.3.2 and (2.13) moreover yield the uniform bound

$$(2.14) \quad \int_0^T \int_M |\nabla^2 u|^2 \, dx\, dt \leq \liminf_{k \to \infty} \int_0^T \int_M |\nabla^2 u_k|^2 \, dx\, dt \leq C(u_0).$$

Uniqueness. Let u and v solve (2.4) on $]0,T[$ (2.14) and satisfying

$$u(t) \to u_0, \quad v(t) \to u_0 \quad \text{in } H^1$$

as $t \downarrow 0$. Then $w = u - v$ solves

$$w_t - \Delta w = A(u)(\nabla u, \nabla u) - A(v)(\nabla v, \nabla v)$$

$$= \big(A(u) - A(v)\big)(\nabla u, \nabla u) + A(v)\big(\nabla w, \nabla(u+v)\big).$$

Multiplying with w and integrating we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2} + \|\nabla w\|^2_{L^2} \leq C\int_M |w|^2 |\nabla u|^2 dx$$

$$+ C\int |\nabla w|\big(|\nabla u| + |\nabla v|\big)|w|\, dx$$

$$\leq C\int_M |w|^2\big(|\nabla u|^2 + |\nabla v|^2\big) dx + \frac{1}{2}\|\nabla w\|^2_{L^2},$$

and integrating also in time in view of $w(t) \to 0$ in H^1 $(t \downarrow 0)$ for any $0 < t_1 \leq T$ we find

$$\beta := \sup_{0<t<t_1} \|w(t)\|^2_{L^2} + \int_0^{t_1}\int_M |\nabla w|^2 dx\, dt$$

$$\leq C\left(\int_0^{t_1}\int_M |w|^4 dx\, dt \int_0^{t_1}\int_M \big(|\nabla u|^4 + |\nabla v|^4\big) dx\, dt\right)^{1/2}$$

$$\leq C\left(\sup_{0<t<t_1}\|w(t)\|^2 \int_0^{t_1}\int_M \big(|\nabla w|^2 + |w|^2\big) dx\, dt\right)^{1/2} \cdot$$

$$\cdot \left(\int_0^{t_1}\int_M \big(|\nabla u|^4 + |\nabla v|^4\big) dx\, dt\right)^{1/2},$$

where we use Gagliardo-Nirenberg interpolation.

But in view of (2.14) by Gagliardo-Nirenberg again we have $\nabla u, \nabla v \in L^4(M \times [0,T])$.

Thus
$$\int_0^{t_1}\int_M \left(|\nabla u|^4 + |\nabla v|^4\right) dx\,dt \to 0 \quad (t_1 \downarrow 0)$$
by absolute continuity of the Lebesgue integral, and for suitably small $t_1 > 0$ we obtain
$$\beta \leq \tfrac{1}{2}\beta\ ; \quad \text{hence } \beta = 0.$$
Thus $w = 0$ on $[0, t_1]$. Shifting time by t_0, we see that the interval
$$I = \{ t \in [0, T[\ ;\ w = 0 \text{ on } [0, t] \}$$
is open. By continuity of the map $t \mapsto w(t) \in L^2(M)$, I also is closed. Trivially, moreover, $0 \in I$.

Hence $I = [0, T[$, and $w \equiv 0$.

$\square$

Proof of Theorem 2.3.1. Given $u_0 \in H^1(M;N)$, the existence of a unique weak solution u of (2.4) which is smooth on a maximal time interval $]0, \bar{t}_1[$ and with $u(t) \xrightarrow{(t \downarrow 0)} u_0$ in H^1 follows from Cor. 2.3.3, and u satisfies (2.6).

Suppose $\bar{t}_1 < \infty$. By Cor. 2.3.2 there are at most finitely many concentration points $\bar{x}_1, \dots, \bar{x}_{i_1}$, $i_1 \leq C\,E(u_0)$, such that

$$\forall R > 0 \quad \limsup_{t \uparrow \bar{t}_1} \int_{B_{R/2}(\bar{x}_i)} |\nabla u(t)|^2 dx \geq \varepsilon_1, \quad 1 \leq i \leq i_0,$$

and by Lemma 2.3.7 there holds

$$\forall R > 0 \quad \liminf_{t \uparrow \bar{t}_1} \int_{B_R(\bar{x}_i)} |\nabla u(t)|^2 dx \geq \frac{\varepsilon_1}{2}, \quad 1 \leq i \leq i_0.$$

In view of Prop. 2.3.1, moreover, we have locally uniform smooth bounds for u near $t = T$ and away from $\bar{x}_1, \dots, \bar{x}_{i_1}$. As $t \uparrow T$, by the energy inequality (2.6) therefore

$$u(t) \to u_1 \quad \text{weakly in } H^1$$

and smoothly away from $\bar{x}_1, \dots, \bar{x}_{i_1}$, and we have

$$E(u_1) = \lim_{R \downarrow 0} \left(\frac{1}{2} \int_{M \setminus \bigcup_{i=1} B_R(\bar{x}_i)} |\nabla u_1|^2 dx \right)$$

$$= \lim_{R \downarrow 0} \lim_{t \uparrow \bar{t}_1} \left(\frac{1}{2} \int_{M \setminus \bigcup_{i=1} B_R(\bar{x}_i)} |\nabla u(t)|^2 dx \right)$$

$$= \lim_{R \downarrow 0} \lim_{t \uparrow \bar{t}_1} \Big(E(u(t)) - \sum_{i=1}^{i_1} \underbrace{\frac{1}{2} \int_{B_R(\bar{x}_i)} |\nabla u(t)|^2 dx}_{\geq \varepsilon_1/4} \Big)$$

$$\leq \lim_{t \uparrow \bar{t}_1} E(u(t)) - \frac{\varepsilon_1 i_1}{4} \leq E(u_0) - \frac{\varepsilon_1 i_1}{4}.$$

Repeating the argument, we may extend u by the unique weak solution of (2.4) on $[\bar{t}_1, \bar{t}_2[$ which is smooth on $]\bar{t}_1, \bar{t}_2[$ and with $u(t) \xrightarrow{(t \downarrow \bar{t}_1)} u_1$ in H^1.

The extended flow u weakly solves (2.4) on $]0, \bar{t}_2[$. Indeed, for any $\varphi \in C_c^\infty(M \times]0, \bar{t}_2[; \mathbb{R}^n)$

$$\int_0^{\bar{t}_2} \int_M u(-\varphi_t - \Delta\varphi)\, dx\, dt = \int_0^{\bar{t}_1} \ldots + \int_{\bar{t}_1}^{\bar{t}_2} \ldots$$

$$= \int_0^{\bar{t}_2} \int_M (u_t - \Delta u)\varphi\, dx\, dt - \int_M u\varphi\, dx \Big|_{t=\bar{t}_1-} + \int_M u\varphi\, dx \Big|_{t=\bar{t}_1+}$$

$$= \int_0^{\bar{t}_2} \int_M A(u)(\nabla u, \nabla u)\varphi\, dx\, dt,$$

since $u(t) \to u_1$ weakly in H^1 as $t \uparrow \bar{t}_1$ and as $t \downarrow \bar{t}_1$.

Moreover, if u blows up at $t = \bar{t}_2$ in points $\bar{x}_1^{(2)}, \ldots, \bar{x}_{i_2}^{(2)}$ as above we have

$$\frac{\varepsilon_1 i_2}{4} \leq E(u_1) \leq E(u_0) - \frac{\varepsilon_1 i_1}{4}.$$

Iterating, there can only be finitely many blow-up times where at least $\varepsilon_1/4$ ~~in~~ energy is lost, and if $\bar{t}_{k_0} < \infty$ is the last such blow-up time we can continue the flow to $]0, \infty[$ by proceeding as above.

Blow-up. Let u smoothly solve (2.4) on a maximal time interval $]0,T[$, where $T<\infty$, and let $\bar{x}\in M$ be a blow-up point.

For a sequence $R_k\downarrow 0$ there then exist points $z_k=(x_k,t_k)$ with $t_k\uparrow T$, $x_k\to\bar{x}$ such that

$$\sup_{|x_0-\bar{x}|<R_0}\ \sup_{t<t_k}\left(\int_{B_{R_k}(x_0)}|\nabla u(t)|^2dx\right)$$

$$\leq \varepsilon_1 = \int_{B_{R_k}(x_k)}|\nabla u(t_k)|^2dx,$$

where

$$0<R_0\leq \frac{1}{2}\min\{|\bar{x}-\bar{y}|;\ \bar{y}\neq\bar{x} \text{ is blow-up point}\}.$$

The scaled function

$$u_k(x,t)=u(x_k+R_kx,\ t_k+R_k^2t)$$

then satisfies

$$\sup_{|x_0|<L}\ \sup_{-1\leq t\leq 0}\int_{B_1(x_0)}|\nabla u_k(t)|^2dx$$

(2.15)

$$\leq \varepsilon_1 = \int_{B_1(0)}|\nabla u_k(0)|^2dx$$

and

$$\int_{-1}^{0}\int_{B_L(0)}|u_{kt}|^2dx\,dt \leq \int_{t_k-R_k^2}^{t_k}\int_M|u_t|^2dx\,dt \xrightarrow{(k\to\infty)} 0.$$

for any $L\in\mathbb{N}$.

Moreover, for all $-1 \leq t \leq 0$ we have

$$E(u_k(t)) = E(u(t_k + R_k^2 t)) \leq E(u_0),$$

for any $k \in \mathbb{N}$.

Since in view of (2.15) the condition (2.8) holds true for each u_k, we obtain smooth bounds for u_k on any domain $B_L(0) \times [\tau, 0]$, for any $L \in \mathbb{N}$ and any $\tau > -1$, uniformly in k.

A subsequence $u_k \to u_\infty$ smoothly, locally on $\mathbb{R}^2 \times]-1, 0]$, where u_∞ satisfies (2.4) with $\partial_t u_\infty = 0$; that is, $u_\infty = u_\infty(x)$ is a smooth harmonic map $u_\infty : \mathbb{R}^2 \to N$.

Moreover, again in view of (2.15) we have

$$\int_{B_1(0)} |\nabla u_\infty|^2 dx = \lim_{k \to \infty} \int_{B_1(0)} |\nabla u_k(0)|^2 dx = \varepsilon_1,$$

and $u_\infty \not\equiv$ constant.

Pulling back u_∞ to a smooth map

$$\bar{u} = \pi^* u_\infty = u_\infty \circ \pi : S^2 \setminus \{p_0\} \to N$$

via stereographic projection π from $p_0 \in S^2$, since π is conformal and since the Dirichlet energy is invariant under conformal changes of coordinates we obtain a non-constant, smooth harmonic map $\bar{u} : S^2 \setminus \{p_0\} \to N$ with $E(\bar{u}) = E(u_\infty)$

$$E(\bar{u}) = E(u_\infty) \leq \liminf_k E(u_k(-1)) \leq E(u_0) < \infty.$$

Since $\{p_0\}$ has vanishing H^1 capacity, then $\bar{u} \in H^1(S^2; N)$ is weakly harmonic as a map from S^2 to N.

By a result of Sacks-Uhlenbeck (1981) or by the regularity result of Hélein (1991), revisited by Rivière (2007), Rivière-Struwe (2008), $\bar{u}$ then, in fact, is smooth.

Asymptotics. i) Let u be a smooth solution of (2.4) on $]0,\infty[$. By (2.6) then we have

$$\int_1^\infty \int_M |u_t|^2 dx dt \leq E(u(1)) < \infty$$

and for any $t_k \uparrow \infty$ there holds

$$(2.16) \quad \int_{t_k}^{t_k+1} \int_M |u_t|^2 dx dt \to 0 \quad (k \to \infty).$$

Suppose that for such (t_k) there is $x_{ik} \to \bar{x}_i$, $1 \leq i \leq i_0$, such that

$$\forall R > 0 : \sup_{t_k < t < t_k+1} \int_{B_R(x_{ik})} |\nabla u(t)|^2 dx \geq \varepsilon_1 .$$

Choose $R > 0$ such that

$$R = \frac{1}{6} \min_{i \neq j} |\bar{x}_i - \bar{x}_j| > 0 .$$

Then for $k \geq k_0$ by (2.12) and (2.16) we have

$$\int_{B_{3R}(\bar{x}_i)} |\nabla u(t_k)|^2 dx$$

$$\geq \int_{B_{2R}(x_{ik})} |\nabla u(t_k)|^2 dx \geq \frac{\varepsilon_1}{2}, \quad 1 \leq i \leq i_0,$$

and

$$\frac{i_0 \varepsilon_1}{4} \leq E(u(t_k)) \leq E(u(1)).$$

Thus for suitable $t_k \to \infty$ there are at most $i_0 \le \frac{2E(u(1))}{\varepsilon_1}$ points $\bar{x}_i$, $1 \le i \le i_0$, where the flows

$$u_k(x,t) = u(x, t - t_k)$$

on $M \times [0,1]$ concentrate energy $\ge \varepsilon_1/2$.

