A CÀDLÀG ROUGH PATH FOUNDATION FOR ROBUST FINANCE

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ABSTRACT. Using rough path theory, we provide a pathwise foundation for stochastic Itô integration, which covers most commonly applied trading strategies and mathematical models of financial markets, including those under Knightian uncertainty. To this end, we introduce the so-called Property (RIE) for càdlàg paths, which is shown to imply the existence of a càdlàg rough path and of quadratic variation in the sense of Föllmer. We prove that the corresponding rough integrals exist as limits of left-point Riemann sums along a suitable sequence of partitions. This allows one to treat integrands of non-gradient type, and gives access to the powerful stability estimates of rough path theory. Additionally, we verify that (path-dependent) functionally generated trading strategies and Cover's universal portfolio are admissible integrands, and that Property (RIE) is satisfied by both (Young) semimartingales and typical price paths.

Key words: Föllmer integration, model uncertainty, semimartingale, pathwise integration, rough path, functionally generated portfolios, universal portfolio.
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1. Introduction

A fundamental pillar of mathematical finance is the theory of stochastic integration initiated by K. Itô in the 1940s. Itô's stochastic integration not only allows for a well-posedness theory for most probabilistic models of financial markets, but also comes with invaluable properties, such as having an integration by parts formula and chain rule, and that of being a continuous operator (with respect to suitable spaces of random variables), which is essential for virtually all applications. However, despite the elegance and success of Itô integration, it also admits some significant drawbacks from both theoretical and practical perspectives.

The construction of the Itô integral requires one to fix a probability measure a priori, and is usually based on a limiting procedure of approximating Riemann sums in probability. While in mathematical finance the Itô integral usually represents the capital gain process from continuous-time trading in a financial market, it lacks a robust pathwise meaning. That is, the stochastic Itô integral does not have a well-defined value on a given "state of the world", e.g. a realized price trajectory of a liquidly traded asset on a stock exchange. This presents a gap between probabilistic models and their financial interpretation. Addressing the pathwise meaning of stochastic integration has led to a stream of literature beginning with the classical works of Bichteler [Bic81] and Willinger and Taqqu [WT89]; see also [Kar95, Nut12].

The requirement of fixing a probability measure to have access to Itô integration becomes an even more severe obstacle when one wants to develop mathematical finance under model risk—also known as Knightian uncertainty. Starting from the seminal works [ALP95, Lyo95], there has been an enormous and on-going effort to treat the challenges posed by model risk

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in mathematical finance, that is, the risk stemming from the possible misspecification of an adopted stochastic model—typically represented by a single fixed probability measure. The majority of the existing robust treatments of financial modelling replace the single probability measure by a family of (potentially singular) probability measures, or even take so-called model-free approaches, whereby no probabilistic structure of the underlying price trajectories is assumed; see for example [Hob11] for classical lecture notes on robust finance. In particular, the latter model-free approaches often require a purely deterministic integration theory sophisticated enough to handle the irregular sample paths of standard continuous-time financial models and commonly employed functionally generated trading strategies.

In the seminal paper [Föl81a], H. Föllmer provided the first deterministic analogue to stochastic Itô integration which had the desired properties required by financial applications. Indeed, assuming that a càdlàg path $S\colon[0,T]\to\mathbb{R}^d$ possesses a suitable notion of quadratic variation along a sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of partitions of the interval $[0,T]$, Föllmer proved that the limit

$$\int_0^t \mathrm{D}f(S_u)\,\mathrm{d}S_u := \lim_{n\to\infty}\sum_{[u,v]\in\mathcal{P}^n}\mathrm{D}f(S_u)(S_{v\wedge t}-S_{u\wedge t}), \qquad t\in[0,T],$$

exists for all twice continuously differentiable functions $f\colon\mathbb{R}^d\to\mathbb{R}$. The resulting pathwise integral $\int_0^t \mathrm{D}f(S_u)\,\mathrm{d}S_u$ is often called the Föllmer integral, and has proved to be a valuable tool in various applications in model-free finance; for some recent examples we refer to [FS13, DOR14, SSV18, CSW19]. In fact, even classical Riemann–Stieltjes integration has been successfully used as a substitution to Itô integration in model-free finance; see e.g. [DS14, HO18].

By now arguably the most general pathwise (stochastic) integration theory is provided by the theory of rough paths, as introduced by T. Lyons [Lyo98], and its recent extension to càdlàg rough paths [FS17, FZ18, CF19]. Rough integration can be viewed as a generalization of Young integration which is able to handle paths of lower regularity. While rough integration allows one to treat the sample paths of numerous stochastic processes as integrators and offers powerful pathwise stability estimates, it comes with a pitfall from a financial perspective: the rough integral is defined as a limit of so-called compensated Riemann sums, and thus apparently does not correspond to the canonical financial interpretation as the capital gain process generated by continuous-time trading.

We overcome this issue by introducing the so-called Property (RIE) for a càdlàg path $S\colon[0,T]\to\mathbb{R}^d$ and a sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of partitions of the interval $[0,T]$. This property is very much in the same spirit as Föllmer's assumption of quadratic variation along a sequence of partitions. Indeed, we show that Property (RIE) implies the existence of quadratic variation in the sense of Föllmer, and even the existence of a càdlàg rough path $\mathbf{S}=(S,\mathbb{S})$. Assuming Property (RIE), we prove that the corresponding rough integrals exist as limits of left-point Riemann sums along the sequence of partitions $(\mathcal{P}^n)_{n\in\mathbb{N}}$. This result restores the canonical financial interpretation for rough integration, and links it to Föllmer integration for càdlàg paths. Property (RIE) was previously introduced in [PP16] for continuous paths, though we emphasize that the present more general càdlàg setting requires quite different techniques compared to the continuous setting of [PP16].

Given the aforementioned results, a càdlàg path which satisfies Property (RIE) permits the path-by-path existence of rough integrals with their desired financial interpretation, and moreover maintains access to their powerful stability results which ensure that the integral is a continuous operator. This appears to be a significant advantage compared to the classical notions of pathwise stochastic integration in [Bic81, WT89, Kar95, Nut12], which do not come with such stability estimates. In particular, the pathwise stability results of rough path theory allow one to prove a model-free version of the so-called fundamental theorem of derivative trading—see [ABBC18]—and may be of interest when investigating discretization errors of continuous-time trading in model-free finance; see [Rig16]. Furthermore, in contrast to Föllmer integration, rough integration allows one to consider general functionally generated integrands $g(S_t)$, where g is a general (sufficiently smooth) function $g\colon \mathbb{R}^d \to \mathbb{R}^d$, and *not* necessarily the gradient of another vector field $f\colon \mathbb{R}^d \to \mathbb{R}$. For instance, model-free portfolio theory constitutes a research direction in which it is beneficial to consider non-gradient trading strategies; see [ACLP21]. Even more generally, rough integration allows one to treat path-dependent functionally generated options in the sense of Duprie [Dup19] and pathwise versions of Cover's universal portfolio [CSW19], as discussed in Section 3.

Of course, it remains to verify that Property (RIE) is a reasonable modelling assumption in mathematical finance, in the sense that it is fulfilled almost surely by sample paths of the commonly used probabilistic models of financial markets. Since it seems natural that continuous-time trading takes place when the underlying price process fluctuates, we employ sequences of partitions based on such a "space discretization". For such sequences of partitions, we show that the sample paths of càdlàg semimartingales almost surely satisfy Property (RIE). This result is then extended to so-called Young semimartingales, which are stochastic possesses given by the sum of a càdlàg local martingale and an adapted càdlàg process of finite q-variation for some $q < 2$. Finally, we prove that Property (RIE) is satisfied by typical price paths in the sense of Vovk [Vov08], which correspond to a model-free version of "no unbounded profit with bounded risk".

Organization of the paper: In Section 2 we introduce Property (RIE) and verify the properties of the associated rough integration as described above. In Section 3 we exhibit functionally generated trading strategies and generalizations thereof which provide valid integrands for rough integration. In Section 4 we prove that (Young) semimartingales and typical price paths satisfy Property (RIE).

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2. Rough integration under Property (RIE)

In this section we develop pathwise integration under Property (RIE). We set up the essential ingredients from rough path theory in Subsection 2.2 and show in Subsection 2.3 that paths satisfying (RIE) serve as suitable integrators in mathematical finance. Finally, in Subsection 2.4 we connect Property (RIE) with the existence of quadratic variation in the sense of Föllmer.

2.1. **Basic notation.** Let $(\mathbb{R}^d, |\cdot|)$ denote standard Euclidean space, and let $D([0,T];\mathbb{R}^d)$ denote the space of all càdlàg (i.e. right-continuous with left-limits) functions from $[0,T] \to \mathbb{R}^d$. A partition $\mathcal{P} = \mathcal{P}([s,t])$ of the interval $[s,t]$ is a set of essentially disjoint intervals covering $[s,t]$, i.e. $\mathcal{P} = \{s = t_0 < t_1 < \cdots < t_N = t\}$ for some $N \in \mathbb{N}$. The mesh size of a partition $\mathcal{P}$ is given by $|\mathcal{P}|:= \max\{|t_{i+1} - t_i| : i = 0,\dots,N-1\}$, and, for a partition $\mathcal{P}$ of the interval $[s,t]$ and a subinterval $[u,v] \subset [s,t]$, we write $\mathcal{P}([u,v]) := (\mathcal{P} \cup \{u,v\}) \cap [u,v] = \{r \in \mathcal{P} \cup \{u,v\} : u \leq r \leq v\}$, for the restriction of the partition $\mathcal{P}$ to the interval $[u,v]$. A sequence $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of partitions is called *nested* if $\mathcal{P}^n \subset \mathcal{P}^{n+1}$ for all $n \in \mathbb{N}$.

Setting $\Delta_{[0,T]} := \{(s,t) \in [0,T]^2 : s \leq t\}$, a *control function* is defined as a function $w \colon \Delta_{[0,T]} \to [0,\infty)$ which is superadditive, in the sense that $w(s,u) + w(u,t) \leq w(s,t)$ for all $0 \leq s \leq u \leq t \leq T$.

Throughout this section we fix a finite time interval $[0,T]$ and the dimension $d \in \mathbb{N}$. We also adopt the convention that, given a path A defined on $[0,T]$, we will write $A_{s,t} := A_t - A_s$ for the increment of A over the interval $[s,t]$. Note however that whenever A is a two-parameter function defined on $\Delta_{[0,T]}$, then the notation $A_{s,t}$ will simply denote the value of A evaluated at the pair of times $(s,t) \in \Delta_{[0,T]}$.

If A denotes either a path from $[0,T] \to E$ or a two-parameter function from $\Delta_{[0,T]} \to E$ for some normed vector space E (which for us will always be either $\mathbb{R}^d$ or $\mathbb{R}^{d\times d}$), then, for any $p \in [1,\infty)$, the p-variation of A over the interval $[s,t]$ is defined by

$$\|A\|_{p,[s,t]} := \Bigg(\sup_{\mathcal{P}([s,t])} \sum_{[u,v]\in\mathcal{P}([s,t])} |A_{u,v}|^p\Bigg)^{\frac{1}{p}}$$

where the supremum is taken over all partitions $\mathcal{P}([s,t])$ of the interval $[s,t] \subseteq [0,T]$, and in the case when A is a path we write $A_{u,v} := A_v - A_u$. If $\|A\|_{p,[0,T]} < \infty$ then A is said to have finite p-variation.

We write $D^p = D^p([0,T];E)$ for the space of all càdlàg paths $A \colon \Delta_{[0,T]} \to E$ of finite p-variation, and we similarly write $D_2^r = D_2^r(\Delta_{[0,T]};E)$ for the space of two-parameter functions $A \colon \Delta_{[0,T]} \to E$ of finite p-variation which are such that the maps $s \mapsto A_{s,t}$ for fixed t, and $t \mapsto A_{s,t}$ for fixed s, are both càdlàg. Note that A having finite p-variation is equivalent to the existence of a control function w such that $|A_{s,t}|^p \leq w(s,t)$ for all $(s,t) \in \Delta_{[0,T]}$. (For instance, one may take $w(s,t) = \|A\|_{p,[s,t]}^p$.)

2.2. **Càdlàg rough path theory and Property (RIE).** While rough path theory has by now been well studied in the case of continuous paths, as exhibited in a number of books, notably [FH20], its extension to càdlàg paths appeared only recently, starting with [FS17]. In this section we mainly rely on results regarding forward integration with respect to càdlàg rough paths as presented in [FZ18].