Prop. 2.3.1 then yields smooth bounds for u_k on $(M \setminus \bigcup_{i=1}^{i_0} B_R(\bar{x}_i)) \times [\frac{1}{2}, 1]$ for any $R > 0$, uniformly in k. A subsequence $u_k \to u_\infty(x,t) \equiv u_\infty(x)$ smoothly away from $\bar{x}_1, .., \bar{x}_{i_0}$, where $u_\infty : M \setminus \{\bar{x}_1, .., \bar{x}_{i_0}\} \to N$ is a smooth harmonic map with

$$E(u_\infty) \le \liminf_{k\to\infty} E(u(t_k)) \le E(u(1)) < \infty .$$

As above, u_∞ then extends to a weakly harmonic map $u_\infty \in H^1(M; N)$, which is smooth by Sacks-Uhlenbeck (1981).

ii) Replacing t_k by $\tilde{t}_k = t_k + 1$, the blow-up analysis near $\bar{x}_i$, $1 \le i \le i_0$, may be carried out as in the case $T < \infty$.

An application. We can apply Theorem 2.3.1 to obtain a proof of the following result of Sacks-Uhlenbeck, Ann. Math. 113 (1981).

Theorem 2.3.2. Suppose N is closed with $\pi_2(N) = 0$. Then for any smooth $u_0 : M \to N$ there is a harmonic map $u_\infty : M \to N$ homotopic to u_0.

Proof. Let

$$\varepsilon_0 = \inf \{ E(u);\ u \in C^\infty(S^2; N) \text{ is non-const. and harmonic} \} > 0,$$

and set

$$[u_0] = \{ u \in C^\infty(M, N);\ u \text{ is homotopic to } u_0 \}.$$

Replacing u_0 by a suitable map $u \in [u_0]$, if necessary, we may assume that

$$E(u_0) \leq \inf \{ E(u);\ u \in [u_0] \} + \frac{\varepsilon_0}{4}.$$

Let u be the smooth solution of (2.4) with initial data $u|_{t=0} = u_0$ guaranteed by Theorem 2.3.1, defined on a maximal time interval $[0, T[$.

<u>Claim 1.</u> $T = \infty$.

<u>Proof.</u> Suppose $T < \infty$. Then there is a blow-up point $(\bar{x}, T)$ and sequences $t_k \uparrow T$, $x_k \to \bar{x}$, $R_k \downarrow 0$ such that

$$u_k(x) = u(x_k + R_k x, t_k) \xrightarrow{C^0} \tilde{u}(x) \text{ in } H^2_{loc}(\mathbb{R}^2; N),$$

where $\tilde{u} \circ \pi : S^2 \setminus \{p_0\} \to N$ may be extended to a non-constant, smooth harmonic map $\bar{u} : S^2 \to N$.

In particular, for large $L > 1$, $k \geq k_0(L)$

$$\int_{B_{2L} \setminus B_L(0)} (|\nabla u_k|^2 + |\nabla \tilde{u}|^2) dx < \frac{\varepsilon_0}{4}.$$

Fix $\tau \in C^\infty(\mathbb{R})$ with $0 \leq \tau \leq 1$, $\tau' \leq 0$, $\tau(s) = \begin{cases} 1, & s \leq 1/2 \\ 0, & s \geq 1 \end{cases}$, and scale $\tau_L(s) = \tau(\frac{s}{2L})$, $L \in \mathbb{N}$. Then setting

$$\bar{u}_k(x) = \begin{cases} u_k(x), & |x| \geq 2L, \\ \pi_N\left((1 - \tau_L(|x|)) u_k(x) + \tau_L(|x|)\, \tilde{u}(x)\right), & L < |x| < 2L, \\ \tilde{u}\left(L^2 \frac{x}{|x|^2}\right), & 0 < |x| \leq L \\ \bar{u}(p_0), & x = 0 \end{cases}$$

we obtain a continuos map $\bar{u}_k \in H^1(M;N)$

with
$$E(\bar{u}_k) \leq E(u_k) - \frac{1}{2}\int_{B_{2L}(0)} |\nabla u_k|^2 dx$$

$$+ \frac{1}{2}\int_{B_{2L}\setminus B_L(0)} \overbrace{|d\pi_N((1-\tau_L)u_k + \tau_L \tilde{u})|^2}^{\leq 1 + C\,\mathrm{dist}(\cdots, N) \leq 2} (|\nabla u_k|^2 + |\nabla \tilde{u}|^2)\, dx$$

$$+ \frac{1}{2}\int_{\mathbb{R}^2\setminus B_L(0)} |\nabla \tilde{u}|^2 dx$$

$$\leq E(u_0) + \frac{\varepsilon_0}{4} - \frac{1}{2}\int_{B_L(0)} |\nabla \tilde{u}|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2\setminus B_L(0)} |\nabla \tilde{u}|^2 dx.$$

$$\leq \inf\{E(v);\, v \in [u_0]\} + \frac{\varepsilon_0}{2} - \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \tilde{u}|^2 dx$$

$$+ \int_{\mathbb{R}^2\setminus B_L(0)} |\nabla \tilde{u}|^2 dx.$$

Moreover, since $\pi_2(N) = 0$, we have $\bar{u}_k \sim u_k \sim u_0$.

But
$$E(\tilde{u}) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \tilde{u}|^2 dx = E(\bar{u}) \geq \varepsilon_0$$

while
$$\int_{\mathbb{R}^2\setminus B_L(0)} |\nabla \tilde{u}|^2 dx \to 0 \quad \text{as } L \to \infty.$$

Hence
$$E(\bar{u}_k) < \inf\{E(v);\, v \in [u_0]\}$$

for large $L > 1$ and $k \geq k_0(L)$. Contradiction. $\square$

By claim 1 the solution u to (2.4) exists globally; moreover, if there is a concentration point $\bar{x}$ as $t \to \infty$, a contradiction results as in the proof of Claim 1. Thus $u(t) \to u_\infty$ smoothly as $t = t_i \to \infty$ suitably.

Finally, letting

$$h(\cdot, s) = \begin{cases} u(2t_i s), & 0 \leq s \leq 1/2 \\ \pi_N\big((2-2s)\, u(t_i) + (2s-1)\, u_\infty\big), & 1/2 \leq s \leq 1 \end{cases}$$

for sufficiently large t_i, we obtain the desired homotopy $u_0 = h(\cdot, 0) \sim h(\cdot, 1) = u_\infty$.

$\square$

2.4 The harmonic map heat flow in higher dimensions

As we saw in Thm. 2.1.2 already, in dimensions $m \geq 3$ the Dirichlet energy gives only insufficient bounds for harmonic maps. As further illustration we have the following result.

Example 2.4.1. i) Let $u: B_{1/4}(0;\mathbb{R}^m) \to S^m \setminus \{(0,\dots,0,1)\}$ be a smooth bijection with

$$u(x) \mapsto p_0 = (0,\dots,0,1) \quad \text{as } |x| \to 1/4,$$

inducing a smooth map $u: \mathbb{T}^m \to S^m$ of topological degree $+1$ by extending

$$u(x) \equiv p_0 \text{ for } x \in \mathbb{T}^m \setminus B_{1/4}(0).$$

Scaling $u_R(x) = u(\frac{x}{R})$ for $|x| \leq R/4$ and letting $u_R(x) = p_0$ else, we obtain a family of maps $u_R \in C^\infty(\mathbb{T}^m; S^m)$ homotopic to $u = u_1$ and hence of topological degree $+1$, $0 < R \leq 1$, with

$$E(u_R) = \frac{1}{2}\int_{\mathbb{T}^m} |\nabla u_R|^2 dx = R\, E(u_1) \xrightarrow{(R \to 0)} 0.$$

ii) More generally, for any closed target manifold N, any $m \geq 3$, if $\pi_m(N) \neq 0$ there are nontrivial $u_0 \in C^\infty(M; N)$ with $\inf_{u \sim u_0} E(u) = 0$.

Monotonicity formula: For harmonic maps $u \in C^\infty(B; N)$ from a ball $B = B_R(0; \mathbb{R}^m)$ to a closed target there is a scaled version of the energy which can be controlled, as follows.

Theorem 2.4.1. Let $u \in C^\infty(B_{R_0}(0); N)$ be harmonic. Then for $0 < r < R < R_0$ there holds

$$\frac{1}{r^{m-2}} \int_{B_r(0)} |\nabla u|^2 dx \leq \frac{1}{R^{m-2}} \int_{B_R(0)} |\nabla u|^2 dx.$$

Proof. Scaling $u_R(x) = u(Rx)$ we have

$$\frac{1}{R^{m-2}} \int_{B_R(0)} |\nabla u|^2 dx = \int_{B_1(0)} |\nabla u_R|^2 dx$$

and

$$\frac{d}{dR}\Big|_{R=1} \int_{B_1(0)} |\nabla u_R|^2 dx = 2 \int_{B_1(0)} \nabla u \, \nabla(x \cdot \nabla u) \, dx$$

$$= \underbrace{2 \int_{\partial B_1(0)} |x \cdot \nabla u|^2 do}_{\geq 0} - 2 \int_{B_1(0)} \underbrace{\langle \Delta u, x \cdot \nabla u \rangle_{\mathbb{R}^n}}_{=0} dx \geq 0.$$

$\square$

Is there an analogue for the flow?

Monotonicity formula for the flow. Control of a scale-invariant quantity related to the energy for solutions of (2.4) can be achieved by means of following result due to S. (1988).

Let

$$G(x,t) = \frac{1}{\sqrt{4\pi|t|}^{\,m}} \exp\left(-\frac{|x|^2}{4|t|}\right), \quad t<0$$

the fundamental solution of the (backward) heat equation and for $z_0 = (x_0, t_0) \in \mathbb{R}^m \times \mathbb{R}$ let

$$G_{z_0}(x,t) = G(x-x_0, t-t_0).$$

Theorem 2.4.2 (S., JDG 28 (1988)).

For any smooth solution $u : \mathbb{T}^m \times [0, T[\to N$ of (2.4) and any $z_0 = (x_0, t_0)$, $R>0$ let

$$\Phi(R) = \frac{R^2}{2} \int_{\mathbb{T}^m} |\nabla u|^2 G_{z_0}\, dx \Big|_{t=t_0-R^2},$$

where $0 < t_0 - R^2 < T$. Then, shifting $z_0 = (0,0)$, there holds

$$R \frac{d}{dR} \Phi(R) = \int_{\mathbb{R}^m} \frac{|x\cdot\nabla u + 2tu_t|^2}{2} G\, dx \Big|_{t=t_0-R^2} \geq 0,$$

where we regard u as $u \in C^\infty(\mathbb{R}^m \times [-T, 0[; N)$ of period 1 in each spatial variable.

Proof. We may regard u as a map $u \in C^\infty(\mathbb{R}^m \times [0,T[; N)$, periodic of period 1 in each spatial variable. Shifting $z_0 = (0,0)$, for $R > 0$ let

$$u_R(x,t) = u(Rx, R^2 t), \quad t < 0.$$

Observe that

$$\Phi(R,u) = \frac{R^2}{2} \int_{\mathbb{R}^m} |\nabla u|^2 G \, dx \Big|_{t=-R^2}$$

$$= \frac{1}{2} \int_{\mathbb{R}^m} |\nabla u_R|^2 G \, dx \Big|_{t=-1} = \Phi(1, u_R).$$

Thus, it suffices to assume $R = 1$.

Computing, we have

$$\frac{d}{dR}\Big|_{R=1} \Phi(R,u) = \frac{d}{dR}\Big|_{R=1} \Phi(1, u_R)$$

$$= \frac{d}{dR}\Big|_{R=1} \left(\frac{1}{2} \int_{\mathbb{R}^m} |\nabla u_R|^2 G \, dx \Big|_{t=-1} \right)$$

$$= \int_{\mathbb{R}^m} \nabla u \, \nabla(x \cdot \nabla u + 2t u_t) G \, dx \Big|_{t=-1}$$

$$= \int_{\mathbb{R}^m} \Big(-\Delta u - \underbrace{\frac{\nabla u \cdot \nabla G}{G}}_{= -\frac{x}{2|t|} \cdot \nabla u} \Big) (x \cdot \nabla u + 2t u_t) G \, dx \Big|_{t=-1}$$

and by orthogonality $\langle A(u)(\nabla u, \nabla u), x\cdot\nabla u + 2tu_t\rangle_{\mathbb{R}^n} = 0$ there results

$$\frac{d}{dR}\Big|_{R=1} \Phi(R,u) =$$

$$= \int_{\mathbb{R}^m} \left(-u_t + \frac{x}{2|t|}\cdot\nabla u\right)\left(x\cdot\nabla u + 2tu_t\right) G\, dx \Big|_{t=-1}$$

$$= \frac{1}{2}\int_{\mathbb{R}^m} |x\cdot\nabla u - 2u_t|^2 G\, dx \Big|_{t=-1} \geq 0 ,$$

and therefore

$$\left(R\frac{d}{dR}\right)\Big|_{R=R_0} \Phi(R,u) = \left(R\frac{d}{dR}\right)\Big|_{R=R_0} \Phi(1,u_R)$$

$$\overset{(R=sR_0)}{=} \frac{d}{ds}\Big|_{s=1} \Phi(s, u_{R_0})$$

$$= \frac{1}{2}\int_{\mathbb{R}^m} |x\cdot\nabla u_{R_0} - 2\partial_t u_{R_0}|^2 G\, dx \Big|_{t=-1}$$

$$= \int_{\mathbb{R}^m} \frac{|x\cdot\nabla u + 2tu_t|^2}{2} G\, dx \Big|_{t=-R_0^2} .$$

□

Remarks 2.4.1. i) For time independent elliptic equations or systems of the type

$$-\Delta u = f(x, u, \nabla u) \text{ in } \Omega \subset \mathbb{R}^m$$

the Green's function (here on $\mathbb{R}^m$)

$$G(x) = \frac{c_m}{|x|^{m-2}}, \quad x \neq 0,$$

has been used repeatedly by Hildebrandt-Widman and others to define a weighted Dirichlet integral

$$\int_\Omega |\nabla u|^2 G \, dx,$$

which scales invariantly.

ii) In their analysis of blow-up of solutions to the equation

$$u_t - \Delta u = u|u|^{p-2} \text{ on } \mathbb{R}^m \times]0,T[, \; m \geq 3,$$

in the sub-critical case $2 < p < 2^* = \frac{2m}{m-2}$, Giga-Kohn (1985), assuming the blow-up to be of type I so that in self-similar variables

$$s = \log\left(\frac{1}{T-t}\right), \; y = \frac{x}{\sqrt{T-t}} = e^{s/2}x, \; w(y,s) = (T-t)^{\frac{1}{p-2}} u(x,t)$$

the function w is bounded, find an energy inequality for w with weight $e^{-\frac{|y|^2}{4}}$; see [Giga-Kohn], Prop. 3, p. 307.

iii) Independently of S. (1988), Gerhard Huisken (1990) obtained a monotonicity formula similar to Theorem 2.4.2 for the mean curvature flow of hypersurfaces in $\mathbb{R}^m$, with kernel $G_{z_0}(x,t)$ at any point x_0 in the ambient space $\mathbb{R}^m$ and any retarded time t_0-t.

In fact, there is a variant of the monotonicity formula in Theorem 2.4.2 for the level set flow introduced by Evans-Spruck (1991-95), Osher-Sethian (1988) by means of which the results of S. (1988) and Huisken (1990) can be related; see

M. S.: Geometric evolution problems, IAS / Park City Series 2, AMS, 1996. (3.1)-(3.4), p. 8,

iv) Perelman (2002), defines an entropy functional $\mathcal{W}$ with weight $G = (4\pi\tau)^{-\frac{m}{2}}e^{-f}$ satisfying $(-\partial_t - \Delta + R)\,G = 0$ and computes $d\mathcal{W}/dt = \int_M 2\tau|\dots|^2 G\,dV$ (3.4).
On p. 4, l. 2, he refers to "... main monotonicity formula (3.4)" which "removes the major stumbling block in Hamilton's approach to geometrization" (p. 4, l. 6 f.).

<u>Partial regularity</u>. With the help of Thm. 2.4.1 we can obtain bounds for (smooth) solutions u of (2.4) away from a closed set of codimension ≥ 2, as follows.

<u>Proposition</u> 2.4.1 (ε_0-regularity). There is $\varepsilon_0 = \varepsilon_0(N, m) > 0$ such that for any smooth solution u of (2.4) on $]-R_0^2, 0[$ with $E(u(t)) \leq E_0 < \infty$ uniformly in t the following holds:

If for some $0 < R < R_0$ there holds

$$\Phi(R) = \Phi(R, u) = \int_{\mathbb{R}^m} |\nabla u|^2 G \, dx \Big|_{t=-R^2} < \varepsilon_0,$$

then

$$\sup_{Q_{\delta R}} |\nabla u|^2 \leq c(\delta R)^{-2}$$

with constants $c = c(N, m) > 0$ and $\delta = \delta(N, m, E_0) > 0$.

<u>Proof</u>. The argument is modelled on Schoen's (1984) proof for the time-independent case.

Let $R_1 = \delta R$ for $0 < \delta < 1$ to be determined.

There exists $0 < r_1 < R_1$ such that (with $e(u) = |\nabla u|^2$)

$$(R_1 - r_1)^2 \sup_{Q_{r_1}} e(u) = \max_{0 \le r \le R_1} \left((R_1 - r)^2 \sup_{Q_r} e(u) \right);$$

and $z_0 = (x_0, t_0) \in \overline{Q_{r_1}}$ with

$$\sup_{Q_{r_1}} e(u) = e(u)(z_0) =: e_0$$

Set $r_0 = \frac{R_1 - r_1}{2} > 0$. Note that $Q_{r_0}(z_0) \subset Q_{r_0 + r_1}$;

hence

$$\sup_{Q_{r_0}(z_0)} e(u) \le \sup_{Q_{r_0 + r_1}} e(u) \le \frac{(R_1 - r_1)^2 \sup_{Q_{r_1}} e(u)}{(R_1 - (r_0 + r_1))^2} = 4 e_0 .$$

Set $\sigma_0 := \sqrt{e_0} \cdot r_0$. Note that if $\sigma_0 \le 1$ we have

$$(R_1/2)^2 \sup_{Q_{R_1/2}} e(u) \le (R_1 - r_1)^2 e_0 = 4 r_0^2 e_0 = 4 \sigma_0^2 \le 4$$

and our claim follows.

The proposition therefore results from

<u>Claim</u> 1. For suitable $\delta = \delta(N, m, E_0) > 0$ there holds $\sigma_0 \le 1$.

<u>Proof</u>: Suppose $\sigma_0 > 1$. Scale

$$v(x,t) = u\left(x_0 + \frac{x}{\sqrt{e_0}}, t_0 + \frac{t}{e_0}\right) : Q_{\sigma_0} \to N .$$

Note that v solves (2.4) with

$$\sup_{Q_{\sigma_0}} e(v) = e_0^{-1} \sup_{Q_{r_0}(z_0)} e(u) \leq 4$$

and

$$e(v)(0,0) = 1.$$

The Bochner formula (2.7) thus yields the differential inequality

$$(\partial_t - \Delta)\, e(v) \leq C e(v) \quad \text{in } Q_{\sigma_0} \supset Q_1$$

with a constant $C = C(N)$. Moser's sup-estimate then gives the bound

$$1 = e(v)(0,0) \leq \sup_{Q_{1/2}} e(v) \leq C \int_{Q_1} e(v)\, dz.$$

But, scaling back, we have

$$\int_{Q_1} e(v)\, dz = e_0^{m/2} \int_{Q_{1/\sqrt{e_0}}(z_0)} e(u)\, dz,$$

where

$$1/\sqrt{e_0} + |x_0| \leq \frac{r_0}{\sigma_0} + r_1 \leq r_0 + r_1 = \frac{R_1 + r_1}{2} \leq R_1$$

$$1/e_0 + |t_0| \leq \frac{r_0^2}{\sigma_0^2} + r_1^2 \leq r_0^2 + r_1^2 \leq R_1^2,$$

and our claim follows from Claim 2 below by choosing $\varepsilon_0 = \frac{1}{3C_1}$, $\varepsilon = \frac{\varepsilon_0}{E_0}$, $\delta = \delta(N, m, \varepsilon)$.

Claim 2. For any $r_1, s > 0$, any $z_0 \in Q_{r_1}$ with $r_1 + s < \delta R$, any $\varepsilon > 0$, for sufficiently small $\delta = \delta(N, m, \varepsilon, \inf\{R, 1\}) > 0$ there holds

$$s^{-m} \int_{Q_s(z_0)} e(u)\, dz \leq C_1 \Phi(R) + \varepsilon E_0 \,,$$

with a constant $C_1 = C_1(N, m) > 0$.

Proof: Estimate for any z_0, $s > 0$ as above

$$s^{-m} \int_{Q_s(z_0)} e(u)\, dz = s^{-m} \int_{t_0 - s^2}^{t_0} \int_{B_s(x_0)} |\nabla u|^2 dx\, dt$$

$$\leq C \int_{t_0-s^2}^{t_0} \frac{t_0 + s^2 - t}{s^2} \Big(\int_{\mathbb{R}^m \times \{t\}} |\nabla u|^2 G_{(x_0, t_0 + s^2)}\, dx \Big)\, dt$$

$$\leq C \int_{t_0-s^2}^{t_0} \frac{dt}{s^2} \cdot \frac{t_0 + s^2 + R^2}{2} \int_{\mathbb{R}^m \times \{-R^2\}} |\nabla u|^2 G_{(x_0, t_0+s^2)}\, dx$$

$$\leq C R^2 \int_{\mathbb{R}^m \times \{-R^2\}} |\nabla u|^2 G_{(x_0, t_0 + s^2)}\, dx \,,$$

where we used Thm. 2.4.2 in the second inequality.

At $t=-R^2$, moreover, we can estimate

$$G_{(x_0,\,t_0+s^2)}(x,t) \leq C R^{-m} \exp\left(-\frac{|x-x_0|^2}{4(t_0+s^2+R^2)}\right)$$

with

$$\exp\left(-\frac{|x-x_0|^2}{4(t_0+s^2+R^2)}\right) \Big/ \exp\left(-\frac{|x|^2}{4R^2}\right)$$

$$= \exp\underbrace{\left(\frac{|x|^2}{4R^2} - \frac{|x-x_0|^2}{4(t_0+s^2+R^2)}\right)}$$

$$= \left(\frac{(t_0+s^2+R^2)\,|x|^2 - (|x|^2 - 2x\cdot x_0 + |x_0|^2)R^2}{4R^2(t_0+s^2+R^2)}\right)$$

$$\leq C\,\frac{|t_0+s^2|\,|x|^2 + 2\overbrace{x\cdot x_0}^{\leq \delta^2|x|^2+\delta^{-2}|x_0|^2 \leq \delta^2|x|^2+R^2}R^2 - |x_0|^2R^2}{R^4}$$

$$\leq C\left(\frac{\delta^2|x|^2}{R^2}+1\right) \leq C, \text{ if } |x| \leq R/\delta,$$

so that

$$G_{(x_0,\,t_0+s^2)}(x,t) \leq C_1\, G(x,t) \quad \text{if } |x| \leq R/\delta,$$

while for $|x| \geq R/\delta$ we can bound

$$G_{(x_0,\,t_0+s^2)}(x,t) \leq C\exp\left(-\frac{|x|^2}{5R^2} - \frac{m}{2}\log R\right)$$

$$\leq C\exp\left(-\frac{|x|^2}{10R^2} - \frac{\delta^{-2}}{10} - \frac{m}{2}\log R\right) \leq \varepsilon\exp\left(-\frac{|x|^2}{10R^2}\right)$$

if $0<\delta<\delta_0(N,m,\varepsilon,R)$ for $R\leq 1$, $0<\delta<\delta_0(N,m,\varepsilon)$ else. $\square$

Corollary 2.4.1 . i) There exists $E_0 > 0$ such that any smooth solution u of (2.4) on $[0, \infty[$ with $E(u(0)) < E_0$ is homotopic to a constant map.

ii) Conversely, if $u_0 \in C^\infty(M, N)$ is not homotopic to a constant map and if $E(u_0) < E_0$, the maximal smooth solution u of (2.4) with $u|_{t=0} = u_0$ must blow up in finite time.

Remark 2.4.1: Chen - Ding (1990) combine Prop. 2.4.1 and Example 2.4.1 to give an example illustrating statement ii) of the above Cor. 2.4.2.

Proof: i) For $R \geq 1 \geq E_0$, Proposition 2.4.1 gives the a-priori bound

$$\sup_{x \in M} |\nabla u(x,t)| \leq C R^{-1}$$

for all $t \geq R^2$ with $C = C(N, m) > 0$, if $E(u(t)) \leq E_0$ for sufficiently small $E_0 > 0$.