In the following we fix $p \in (2,3)$ and $q \geq p$ such that

$$\frac{2}{p} + \frac{1}{q} > 1, \tag{2.1}$$

and define $r > 1$ by the relation

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{2.2}$$

This means in particular that $1 < p/2 \leq r < p \leq q < \infty$.

Throughout the paper, we will use the symbol $\lesssim$ to denote inequality up to a multiplicative constant which depends only on the numbers p, q and r as chosen above.

We begin by recalling the definition of a càdlàg rough path, as well as the corresponding notion of controlled paths. In the following we will write $A \otimes B$ for the tensor product of two vectors $A, B \in \mathbb{R}^d$, i.e. the $d \times d$-matrix with (i,j)-component given by $[A \otimes B]^{ij} = A^i B^j$ for $1 \leq i, j \leq d$.

Definition 2.1. *We say that a triplet* $\mathbf{X} = (X, Z, \mathbb{X})$ *is a* (càdlàg) p-rough path *(over* $\mathbb{R}^d$*) if* $X \in D^p([0,T];\mathbb{R}^d)$*,* $Z \in D^p([0,T];\mathbb{R}^d)$ *and* $\mathbb{X} \in D_2^{p/2}(\Delta_{[0,T]};\mathbb{R}^{d\times d})$*, and if Chen's relation:*

$$\mathbb{X}_{s,t} = \mathbb{X}_{s,u} + \mathbb{X}_{u,t} + Z_{s,u} \otimes X_{u,t} \tag{2.3}$$

holds for all times $0 \leq s \leq u \leq t \leq T$*. We denote the space of càdlàg rough paths by* $\mathscr{V}^p$*.*

The unfamiliar reader is encouraged to check that, given càdlàg paths X and Z of bounded variation, setting $\mathbb{X}_{s,t} = \int_s^t Z_{s,r} \otimes \mathrm{d}X_r \equiv \int_s^t Z_r \otimes \mathrm{d}X_r - Z_s \otimes X_{s,t}$ for $(s,t) \in \Delta_{[0,T]}$, with the integral defined as a limit of left-point Riemann sums, gives a p-rough path. Although the integral $\int_s^t Z_{s,r} \otimes \mathrm{d}X_r$ is not in general well-defined when X and Z are not of bounded variation, given a rough path $(X, Z, \mathbb{X})$, we may think of $\mathbb{X}$ as postulating a "candidate" for the value of such integrals.

Remark 2.2. *The definition of rough paths we have introduced above looks slightly different to the standard definition, in which one takes* $X = Z$*. Our definition is slightly more general, but the corresponding theory works in exactly the same way, and turns out to be more convenient in the context of Property* (RIE) *as we will see later.*

More precisely, later the matrix $\mathbb{X}_{s,t}$ *will for us represent the (a priori ill-defined) 'integral'* $\int_s^t S_{s,u} \otimes \mathrm{d}S_u$*, which will be defined as the limit as* $n \to \infty$ *of the Riemann sums* $(\int_s^t S^n_{s,u} \otimes \mathrm{d}S_u)_{n\in\mathbb{N}}$ *appearing in Property* (RIE) *below. In the continuous (i.e. without jumps) setting of* [PP16]*, a linear interpolation is used to provide a continuous approximation of* S^n*, leading to a Stratonovich type integral in the limit, which is subsequently converted back into an Itô type integral. Thanks to the recently developed theory of càdlàg rough paths, here we can use a more direct argument which avoids this detour. This means working directly with the integral* $\int_s^t S^n_{s,u} \otimes \mathrm{d}S_u$*, which corresponds to taking* $X = S$ *and* $Z = S^n$ *in Definition 2.1, thus requiring* $X \neq Z$*.*

For two rough paths, $\mathbf{X} = (X, Z, \mathbb{X})$ and $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{Z}, \tilde{\mathbb{X}})$, we use the seminorm

$$\|\mathbf{X}\|_{p,[s,t]} := \|X\|_{p,[s,t]} + \|Z\|_{p,[s,t]} + \|\mathbb{X}\|_{\frac{p}{2},[s,t]},$$

and the pseudometric

$$\|\mathbf{X};\tilde{\mathbf{X}}\|_{p,[s,t]} := \|X - \tilde{X}\|_{p,[s,t]} + \|Z - \tilde{Z}\|_{p,[s,t]} + \|\mathbb{X} - \tilde{\mathbb{X}}\|_{\frac{p}{2},[s,t]},$$

for $[s,t] \subseteq [0,T]$.

Definition 2.3. *Let* $Z \in D^p([0,T];\mathbb{R}^d)$*. We say that a pair* (Y, Y') *is a* controlled path *(with respect to* Z*), if* $Y \in D^p([0,T];\mathbb{R}^d)$*,* $Y' \in D^q([0,T];\mathcal{L}(\mathbb{R}^d;\mathbb{R}^d))$ *and* $R^Y \in D_2^r(\Delta_{[0,T]};\mathbb{R}^d)$*, where the remainder* R^Y *is defined implicitly by the relation*

$$Y_{s,t} = Y'_s Z_{s,t} + R^Y_{s,t}, \qquad (s,t) \in \Delta_{[0,T]}.$$

We refer to Y' *as the* Gubinelli derivative *of* Y *(with respect to* Z*), and denote the space of controlled paths by* $\mathcal{V}^{q,r}_Z$*.*

Given a path $Z \in D^p([0,T];\mathbb{R}^d)$, the space of controlled paths $\mathcal{V}^{q,r}_Z$ becomes a Banach space when equipped with the norm

$$(Y, Y') \mapsto |Y_0| + |Y'_0| + \|Y'\|_q + \|R^Y\|_r.$$

With the concepts of rough paths and controlled paths at hand we are ready to introduce rough integration.

Proposition 2.4. *Let* $\mathbf{X} = (X, Z, \mathbb{X}) \in \mathscr{V}^p$ *be a rough path and let* $(Y, Y') \in \mathcal{V}^{q,r}_Z$ *be a controlled path with remainder* R^Y*. Then, for each* $t \in [0,T]$*, the limit*

$$\int_0^t Y_u \,\mathrm{d}\mathbf{X}_u := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} Y_u X_{u,v} + Y'_u \mathbb{X}_{u,v} \tag{2.4}$$

exists along every sequence of partitions $\mathcal{P}$ *of the interval* $[0,t]$ *with mesh size* $|\mathcal{P}|$ *tending to zero. We call this limit the* rough integral *of* (Y,Y') *against* $\mathbf{X}$*, which moreover comes with the estimate*

$$\left| \int_s^t Y_u \,\mathrm{d}\mathbf{X}_u - Y_s X_{s,t} - Y'_s \mathbb{X}_{s,t} \right| \leq C(\|R^Y\|_{r,[s,t]}\|X\|_{p,[s,t]} + \|Y'\|_{q,[s,t]}\|\mathbb{X}\|_{\frac{p}{2},[s,t]}), \tag{2.5}$$

where the constant C *depends only on* p, q *and* r*.*

Proof. Let $\Xi_{s,t} := Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}$ and define $\delta\Xi_{s,u,t} := \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}$. It is straightforward, using Chen's relation (2.3), to show that $\delta\Xi_{s,u,t} = -R^Y_{s,u}X_{u,t} - Y'_{s,u}\mathbb{X}_{u,t}$. Defining $w_{R^Y}(s,t) := \|R^Y\|^r_{r,[s,t]}$, $w_X(s,t) := \|X\|^p_{p,[s,t]}$, $w_{Y'}(s,t) := \|Y'\|^q_{q,[s,t]}$ and $w_{\mathbb{X}}(s,t) := \|\mathbb{X}\|^{\frac{p}{2}}_{\frac{p}{2},[s,t]}$, we see that $w_{R^Y}, w_X, w_{Y'}$ and $w_{\mathbb{X}}$ are control functions, and that

$$|\delta\Xi_{s,u,t}| \leq w_{R^Y}(s,u)^{\frac{1}{r}} w_X(u,t)^{\frac{1}{p}} + w_{Y'}(s,u)^{\frac{1}{q}} w_{\mathbb{X}}(u,t)^{\frac{2}{p}}.$$

From the relations (2.1) and (2.2), we see that $\frac{1}{r} + \frac{1}{p} > 1$ and $\frac{1}{q} + \frac{2}{p} > 1$. The result then follows from the generalized sewing lemma [FZ18, Theorem 2.5]. □

Remark 2.5. *It follows from the estimate in* (2.5) *that the rough integral* $\int_0^\cdot Y_u \,\mathrm{d}\mathbf{X}_u$ *is itself a controlled path with respect to* X*, with Gubinelli derivative* Y*, so that* $(\int_0^\cdot Y_u \,\mathrm{d}\mathbf{X}_u, Y) \in \mathcal{V}^{q,r}_X$*.*

Notice that the construction of the rough integral in (2.4) is based on so-called compensated Riemann sums $\sum_{[u,v]\in\mathcal{P}} Y_u X_{u,v} + Y'_u \mathbb{X}_{u,v}$ instead of classical left-point Riemann sums $\sum_{[u,v]\in\mathcal{P}} Y_u X_{u,v}$. While the classical Riemann sums come with a natural interpretation as capital (gain) processes in the context of mathematical finance, the interpretation of compensated Riemann sums is by no means obvious. However, one advantage of rough integration is that it provides rather powerful stability estimates, for instance as presented in the next proposition.

Proposition 2.6. *Let* $\mathbf{X} = (X, Z, \mathbb{X}), \tilde{\mathbf{X}} = (\tilde{X}, \tilde{Z}, \tilde{\mathbb{X}}) \in \mathscr{V}^p$ *be rough paths, and let* $(Y, Y') \in \mathcal{V}^{q,r}_Z$ *and* $(\tilde{Y}, \tilde{Y}') \in \mathcal{V}^{q,r}_{\tilde{Z}}$ *be controlled paths with remainders* R^Y *and* $R^{\tilde{Y}}$*, respectively. Then, we have the estimate*

$$\begin{aligned}
&\left\| \int_0^\cdot Y_u \,\mathrm{d}\mathbf{X}_u - \int_0^\cdot \tilde{Y}_u \,\mathrm{d}\tilde{\mathbf{X}}_u \right\|_{p;[0,T]} \\
&\quad \leq C\bigg((|\tilde{Y}_0| + |\tilde{Y}'_0| + \|\tilde{Y}'\|_{q;[0,T]} + \|R^{\tilde{Y}}\|_{r;[0,T]})(1 + \|X\|_{p;[0,T]} + \|\tilde{Z}\|_{p;[0,T]})\|\mathbf{X}; \tilde{\mathbf{X}}\|_{p;[0,T]} \\
&\qquad + (|Y_0 - \tilde{Y}_0| + |Y'_0 - \tilde{Y}'_0| + \|Y' - \tilde{Y}'\|_{q;[0,T]} + \|R^Y - R^{\tilde{Y}}\|_{r;[0,T]})(1 + \|Z\|_{p;[0,T]})\|\mathbf{X}\|_{p;[0,T]} \bigg),
\end{aligned}$$

where the constant C *depends only on* p, q *and* r*.*

Proof. Following the proof of [FZ18, Lemma 3.4], in our slightly more general setting one deduces the estimates

$$\begin{aligned}\|Y-\tilde{Y}\|_{p;[0,T]} &\lesssim (|Y_0'-\tilde{Y}_0'|+\|Y'-\tilde{Y}'\|_{q;[0,T]})\|Z\|_{p;[0,T]}\\ &\quad + (|\tilde{Y}_0'|+\|\tilde{Y}'\|_{q;[0,T]})\|Z-\tilde{Z}\|_{p;[0,T]}+\|R^Y-R^{\tilde{Y}}\|_{r;[0,T]},\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}&\|R^{\int_0^\cdot Y_u\,\mathrm{d}\mathbf{X}_u}-R^{\int_0^\cdot \tilde{Y}_u\,\mathrm{d}\tilde{\mathbf{X}}_u}\|_{r;[0,T]}\\ &\quad\lesssim (|\tilde{Y}_0'|+\|\tilde{Y}'\|_{q;[0,T]}+\|R^{\tilde{Y}}\|_{r;[0,T]})(\|X-\tilde{X}\|_{p;[0,T]}+\|\mathbb{X}-\tilde{\mathbb{X}}\|_{\frac{p}{2};[0,T]})\\ &\qquad + (|Y_0'-\tilde{Y}_0'|+\|Y'-\tilde{Y}'\|_{q;[0,T]}+\|R^Y-R^{\tilde{Y}}\|_{r;[0,T]})(\|X\|_p+\|\mathbb{X}\|_{\frac{p}{2};[0,T]}).\end{aligned} \tag{2.7}$$

From the controlled path structure of $(\tilde{Y},\tilde{Y}')$, i.e. $\tilde{Y}_{s,t}=\tilde{Y}_s'\tilde{Z}_{s,t}+R^{\tilde{Y}}_{s,t}$, it is easy to obtain the inequality

$$\|\tilde{Y}\|_{p;[0,T]}\lesssim (|\tilde{Y}_0'|+\|\tilde{Y}'\|_q)\|\tilde{Z}\|_{p;[0,T]}+\|R^{\tilde{Y}}\|_{r;[0,T]}. \tag{2.8}$$

Recalling Remark 2.5, we similarly find, using the controlled path structure of the rough integrals, that

$$\begin{aligned}&\left\|\int_0^\cdot Y_u\,\mathrm{d}\mathbf{X}_u-\int_0^\cdot \tilde{Y}_u\,\mathrm{d}\tilde{\mathbf{X}}_u\right\|_{p;[0,T]}\\ &\quad\lesssim (|Y_0-\tilde{Y}_0|+\|Y-\tilde{Y}\|_{p;[0,T]})\|X\|_{p;[0,T]}\\ &\qquad + (|\tilde{Y}_0|+\|\tilde{Y}\|_{p;[0,T]})\|X-\tilde{X}\|_{p;[0,T]}+\|R^{\int_0^\cdot Y_u\,\mathrm{d}\mathbf{X}_u}-R^{\int_0^\cdot \tilde{Y}_u\,\mathrm{d}\tilde{\mathbf{X}}_u}\|_{r;[0,T]}.\end{aligned} \tag{2.9}$$

The result then follows upon substituting the estimates (2.6), (2.7) and (2.8) into (2.9). □

In the spirit of Föllmer's assumption of quadratic variation along a sequence of partitions [Föl81a], we introduce the following property.

Property (RIE). *Let $p\in(2,3)$ and let $\mathcal{P}^n=\{0=t_0^n<t_1^n<\cdots<t_{N_n}^n=T\}$, $n\in\mathbb{N}$, be a sequence of nested partitions of the interval $[0,T]$ such that $|\mathcal{P}^n|\to 0$ as $n\to\infty$. For $S\in D([0,T];\mathbb{R}^d)$, we define $S^n\colon[0,T]\to\mathbb{R}^d$ by*

$$S_t^n=S_T\mathbf{1}_{\{T\}}(t)+\sum_{k=0}^{N_n-1}S_{t_k^n}\mathbf{1}_{[t_k^n,t_{k+1}^n)}(t),\qquad t\in[0,T],$$

for each $n\in\mathbb{N}$. We assume that:

- *the sequence of paths $(S^n)_{n\in\mathbb{N}}$ converges uniformly to S as $n\to\infty$,*
- *the Riemann sums $\int_0^t S_u^n\otimes\mathrm{d}S_u=\sum_{k=0}^{N_n-1}S_{t_k^n}\otimes S_{t_k^n\wedge t,t_{k+1}^n\wedge t}$ converge uniformly as $n\to\infty$ to a limit, which we denote by $\int_0^t S_u\otimes\mathrm{d}S_u$, $t\in[0,T]$,*
- *and that there exists a control function w such that*[1]

$$\sup_{(s,t)\in\Delta_{[0,T]}}\frac{|S_{s,t}|^p}{w(s,t)}+\sup_{n\in\mathbb{N}}\sup_{0\le k<\ell\le N_n}\frac{|\int_{t_k^n}^{t_\ell^n}S_u^n\otimes\mathrm{d}S_u-S_{t_k^n}\otimes S_{t_k^n,t_\ell^n}|^{\frac{p}{2}}}{w(t_k^n,t_\ell^n)}\le 1. \tag{2.10}$$

Definition 2.7. *A path $S\in D([0,T];\mathbb{R}^d)$ is said to satisfy* (RIE) *with respect to p and $(\mathcal{P}^n)_{n\in\mathbb{N}}$, if p, $(\mathcal{P}^n)_{n\in\mathbb{N}}$ and S together satisfy Property* (RIE).

[1] Here and throughout, we adopt the convention that $\frac{0}{0}:=0$.

Property (RIE) is a stronger assumption than the existence of quadratic variation in the sense of Föllmer and, as we will see, is even enough to allow us to lift S in a canonical way to a rough path, giving us access to the powerful stability results of rough path theory, given in this setting by Proposition 2.6. Moreover, Property (RIE) can be verified for most typical stochastic processes in mathematical finance, as we will see in Section 4.

Remark 2.8. *We will see in Proposition 2.11 below that it is actually enough in Property* (RIE) *to assume that the sequence $(S^n)_{n\in\mathbb{N}}$ converges only pointwise to S, since the uniformity of this convergence then immediately follows.*

Next we shall verify that Property (RIE) ensures the existence of a càdlàg rough path. For this purpose, we consider a suitable approximating sequence of the so-called 'area process', which is represented by $\mathbb{X}$ in Definition 2.1.

Lemma 2.9. *Suppose $S \in D([0,T];\mathbb{R}^d)$ satisfies Property* (RIE) *with respect to p and $(\mathcal{P}^n)_{n\in\mathbb{N}}$ (as in Definition 2.7). If for each $n \geq 1$ we define $A^n\colon \Delta_{[0,T]} \to \mathbb{R}^{d\times d}$ by*

$$A^n_{s,t} := \int_s^t S^n_{s,u} \otimes \mathrm{d}S_u = \int_s^t S^n_u \otimes \mathrm{d}S_u - S^n_s \otimes S_{s,t}, \qquad (s,t) \in \Delta_{[0,T]}, \tag{2.11}$$

then there exists a constant C depending only on p, such that

$$\|A^n\|_{\frac{p}{2};[0,T]} \leq Cw(0,T)^{\frac{2}{p}} \qquad \text{for every} \quad n \geq 1. \tag{2.12}$$

Proof. Let $n \geq 1$ and $(s,t) \in \Delta_{[0,T]}$. If there exists a k such that $t^n_k \leq s < t \leq t^n_{k+1}$ then we simply have that $A^n_{s,t} = S_{t^n_k} \otimes S_{s,t} - S_{t^n_k} \otimes S_{s,t} = 0$. Otherwise, let k_0 be the smallest k such that $t^n_k \in (s,t)$, and let k_1 be the largest such k. It is easy to see that the triplet (S, S^n, A^n) satisfies Chen's relation (2.3), from which it follows that

$$A^n_{s,t} = A^n_{s,t^n_{k_0}} + A^n_{t^n_{k_0},t^n_{k_1}} + A^n_{t^n_{k_1},t} + S^n_{s,t^n_{k_0}} \otimes S_{t^n_{k_0},t^n_{k_1}} + S^n_{s,t^n_{k_1}} \otimes S_{t^n_{k_1},t}.$$

As we have already observed, we have that $A^n_{s,t^n_{k_0}} = A^n_{t^n_{k_1},t} = 0$. By the inequality (2.10), we have

$$|A^n_{t^n_{k_0},t^n_{k_1}}|^{\frac{p}{2}} \leq w(t^n_{k_0}, t^n_{k_1}) \leq w(t^n_{k_0-1}, t).$$

We estimate the remaining terms as

$$\begin{aligned}
|S^n_{s,t^n_{k_0}} \otimes S_{t^n_{k_0},t^n_{k_1}}|^{\frac{p}{2}} + |S^n_{s,t^n_{k_1}} \otimes S_{t^n_{k_1},t}|^{\frac{p}{2}} &\lesssim |S^n_{s,t^n_{k_0}}|^p + |S_{t^n_{k_0},t^n_{k_1}}|^p + |S^n_{s,t^n_{k_1}}|^p + |S_{t^n_{k_1},t}|^p \\
&= |S_{t^n_{k_0-1},t^n_{k_0}}|^p + |S_{t^n_{k_0},t^n_{k_1}}|^p + |S_{t^n_{k_0-1},t^n_{k_1}}|^p + |S_{t^n_{k_1},t}|^p \\
&\leq w(t^n_{k_0-1}, t^n_{k_0}) + w(t^n_{k_0}, t^n_{k_1}) + w(t^n_{k_0-1}, t^n_{k_1}) + w(t^n_{k_1}, t) \\
&\leq 2w(t^n_{k_0-1}, t),
\end{aligned}$$

so that, putting this all together, we deduce the existence of a constant $\tilde{C} > 0$ such that $|A^n_{s,t}|^{\frac{p}{2}} \leq \tilde{C}w(t^n_{k_0-1}, t)$. Taking an arbitrary partition $\mathcal{P}$ of the interval $[0,T]$, it follows that $\sum_{[s,t]\in\mathcal{P}} |A^n_{s,t}|^{\frac{p}{2}} \leq 2\tilde{C}w(0,T)$. We thus conclude that (2.12) holds with $C = (2\tilde{C})^{\frac{2}{p}}$. □

Lemma 2.10. *Suppose that $S \in D([0,T];\mathbb{R}^d)$ satisfies Property* (RIE) *with respect to p and $(\mathcal{P}^n)_{n\in\mathbb{N}}$. With the natural notation $\int_s^t S_u \otimes \mathrm{d}S_u := \int_0^t S_u \otimes \mathrm{d}S_u - \int_0^s S_u \otimes \mathrm{d}S_u$, we define $A\colon \Delta_{[0,T]} \to \mathbb{R}^{d\times d}$ by*

$$A_{s,t} = \int_s^t S_u \otimes \mathrm{d}S_u - S_s \otimes S_{s,t}, \qquad (s,t) \in \Delta_{[0,T]}.$$

Then, the triplet $\mathbf{S} = (S, \mathbb{S}, A)$ *is a càdlàg* p*-rough path.*

Proof. It is straightforward to verify Chen's relation (2.3), i.e. that

$$A_{s,t} = A_{s,u} + A_{u,t} + S_{s,u} \otimes S_{u,t}, \qquad (s,t) \in \Delta_{[0,T]}.$$

By Property (RIE), we know that $\lim_{n\to\infty} A^n_{s,t} = A_{s,t}$, where the convergence is uniform in (s,t), and thus, being a uniform limit of càdlàg functions, A is itself càdlàg. By the lower semi-continuity of the $\frac{p}{2}$-variation norm, and the result of Lemma 2.9, we have that

$$\|A\|_{\frac{p}{2},[0,T]} \leq \liminf_{n\to\infty} \|A^n\|_{\frac{p}{2},[0,T]} \leq Cw(0,T)^{\frac{2}{p}} < \infty.$$

It follows that $(S, \mathbb{S}, A)$ is a càdlàg p-rough path. □

2.3. **The rough integral as a limit of Riemann sums.** While the rough integral in (2.4) is a powerful tool to study various differential equations, it lacks the natural interpretation as the capital gain process in the context of mathematical finance. The aim of this subsection is to restore this interpretation by showing that the rough integral can be obtained as the limit of left-point Riemann sums provided that the integrator satisfies Property (RIE). As preparation we need the following approximation result.

Proposition 2.11. *Let* $\mathcal{P}^n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_{N_n} = T\}$*,* $n \in \mathbb{N}$*, be a sequence of nested partitions with vanishing mesh size, so that* $\mathcal{P}^n \subset \mathcal{P}^{n+1}$ *for all* n*, and* $|\mathcal{P}^n| \to 0$ *as* $n \to \infty$ *(as in the setting of Property* (RIE)*). Let* $F \colon [0,T] \to \mathbb{R}^d$ *be a càdlàg path, and define*

$$F^n_t = F_T \mathbf{1}_{\{T\}}(t) + \sum_{k=0}^{N_n-1} F_{t^n_k} \mathbf{1}_{[t^n_k, t^n_{k+1})}(t), \qquad t \in [0,T]. \tag{2.13}$$

Let

$$J_F := \{t \in (0,T] : F_{t-,t} \neq 0\} \tag{2.14}$$

be the set of jump times of F*. The following are equivalent:*

(i) $J_F \subseteq \cup_{n\in\mathbb{N}} \mathcal{P}^n$*,*
(ii) *The sequence* $(F^n)_{n\in\mathbb{N}}$ *converges pointwise to* F*,*
(iii) *The sequence* $(F^n)_{n\in\mathbb{N}}$ *converges uniformly to* F*.*

Proof. We first show that conditions (i) and (ii) are equivalent. To this end, suppose that $J_F \subseteq \cup_{n\geq1} \mathcal{P}^n$ and let $t \in (0,T]$. If $t \in J_F$, then there exists $m \geq 1$ such that $t \in \mathcal{P}^n$ for all $n \geq m$. In this case we then have that $F^n_t = F_t$ for all $n \geq m$. If $t \notin J_F$, then $F_{t-} = F_t$, and since the mesh size $|\mathcal{P}^n| \to 0$, it follows that $F^n_t \to F_{t-} = F_t$ as $n \to \infty$.