Indeed, we can estimate

$$\Phi_z(R, u) = \frac{R^2}{2} \int_{\mathbb{R}^m} |\nabla u|^2 G_z \, dx$$

$$\leq C R^{2-m} \int_{\mathbb{R}^m} |\nabla u|^2 \exp\left(-\frac{|x-x_0|^2}{4}\right) dx$$

$$\leq C R^{2-m} E(u) \leq C E_0 < \varepsilon_0$$

uniformly for all $z = (x, t)$ with $t \geq R^2 \geq 1$, with constants $C = C(N, m)$, for any u with $E(u) \leq E_0$, where $\varepsilon_0 > 0$ is as determined in Prop. 2.4.1.

Note that for $R \geq 1 \geq E_0$, the constant $\delta = \delta(N, m, E_0, \inf\{R, 1\}) = \delta(N, m) > 0$.

For sufficiently large $R_0 = R_0(N) \geq 1$ and any $t_0 \geq R_0^2$ then the image of $u(t)$ will be contained in a contractible coordinate ball on N; hence $u(t_0)$ is homotopic to a constant, and therefore also any map $u(t)$, $0 \leq t < \infty$.

ii) This is an immediate consequence of i). $\square$

Corollary 2.4.2. Let $(u_k) \subset C^\infty(\mathbb{T}^m \times [0,T[; N)$ be a sequence of smooth sol's to (2.4) with $E(u_k(t)) \leq E_0 < \infty$ uniformly in $0 \leq t < T$, $k \in \mathbb{N}$. Then exists a rel. closed set $S \subset \mathbb{T}^m \times]0,T[$ with $\mathcal{H}^m(S \cap Q) < \infty$ for every $Q \subset \bar{Q} \subset M \times]0,T[$ such that a subsequence $u_k \to u$ smoothly locally away from S, where u weakly solves (2.4) with $E(u(t)) \leq E_0$, $0 \leq t < T$.

Proof: Following Schoen's (1984) argument for the time-independent case, we let

$$S = \bigcap_{R>0} \{ z_0 = (x_0, t_0) \in \mathbb{T}^m \times [0,T[; \liminf_{k\to\infty} \Phi_{z_0}(R; u_k) \geq \varepsilon_0 \},$$

where $\varepsilon_0 > 0$ is as determined in Prop. 2.4.1.

Claim 1. S is closed.

Proof: Let $(z_l) \subset S$ with $z_l = (x_l, t_l) \to z_0 = (x_0, t_0)$ as $l \to \infty$. Then for any $R > 0$ we have

$$\Phi_{z_0}(R, u_k) = \frac{R^2}{2} \int_{\mathbb{R}^m} |\nabla u_k|^2 G_{z_0} \, dx \Big|_{t = t_0 - R^2}$$

$$= \frac{R^2}{2} \int_{\mathbb{R}^m} |\nabla u_k|^2 \lim_{l\to\infty} G_{z_l} \, dx \Big|_{t = t_0 - R^2}$$

$$\geq \lim_{l\to\infty} \Phi_{z_l}(R_l, u_k) \geq \varepsilon_0$$

for suitable $R_\ell \uparrow R$ $(\ell \to \infty)$ by definition of S, and $z_0 \in S$. $\square$

Claim 2. (For any $Q \subset \overline{Q} \subset \mathbb{T}^m \times]0,T[$ there holds) $\mathcal{H}^m(S \cap Q) < \infty$ (with respect to the metric $d((x,t),(y,s)) = |x-y| + |t-s|^{1/2}$).

Proof. By definition
$$\mathcal{H}^m(S) = \lim_{\delta \downarrow 0} \mathcal{H}^m_\delta(S),$$
where for $\delta > 0$
$$\mathcal{H}^m_\delta(S) = \inf\{\sum_{j \in J} r_j^m;\ S \subset \bigcup_{j \in J} P_{r_j}(z_j),\ r_j < \delta\}$$
with
$$P_r(z_0) = \{z = (x,t);\ |x - x_0| < r,\ |t - t_0| < r^2\}.$$

Fix $L, \delta > 0$, and let $S \cap \overline{Q} \subset \bigcup_{j \in J} P_{Lr_j}(z_j)$ with $Lr_j < \delta$ for all $j \in J$. (By compactness, $|J| < \infty$.) Replacing r_j by $2r_j$, if necessary, we may assume that $z_j \in S \cap \overline{Q}$, $j \in J$, and that, given $L \in \mathbb{N}$, $P_{Lr_i}(z_i) \cap P_{Lr_j}(z_j) = \phi$ for $i \neq j$ while $S \cap \overline{Q} \subset \bigcup_{j \in J} P_{5Lr_j}(z_j)$, where we use Vitali's covering lemma.

For each $j \in J$ then we have
$$\varepsilon_0 \leq \underbrace{\liminf_{k \to \infty} \left(\frac{2}{r_j} \int_{r_j/2}^{r_j} \Phi_{z_j}(R, u_k)\, dR \right)}_{\leq \int_{r_j/2}^{r_j} \left(\int_{\mathbb{R}^m} |\nabla u_k|^2 G_{z_j}\, dx \right) \Big|_{t = t_j - R^2} R\, dR} \leq \liminf_{k \to \infty} \left(\int_{t_j - r_j^2}^{t_j - (r_j/2)^2} \int_{\mathbb{R}^m} |\nabla u_k|^2 G_{z_j}\, dz \right).$$

Given $\varepsilon > 0$ there is $L \in \mathbb{N}$ such that

$$G_{z_j}(x,t) = \frac{1}{(4\pi|t_j - t|)^{m/2}} \exp\left(-\frac{|x-x_j|^2}{4|t_j-t|}\right)$$

$$\leq \varepsilon\, G_{(z_j + (0, r_j^2))}(x,t)$$

for $|x-x_j| \geq L r_j$, $(r_j/2)^2 \leq t_j - t \leq r_j^2$. Thus we have

$$\int_{t_j - r_j^2}^{t_j - (r_j/2)^2} \int_{\mathbb{R}^m} |\nabla u_k|^2 G_{z_j}\, dx\, dt$$

$$\leq \int_{t_j - r_j^2}^{t_j - (r_j/2)^2} \int_{B_{Lr_j}(x_j)} |\nabla u_k|^2 G_{z_j}\, dx\, dt$$

$$+ \varepsilon \underbrace{\int_{t_j - r_j^2}^{t_j} r_j^2 \int_{\mathbb{R}^m} |\nabla u_k|^2 G_{z_j + (0, r_j^2)}\, dx \frac{dt}{r_j^2}}_{\leq \Phi(R_j, u_k) \leq C(Q) E_0,}$$

where $R_j^2 \doteq t_j + 2r_j^2$,

$$\leq C r_j^{-m} \int_{P_{Lr_j}(z_j)} |\nabla u|^2 dz + \varepsilon_0 / 2,$$

if $\varepsilon = \varepsilon(Q) > 0$ is sufficiently small.

It follows that for $j \in \mathcal{J}$ there holds

$$\varepsilon_0/2 \leq \liminf_{k \to \infty} C r_j^{-m} \int_{P_{Lr_j}(z_j)} |\nabla u|^2 dz .$$

For sufficiently large $k \in \mathbb{N}$ then we obtain

$$\sum_{j \in \mathcal{J}} r_j^m \leq C \varepsilon_0^{-1} \int_{\bigcup_{j \in \mathcal{J}} P_{Lr_j}(z_j)} |\nabla u_k|^2 dz \leq C T E_0 ,$$

independently of $\delta > 0$, and

$$\mathcal{H}^m_\delta (S \cap Q) \leq (5L)^m \sum_{j \in \mathcal{J}} r_j^m \leq C L^m T E_0 ,$$

where $L = L(\varepsilon) = L(Q)$. □

By the energy inequality we may assume that $u_k \rightarrow u$ weakly in $H^1(\mathbb{T}^m \times [0,T[)$. Moreover, for any $z_0 \notin S$ there is $R > 0$ and a sequence $k \to \infty$ such that

$$\limsup_{k \to \infty} \Phi_{z_0}(R, u_k) < \varepsilon_0$$

and by Prop. 2.4.1 there holds

$$\sup_{Q_{\delta R}(z_0)} |\nabla u_k|^2 \leq C < \infty ,$$

uniformly in k, and $u_k \rightarrow u$ smoothly near z_0.

Thus u smoothly solves (2.4) away from S.

Claim 3. u weakly solves (2.4).

Proof: Given $\varphi \in C_c^\infty(\Pi^m \times]0,T[)$ let $Q \subset \overline{Q} \subset \Pi^m \times]0,T[$ with $\mathrm{supp}(\varphi) \subset Q$ and let $(P_{r_j}(z_j))_{j\in J}$ (be disjoint) with $S \cap \overline{Q} \subset \bigcup_{j\in J} P_{5r_j}(z_j)$, $r_j < \delta$, and

$$\sum_{j\in J} r_j^m \leq C = C(Q, N, u, E_0) < \infty .$$

Fix $\eta \in C_c^\infty(P_2)$ with $0 \leq \eta \leq 1$; $\eta \equiv 1$ on P_1, and scale

$$\eta_{r,z_0}(x,t) = \eta\left(\frac{x-x_0}{r}, \frac{t-t_0}{r^2}\right) \in C_c^\infty(P_r(z_0)).$$

Then we have, with $\eta_j = \eta_{5r_j, z_j}$, $j \in J$:

$$0 = \int_{\Pi^m \times]0,T[} \big(u_t - \Delta u - A(u)(\nabla u, \nabla u)\big)\, \varphi \inf_{j\in J}(1-\eta_j)\, dz$$

$$= \int_{\Pi^m \times]0,T[} \big(u_t - A(u)(\nabla u, \nabla u)\big)\, \varphi \inf_{j\in J}(1-\eta_j)\, dz$$

$$+ \int_{\Pi^m \times]0,T[} \nabla u \Big(\nabla\varphi \inf_{j\in J}(1-\eta_j) - \varphi \sum_{1-\eta_i = \inf_{j\in J}(1-\eta_j)} \nabla\eta_i\Big)\, dz$$

$$= \int_{\Pi^m \times]0,T[} \Big(\big(u_t - A(u)(\nabla u, \nabla u)\big)\varphi + \nabla u \nabla\varphi\Big)\, dz + I ,$$

with error

$$|\mathrm{I}| \leq \|u_t\|_{L^1(Q_\delta)} + C\|\nabla u\|^2_{L^2(Q_\delta)}$$

$$+ C \int_Q |\nabla u| \sum_{1-\eta_i = \inf_j (1-\eta_j)} |\nabla \eta_i| \, dz,$$

where $Q_\delta = \bigcup_{j \in J} \operatorname{supp}(\eta_j)$, $\mathcal{L}^{m+1}(Q_\delta) \to 0 \ (\delta \downarrow 0)$.

Moreover, by Hölder's inequality and Cauchy-Schwarz

$$\int_Q |\nabla u| \sum_{1-\eta_i = \inf_j (1-\eta_j)} \overbrace{|\nabla \eta_i|}^{\leq C/r_i} \, dz \leq C \sum_{i \in J} \left(r_i^{-1} \int_{B_{5r_i}(z_i) \cap \{1-\eta_i = \inf_j (1-\eta_j)\}} |\nabla u| \, dz \right)$$

$$\leq C \sum_{i \in J} r_i^{\frac{m}{2}} \left(\int_{B_{5r_i}(z_i) \cap \{1-\eta_i = \inf_j (1-\eta_j)\}} |\nabla u|^2 dz \right)^{\frac{1}{2}}$$

$$\leq C \underbrace{\left(\sum_{i \in J} r_i^m \right)^{\frac{1}{2}}}_{\leq C} \left(\sum_{i \in J} \int_{B_{5r_i}(z_i) \cap \{1-\eta_i = \inf_j (1-\eta_j)\}} |\nabla u|^2 dz \right)^{\frac{1}{2}}$$

$$\leq C \left(\int_{\bigcup_{i \in J} B_{5r_i}(z_i)} |\nabla u|^2 dz \right)^{\frac{1}{2}} \to 0 \ (\delta \downarrow 0)$$

Thus, $|\mathrm{I}| \to 0$ as $\delta \downarrow 0$, proving the claim. $\square$

By means of a suitable approximation of (2.4) for which a result analogous to Cor. 2.4.1 holds we then obtain the following global existence result.

Theorem 2.4.3 (Chen - S., Math. Z. (1989))

For any $u_0 \in H^1(M; N)$ there exists a weak solution $u: M \times [0, \infty[\to N$ of (2.4) which satisfies the energy inequality

$$\int_0^T \int_M |u_t|^2 dx\, dt + E(u(T)) \leq E(u_0)$$

for all $T > 0$ and with $u(t) \xrightarrow{(t \downarrow 0)} u_0$ in H^1.

Moreover, u is smooth away from a closed singular set S of locally finite m-dim. Hausdorff measure on $M \times]0, \infty[$, and there holds

(2.17) $$\Phi_{z_0}(R, u) \leq \frac{R_0^2}{2} \int_{\mathbb{R}^m} |\nabla u_0|^2 G_{z_0}\, dx$$

for any $z_0 = (x_0, t_0) \in M \times]0, \infty[$, any $0 < R < R_0 = \sqrt{t_0}$.