Now suppose instead that there exists a $t \in J_F$ such that $t \notin \cup_{n\geq1} \mathcal{P}^n$. We then observe that $F^n_t \to F_{t-} \neq F_t$, so that $F^n_t \nrightarrow F_t$. This establishes the equivalence of (i) and (ii).

Since (iii) clearly implies (ii), it only remains to show that (ii) implies (iii). By [Fra19, Theorem 3.3], it is enough to show that the family of paths $\{F^n : n \geq 1\}$ is equiregulated in the sense of [Fra19, Definition 3.1].

Step 1. Let $t \in (0,T]$ and $\varepsilon > 0$. Since the left limit F_{t-} exists, there exists $\delta > 0$ with $t - \delta > 0$, such that

$$|F_{s,t-}| < \frac{\varepsilon}{2} \qquad \text{for all} \quad s \in (t-\delta, t).$$

Let

$$m = \min\{n \geq 1 : \exists k \ \text{ such that } \ t^n_k \in (t-\delta, t)\}.$$

Since $|\mathcal{P}^n| \to 0$ as $n \to \infty$, we know that $m < \infty$. Moreover, since the sequence of partitions is nested, we immediately have that, for all $n \geq m$, there exists a k such that $t_k^n \in (t - \delta, t)$. We define

$$u = \min\{t_k^m \in \mathcal{P}^m : t_k^m \in (t - \delta, t)\} \equiv \min(\mathcal{P}^m \cap (t - \delta, t)),$$

and let $s \in [u, t)$ and $n \geq 1$.

If $n < m$, then there does not exist a k such that $t_k^n \in (t - \delta, t)$, which implies that F^n is constant on the interval $(t - \delta, t)$, and hence that $F_s^n = F_{t-}^n$.

Suppose instead that $n \geq m$. Let $i = \max\{k : t_k^n \leq s\}$ and $j = \max\{k : t_k^n < t\}$. By the definition of u, we see that $t_i^n \in [u, t)$ and $t_j^n \in [u, t)$. Then

$$|F_s^n - F_{t-}^n| = |F_{t_i^n} - F_{t_j^n}| \leq |F_{t_i^n, t-}| + |F_{t_j^n, t-}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we have that $|F_s^n - F_{t-}^n| < \varepsilon$ for all $s \in [u, t)$ and all $n \geq 1$.

Step 2. Let $t \in (J_F \cup \{0\}) \setminus \{T\}$ and $\varepsilon > 0$. Since F is right-continuous, there exists a $\delta > 0$ with $t + \delta < T$, such that

$$|F_{t,s}| < \varepsilon \qquad \text{for all} \quad s \in [t, t + \delta).$$

By part (i), we know that $t \in \cup_{n \geq 1} \mathcal{P}^n$. Let

$$m = \min\{n \geq 1 : \exists k \ \text{ such that } \ t_k^n = t\}.$$

Since $t \in \cup_{n \geq 1} \mathcal{P}^n$, it is clear that $m < \infty$. We define

$$u = \min\{t_k^m \in \mathcal{P}^m : t_k^m > t\} \equiv \min(\mathcal{P}^m \cap (t, T]).$$

We then let $v \in (t, u \wedge (t + \delta))$, $s \in (t, v]$, and $n \geq 1$.

If $n < m$, then, since $v < u$, there does not exist a k such that $t_k^n \in [t, v]$. Hence, F^n is constant on the interval $[t, v]$, so that in particular $F_s^n = F_t^n$.

Suppose instead that $n \geq m$. By the definition of m, there exists a j such that $t_j^n = t$. Let $i = \max\{k : t_k^n \leq s\}$. In particular, we then have that $t = t_j^n \leq t_i^n \leq s \leq v < t + \delta$, and hence that

$$|F_s^n - F_t^n| = |F_{t_i^n} - F_{t_j^n}| = |F_{t, t_i^n}| < \varepsilon.$$

Thus, we have that $|F_s^n - F_t^n| < \varepsilon$ for all $s \in (t, v]$ and all $n \geq 1$.

Step 3. Let $t \in (0, T) \setminus J_F$ and $\varepsilon > 0$. Since F is continuous at time t, there exists a $\delta > 0$ with $0 < t - \delta$ and $t + \delta < T$, such that

$$|F_{s,t}| < \frac{\varepsilon}{2} \qquad \text{for all} \quad s \in (t - \delta, t + \delta).$$

Let

$$m = \min\{n \geq 1 : \exists k \ \text{ such that } \ t_k^n \in (t - \delta, t]\}.$$

Since $|\mathcal{P}^n| \to 0$ as $n \to \infty$, we know that $m < \infty$. We define

$$u = \min\{t_k^m \in \mathcal{P}^m : t_k^m > t\} \equiv \min(\mathcal{P}^m \cap (t, T]).$$

We then let $v \in (t, u \wedge (t + \delta))$, $s \in (t, v]$ and $n \geq 1$.

If $n < m$, then, since $v < u$, there does not exist a k such that $t_k^n \in (t, v]$. Hence, F^n is constant on the interval $[t, v]$, so that in particular $F_s^n = F_t^n$.

Suppose instead that $n \geq m$. Let $i = \max\{k : t_k^n \leq s\}$ and $j = \max\{k : t_k^n \leq t\}$. Since, by the definition of m, there exists at least one k such that $t_k^n \in (t - \delta, t]$, and since $t < s \leq v < t + \delta$, it follows that $t_i^n \in (t - \delta, t + \delta)$ and $t_j^n \in (t - \delta, t]$. Then

$$|F_s^n - F_t^n| = |F_{t_i^n} - F_{t_j^n}| \leq |F_{t_i^n, t}| + |F_{t_j^n, t}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we have that $|F^n_s - F^n_t| < \varepsilon$ for all $s \in (t, v]$ and all $n \geq 1$. It follows that the family of paths $\{F^n : n \geq 1\}$ is indeed equiregulated. □

The next theorem is the main result of this section, stating that the rough integral can be approximated by left-point Riemann sums along a suitable sequence of partitions, in the spirit of Föllmer's pathwise integration.

Theorem 2.12. *Let $q \geq p$ such that $\frac{2}{p} + \frac{1}{q} > 1$, and let $r > 1$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Suppose that $S \in D([0,T];\mathbb{R}^d)$ satisfies Property* (RIE) *with respect to p and $(\mathcal{P}^n)_{n\in\mathbb{N}}$. Let $(F, F') \in \mathcal{V}^{q,r}_S$ be a controlled path with respect to S, such that $J_F \subseteq \cup_{n\in\mathbb{N}} \mathcal{P}^n$, where J_F is the set of jump times of F, as in* (2.14)*. Then the rough integral of (F, F') against the rough path $\mathbf{S} = (S, \mathbb{S}, A)$, as defined in* (2.4)*, is given by*

$$\int_0^t F_u \, \mathrm{d}\mathbf{S}_u = \lim_{n\to\infty} \sum_{k=0}^{N_n-1} F_{t^n_k} S_{t^n_k\wedge t, t^n_{k+1}\wedge t},$$

where the convergence is uniform in $t \in [0, T]$.

Proof. We recall from Lemma 2.10 that $\mathbf{S} = (S, \mathbb{S}, A)$ is a p-rough path, so that, by Proposition 2.4, the rough integral of the controlled path (F, F') against $\mathbf{S}$ exists. It is also clear that $\mathbf{S}^n := (S, \mathbb{S}^n, A^n)$ is a p-rough path, where A^n was defined in (2.11). Moreover, by Property (RIE), we immediately have that S^n and A^n converge uniformly to S and A respectively as $n \to \infty$.

For each $n \geq 1$, we let F^n be the path defined in (2.13). We consider the pair (F^n, F') as a controlled path with respect to S^n, defining the remainder term R^n by the usual relation:

$$F^n_{s,t} = F'_s S^n_{s,t} + R^n_{s,t}, \qquad (s,t) \in \Delta_{[0,T]}.$$

Since S^n converges uniformly to S and, by Proposition 2.11, F^n converges uniformly to F, it follows that R^n also converges uniformly to the remainder term R corresponding to the S-controlled path (F, F').

We observe that $\|S^n\|_{p,[0,T]} \leq \|S\|_{p,[0,T]}$ and $\|F^n\|_{p,[0,T]} \leq \|F\|_{p,[0,T]}$, and we have from Lemma 2.9 that $\|A^n\|_{\frac{p}{2},[0,T]} \leq C w(0,T)^{\frac{2}{p}}$ for every $n \geq 1$. It remains to show that R^n is bounded in r-variation, uniformly in n.

Let $n \geq 1$ and $(s,t) \in \Delta_{[0,T]}$. If there exists a k such that $t^n_k \leq s < t < t^n_{k+1}$, then

$$R^n_{s,t} = F^n_{s,t} - F'_s S^n_{s,t} = F_{t^n_k,t^n_k} - F'_s S_{t^n_k,t^n_k} = 0.$$

If there exists a k such that $t^n_k \leq s < t = t^n_{k+1}$, then

$$\begin{aligned}|R^n_{s,t}|^r &= |F^n_{s,t} - F'_s S^n_{s,t}|^r = |F_{t^n_k,t^n_{k+1}} - F'_s S_{t^n_k,t^n_{k+1}}|^r \\ &\lesssim |F_{t^n_k,t^n_{k+1}} - F'_{t^n_k} S_{t^n_k,t^n_{k+1}}|^r + |F'_{t^n_k,s} S_{t^n_k,t^n_{k+1}}|^r \\ &\lesssim |R_{t^n_k,t^n_{k+1}}|^r + |F'_{t^n_k,s}|^q + |S_{t^n_k,t^n_{k+1}}|^p,\end{aligned}$$

where in the last line we used Young's inequality, recalling that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Otherwise, let k_0 be the smallest k such that $t^n_k \in [s,t]$, and let k_1 be the largest such k. After a short calculation, we find that

$$R^n_{s,t} = R^n_{s,t^n_{k_0}} + R^n_{t^n_{k_0},t^n_{k_1}} + R^n_{t^n_{k_1},t} + F'_{s,t^n_{k_0}} S_{t^n_{k_0},t^n_{k_1}} + F'_{s,t^n_{k_1}} S^n_{t^n_{k_1},t}.$$

We observe that $S^n_{t^n_{k_1},t} = 0$ and $R^n_{t^n_{k_0},t^n_{k_1}} = R_{t^n_{k_0},t^n_{k_1}}$. We can deal with the terms $R^n_{s,t^n_{k_0}}$ and $R^n_{t^n_{k_1},t}$ using the above, and we bound $|F'_{s,t^n_{k_0}} S_{t^n_{k_0},t^n_{k_1}}|^r \lesssim |F'_{s,t^n_{k_0}}|^q + |S_{t^n_{k_0},t^n_{k_1}}|^p$. Putting this all together, we have that

$$|R^n_{s,t}|^r \leq C(|R_{t^n_{k_0-1},t^n_{k_0}}|^r + |F'_{t^n_{k_0-1},s}|^q + |S_{t^n_{k_0-1},t^n_{k_0}}|^p + |R_{t^n_{k_0},t^n_{k_1}}|^r + |F'_{s,t^n_{k_0}}|^q + |S_{t^n_{k_0},t^n_{k_1}}|^p),$$

where the constant C depends only on p, q and r. Taking an arbitrary partition $\mathcal{P}$ of the interval $[0,T]$, we deduce that $\sum_{[s,t]\in\mathcal{P}} |R^n_{s,t}|^r \leq 2C(\|R\|^r_{r,[0,T]} + \|F'\|^q_{q,[0,T]} + \|S\|^p_{p,[0,T]})$. Thus, $\|R^n\|_{r,[0,T]}$ is bounded uniformly in $n \geq 1$.

Let $p' > p$, $q' > q$ and $r' > r$, such that $p' \in (2,3)$, $q' \geq p'$, $\frac{2}{p'} + \frac{1}{q'} > 1$, and $\frac{1}{r'} = \frac{1}{p'} + \frac{1}{q'}$. Since the sequence $(S^n)_{n\geq 1}$ has uniformly bounded p-variation, and S^n converges uniformly to S as $n \to \infty$, it follows by interpolation that S^n converges to S with respect to the p'-variation norm, i.e. $\|S^n - S\|_{p'} \to 0$ as $n \to \infty$. It follows similarly that $\|A^n - A\|_{\frac{p'}{2}} \to 0$ and $\|R^n - R\|_{r'} \to 0$ as $n \to \infty$. It thus follows from Proposition 2.6 that

$$\int_0^t F^n_u \,\mathrm{d}\mathbf{S}^n_u \longrightarrow \int_0^t F_u \,\mathrm{d}\mathbf{S}_u \qquad \text{as} \quad n \longrightarrow \infty, \tag{2.15}$$

where the convergence is uniform in $t \in [0,T]$.

We recall from Proposition 2.4 that the rough integral of the controlled path (F^n, F') against the rough path $\mathbf{S}^n = (S, S^n, A^n)$ is given by the limit

$$\int_0^t F^n_u \,\mathrm{d}\mathbf{S}^n_u = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} F^n_u S_{u,v} + F'_u A^n_{u,v},$$

where the limit is taken over any sequence of partitions of the interval $[0,t]$ with vanishing mesh size. Take any refinement $\tilde{\mathcal{P}}$ of the partition $(\mathcal{P}^n \cup \{t\}) \cap [0,t]$ (where as usual $\mathcal{P}^n$ is the partition given in Property (RIE)), and let $[u,v] \in \tilde{\mathcal{P}}$. By the choice of the partition $\tilde{\mathcal{P}}$, there exists a k such that $t^n_k \leq u < v \leq t^n_{k+1}$, which, recalling (2.11), implies that $A^n_{u,v} = 0$. Since the mesh size of $\tilde{\mathcal{P}}$ may be arbitrarily small, it follows that

$$\lim_{|\tilde{\mathcal{P}}|\to 0} \sum_{[u,v]\in\tilde{\mathcal{P}}} F'_u A^n_{u,v} = 0.$$

To conclude, we then simply recall (2.15), and note that

$$\int_0^t F^n_u \,\mathrm{d}\mathbf{S}^n_u = \lim_{|\tilde{\mathcal{P}}|\to 0} \sum_{[u,v]\in\tilde{\mathcal{P}}} F^n_u S_{u,v} = \sum_{k=0}^{N_n-1} F_{t^n_k} S_{t^n_k\wedge t, t^n_{k+1}\wedge t}.$$

□

We can actually generalize the result of Theorem 2.12 to a slightly larger class of integrands.

Corollary 2.13. *Recall the assumptions of Theorem 2.12, and let $\gamma \in D^r([0,T];\mathbb{R}^d)$. Then, the rough integral of the controlled path $G = F + \gamma$, given by $(G, G') := (F + \gamma, F') \in \mathcal{V}^{q,r}_S$, against $\mathbf{S}$ is given by*

$$\int_0^t G_u \,\mathrm{d}\mathbf{S}_u = \lim_{n\to\infty} \sum_{k=0}^{N_n-1} G_{t^n_k} S_{t^n_k\wedge t, t^n_{k+1}\wedge t}$$

for every $t \in [0,T]$.

The point here is that the path γ may have jump times which do not belong to the set $\cup_{n\in\mathbb{N}}\mathcal{P}^n$.

Proof. Since γ has finite r-variation, we immediately have that γ is a controlled path with Gubinelli derivative simply given by $\gamma' = 0$. By linearity, it is then clear that $(G, G') = (F + \gamma, F')$ is indeed a controlled path with respect to S. Since $\gamma' = 0$, we have from Proposition 2.4 that

$$\int_0^t \gamma_u \,\mathrm{d}\mathbf{S}_u = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \gamma_u S_{u,v} = \lim_{n\to\infty} \sum_{k=0}^{N_n-1} \gamma_{t_k^n} S_{t_k^n\wedge t, t_{k+1}^n\wedge t}.$$

By linearity, we have that $\int_0^t G_u \,\mathrm{d}\mathbf{S}_u = \int_0^t F_u \,\mathrm{d}\mathbf{S}_u + \int_0^t \gamma_u \,\mathrm{d}\mathbf{S}_u$, and the result then follows from Theorem 2.12. □

2.4. **Link to Föllmer integration.** In his seminal paper [Föl81a], Föllmer introduced a notion of pathwise integration based on the concept of quadratic variation, and derived a corresponding pathwise Itô formula, which have proved to be useful tools in robust approaches to mathematical finance.

In the following we will write $\mathcal{B}[0,T]$ for the Borel σ-algebra on $[0,T]$.

Definition 2.14. *Let $S \in D([0,T];\mathbb{R})$ and let $\mathcal{P}^n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T\}$, $n \geq 1$, be a sequence of partitions with vanishing mesh size. We say that S has* quadratic variation along $(\mathcal{P}^n)_{n\in\mathbb{N}}$ in the sense of Föllmer *if the sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ on $([0,T], \mathcal{B}[0,T])$ defined by*

$$\mu_n := \sum_{k=0}^{N_n-1} |S_{t_k^n, t_{k+1}^n}|^2 \delta_{t_k^n},$$

converges weakly to a measure μ, such that the map $t \mapsto [S]^c_t := \mu([0,t]) - \sum_{0<s\leq t} |S_{s-,s}|^2$ is continuous and increasing. In this case we call the function $[S]$, given by $[S]_t = \mu([0,t])$, the quadratic variation *of S along $(\mathcal{P}^n)_{n\in\mathbb{N}}$.*

We say that a path $S \in D([0,T];\mathbb{R}^d)$ has quadratic variation along $(\mathcal{P}^n)_{n\in\mathbb{N}}$ in the sense of Föllmer *if the condition above holds for S^i and $S^i + S^j$ for every (i,j), and in this case we write*

$$[S^i, S^j] := \frac{1}{2}([S^i + S^j] - [S^i] - [S^j]). \tag{2.16}$$

Assuming that a path $S \in D([0,T];\mathbb{R}^d)$ has quadratic variation along $(\mathcal{P}^n)_{n\in\mathbb{N}}$ and $f \in C^2(\mathbb{R}^d;\mathbb{R})$, Föllmer showed that the limit

$$\int_0^T \mathrm{D}f(S_u)\,\mathrm{d}S_u := \lim_{n\to\infty} \sum_{[s,t]\in\mathcal{P}^n} \mathrm{D}f(S_s) S_{s,t}$$

exists and the resulting integral $\int_0^T \mathrm{D}f(S_u)\,\mathrm{d}S_u$ satisfies a pathwise Itô formula, see [Föl81a, THÉORÈME]. Let us remark that the Föllmer integral $\int_0^T \mathrm{D}f(S_u)\,\mathrm{d}S_u$ is only well-defined for gradients $\mathrm{D}f$ and not for general functions, as its existence is given by the corresponding pathwise Itô formula. This result can also be explained via the language rough path theory, see [FH20, Chapter 5.3].

In the following we relate Property (RIE) to the existence of quadratic variation in the sense of Föllmer. To this end, for each $i = 1, \dots, d$, we introduce

$$S_t^{n,i} = S_T^i \mathbf{1}_{\{T\}}(t) + \sum_{k=0}^{N_n-1} S_{t_k^n}^i \mathbf{1}_{[t_k^n, t_{k+1}^n)}(t)$$

and the discrete quadratic variation $\langle S^i, S^j \rangle^n$ by

$$\langle S^i, S^j \rangle_t^n = \sum_{k=0}^{N_n-1} S^i_{t_k^n \wedge t, t_{k+1}^n \wedge t} S^j_{t_k^n \wedge t, t_{k+1}^n \wedge t}, \qquad t \in [0,T].$$

Proposition 2.15. *Let $S \in D([0,T];\mathbb{R}^d)$ and let $\mathcal{P}^n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = T\}$, $n \in \mathbb{N}$, be a sequence of nested partitions with vanishing mesh size. The following conditions are equivalent:*

(i) *For every pair (i,j), the Riemann sums $\int_0^t S_u^{n,i} \,\mathrm{d}S_u^j + \int_0^t S_u^{n,j} \,\mathrm{d}S_u^i$ converge uniformly to a limit, which we denote by $\int_0^t S_u^i \,\mathrm{d}S_u^j + \int_0^t S_u^j \,\mathrm{d}S_u^i$.*

(ii) *For every pair (i,j), the discrete quadratic variation $\langle S^i, S^j \rangle^n$ converges uniformly to a càdlàg path, which we denote by $\langle S^i, S^j \rangle$.*

(iii) *The path S has quadratic variation along $(\mathcal{P}^n)_{n \in \mathbb{N}}$ in the sense of Föllmer.*

Moreover, if these conditions hold then the path $\langle S^i, S^j \rangle$ has finite total variation, and, for every (i,j), we have that $[S^i, S^j] = \langle S^i, S^j \rangle$ and the equality

$$S_t^i S_t^j = S_0^i S_0^j + \int_0^t S_u^i \,\mathrm{d}S_u^j + \int_0^t S_u^j \,\mathrm{d}S_u^i + \langle S^i, S^j \rangle_t \tag{2.17}$$

holds for every $t \in [0,T]$.

Proof. We have

$$\begin{aligned}
S_t^i S_t^j - S_0^i S_0^j &= \sum_{k=0}^{N_n-1} (S^i_{t_{k+1}^n \wedge t} S^j_{t_{k+1}^n \wedge t} - S^i_{t_k^n \wedge t} S^j_{t_k^n \wedge t}) \\
&= \sum_{k=0}^{N_n-1} (S^i_{t_k^n \wedge t} S^j_{t_k^n \wedge t, t_{k+1}^n \wedge t} + S^j_{t_k^n \wedge t} S^i_{t_k^n \wedge t, t_{k+1}^n \wedge t}) + \sum_{k=0}^{N_n-1} S^i_{t_k^n \wedge t, t_{k+1}^n \wedge t} S^j_{t_k^n \wedge t, t_{k+1}^n \wedge t} \\
&= \int_0^t S_u^{n,i} \,\mathrm{d}S_u^j + \int_0^t S_u^{n,j} \,\mathrm{d}S_u^i + \langle S^i, S^j \rangle_t^n,
\end{aligned}$$

from which it follows that conditions (i) and (ii) are equivalent, and that (2.17) then also holds. In this case, we also have that

$$\langle S^i, S^j \rangle_t = \frac{1}{4}\Big(\langle S^i + S^j, S^i + S^j \rangle_t - \langle S^i - S^j, S^i - S^j \rangle_t \Big),$$

so that, as the difference of two non-decreasing functions, $\langle S^i, S^j \rangle$ has finite total variation.

For one-dimensional paths S, the equivalence of conditions (ii) and (iii) follows from [Vov15, Propositions 3 and 4]. The extension of this to d-dimensional paths S and the equality $[S^i, S^j] = \langle S^i, S^j \rangle$ then follow from the polarization identity

$$\langle S^i, S^j \rangle_t^n = \frac{1}{2}\Big(\langle S^i + S^j, S^i + S^j \rangle_t^n - \langle S^i, S^i \rangle_t^n - \langle S^j, S^j \rangle_t^n \Big)$$

and the definition of $[S^i, S^j]$ in (2.16). □

Remark 2.16. *As an immediate consequence of Proposition 2.15, we have that if a path S satisfies* (RIE) *along $(\mathcal{P}^n)_{n\in\mathbb{N}}$, then it has quadratic variation along $(\mathcal{P}^n)_{n\in\mathbb{N}}$ in the sense of Föllmer, thus allowing one to apply all the known results regarding Föllmer integration.*

In particular, if a vector field $f\colon\mathbb{R}^d\to\mathbb{R}$ is of class C^3, then, by Theorem 2.12, the Föllmer integral $\int_0^\cdot \mathrm{D}f(S_u)\,\mathrm{d}S_u$ coincides with the rough integral $\int_0^\cdot \mathrm{D}f(S_u)\,\mathrm{d}\mathbf{S}_u$. We thus obtain the rough Itô formula:

$$f(S_t)-f(S_0)=\int_0^t \mathrm{D}f(S_u)\,\mathrm{d}\mathbf{S}_u+\frac{1}{2}\int_0^t \mathrm{D}^2 f(S_u)\,\mathrm{d}[S]_u$$
$$+\sum_{0<u\le t}\Big(f(S_u)-f(S_{u-})-\mathrm{D}f(S_{u-})\Delta_u S-\frac{1}{2}\mathrm{D}^2 f(S_{u-})(\Delta_u S\otimes\Delta_u S)\Big),$$

which holds for every $t\in[0,T]$, where $[S]=([S^i,S^j])_{1\le i,j\le d}$ denotes the quadratic variation matrix and $\Delta_u S:=\lim_{s\to u,s<u} S_{s,u}$. We note that the formula above is precisely the Itô formula for rough paths derived in [FZ18].

3. Functionally generated trading strategies and their generalizations

Given Property (RIE), we can introduce a model-free framework for continuous-time financial markets with a possibly infinite time horizon. In this section we shall verify that most relevant trading strategies from a practical perspective, such as delta-hedging strategies and functionally generated strategies, are admissible integrands for price paths satisfying Property (RIE). Furthermore, the underlying rough integration allows us to deduce stability estimates for admissible strategies.