Remarks 2.4.1. i). Coron (1990) observed that Theorem 2.4.3 implies non-uniqueness of weak solutions of class H^1 to (2.4) for certain initial data u_0.

Indeed, if $u_0 \in H^1(M;N)$ is weakly harmonic but not "stationary" in the sense of Evans (1991), the time-constant weak solution $u(t) \equiv u_0$ of (2.4) does not satisfy (2.17)

ii) Feldman (1994) defined a notion of "stationary" (time-dependent) solution of (2.4), incorporation the monotonicity requirement from Thm. 2.4.2.

iii) Germain - Ghoul - Miura (arXiv 2016) analyse uniqueness / non-uniqueness for $m > 2$ in the co-rotational equivariant setting.

Proof of Theorem 2.4.3 (sketch): Following ideas of Chen (1989), Keller-Rubinstein-Sternberg (1989) Shatah (1988) for harmonic flows/wave maps into $N = S^k$, we employ a Ginzburg-Landau approximation.

Let $\delta > 0$ such that $\pi_N : \mathcal{U}_{2\delta}(N) \to N$ is smooth. For $\chi \in C^\infty(\mathbb{R})$ with $\chi' \geq 0$, $\chi(s) = \delta^2$ for $s \geq 2\delta^2$, $\chi(s) = s$ for $s \leq \delta^2/2$, $K \in \mathbb{N}$, $u \in C^1(M; \mathbb{R}^n)$ let

$$(2.18) \quad E_K(u) = \frac{1}{2} \int_M \left(|\nabla u|^2 + 2K \chi\left(\frac{\mathrm{dist}^2(u, N)}{2} \right) \right) dx,$$

and consider the corresponding L^2-gradient flow

$$(2.19) \quad u_t - \Delta u + K \chi'\left(\frac{\mathrm{dist}^2(u, N)}{2} \right) (u - \pi_N(u)) = 0,$$

where we note that

$$u - \pi_N(u) = \nabla\left(\frac{\mathrm{dist}^2(u, N)}{2} \right), \quad u \in \mathcal{U}_{2\delta}(N) \setminus N,$$

with initial condition

$$u|_{t=0} = u_0 \in C^\infty(M; N).$$

By Thm. 1.6.1 there is a unique sol. u_K of this problem. The analogues of the energy inequality and monotonicity formula hold for u_K.

<u>Energy inequality</u>. Compute, for $u = u_K$:

$$\frac{d}{dt} E_K(u_K(t)) = \int_M \Big(\nabla u \cdot \nabla(\partial_t u) + K \chi' \langle u - \pi_N(u), u_t \rangle \Big) dx$$

$$= \int_M \partial_t u \left(-\Delta u + K \chi' (u - \pi_N(u)) \right) dx$$

$$= - \int_M |\partial_t u|^2 dx.$$

<u>Bochner formula</u>. Differentiate (2.19) and multiply with $\nabla u = \nabla u_K$ to obtain (for $u = u_K$):

$$(\partial_t - \Delta)\left(\frac{|\nabla u|^2}{2}\right) + |\nabla^2 u|^2$$

$$+ K \underbrace{\nabla\left(\chi'\left(\frac{\mathrm{dist}^2(u,N)}{2}\right)(u - \pi_N(u))\right) \nabla u}_{} = 0$$

$$= \nabla \chi'\left(\frac{\mathrm{dist}^2(u,N)}{2}\right) \nabla\left(\frac{\mathrm{dist}^2(u,N)}{2}\right)$$

$$+ \chi'\left(\frac{\mathrm{dist}^2(u,N)}{2}\right) \left((\mathbb{1} - d\pi_N(u)) \nabla u, \nabla u \right)$$

$$= \chi''\left(\frac{\mathrm{dist}^2(u,N)}{2}\right) \left| \nabla\left(\frac{\mathrm{dist}^2(u,N)}{2}\right) \right|^2$$

$$+ \chi'\left(\frac{\mathrm{dist}^2(u,N)}{2}\right) |(\mathbb{1} - d\pi_N(u)) \nabla u|^2$$

Moreover, we have

$$
\begin{aligned}
(\partial_t - \Delta)\,\chi\!\left(\frac{\operatorname{dist}^2(u,N)}{2}\right) &= \\
&= \chi'\!\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)\overbrace{(\partial_t-\Delta)\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)}^{=\langle u-\pi_N(u),\, u_t-\Delta u\rangle_{\mathbb{R}^n} - \langle \nabla(u-\pi_N(u)),\, \nabla u\rangle_{\mathbb{R}^n}} \\
&\quad - \chi''\!\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)\left|\nabla\!\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)\right|^2
\end{aligned}
$$

$$
\begin{aligned}
&= -K\left(\chi'\!\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)|u-\pi_N(u)|\right)^2 - \chi'(\dots)\,|(\mathbb{1}-d\pi_N(u))\nabla u|^2 \\
&\quad - \chi''(\dots)\left|\nabla\!\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)\right|^2.
\end{aligned}
$$

Letting

$$
e_K = e_K(u) = \frac{1}{2}|\nabla u|^2 + K\chi\!\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)
$$

thus there results

$$
\begin{aligned}
(\partial_t - \Delta)e_K + |\nabla^2 u|^2 + K^2|\chi'(\dots)|^2 \operatorname{dist}^2(u,N) \\
+ 2K\chi'(\dots)\,|(\mathbb{1}-d\pi_N(u))\nabla u|^2 + 2K\chi''(\dots)\left|\nabla\!\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)\right|^2 = 0
\end{aligned}
$$

and since $\chi' \geq 0$ we have the equation

$$
\begin{aligned}
(\partial_t - \Delta)\,e_K &\leq CK|\chi''|\left|\nabla\!\left(\frac{\operatorname{dist}^2(u,N)}{2}\right)\right|^2 = CK|\chi''|\,|\langle u-\pi_N(u), \nabla u\rangle_{\mathbb{R}^n}|^2 \\
&\leq CK|\chi''|\,|\nabla u|^2 \operatorname{dist}^2(u,N) \leq C e_K^2,
\end{aligned}
$$

where we note that $\chi''(\dots) \neq 0$ only if $\delta \leq \operatorname{dist}(u,N) \leq 2\delta$, in which case $K \operatorname{dist}^2(u,N) \leq CK\chi\!\left(\frac{\operatorname{dist}^2(u,N)}{2}\right) \leq C e_K$.

Monotonicity formula. Observe that scaling

$$u_K^R(x,t) = u_K(Rx, R^2 t),$$

from u_K we obtain a solution of (2.19) with $K_R = R^2 K$.

For a solution u_K of (2.19) on $\mathbb{R}^m \times]-R_0^2, 0[$, letting

$$\Phi_K(R, u_K) = \int_{\mathbb{R}^m} e_K(u_K)\, G\, dx \Big|_{t=-R^2}$$

we have

$$\Phi_K(R, u_K) = \int_{\mathbb{R}^m} e_{K_R}(u_K^R)\, G\, dx \Big|_{t=-1} = \Phi_{K_R}(1, u_K^R)$$

with u as above.

Hence

$$\frac{d}{dR}\Big|_{R=1} \Phi_K(R, u_K) = \frac{d}{dR}\Big|_{R=1} \Phi_{K_R}(1, u_K^R)$$

$$= \frac{d}{dR}\Big|_{R=1} \left(\frac{1}{2}\int_{\mathbb{R}^m} |\nabla u_K^R|^2 G\, dx + R^2 K \int_{\mathbb{R}^m} \chi(\dots)\, G\, dx \right)$$

$$= \int_{\mathbb{R}^m} \left(-\Delta u_K + K\chi'(\dots)(u_K - \pi_N(u_K)) - \frac{\nabla u_K \cdot \nabla G}{G} \right) \frac{d}{dR}\Big|_{R=1} u_K^R\, G\, dx$$

$$+ 2K \int_{\mathbb{R}^m} \chi(\dots)\, G\, dx$$

$$= \int_{\mathbb{R}^m} \left(\left| \frac{x \cdot \nabla u_K + 2t\, \partial_t u_K}{2} \right|^2 + 2K\chi(\dots) \right) G\, dx \geq 0.$$

3. The Navier-Stokes equations

An inviscid, incompressible fluid in a container $\Omega \subset \mathbb{R}^n$ is modelled by the system of equations

$$(3.1) \quad u_t - \Delta u + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \Omega \times [0, T[,$$

$$(3.2) \quad \operatorname{div} u = 0 \quad \text{in } \Omega \times [0, T[,$$

with

$$u|_{t=0} = u_0$$

for some $u_0 \in C^\infty(\bar{\Omega}; \mathbb{R}^n)$ with $\operatorname{div} u_0 = 0$ and suitable boundary conditions.

Here, we consider the periodic case and seek solutions u, p to (3.1), (3.2) on $\mathbb{T}^n \times [0, T[$.

3.1 Basic facts.

The energy inequality for (3.1), (3.2) and Helmholtz decomposition of vector fields will play a key role in the following.

Energy inequality. For a (smooth) solution u, p of (3.1), (3.2) we have the Bochner-type identity

$$(3.3) \quad (\partial_t - \Delta)\left(\frac{|u|^2}{2}\right) + |\nabla u|^2 + u \cdot \nabla\left(\frac{|u|^2}{2} + p\right) = 0.$$

Integrating over $\mathbb{T}^n \times [0, t_1]$, we find

$$(3.4) \quad \sup_{0 \leq t_1 \leq T} \frac{\|u(t_1)\|^2_{L^2}}{2} + \int_0^{t_1}\int_{\mathbb{T}^n} |\nabla u|^2 \, dx\, dt \leq \frac{\|u_0\|^2_{L^2}}{2},$$

where we used the fact that

$$u \cdot \nabla\left(\frac{|u|^2}{2} + p\right) = \operatorname{div}\left(u\left(\frac{|u|^2}{2} + p\right)\right)$$

in view of (3.2).

<u>Sobolev</u> <u>embedding</u>. For $n=2$ the energy inequality by Gagliardo–Nirenberg / Ladyzhenskaya interpolation gives an L^4 bound for u in space-time

$$\|u\|^4_{L^4} \leqslant C\|u\|^2_{L^{2,\infty}_{x,t}} \int_0^T \|u(t)\|^2_{H^1} dt \leqslant$$

$$\leqslant C\|u\|^2_{L^{2,\infty}_{x,t}}\left(\|\nabla u\|^2_{L^2} + T\|u\|^2_{L^{2,\infty}_{x,t}}\right)$$

for any $T>0$.

Likewise, for $n \geqslant 3$ we have a space-time bound in $L^{2\frac{n+2}{n}}$ by Sobolev's embedding $H^1 \hookrightarrow L^{\frac{2n}{n-2}}$ and Hölder's inequality, estimating

$$\int_0^T \int_{\mathbb{T}^n} |u(t)|^{2\frac{n+2}{n}} dx\, dt \leqslant \int_0^T \Big(\int_{\mathbb{T}^n} \underbrace{|u(t)|^{\frac{4}{n}}}_{\in L^{\frac{n}{2}}} \underbrace{|u(t)|^2}_{\in L^{\frac{n}{n-2}}} dx \Big) dt$$

$$\leqslant C\|u\|^{4/n}_{L^{2,\infty}_{x,t}} \int_0^T \|u(t)\|^2_{H^1} dt \leqslant C\|u\|_{L^{2,\infty}_{x,t}}\left(\|\nabla u\|^2_{L^2} + T\|u\|^2_{L^{2,\infty}_{x,t}}\right).$$

Helmholtz decomposition. As a special case of Hodge decomposition, we can write any (smooth) vector field φ on $\mathbb{T}^n$ as the sum of an irrotational (curl-free) and a solenoidal (divergence-free) component. For ease of exposition, let $n=3$.

Theorem 3.1.1. For any $\varphi \in C^2(\mathbb{T}^3; \mathbb{R}^3)$ there exists a unique L^2-orthogonal decomposition

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} = \operatorname{curl} \psi + \nabla q + h$$

where $\varphi^{(0)} = h \equiv \text{const}$, $\operatorname{div} \varphi_1 = 0$, $\operatorname{curl} \varphi_2 = 0$.

Proof: Let $f := -\Delta \varphi$. Observe that

$$\Delta \varphi = -\operatorname{curl} \operatorname{curl} \varphi + \nabla \operatorname{div} \varphi .$$

Let

$$\psi = -\Delta^{-1}(\operatorname{curl} \varphi) \quad \text{on } \mathbb{T}^3$$

$$q = -\Delta^{-1}(\operatorname{div} \varphi) \quad \text{on } \mathbb{T}^3$$

Then $h = \varphi - (\operatorname{curl} \psi + \nabla q)$ is harmonic and therefore $h \equiv \text{const}$. Let $\varphi^{(1)} = \operatorname{curl} \psi$, $\varphi^{(2)} = \nabla q$.