For a path $S\colon[0,\infty)\to\mathbb{R}^d$, we denote by $S|_{[0,T]}$ the restriction of S to the interval $[0,T]$.

Definition 3.1. *For a fixed $p\in(2,3)$, we say that a path $S\in D([0,\infty);\mathbb{R}^d)$ is a* price path, *if there exists a nested sequence of locally finite partitions $(\mathcal{P}^n)_{n\in\mathbb{N}}$ of the interval $[0,\infty)$, with vanishing mesh size on compacts, such that, for all $T>0$, the restriction $S|_{[0,T]}$ satisfies* (RIE) *with respect to p and $(\mathcal{P}^n([0,T]))_{n\in\mathbb{N}}$.*

We denote the family of all such price paths by Ω_p.

Note that the sequence of partitions $(\mathcal{P}^n)_{n\in\mathbb{N}}$ may depend on the choice of price path $S\in\Omega_p$, consistent with the stochastic framework where this sequence will naturally be defined in terms of (probabilistic) stopping times.

Having fixed the model-free structure of the underlying price paths, we can introduce the class of admissible strategies and the corresponding capital process.

Definition 3.2. *Let $p\in(2,3)$ and let $S\in\Omega_p$ be a price path. We say that a path $\varphi\colon[0,\infty)\to\mathbb{R}^d$ is an* admissible strategy (with respect to S), *if*

- *there exist $q\ge p$ and $r>1$ with $2/p+1/q>1$ and $1/r=1/p+1/q$, such that for every $T>0$, there exists a path $\varphi'\colon[0,T]\to\mathcal{L}(\mathbb{R}^d;\mathbb{R}^d)$ such that the pair $(\varphi,\varphi')\in\mathcal{V}_S^{q,r}$ is a controlled path with respect to S in the sense of Definition 2.3,*
- *and $J_\varphi\subset\cup_{n\in\mathbb{N}}\mathcal{P}^n$, where J_φ is the set of jump times of φ in $(0,\infty)$, and $(\mathcal{P}^n)_{n\in\mathbb{N}}$ is the sequence of partitions associated with the price path $S\in\Omega_p$.*

We denote the space of all admissible strategies (with respect to S) by $\mathcal{A}_S$.

We define the capital process *associated with* φ *and* S *as the path* $V^{\varphi}(S)\colon [0,\infty) \to \mathbb{R}$ *given by*

$$V_t^{\varphi}(S) := \lim_{n\to\infty} \sum_{k=0}^{N_n-1} \sum_{i=1}^{d} \varphi^i_{t^n_k}(S^i_{t^n_{k+1}\wedge t} - S^i_{t^n_k \wedge t}), \qquad t \in [0,\infty), \tag{3.1}$$

where $\mathcal{P}^n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_{N_n} = T\}$ *is the sequence of partitions specified in Property* (RIE)*.*

Proposition 3.3. *Let* $p \in (2,3)$*, let* $S \in \Omega_p$ *be a price path, and let* $\varphi \in \mathcal{A}_S$ *be an admissible strategy (in the sense of Definition 3.2). Then, the capital process* $V^{\varphi}(S)$ *as defined in* (3.1) *exists as a locally uniform limit, and is actually given by*

$$V_t^{\varphi}(S) = \int_0^t \varphi_s \,\mathrm{d}\mathbf{S}_s, \qquad t \in [0,\infty), \tag{3.2}$$

that is, the rough integral of the controlled path $(\varphi,\varphi') \in \mathcal{V}^{q,r}_S$ *against the rough path* $\mathbf{S}$ *as defined in Lemma 2.10.*

Moreover, given another price path $\tilde{S} \in \Omega_p$ *and an admissible strategy* $\tilde{\varphi}$ *with respect to* $\tilde{S}$ *such that* $(\tilde{\varphi},\tilde{\varphi}') \in \mathcal{V}^{q,r}_{\tilde{S}}$*, we have that*

$$\begin{aligned}
&|V_T^{\varphi}(S) - V_T^{\tilde{\varphi}}(\tilde{S})| \\
&\leq C\Big((|\tilde{\varphi}_0| + |\tilde{\varphi}'_0| + \|\tilde{\varphi}'\|_{q;[0,T]} + \|R^{\tilde{\varphi}}\|_{r;[0,T]})(1 + \|S\|_{p;[0,T]} + \|\tilde{S}\|_{p;[0,T]})\|\mathbf{S};\tilde{\mathbf{S}}\|_{p;[0,T]} \\
&\qquad + (|\varphi_0 - \tilde{\varphi}_0| + |\varphi'_0 - \tilde{\varphi}'_0| + \|\varphi' - \tilde{\varphi}'\|_{q;[0,T]} + \|R^{\varphi} - R^{\tilde{\varphi}}\|_{r;[0,T]})(1 + \|S\|_{p;[0,T]})\|\mathbf{S}\|_{p;[0,T]}\Big),
\end{aligned}$$

for every $T > 0$*, where the constant* C *depends only on* p, q *and* r*.*

Remark 3.4. *Recall from Proposition 2.4 that the rough integral in* (3.2) *is defined by the limit* $\int_0^t \varphi_s \,\mathrm{d}\mathbf{S}_s = \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} \varphi_u S_{u,v} + \varphi'_u A_{u,v}$*, where the limit is taken over any sequence of partitions of the interval* $[0,t]$ *with vanishing mesh size. Here,* φ_u *and* $S_{u,v}$ *both take values in* $\mathbb{R}^d$*, and we interpret their multiplication as the Euclidean inner product. The derivative* φ'_u *takes values in* $\mathcal{L}(\mathbb{R}^d;\mathbb{R}^d)$*, which we can also identify with* $\mathcal{L}(\mathbb{R}^{d\times d};\mathbb{R})$*. Since* $A_{u,v} \in \mathbb{R}^{d\times d}$*, the product* $\varphi'_u A_{u,v}$ *also takes values in* $\mathbb{R}$*.*

Proof of Proposition 3.3. Let $T > 0$, and let $(\mathcal{P}^n)_{n\in\mathbb{N}}$ be a sequence of nested partitions such that p, $(\mathcal{P}^n)_{n\in\mathbb{N}}$ and S satisfy Property (RIE) on the interval $[0,T]$. Recall from Property (RIE) the existence of the limit $\int_0^t S_u \otimes \mathrm{d}S_u$ for every $t \in [0,T]$. By Lemma 2.10, defining the function $A\colon \Delta_{[0,T]} \to \mathbb{R}^{d\times d}$ by

$$A_{s,t} := \int_s^t S_u \otimes \mathrm{d}S_u - S_s \otimes S_{s,t},$$

we have that the triplet $\mathbf{S} = (S,S,A)$ is a càdlàg rough path (in the sense of Definition 2.1). Hence, the rough integral in (3.2) is well-defined by Proposition 2.4, and satisfies (3.1) as a locally uniform limit by Theorem 2.12.

For the stability estimate we simply note that

$$|V_T^{\varphi}(S) - V_T^{\tilde{\varphi}}(\tilde{S})| = \left|\int_0^T \varphi_s \,\mathrm{d}\mathbf{S}_s - \int_0^T \tilde{\varphi}_s \,\mathrm{d}\tilde{\mathbf{S}}_s\right| \leq \left\|\int_0^{\cdot} \varphi_s \,\mathrm{d}\mathbf{S}_s - \int_0^{\cdot} \tilde{\varphi}_s \,\mathrm{d}\tilde{\mathbf{S}}_s\right\|_{p,[0,T]}$$

and apply the result of Proposition 2.6. □

In the following we show that the most relevant trading strategies from a practical viewpoint belong to the class of admissible strategies in the sense of Definition 3.2.

3.1. **Functionally generated trading strategies.** Having fixed the set Ω_p of underlying price paths, we start by introducing functionally generated portfolios. For this purpose, for some $d_A \in \mathbb{N}$, we fix a càdlàg path $A\colon [0,\infty) \to \mathbb{R}^{d_A}$ of locally bounded variation and assume that the jump times of A belong to the union of the partitions $(\mathcal{P}^n)_{n\in\mathbb{N}}$ appearing in Property (RIE); that is, we assume that $J_A \subseteq \cup_{n\in\mathbb{N}}\mathcal{P}^n$, where $J_A := \{t \in (0,\infty) : A_{t-,t} \neq 0\}$. The path A is supposed to include additional information pertaining to the market which a trader would like to include in their trading decisions. For instance, the components of the path $A = (A^1,\dots,A^{d_A})$ could include time $t \mapsto t$, the running maximum $t \mapsto \max_{u\in[0,t]} S^i_u$, or the integral $t \mapsto \int_0^t S^i_u \,\mathrm{d}u$ for some (or all) $i = 1,\dots,d$. A more detailed discussion on practical choices of the path A can be found in [SSV18].

For $\ell = d + d_A$, we denote by $C^2_b(\mathbb{R}^\ell;\mathbb{R}^d)$ the space of twice continuously differentiable (in the Fréchet sense) functions $f\colon \mathbb{R}^\ell \to \mathbb{R}^d$ such that f and all its derivatives up to order 2 are uniformly bounded; that is

$$C^2_b(\mathbb{R}^\ell;\mathbb{R}^d) := \{f\colon \mathbb{R}^\ell \to \mathbb{R}^d : \|f\|_{C^2_b} < \infty\}$$

with

$$\|f\|_{C^2_b} := \|f\|_\infty + \|\mathrm{D}f\|_\infty + \|\mathrm{D}^2 f\|_\infty.$$

For $S \in \Omega_p$ we introduce the set $\mathcal{G}^2_S$ of all generalized functionally generated trading strategies φ^f which are all strategies of the form

$$\varphi^f_t = (f^1_t,\dots,f^d_t) := f(S_t, A_t), \qquad t \in [0,\infty), \tag{3.3}$$

for some $f \in C^2_b(\mathbb{R}^\ell;\mathbb{R}^d)$. For $\varphi^f \in \mathcal{G}^2_S$ the corresponding capital process is given by

$$V^f_t(S) = \lim_{n\to\infty} \sum_{k=0}^{N_n-1} \sum_{i=1}^{d} f^i_{t^n_k}(S^i_{t^n_{k+1}\wedge t} - S^i_{t^n_k\wedge t}), \qquad t \in [0,\infty). \tag{3.4}$$

Proposition 3.5. *Let $p \in (2,3)$ and $S \in \Omega_p$, and let $\varphi^f, \varphi^{\tilde f} \in \mathcal{G}^2_S$. Then, $\varphi^f \in \mathcal{A}_S$ is an admissible strategy, and the capital process $(V^f_t(S))_{t\in[0,\infty)}$ given in (3.4) is well-defined as a locally uniform limit in $t \in [0,\infty)$. Moreover, for every $T \in [0,\infty)$, we have the stability estimate*

$$|V^f_T(S) - V^{\tilde f}_T(S)| \leq C\|f - \tilde f\|_{C^2_b}(1 + \|S\|^2_{p,[0,T]} + \|A\|_{1,[0,T]})(1 + \|S\|_{p,[0,T]})\|\mathbf{S}\|_{p,[0,T]}, \tag{3.5}$$

where the constant C depends only on p, and the triplet $\mathbf{S} = (S,S,A)$ is the càdlàg rough path defined in Lemma 2.10.