Moreover, since $\operatorname{curl}^* = \operatorname{curl}$, we have

$$\int_{\mathbb{T}^3} \operatorname{curl} \psi \cdot \nabla q \, dx = -\int \psi \operatorname{curl} \nabla q \, dx = 0,$$

and the decomposition is L^2-orthogonal, as claimed. In particular, the above decomposition $\varphi = \varphi_1 + \varphi_2 + \varphi_0$ is unique. $\square$

From an L^2-orthonormal basis (φ_i) of eigenfunctions of the Laplacian via Theorem 3.1.1 we then obtain an L^2-orthonormal basis $(\varphi_i^{(1)})$ of the set of solenoidal vector fields, satisfying

$$-\Delta \varphi_i^{(1)} = \lambda_i \varphi_i^{(1)}, \quad \operatorname{div} \varphi_i = 0$$

for each $i \in \mathbb{N}$ in view of the decomposition

$$\varphi_i^{(1)} = \operatorname{curl}(-\Delta^{-1}(\operatorname{curl} \varphi_i))$$

and the fact that Δ and curl commute.

3.2 Local existence, Hopf solutions

We can adapt the method of proof of Thm. 1.4.4 to obtain global weak solutions of energy class, and classical solutions for short time, for any (smooth) data u_0. Let $n = 2$ or 3.

<u>Theorem</u> 3.2.1 (E. Hopf (1951), J. Leray (19)).
For any $u_0 \in L^2(\mathbb{T}^n; \mathbb{R}^n)$ with $\operatorname{div} u_0 = 0$ there exists a function $u \in V^{1,0}(\mathbb{T}^n \times [0, \infty[; \mathbb{R}^n)$ weakly solving (3.1), (3.2) for some $p \in L^2$.

<u>Proof</u>: Let (φ_i) be an L^2-orthonormal basis of eigenfunctions of the Laplacian, $\psi_i = \varphi_i^{(n)}$ the corresponding smooth solenoidal vector fields, $i \in \mathbb{N}$.
Given $I \in \mathbb{N}$ let $u^{(I)}(x,t) = \sum_{i=1}^{I} \alpha_i(t) \psi_i(x)$, where the coefficients $\alpha_i = \alpha_i^{(I)}$, $1 \le i \le I$, are determined so that

$$0 = \int_{\mathbb{T}^n} \langle \partial_t u^{(I)} - \Delta u^{(I)} + u^{(I)} \cdot \nabla u^{(I)}, \psi \rangle_{\mathbb{R}^n} dx, \quad t > 0,$$

$$0 = \int_{\mathbb{T}^n} \langle u^{(I)}(0) - u_0, \psi \rangle_{\mathbb{R}^n}$$

for all $\psi \in \operatorname{span}\{\psi_1, \dots, \psi_I\}$. Choosing $\psi = \psi_i$, then we have

$$\partial_t \alpha_i(t) = \int_{\mathbb{T}^n} \langle \partial_t u^{(I)}, \psi_i \rangle_{\mathbb{R}^n} dx$$
$$= \int_{\mathbb{T}^n} \langle \Delta u^{(I)} - u^{(I)} \cdot \nabla u^{(I)}, \psi_i \rangle_{\mathbb{R}^n} dx$$

for all $t > 0$, with

$$\alpha_i(0) = \int_{\mathbb{T}^n} \langle u_0, \psi_i \rangle_{\mathbb{R}^n} dx =: \alpha_{i0}$$

But

$$\Delta u^{(I)} = \sum_{i=1}^{I} \alpha_i \Delta \psi_i = - \sum_{i=1}^{I} \alpha_i \lambda_i \psi_i$$

and therefore

$$\int_{\mathbb{T}^n} \langle \Delta u^{(I)}, \psi_i \rangle_{\mathbb{R}^n} dx = - \alpha_i \lambda_i, \quad 1 \leq i \leq I.$$

Finally,

$$\int_{\mathbb{T}^n} \langle u^{(I)} \cdot \nabla u^{(I)}, \psi_i \rangle_{\mathbb{R}^n} dx = \sum_{j,k=1}^{I} \alpha_j \alpha_k \underbrace{\int_{\mathbb{T}^n} \langle \psi_j \cdot \nabla \psi_k, \psi_i \rangle_{\mathbb{R}^n} dx}_{=: B_i^{jk}}$$

Thus the coefficients $\alpha_i = \alpha_i(t)$ are solutions of the initial value problem

$$\partial_t \alpha_i + \lambda_i \alpha_i + \sum_{j,k=1}^{I} B_i^{jk} \alpha_j \alpha_k = 0,$$

$$\alpha_i(0) = \alpha_{i0},$$

with constant coefficients $(B_i^{jk})_{1 \le i,j,k \le I}$.

<u>Claim 1.</u> There holds the a-priori bound

$$\sum_{i=1}^{I} \alpha_i^2(t) \le \sum_{i=1}^{I} \alpha_{i0}^2 \le \|u_0\|_{L^2}^2$$

for all $t \ge 0$ and all $I \in \mathbb{N}$.

<u>Proof</u>: Choose as test function $\psi = u^{(I)}(t)$ for each $t \ge 0$ to obtain

$$0 = \int_{\mathbb{T}^n} \langle \partial_t u^{(I)} - \Delta u^{(I)} + u^{(I)} \cdot \nabla u^{(I)}, u^{(I)} \rangle_{\mathbb{R}^n} \, dx$$

$$= \frac{1}{2} \frac{d}{dt} \|u^{(I)}(t)\|_{L^2}^2 + \|\nabla u^{(I)}(t)\|_{L^2}^2 + \underbrace{\int_{\mathbb{T}^n} u^{(I)} \cdot \nabla \frac{|u^{(I)}|^2}{2} \, dx}_{=0}$$

and hence

$$\sum_{i=1}^{I} |\alpha_i(T)|^2 \le \|u^{(I)}(T)\|_{L^2}^2 + 2\int_0^T \|\nabla u^{(I)}(t)\|_{L^2}^2 \, dt$$

$$\le \|u^{(I)}(0)\|_{L^2}^2 = \sum_{i=1}^{I} |\alpha_{i0}|^2 \le \|u_0\|_{L^2}^2 .$$

$\square$

It follows that $(u^{(I)})_{I\in\mathbb{N}}$ is bounded in L^2 and a subsequence $u^{(I)} \xrightarrow{w} u$, $\nabla u^{(I)} \xrightarrow{w} \nabla u$ in $L^2(\mathbb{T}^n \times [0,T])$ for any fixed $T>0$.

<u>Claim 2.</u> As $I \to \infty$ suitably, for any $T>0$ we have $\forall i \in \mathbb{N}$: $\alpha_i^{(I)}(t) \longrightarrow \alpha_i(t) = \int_{\mathbb{T}^n} \langle u(t), \psi_i \rangle_{\mathbb{R}^n} dx$ uniformly in $0\le t\le T$.

<u>Proof:</u> (See Ladyzhenskaya, p. 175.) For every $i\in\mathbb{N}$ the sequence $(\alpha_i^{(I)})_{I\in\mathbb{N}}$ is uniformly bounded in view of Claim 1 and equi-continuous with

$$|\partial_t \alpha_i^{(I)}| \le \lambda_i |\alpha_i^{(I)}| + \sup_{i,j,k} |B_i^{jk}| \underbrace{\left|\sum_{j=1}^{I} \alpha_j^{(I)}\right|^2}_{\le \|u_0\|^2_{L^2}} \le C,$$

again by Claim 1.

The claim now follows from Arzelà-Ascoli's theorem. $\square$

Claim 3. For every $\varepsilon > 0$ and any $\psi \in C^1(\mathbb{T}^n; \mathbb{R}^n)$ we can bound

$$\Big| \int_{\mathbb{T}^n} \langle u^{(I)} \cdot \nabla u^{(I)} - u \cdot \nabla u, \psi \rangle_{\mathbb{R}^n} \, dx \Big|$$

$$+ \Big| \int_{\mathbb{T}^n} \Big(\langle u^{(I)} \cdot \nabla u^{(I)}, \psi \rangle_{\mathbb{R}^n} + u^k u^j \frac{\partial \psi^k}{\partial x^j} \Big) dx$$

$$\leq \varepsilon \left(\| \nabla u^{(I)} \|^2_{L^2} + \| \nabla u \|^2_{L^2} \right)$$

if $I \geq I_0(\varepsilon)$ is sufficiently large, with error $\int_0^T o(t)\, dt \to 0$ as $I \to \infty$.

Proof. Note that for every $t \in [0,T]$ we have

$$\| \nabla u^{(I)} - \nabla u \|^2_{L^2} = \sum_{k=1}^{\infty} \lambda_k | \alpha_k^{(I)} - \alpha_k |^2 ,$$

where $\lambda_k \to \infty$ $(k \to \infty)$. Choose $K \in \mathbb{N}$ such that $\lambda_k \geq 1/\varepsilon^2$ for $k \geq K$.

Write

$$u^{(I)} \cdot \nabla u^{(I)} - u \cdot \nabla u = u^{(I)} \cdot \nabla (u^{(I)} - u) + (u^{(I)} - u) \cdot \nabla u$$

and estimate

$$\left| \int_{\mathbb{T}^n} \langle (u^{(I)} - u) \cdot \nabla u, \psi \rangle_{\mathbb{R}^n} dx \right|$$

$$\leq \|\psi\|_{L^\infty} \|\nabla u\|_{L^2} \|u^{(I)} - u\|_{L^2}$$

$$\leq \varepsilon \|\nabla u\|^2_{L^2} + C\varepsilon^{-1} \|u^{(I)} - u\|^2_{L^2} .$$

Estimating

$$\|u^{(I)} - u\|^2_{L^2} = \sum_{k=1}^{K} |\alpha_k^{(I)} - \alpha_k|^2 + \sum_{k \geq K} |\alpha_k^{(I)} - \alpha_k|^2$$

$$\leq o(1) + \lambda_K^{-1} \|\nabla(u^{(I)} - u)\|^2_{L^2}$$

with $o(1) \to 0$ uniformly as $I \to \infty$, and recalling that we have $\lambda_K^{-1} \leq 1/\varepsilon^2$ by choice of K, we then obtain the bound

$$\left| \int_{\mathbb{T}^n} \langle (u^{(I)} - u) \cdot \nabla u, \psi \rangle_{\mathbb{R}^n} dx \right| \leq$$

$$\leq C\varepsilon \left(\|\nabla u^{(I)}\|^2_{L^2} + \|\nabla u\|^2_{L^2} \right) + o(1)$$

for this term.

Using that $\operatorname{div} u^{(I)} = 0$ to write

$$\left| \int_{\mathbb{T}^n} \langle u^{(I)} \cdot \nabla(u^{(I)} - u), \psi \rangle_{\mathbb{R}^n} dx \right|$$

$$= \left| \int_{\mathbb{T}^n} \langle u^{(I)} - u, u^{(I)} \cdot \nabla \psi \rangle_{\mathbb{R}^n} dx \right|$$

$$\leq \|\nabla \psi\|_{L^\infty} \|u^{(I)}\|_{L^2} \|u^{(I)} - u\|_{L^2}$$

we can estimate the second term in a similar fashion, and the first estimate follows.

With $\operatorname{div} u^{(I)} = 0$ and

$$\int_{\mathbb{T}^n} \langle u^{(I)} \cdot \nabla u^{(I)}, \psi \rangle_{\mathbb{R}^n} dx = - \int_{\mathbb{T}^n} \langle u^{(I)}, u^{(I)} \cdot \nabla \psi \rangle_{\mathbb{R}^n} dx$$

the second estimate follows in the same manner,

$\square$

Claim 4. For every $\psi \in C_0^\infty(\mathbb{T}^n \times]0,\infty[; \mathbb{R}^n)$ with $\operatorname{div}\psi = 0$ there holds

$$0 = \int_0^\infty \int_{\mathbb{T}^n} \left(\langle \nabla u; \nabla \psi \rangle_{\mathbb{R}^{n\times n}} - \langle u, \psi_t \rangle_{\mathbb{R}^n} + \langle u \cdot \nabla u, \psi \rangle_{\mathbb{R}^n} \right) dx\,dt.$$

Proof: Integrating by parts, for any $\psi = \sum_{k=1}^K \beta_k(t)\psi_k(x)$ and any $I \geq K$ we have

$$0 = \int_0^\infty \int_{\mathbb{T}^n} \left(\langle \nabla u^{(I)}, \nabla \psi \rangle_{\mathbb{R}^{n\times n}} - \langle u^{(I)}, \psi_t \rangle_{\mathbb{R}^n} + \langle u^{(I)} \cdot \nabla u^{(I)}, \psi \rangle_{\mathbb{R}^n} \right) dx\,dt.$$

Passing to the limit $I \to \infty$ for fixed ψ, in view of Claim 3 we conclude

$$\left| \int_0^\infty \int_{\mathbb{T}^n} \left(\langle \nabla u, \nabla \psi \rangle_{\mathbb{R}^{n\times n}} - \langle u, \psi_t \rangle_{\mathbb{R}^n} + \langle u \cdot \nabla u, \psi \rangle_{\mathbb{R}^n} \right) dx\,dt \right|$$

$$\leq \liminf_{I\to\infty} \left| \int_0^\infty \int_{\mathbb{T}^n} \langle u \cdot \nabla u - u^{(I)} \cdot \nabla u^{(I)}, \psi \rangle_{\mathbb{R}^n} dx\,dt \right|$$

$$\leq \varepsilon \liminf_{I\to\infty} \int_0^\infty \int_{\mathbb{T}^n} \left(|\nabla u^{(I)}|^2 + |\nabla u|^2 \right) dx\,dt \leq 2\varepsilon \|u_0\|_{L^2}^2.$$

Since $\varepsilon > 0$ is arbitrary, our claim follows for any $\psi = \sum_{k=1}^{K} \beta_k(t)\psi_k(x)$.

Given any $\psi \in C_c^\infty(\mathbb{T}^n \times]0,T[; \mathbb{R}^n)$ with $\operatorname{div} \psi = 0$, letting

$$\beta_k = \beta_k(t) = \int_{\mathbb{T}^n} \langle \psi, \psi_k \rangle_{\mathbb{R}^n} \, dx \in C_c^\infty(]0,T[),\ k \in \mathbb{N},$$

by completeness of $(\psi_k)_{k\in\mathbb{N}}$ we have

$$\psi^{(K)} = \sum_{k=1}^{K} \beta_k \psi_k \xrightarrow{(K\to\infty)} \psi \quad \text{in } H^1 \cap L^\infty(\mathbb{T}^n \times]0,T[);$$

hence we can pass to the limit $K \to \infty$ in the identity

$$0 = \int_0^T \int_{\mathbb{T}^n} \Big(\langle \nabla u, \nabla \psi^{(K)} \rangle_{\mathbb{R}^{n\times n}} - \langle u, \psi_t^{(K)} \rangle_{\mathbb{R}^n}$$

$$+ \langle u \cdot \nabla u, \psi^{(K)} \rangle_{\mathbb{R}^n} \Big) \, dx\, dt$$

to obtain the claim. $\square$

Finally, let $p = p(t)$ satisfy $\int_{\mathbb{T}^n} p\,dx = 0$ and

$$-\Delta p = \operatorname{div}(u \cdot \nabla u) = \sum_{i,j} \frac{\partial u^i}{\partial x_j} \frac{\partial u^j}{\partial x_i}$$

$$= \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}(u^i u^j).$$

Then $p \in L^2_{loc}(\mathbb{T}^2 \times]0,\infty[)$ if $n=2$,

$p \in L^{5/3}_{loc}(\mathbb{T}^3 \times]0,\infty[)$ if $n=3$, respectively

in view of

and the Calderón-Zygmund inequality.

It then follows that

$$\operatorname{div}(u \cdot \nabla u + \nabla p) = 0$$

in the distribution sense.

Splitting an arbitrary testing function $\psi \in C_c^\infty(\mathbb{T}^n \times]0,\infty[)$ into a gradient and a solenoidal term by letting

$$\psi = -\operatorname{curl}\operatorname{curl} \Delta^{-1}\psi + \nabla \operatorname{div} \Delta^{-1}\psi$$

for every t, we then see that u satisfies (3.1), (3.2) in the distribution sense.

$\square$

Regularity. For smooth initial data, $2 \leq n \leq 3$, we obtain smooth bounds for the Hopf approximations $u^{(I)}$ for short time, and global smooth bounds for $u^{(I)}$ for all time, if $n = 2$.

Passing to the limit $I \to \infty$, we therefore obtain a local smooth solution (u, p) to (3.1), (3.2) for any smooth data u_0 with $\operatorname{div} u_0 = 0$ for $0 \leq t < T = T(u_0)$ for any $n \geq 2$, with $T = \infty$ if $n = 2$.

In the latter two-dimensional case, moreover, we are able to show that any weak Hopf-type solution agrees with this global smooth solution and thus is smooth.

Lemma 3.2.1. Let $n = 2$ and let $u_0 \in H^1(\mathbb{T}^2; \mathbb{R}^2)$ with $\operatorname{div} u_0 = 0$. There exists a constant $C = C(\|u_0\|_{L^2}) > 0$ such that for any $I \in \mathbb{N}$ for $u^{(I)}$ as in the proof of Thm. 3.2.1 there holds
$$\sup_t \|\nabla u^{(I)}(t)\|_{L^2} \leq C \|\nabla u_0\|_{L^2}.$$

Proof: Writing $u = u^{(I)}$ for brevity, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2\right) + \int_{\mathbb{T}^2} |\nabla^2 u|^2 dx =$$

$$= -\int_{\mathbb{T}^2} \langle \nabla(u\cdot\nabla u), \nabla u\rangle_{\mathbb{R}^{2\times 2}} dx$$

$$= \underbrace{-\int_{\mathbb{T}^2} u\cdot\nabla\left(\frac{|\nabla u|^2}{2}\right)dx}_{=0} - \int_{\mathbb{T}^2}\sum_{i,j,k} \partial_k u^j \partial_j u^i \partial_k u^i dx$$

$$\leq \|\nabla u\|_{L^3}^3 \leq \|\nabla u\|_{L^2} \|\nabla u\|_{L^4}^2 ,$$

Let $I \subset [0,\infty[$ such that

$$\int_I \int_{\mathbb{T}^2} |\nabla u|^2 dx\, dt \leq \min\{\|\nabla u_0\|_{L^2}^2, \varepsilon^2\} =: \varepsilon_0^2$$

for $\varepsilon > 0$ to be defined. Letting $E_0 = \|u_0\|_{L^2}^2$, we can cover $[0,\infty[$ with finitely many such I_k, $1\leq k\leq K$, where $K \leq 2E_0/\varepsilon^2$. For such $I = [t_1, t_2]$ integrate in time to find

$$\sup_{t_1\leq t\leq t_2} \|\nabla u(t)\|_{L^2(\mathbb{T}^2)}^2 + \|\nabla^2 u\|_{L^2(\mathbb{T}^2\times I)}^2$$

$$\leq \|\nabla u\|_{L^2(\mathbb{T}^2\times I)} \|\nabla u\|_{L^4(\mathbb{T}^2\times I)}^2 + \|\nabla u(t_1)\|_{L^2(\mathbb{T}^2)}^2$$

$$\leq \|\nabla u(t_1)\|_{L^2}^2 + C_1\varepsilon \sup_{t_1<t<t_2} \|\nabla u(t)\|_{L^2} \cdot \left(\|\nabla^2 u\|_{L^2(\mathbb{T}^2\times I)}^2 + \varepsilon_0^2\right)^{1/2},$$

with $C_1 > 0$ as in the Gagliardo-Nirenberg-Ladyzhenskaya interpolation estimate. For $\varepsilon = \frac{1}{2C_1} > 0$ we can absorb the last term to obtain

$$\sup_{t_1 \le t \le t_2} \|\nabla u(t)\|^2_{L^2(\mathbb{T}^2)} + \|\nabla^2 u\|^2_{L^2(\mathbb{T}^2 \times I)}$$
$$\le 2 \|\nabla u(t_1)\|^2_{L^2(\mathbb{T}^2)} + \varepsilon_0^2 .$$

Hence, by iteration we find

$$\sup_{0 < t < \infty} \|\nabla u(t)\|^2_{L^2(\mathbb{T}^2)} + \|\nabla^2 u\|^2_{L^2(\mathbb{T}^2 \times [0,\infty[)}$$
$$\le 2^K \|\nabla u_0\|^2_{L^2(\mathbb{T}^2)} + \varepsilon_0^2 \le (2^K + 1) \|\nabla u_0\|^2_{L^2(\mathbb{T}^2)} .$$

$\square$

For $u_0 \in H^1(\mathbb{T}^2; \mathbb{R}^2)$ with $\operatorname{div} u_0 = 0$, in view of Lemma 3.2.1 and weak lower semi-continuity of the L^2-norm, the Hopf solution constructed in Thm. 3.2.1 satisfies $\|\nabla u(t)\|_{L^2} \le C < \infty$.

It is not difficult to see that the corresponding pressure p then satisfies $\nabla p(t) \in L^q(\mathbb{T}^2)$ for any $q < \infty$, uniformly in $t \geq 0$, and higher regularity of u and p follows from (3.1), (3.2) in a standard fashion.

We thus obtain

<u>Corollary</u> 3.2.1. Let $n = 2$ and let $u_0 \in H^1(\mathbb{T}^2; \mathbb{R}^2)$ with $\operatorname{div} u_0 = 0$. Then there exists a global solution (u, p) of (3.1), (3.2) which is smooth for $t > 0$.

The above solution is unique even among Hopf-type solutions. In fact, the following "weak-strong uniqueness" result holds.

Proposition 3.2.1. Let $n \geq 2$ and suppose u_1 with corresponding p_1 smoothly solves (3.1), (3.2) while (u_2, p_2) is an energy-class weak solution of (3.1), (3.2) with $u_2 \in L^4$,[1] and suppose $u_1|_{t=0} = u_2|_{t=0} = u_0$. Then $u_1 \equiv u_2$.

Proof. Let $u = u_1 - u_2$ satisfying $u|_{t=0} = 0$ and

$$(\partial_t - \Delta) u + u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2 + \nabla q = 0$$

in the distribution sense, where $q = p_1 - p_2$.

Write

$$\begin{aligned} u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2 &= (u_1 - u_2) \cdot \nabla u_1 + u_2 \cdot \nabla (u_1 - u_2) \\ &= u \cdot \nabla u_1 + u_2 \cdot \nabla u \end{aligned}$$

and observe that

$$\int_{\mathbb{T}^2} \langle u_2 \cdot \nabla u, u \rangle_{\mathbb{R}^2} \, dx = \int_{\mathbb{T}^2} u_2 \cdot \nabla \left(\frac{|u|^2}{2} \right) dx = 0$$

for a.e. $t > 0$ in view of (3.2). Moreover,

$$\begin{aligned} \int_{\mathbb{T}^2} \langle u \cdot \nabla u_1, u \rangle_{\mathbb{R}^2} \, dx &\leq \| \nabla u_1 \|_{L^2} \| u \|_{L^4}^2 \\ &\leq C \| \nabla u_1 \|_{L^2} \| u \|_{L^2} \| u \|_{H^1}, \quad \text{if } n = 2, \end{aligned}$$

and

$$\leq \| \nabla u_1 \|_{L^\infty} \| u \|_{L^2}^2, \quad \text{if } n \geq 3.$$

---

1) Temam, p. 22: Ok also for Hopf sol. u_2, if $n=3$ by Serrin (1963)

If $n=3$, we only have the following local bounds.

Lemma 3.2.2. Let $n=3$ and let $u_0 \in H^1(\mathbb{T}^3;\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. There exist constants $C_1 > 0$, $T_0 = \frac{1}{2C_1(\|\nabla u_0\|_{L^2}^2+1)^2} > 0$ such that for any $I \in \mathbb{N}$ for $u^{(I)}$ as in the proof of Thm. 3.2.1 there holds

$$\|\nabla u^{(I)}(t)\|_{L^2}^2 \leq \frac{\|\nabla u_0\|_{L^2}^2+1}{\sqrt{1-2C_1 t(\|\nabla u_0\|_{L^2}^2+1)^2}}, \quad 0 \leq t < T_0$$

uniformly in $I \in \mathbb{N}$.

Proof: For $u = u^{(I)}$ as in (3.) we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2\right) + \|\nabla^2 u\|_{L^2}^2 \leq \|\nabla u\|_{L^3}^3$$

$$\leq \|\nabla u\|_{L^2}^{3/2} \|\nabla u\|_{L^6}^{3/2}$$

by interpolation, noting that $\frac{1}{3} = \frac{1/2}{2} + \frac{1/2}{6}$.