Proof. Admissibility: Let $\varphi = \varphi^f \in \mathcal{G}^2_S$ be a functionally generated strategy $\varphi = (\varphi^1,\dots,\varphi^d)$ of the form (3.3) for some $f \in C^2_b(\mathbb{R}^\ell;\mathbb{R}^d)$. Fix a $T \in [0,\infty)$. We claim that $(\varphi,\varphi') \in \mathcal{V}^{p,\frac{p}{2}}_S$ is a controlled path with respect to S in the sense of Definition 2.3 (with $q = p$ and $r = p/2$), where

$$\varphi'_t := \mathrm{D}_S f(S_t, A_t), \qquad t \in [0,T],$$

and $\mathrm{D}_S f$ denotes the derivative of f with respect to its first d components. To see this, we first note that

$$|\varphi'_{s,t}| = |\mathrm{D}_S f(S_t,A_t) - \mathrm{D}_S f(S_s,A_s)| \leq \|f\|_{C^2_b}(|S_{s,t}| + |A_{s,t}|),$$

so that

$$\|\varphi'\|_{p,[0,T]} \lesssim \|f\|_{C_b^2}(\|S\|_{p,[0,T]} + \|A\|_{1,[0,T]}) < \infty, \tag{3.6}$$

and hence $\varphi' \in D^p([0,T];\mathcal{L}(\mathbb{R}^d;\mathbb{R}^d))$. We moreover have that

$$\begin{aligned} R^\varphi_{s,t} &:= \varphi_{s,t} - \varphi'_s S_{s,t} \\ &= f(S_t,A_t) - f(S_s,A_s) - \mathrm{D}_S f(S_s,A_s) S_{s,t} \\ &= f(S_t,A_s) - f(S_s,A_s) - \mathrm{D}_S f(S_s,A_s) S_{s,t} + f(S_t,A_t) - f(S_t,A_s) \\ &= \int_0^1 \Big(\mathrm{D}_S f(S_s + \tau S_{s,t}, A_s) - \mathrm{D}_S f(S_s,A_s)\Big) S_{s,t} \,\mathrm{d}\tau + f(S_t,A_t) - f(S_t,A_s), \end{aligned}$$

so that $|R^\varphi_{s,t}| \leq \|f\|_{C_b^2}(|S_{s,t}|^2 + |A_{s,t}|)$. It follows that

$$\|R^\varphi\|_{\frac{p}{2},[0,T]} \lesssim \|f\|_{C_b^2}(\|S\|^2_{p,[0,T]} + \|A\|_{1,[0,T]}) < \infty, \tag{3.7}$$

so that $R^\varphi \in D^{p/2}(\Delta_{[0,T]};\mathbb{R}^d)$, and thus the conditions of Definition 2.3 are satisfied. Thus, by Proposition 2.4, we have the existence for each $t \in [0,T]$ of the ($\mathbb{R}$-valued) rough integral

$$\int_0^t \varphi_s \,\mathrm{d}\mathbf{S}_s = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \varphi_u S_{u,v} + \varphi'_u A_{u,v}.$$

For a given path F, let $J_F = \{t \in (0,T] : F_{t-,t} \neq 0\}$ denote the jump times of F. It follows from Property (RIE) and Proposition 2.11 that $J_S \subset \cup_{n \in \mathbb{N}} \mathcal{P}^n$. Since we also assumed that $J_A \subset \cup_{n \in \mathbb{N}} \mathcal{P}^n$, it then follows from (3.3) that $J_\varphi \subset \cup_{n \in \mathbb{N}} \mathcal{P}^n$. Thus, by Theorem 2.12, we have that

$$\int_0^t \varphi_s \,\mathrm{d}\mathbf{S}_s = \lim_{n \to \infty} \sum_{k=0}^{N_n - 1} \varphi_{t^n_k}(S_{t^n_{k+1} \wedge t} - S_{t^n_k \wedge t}) = \lim_{n \to \infty} \sum_{k=0}^{N_n - 1} \sum_{i=1}^d f^i_{t^n_k}(S^i_{t^n_{k+1} \wedge t} - S^i_{t^n_k \wedge t}) = V^f_t(S),$$

and that this limit is uniform in $t \in [0,T]$.

Stability estimate: Let φ and $\tilde{\varphi}$ be the strategies generated by f and $\tilde{f}$ respectively, as defined in (3.3). By Proposition 2.6, we have the estimate

$$\begin{aligned} &|V^f_T(S) - V^{\tilde{f}}_T(S)| \\ &= \left| \int_0^T \varphi_s \,\mathrm{d}\mathbf{S}_s - \int_0^T \tilde{\varphi}_s \,\mathrm{d}\mathbf{S}_s \right| \leq \left\| \int_0^\cdot \varphi_s \,\mathrm{d}\mathbf{S}_s - \int_0^\cdot \tilde{\varphi}_s \,\mathrm{d}\mathbf{S}_s \right\|_{p,[0,T]} \\ &\lesssim (|\varphi_0 - \tilde{\varphi}_0| + |\varphi'_0 - \tilde{\varphi}'_0| + \|\varphi' - \tilde{\varphi}'\|_{p;[0,T]} + \|R^\varphi - R^{\tilde{\varphi}}\|_{\frac{p}{2};[0,T]})(1 + \|S\|_{p,[0,T]})\|\mathbf{S}\|_{p,[0,T]}. \end{aligned} \tag{3.8}$$

As above, here

$$\varphi'_t = \mathrm{D}_S f(S_t,A_t) \qquad \text{and} \qquad R^\varphi_{s,t} = \varphi_{s,t} - \varphi'_s S_{s,t},$$

with $\tilde{\varphi}'$ and $R^{\tilde{\varphi}}$ defined similarly. We will now aim to estimate each term on the right-hand side of (3.8).

We have that

$$|\varphi_0 - \tilde{\varphi}_0| = |f(S_0,A_0) - \tilde{f}(S_0,A_0)| \leq \|f - \tilde{f}\|_\infty \leq \|f - \tilde{f}\|_{C_b^2}, \tag{3.9}$$

and similarly

$$|\varphi'_0 - \tilde{\varphi}'_0| = |\mathrm{D}_S f(S_0,A_0) - \mathrm{D}_S \tilde{f}(S_0,A_0)| \leq \|\mathrm{D}_S f - \mathrm{D}_S \tilde{f}\|_\infty \leq \|f - \tilde{f}\|_{C_b^2}.$$

Next, for $[s,t] \subseteq [0,T]$, we compute

$$\begin{aligned}(\varphi' - \tilde{\varphi}')_{s,t} &= \mathrm{D}_S f(S_t, A_t) - \mathrm{D}_S f(S_s, A_s) - \mathrm{D}_S \tilde{f}(S_t, A_t) + \mathrm{D}_S \tilde{f}(S_s, A_s) \\ &= \mathrm{D}_S f(S_t, A_s) - \mathrm{D}_S f(S_s, A_s) - \mathrm{D}_S \tilde{f}(S_t, A_s) + \mathrm{D}_S \tilde{f}(S_s, A_s) \\ &\quad + \mathrm{D}_S f(S_t, A_t) - \mathrm{D}_S f(S_t, A_s) - \mathrm{D}_S \tilde{f}(S_t, A_t) + \mathrm{D}_S \tilde{f}(S_t, A_s) \\ &= \int_0^1 \Big(\mathrm{D}^2_{SS} f(S_s + \tau S_{s,t}, A_s) - \mathrm{D}^2_{SS} \tilde{f}(S_s + \tau S_{s,t}, A_s)\Big) S_{s,t} \, \mathrm{d}\tau \\ &\quad + \int_0^1 \Big(\mathrm{D}^2_{SA} f(S_t, A_s + \tau A_{s,t}) - \mathrm{D}^2_{SA} \tilde{f}(S_t, A_s + \tau A_{s,t})\Big) A_{s,t} \, \mathrm{d}\tau,\end{aligned}$$

so that

$$|(\varphi' - \tilde{\varphi}')_{s,t}| \leq \|\mathrm{D}^2_{SS} f - \mathrm{D}^2_{SS} \tilde{f}\|_\infty |S_{s,t}| + \|\mathrm{D}^2_{SA} f - \mathrm{D}^2_{SA} \tilde{f}\|_\infty |A_{s,t}| \leq \|f - \tilde{f}\|_{C^2_b} (|S_{s,t}| + |A_{s,t}|),$$

and thus

$$\|\varphi' - \tilde{\varphi}'\|_{p,[0,T]} \lesssim \|f - \tilde{f}\|_{C^2_b} (\|S\|_{p,[0,T]} + \|A\|_{1,[0,T]}).$$

Finally, we have that

$$\begin{aligned}&(R^\varphi - R^{\tilde{\varphi}})_{s,t} \\ &= f(S_t, A_t) - f(S_s, A_s) - \mathrm{D}_S f(S_s, A_s) S_{s,t} - \tilde{f}(S_t, A_t) + \tilde{f}(S_s, A_s) + \mathrm{D}_S \tilde{f}(S_s, A_s) S_{s,t} \\ &= f(S_t, A_s) - f(S_s, A_s) - \mathrm{D}_S f(S_s, A_s) S_{s,t} - \tilde{f}(S_t, A_s) + \tilde{f}(S_s, A_s) + \mathrm{D}_S \tilde{f}(S_s, A_s) S_{s,t} \\ &\quad + f(S_t, A_t) - f(S_t, A_s) - \tilde{f}(S_t, A_t) + \tilde{f}(S_t, A_s) \\ &= \int_0^1 \int_0^1 \Big(\mathrm{D}^2_{SS} f(S_s + \tau_1 \tau_2 S_{s,t}, A_s) - \mathrm{D}^2_{SS} \tilde{f}(S_s + \tau_1 \tau_2 S_{s,t}, A_s)\Big) S^{\otimes 2}_{s,t} \, \tau_1 \, \mathrm{d}\tau_2 \, \mathrm{d}\tau_1 \\ &\quad + \int_0^1 \Big(\mathrm{D}_A f(S_t, A_s + \tau A_{s,t}) - \mathrm{D}_A \tilde{f}(S_t, A_s + \tau A_{s,t})\Big) A_{s,t} \, \mathrm{d}\tau,\end{aligned}$$

so that

$$|(R^\varphi - R^{\tilde{\varphi}})_{s,t}| \leq \|\mathrm{D}^2_{SS} f - \mathrm{D}^2_{SS} \tilde{f}\|_\infty |S_{s,t}|^2 + \|\mathrm{D}_A f - \mathrm{D}_A \tilde{f}\|_\infty |A_{s,t}| \leq \|f - \tilde{f}\|_{C^2_b} (|S_{s,t}|^2 + |A_{s,t}|),$$

and hence

$$\|R^\varphi - R^{\tilde{\varphi}}\|_{\frac{p}{2},[0,T]} \lesssim \|f - \tilde{f}\|_{C^2_b} (\|S\|^2_{p,[0,T]} + \|A\|_{1,[0,T]}). \tag{3.10}$$

Substituting (3.9)–(3.10) into (3.8), we deduce that the estimate in (3.5) holds. □

3.2. **Path-dependent functionally generated traded strategies.** The functionally generated trading strategies considered in Subsection 3.1 could only depend on the past prices through a process of finite variation. In some contexts it is beneficial to work with trading strategies possessing a more general path-dependent structure; see e.g. [SV16, SSV18] for more detailed discussions in this direction. A common way to treat path-dependent and non-anticipating trading strategies is the calculus initiated by Dupire [Dup19]; see also [CF10]. For the sake of brevity we use in the following corollary the notation and definitions as introduced in [Ana20, Section 3.1].

Corollary 3.6. *If $F \in \mathbb{C}^{0,1}_b(\Lambda^d_T, \mathbb{R}^n)$ with F and $\nabla_x F$ in $Lip(\Lambda^d_T, d_\infty)$, and $S \in \Omega_p$, then the path-dependent functionally generated traded strategy $F(\cdot, S)$ is an admissible strategy in the sense of Definition 3.2.*

Corollary 3.6 is an immediate consequence of [Ana19, Lemma 5.12].

3.3. **Cover's universal portfolio.** While functionally generated trading strategies are most prominent in the literature regarding hedging and control problems in mathematical finance, various other trading strategies with desirable properties have been considered. One example coming from portfolio theory is Cover's universal portfolio, as introduced in [Cov91]. The basic idea is to invest, not according to one specific trading strategy, but according to a mixture of all admissible strategies. Following [CSW19], we introduce here a model-free analogue of Cover's universal portfolio.

Let $\mathcal{Z}$ be a Borel measurable subset of $C_b^2(\mathbb{R}^d;\mathbb{R}^d)$, and suppose that ν is a probability measure on $\mathcal{Z}$. A model-free version of Cover's universal portfolio φ^ν is then given by

$$\varphi_t^\nu := \int_{\mathcal{Z}} \varphi_t^f \,\mathrm{d}\nu(f), \qquad t \in [0,\infty), \tag{3.11}$$

where $\varphi^f = f(S)$ is the portfolio generated by f, for some fixed $S \in \Omega_p$.