By Sobolev's embedding $H^1 \hookrightarrow L^6$ on $\mathbb{T}^3$ and Young's inequality then we find that

$$\|\nabla u\|_{L^2}^{3/2} \|\nabla u\|_{L^6}^{3/2} \leq C \|\nabla u\|_{L^2}^{3/2} \|\nabla u\|_{H^1}^{3/2}$$

$$\leq C \|\nabla u\|_{L^2}^{3} + C \|\nabla u\|_{L^2}^{3/2} \|\nabla^2 u\|_{L^2}^{3/2}$$

$$\leq C \|\nabla u\|_{L^2}^{3} + C \|\nabla u\|_{L^2}^{6} + \frac{1}{2} \|\nabla^2 u\|_{L^2}^{2},$$

and there results the bound

$$\frac{d}{dt} f \leq C\left(f^{3/2} + f^3\right) \leq C_1 (f+1)^3$$

for $f = \|\nabla u\|_{L^2}^2$, with an absolute constant $C_1 > 0$.

Integrating, we obtain

$$(f+1)^2(t) \leq \frac{(f+1)^2(0)}{1 - 2C_1 t (f+1)^2(0)}$$

and our claim follows with

$$T = T(u_0) = \frac{1}{2C_1 (\|\nabla u_0\|_{L^2}^2 + 1)^2} > 0.$$

$\square$

Corollary 3.2.2. Let $n = 3$, $u_0 \in C^\infty(\mathbb{T}^3; \mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. Then there exists $T_0 > 0$ and a smooth solution (u, p) of (3.1), (3.2) for $0 \leq t < T_0$ with $u|_{t=0} = u_0$.

Remark 3.2.1. Estimating

$$\inf_{0<t<T} \|\nabla u(t)\|_{L^2}^2 \leq \frac{1}{T}\int_0^T\int_{\mathbb{T}^n} |\nabla u|^2 dx\,dt \leq \frac{E_0}{2T},$$

for any $T>0$, if $\|u_0\|_{L^2}^2 = E_0 > 0$ is so small that there is $T_1 > 0$ with

$$T_1 < \frac{1}{2C_1\left(\frac{E_0}{2T_1}+1\right)^2},$$

so that

$$1 > 2C_1 T_1\left(\frac{E_0}{2T_1}+1\right)^2 = 2C_1\left(\frac{E_0}{4T_1}+E_0+T_1\right),$$

and if $u_0 \in H^1$ with $\|\nabla u_0\|_{L^2}^2 \leq \frac{E_0}{2T_1}$,

from Lemma 3.2.2 we obtain global a-priori bounds $\sup_{0<t<\infty} \|\nabla u^{(I)}(t)\|_{L^2} \leq C < \infty$, uniformly in I. Hence, $\sup_{0<t<\infty} \|\nabla u(t)\|_{L^2} \leq C < \infty$, and u is smooth, if $n=3$.

3.3 Partial regularity, $n = 3$

There are a number of "conditional" regularity results for $n=3$. For instance, if we assume that

$$\|u\|_{L^{q,r}} = \left(\int_0^T \left(\int_{\mathbb{T}^3} |u(x,t)|^q dx \right)^{r/q} dt \right)^{\frac{1}{r}} < \infty,$$

where

$$\frac{3}{q} + \frac{2}{r} \leq 1,$$

then u is regular; see for instance Serrin (1962), Sohr (1983) and Escauriaza-Seregin-Sverak (2003) for the limit case $q=3$, $r=\infty$.

From experience with harmonic map heat flow we may conjecture a "suitable" weak solution to be regular near a point $z_0 = (x_0, t_0)$ if a "dimensionless" quantity is small near z_0.

Scaling property. Let (u,p) solve (3.1), (3.2) on $\mathbb{R}^n \times]-1,0[$, and for $R>0$ let

$$(3.5) \quad \begin{aligned} u_R(x,t) &= R\, u(Rx, R^2 t), \\ p_R(x,t) &= R^2 p(Rx, R^2 t). \end{aligned}$$

Then (u_R, p_R) solves (3.1), (3.2), as well, Observe that for any (q,r)

$$\|u_R\|_{L^{q,r}(Q_1)} = R^{1-(\frac{3}{q}+\frac{2}{r})}\|u\|_{L^{q,r}(Q_R)};$$

hence $\|u\|_{L^{q,r}(Q_R)}$ scales "dimensionless" if

$$\frac{3}{q}+\frac{2}{r}=1,$$

and similarly in higher dimensions. Also the scaled Dirichlet energy

$$\frac{1}{R}\int_{Q_R}|\nabla u|^2 dz = \int_{Q_1}|\nabla u_R|^2 dz$$

or

$$\frac{1}{R^2}\int_{Q_R}(|u|^3+|p|^{3/2})\,dz$$

are dimensionless.

The following result is the best known "global" result for $n=3$ so far.

<u>Theorem</u> 3.3.1 (Caffarelli-Kohn-Nirenberg (1982))
There is $\varepsilon_0>0$ with the following property. If (u,p) is a suitable weak solution of (3.1) (3.2) near $z_0=(x_0,t_0)$ and if

$$\limsup_{R\to 0}\left(\frac{1}{R}\int_{P_R(z_0)}|\nabla u|^2 dz\right)\le\varepsilon_0,$$

then u is "regular" near z_0.

Corollary 3.3.1 ([C-K-N], Thm. B)
For any "suitable" weak solution of (3.1), (3.2) on an open set in space-time with singular set S there holds $\mathcal{H}^1(S) = 0$, where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure with respect to the parabolic metric.

Thm. 3.3.1 and its corollary improve previous results of Scheffer (1976 f.), who showed the existence of a global weak solution (u, p) to (3.1), (3.2) with

$$\mathcal{H}^{5/3}(S) = 0,$$

where S is the (closed) singular set of u.

Remark 3.3.1 In view of the non-local effect of the pressure, for a general domain the local regularity asserted in Thm. 3.3.1 only refers to spatial regularity.

Indeed, on $\mathbb{R}^3$ we have the following example of Serrin:

Let $u(t,x) = a(t)\nabla h(x)$, where $h : \mathbb{R}^3 \to \mathbb{R}$ is harmonic, $a(t) \in W^{1,1}(\mathbb{R})$.

Then

$$u \cdot \nabla u = a^2 \nabla\left(\frac{|\nabla h|^2}{2}\right) = \nabla\left(a^2 \frac{|\nabla h|^2}{2}\right)$$

$$u_t = a_t \nabla h = \nabla(a_t h)$$

$$\Delta u = 0$$

and u satisfies (3.1), (3.2) with

$$p = -a_t h + \frac{1}{2} a^2 |\nabla h|^2 .$$

On the torus, that is, in the periodic case, by the maximum principle any harmonic function is constant, and local spatial regularity implies smoothness in x and t.

(Nonexistence of) self-similar solutions.

In view of the scaling property (3.5) Leray proposed to study solutions u of the form

$$u(x,t) = \frac{1}{\sqrt{2a(t_0-t)}}\, v\left(\frac{x}{\sqrt{2a(t_0-t)}}\right), \quad x \in \mathbb{R}^3,$$

as singularity models, where $a > 0$.

Inserting such u in (3.1), we obtain

$$u_t - \Delta u + u\cdot\nabla u =$$

$$= \frac{1}{\sqrt{2a(t_0-t)}^{\,3}}\left(av + \frac{a x\cdot\nabla v}{\sqrt{2a(t_0-t)}} - \Delta v + v\cdot\nabla v\right)$$

and

$$\operatorname{div} u = \frac{1}{2a(t_0-t)} \operatorname{div} v .$$

Thus u as above satisfies (3.1), (3.2) if v satisfies $\operatorname{div} v = 0$ and solves the equation

$$(3.6) \quad av + a y\cdot\nabla v - \Delta v + v\cdot\nabla v + \nabla p = 0$$

for some $p : \mathbb{R}^3 \to \mathbb{R}$ and $a > 0$. Normalize

$$(3.7) \qquad p = \sum_{i,j} \partial_i \partial_j (-\Delta^{-1}) v^i v^j ,$$

so that $p \in L^s(\mathbb{R}^3)$ if $v \in L^{2s}(\mathbb{R}^3)$, $s > 1$.

Moreover, the energy inequality (3.4) motivates imposing the condition $v \in H^1(\mathbb{R}^3;\mathbb{R}^3) \hookrightarrow L^2 \cap L^6(\mathbb{R}^3;\mathbb{R}^3)$.

<u>Theorem</u> 3.3.2 (Nečas - Růžička - Šverák (1996))
Let $v \in L^3(\mathbb{R}^3;\mathbb{R}^3)$ weakly solve (3.6), (3.2) for some $a > 0$. Then $v \equiv 0$.

<u>Key</u> <u>observation</u>. Take $a = 1$ for simplicity. Defining $w(y) = v(y) + y$, we have

$$-\Delta w + w \cdot \nabla w =$$

$$= -\Delta v + v \cdot \nabla v + y \cdot \nabla v + v + y = -\nabla\left(p - \frac{|y|^2}{2}\right) \tag{3.8}$$

and

$$\operatorname{div} w = 3;$$

that is, $(w, q = p - \frac{|y|^2}{2})$ is a time-independent solution of (3.1) and (3.2).

Multiplying (3.8) with w, and taking the divergence of the terms in (3.1), respectively, gives

$$-\Delta\left(\frac{|w|^2}{2}\right) + |\nabla w|^2 + w\cdot\nabla\left(\frac{|w|^2}{2}+q\right) = 0$$

and

$$-\Delta q = \operatorname{div}(w\cdot\nabla w) = \sum_{i,j} \partial_i w^j \partial_j w^i$$

so that

$$-\Delta\left(\frac{|w|^2}{2}+q\right) + w\cdot\nabla\left(\frac{|w|^2}{2}+q\right)$$

$$(3.9) \qquad = \sum_{i,j=1}^{3} \partial_i w^j \partial_j w^i - |\nabla w|^2 \leq 0 .$$

Hence the maximum principle applies.

Remark 3.3.1. The identity (3.9) for time-independent solutions u of (3.1), (3.2) was observed by Gilbarg-Weinberger (1974), Amick-Fraenkel (1980). Rediscovered by M.S. in the early 1990's, it lead to existence results for smooth sol's of the time-independent eq. (3.1), (3.2) (for smooth forces f) in independent work of Frehse-Růžička, M.S. (1995).

Lemma 3.3.1 For $F_{jk} \in L^1 \cap L^2(\mathbb{R}^n)$ let

$$P = \sum_{i,j=1}^{n} \partial_i \partial_j (-\Delta)^{-1} F_{jk} = \sum_{i,j=1}^{n} R_i R_j F_{ij}$$

where R_j is the Riesz transform with Fourier multiplier $-i\xi_j/|\xi|$, $\xi \in \mathbb{R}^n$. Also let $\varphi = \varphi(|x|) \in C_c^\infty(\mathbb{R}^n)$ with $\varphi(0) = 1$.

Then

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} P(x)\varphi(\varepsilon x)\, dx = -\frac{1}{n} \int_{\mathbb{R}^n} \sum_{j=1}^{n} F_{jj}\, dx .$$

Proof: See [N-R-S], Lemma 3.1.

Moreover, Thm. 3.3.1 and the scaling property (3.5) give the following asymptotic bounds for v and p; see [N-R-S], Lemma 3.2.

Lemma 3.3.2. For v as in Thm. 3.3.1 with p given by (3.7) there holds

$$|\nabla^k v(y)| \leq C|y|^{-3-k}, \quad |y| \gg 1,$$

$$|\nabla^k p(y)| \leq C|y|^{-2-k}, \quad |y| \gg 1.$$

Proof of Thm 3.3.2. By Lemma 3.3.2

$$\left(\frac{|w|^2}{2}+q\right)(y) = \left(\frac{|v|^2}{2}+p\right)(y) + y\cdot v(y) = O(|y|^{-2}),$$

and the maximum principle applied to (3.9) gives the bound

$$Q := \frac{|w|^2}{2} + q \leq 0 .$$

Let $0 \leq \varphi = \varphi(|x|) \in C_c^\infty(\mathbb{R}^3)$ with $\varphi(0) = 1$. Note that for each $\varepsilon > 0$ we have

$$\int_{\mathbb{R}^3} y\cdot v(y)\, \varphi(\varepsilon y)\, dy =$$

$$= \int_0^\infty r\varphi(\varepsilon r) \Big(\underbrace{\int_{\partial B_r(0)} v\cdot\vec{n}\, do}_{= \int_{B_r(0)} \operatorname{div} v\, dx = 0} \Big) dr = 0$$

in view of (3.2).

Hence by Lemma 3.3.1 we find

$$0 \geqslant \int_{\mathbb{R}^3} Q\, \varphi(\varepsilon y)\, dy = \int_{\mathbb{R}^3} \left(\frac{|v|^2}{2} + p\right)(y)\, \varphi(\varepsilon y)\, dy$$

$$\xrightarrow{(\varepsilon \downarrow 0)} \int_{\mathbb{R}^3} \frac{|v|^2}{2}\, dy - \frac{1}{3}\int_{\mathbb{R}^3} |v|^2 dy = \frac{1}{6}\int_{\mathbb{R}^3} |v|^2 dy,$$

and the claim follows. $\square$