Lemma 3.7. *Let $S \in \Omega_p$ and let ν be a probability measure on $\mathcal{Z}$, as above. If*

$$\int_{\mathcal{Z}} \|f\|_{C_b^2} \,\mathrm{d}\nu(f) < \infty,$$

then Cover's universal portfolio φ^ν, as defined in (3.11)*, is an admissible strategy in the sense of Definition 3.2.*

Proof. Let $T > 0$. We know from Proposition 3.5 that, for each $f \in \mathcal{Z}$, the corresponding functionally generated portfolio $\varphi = \varphi^f$ is an admissible strategy. Let φ' and R^φ denote the corresponding Gubinelli derivative and remainder term. It follows from the inequalities in (3.6) and (3.7) that

$$|\varphi_0| + |\varphi_0'| + \|\varphi'\|_{p,[0,T]} + \|R^\varphi\|_{\frac{p}{2},[0,T]} \lesssim \|f\|_{C_b^2}(1 + \|S\|_{p,[0,T]}^2),$$

and hence that

$$\int_{\mathcal{Z}} (|\varphi_0| + |\varphi_0'| + \|\varphi'\|_{p,[0,T]} + \|R^\varphi\|_{\frac{p}{2},[0,T]}) \,\mathrm{d}\nu \lesssim (1 + \|S\|_{p,[0,T]}^2) \int_{\mathcal{Z}} \|f\|_{C_b^2} \,\mathrm{d}\nu < \infty. \tag{3.12}$$

Recall that the map $(\varphi, \varphi') \mapsto |\varphi_0| + |\varphi_0'| + \|\varphi'\|_{p,[0,T]} + \|R^\varphi\|_{\frac{p}{2},[0,T]}$ is a norm on the Banach space of controlled paths $\mathcal{V}_S^{p,\frac{p}{2}}$. Note moreover that the subset of controlled paths $(\varphi, \varphi') \in \mathcal{V}_S^{p,\frac{p}{2}}$ satisfying $J_\varphi \subset \cup_{n \in \mathbb{N}} \mathcal{P}^n$ is a closed linear subspace of $\mathcal{V}_S^{p,\frac{p}{2}}$, and thus is itself a Banach space.

It follows from the integrability condition in (3.12) that the integral in (3.11) exists as a well-defined Bochner integral, and defines a controlled path $(\varphi^\nu, (\varphi^\nu)') \in \mathcal{V}_S^{p,\frac{p}{2}}$ satisfying $J_{\varphi^\nu} \subset \cup_{n \in \mathbb{N}} \mathcal{P}^n$. □

4. Semimartingales and typical price paths satisfy Property (RIE)

In this section we show that many stochastic processes commonly used to model the price evolutions on financial markets satisfy Property (RIE) along a suitable sequence of partitions. In particular, we verify that Property (RIE) holds for semimartingales and for typical price paths in the sense of Vovk [Vov12].

4.1. **Semimartingales.** Semimartingales, such as geometric Brownian motion and Markov jump-diffusion processes, serve as the most frequently used stochastic processes to model price evolutions on financial markets. For more details on semimartingales and Itô integration we refer to the standard textbook [Pro04].

Usually the considered class of semimartingales is restricted to those satisfying the condition "no free lunch with vanishing risk" in classical mathematical finance, e.g. [DS94], or the condition "no unbounded profit with bounded risk" (NUPBR) in stochastic portfolio theory, e.g. [KK07]. Such a restriction is not required here. Property (RIE) is fulfilled by general càdlàg semimartingales with respect to any $p \in (2,3)$ and a suitable (random) sequence of partitions.

Let us fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ and assume that the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ satisfies the usual conditions, i.e. completeness and right-continuity. On this probability space we consider a d-dimensional càdlàg semimartingale $X = (X_t)_{t\in[0,\infty)}$.

For each $n \in \mathbb{N}$, we introduce stopping times $(\tau_k^n)_{k\in\mathbb{N}\cup\{0\}}$ such that $\tau_0^n = 0$, and

$$\tau_k^n := \inf\{t > \tau_{k-1}^n : |X_t - X_{\tau_{k-1}^n}| \geq 2^{-n}\}, \qquad k \in \mathbb{N}.$$

We then define a sequence of partitions $(\mathcal{P}_X^n)_{n\in\mathbb{N}}$ by

$$\mathcal{P}_X^n := \{\tau_k^m : m \leq n,\, k \in \mathbb{N} \cup \{0\}\}.$$

Note that $(\mathcal{P}_X^n)_{n\in\mathbb{N}}$ is a nested sequence of adapted partitions. However, this sequence of partitions will not have vanishing mesh size if the path X has an interval of constancy. To amend this, we proceed as follows. For each $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, we set

$$\begin{aligned}\sigma_k^n &:= \tau_{k+1}^n \wedge \inf\{t > \tau_k^n : \exists \delta > 0 \text{ such that } X_s = X_t \ \forall s \in [t, t+\delta]\},\\ \varsigma_k^n &:= \tau_{k+1}^n \wedge \inf\{t > \sigma_k^n : X_t \neq X_{\sigma_k^n}\},\end{aligned}$$

that is, the beginning and end points of the first interval of constancy of X within the interval $[\tau_k^n, \tau_{k+1}^n]$. For each $i \in \mathbb{N}$, we then define $\rho_{k,i}^n := (\sigma_k^n + i2^{-n}) \wedge \varsigma_k^n$. Clearly, for each n, k, we will have that $\rho_{k,i}^n = \varsigma_k^n$ for all but finitely many $i \in \mathbb{N}$ (provided that $\varsigma_k^n < \infty$). Finally, we define

$$\mathcal{Q}_X^n := \mathcal{P}_X^n \cup \{\sigma_k^m : m \leq n,\, k \in \mathbb{N} \cup \{0\}\} \cup \{\rho_{k,i}^m : m \leq n,\, k \in \mathbb{N} \cup \{0\},\, i \in \mathbb{N}\}.$$

The sequence of partitions $(\mathcal{Q}_X^n)_{n\in\mathbb{N}}$ is still nested, and moreover has vanishing mesh size on every compact time interval. Moreover, it is straightforward to see that all the points τ_k^n, σ_k^n and $\rho_{k,i}^n$ appearing in the partitions $(\mathcal{Q}_X^n)_{n\in\mathbb{N}}$ are stopping times with respect to the (right-continuous) filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$. In the next proposition we will show that X satisfies Property (RIE) with respect to any $p \in (2,3)$ and the sequence of partitions $(\mathcal{Q}_X^n)_{n\in\mathbb{N}}$.

Proposition 4.1. *Let $p \in (2,3)$, and let X be a d-dimensional càdlàg semimartingale. Then almost every sample path of X is a price path in the sense of Definition 3.1.*

Proof. Step 1. Fix a $T > 0$. Let $\int_0^\cdot X_{u-} \otimes \mathrm{d}X_u$ denote the Itô integral of X with respect to itself. For each $n \in \mathbb{N}$, we define the discretized process $X^n = (X_t^n)_{t\in[0,T]}$ by

$$X_t^n := \sum_{[u,v]\in\mathcal{Q}_X^n([0,T])} X_u \mathbf{1}_{[u,v)}(t), \qquad t \in [0,T], \tag{4.1}$$

so that in particular $\int_0^t X^n_{u-} \otimes \mathrm{d}X_u = \sum_{[u,v]\in\mathcal{Q}^n_X([0,T])} X_u \otimes X_{u\wedge t, v\wedge t}$ for $t \in [0,T]$. By the definition of the partition $\mathcal{Q}^n_X$, we have that

$$\|X^n_{\cdot-} - X_{\cdot-}\|_{\infty,[0,T]} \leq 2^{1-n} \qquad \text{for all} \qquad n \geq 1.$$

An application of the Burkholder–Davis–Gundy inequality and the Borel–Cantelli lemma, as in the proof of [LP18, Proposition 3.4], then yields the existence of a measurable set $\Omega' \subseteq \Omega$ with full measure such that, for every $\omega \in \Omega'$ and every $\varepsilon \in (0,1)$, there exists a constant $C = C(\varepsilon, \omega)$ such that

$$\left\| \left(\int_0^\cdot X^n_{u-} \otimes \mathrm{d}X_u - \int_0^\cdot X_{u-} \otimes \mathrm{d}X_u \right)(\omega) \right\|_{\infty,[0,T]} \leq C 2^{-n(1-\varepsilon)}, \qquad \forall n \geq 1. \tag{4.2}$$

Thus, we have that $X^n(\omega) \to X(\omega)$ and $\int_0^\cdot X^n_{u-} \otimes \mathrm{d}X_u(\omega) \to \int_0^\cdot X_{u-} \otimes \mathrm{d}X_u(\omega)$ uniformly as $n \to \infty$, for every $\omega \in \Omega'$.

Step 2. We choose a $q_0 \in (2,3)$ close enough to 2 and an $\varepsilon \in (0,1)$ small enough such that

$$\frac{p}{2} > \max\left\{\frac{q_0}{2}, q_0 - 1\right\} \qquad \text{and} \qquad \frac{p}{2} \geq \frac{q_0 - 1 - \varepsilon}{1 - \varepsilon}. \tag{4.3}$$

We also fix a control function w_{X,q_0} such that

$$|X_{s,t}|^{q_0} \leq w_{X,q_0}(s,t) \qquad \text{for all} \quad (s,t) \in \Delta_{[0,T]}. \tag{4.4}$$

This is always possible as X has almost surely finite q-variation for every $q > 2$, and so without loss of generality we may assume that $X(\omega)$ has finite q_0-variation for every $\omega \in \Omega'$. Note that by (4.3) we have $p > q_0$ and consequently (by increasing the values of w_{X,q_0} by a multiplicative constant if necessary) we can also assume that

$$\sup_{(s,t)\in\Delta_{[0,T]}} \frac{|X_{s,t}|^p}{w_{X,q_0}(s,t)} \leq 1. \tag{4.5}$$

Let $0 \leq s < t \leq T$ be such that $s = \tau^n_{k_0}$ and $t = \tau^n_{k_0+N}$ for some $n \in \mathbb{N}$, $k_0 \in \mathbb{N} \cup \{0\}$ and $N \geq 1$. By the superadditivity of the control function w_{X,q_0}, there must exist an $l \in \{1, 2, \ldots, N-1\}$ such that

$$w_{X,q_0}(\tau^n_{k_0+l-1}, \tau^n_{k_0+l+1}) \leq \frac{2}{N-1} w_{X,q_0}(s,t).$$

Thus, by (4.4),

$$\begin{aligned}
&|X_{\tau^n_{k_0+l-1}} \otimes X_{\tau^n_{k_0+l-1},\tau^n_{k_0+l}} + X_{\tau^n_{k_0+l}} \otimes X_{\tau^n_{k_0+l},\tau^n_{k_0+l+1}} - X_{\tau^n_{k_0+l-1}} \otimes X_{\tau^n_{k_0+l-1},\tau^n_{k_0+l+1}}| \\
&\quad = |X_{\tau^n_{k_0+l-1},\tau^n_{k_0+l}} \otimes X_{\tau^n_{k_0+l},\tau^n_{k_0+l+1}}| \leq w_{X,q_0}(\tau^n_{k_0+l-1}, \tau^n_{k_0+l+1})^{2/q_0} \\
&\quad \leq \left(\frac{2}{N-1} w_{X,q_0}(s,t)\right)^{2/q_0}.
\end{aligned}$$

By successively removing in this manner all the intermediate points from the partition $\{s = \tau^n_{k_0}, \tau^n_{k_0+1}, \ldots, \tau^n_{k_0+N} = t\}$, we obtain the estimate

$$\begin{aligned}
\left| \sum_{i=0}^{N-1} X_{\tau^n_{k_0+i}} \otimes X_{\tau^n_{k_0+i},\tau^n_{k_0+i+1}} - X_s \otimes X_{s,t} \right| &\leq \sum_{j=2}^{N} \left(\frac{2}{j-1} w_{X,q_0}(s,t)\right)^{2/q_0} \\
&\lesssim N^{1-2/q_0} w_{X,q_0}(s,t)^{2/q_0}.
\end{aligned}$$

Since $w_{X,q_0}(s,t) \geq \sum_{i=0}^{N-1} w_{X,q_0}(\tau^n_{k_0+i}, \tau^n_{k_0+i+1}) \geq \sum_{i=0}^{N-1} |X_{\tau^n_{k_0+i}, \tau^n_{k_0+i+1}}|^{q_0} \geq N 2^{-nq_0}$, we have that $N \leq 2^{nq_0} w_{X,q_0}(s,t)$. Substituting this into the above, we have that

$$\left| \int_s^t X^n_{u-} \otimes \mathrm{d}X_u - X_s \otimes X_{s,t} \right| \lesssim 2^{n(q_0-2)} w_{X,q_0}(s,t), \tag{4.6}$$

where here the discretized process X^n is defined relative to the partition $\{\tau^n_0, \tau^n_1, \tau^n_2, \ldots\}$.

If, more generally, $0 \leq s < t \leq T$ are such that $s, t \in \mathcal{P}^n_X$, then $s = \tau^{m_1}_{k_1}$ and $t = \tau^{m_2}_{k_2}$ for some $m_1, m_2 \leq n$ and $k_1, k_2 \in \mathbb{N} \cup \{0\}$. In this case, the number of partition points N above satisfies $N \leq \sum_{m=1}^n 2^{mq_0} w_{X,q_0}(s,t) \lesssim 2^{nq_0} w_{X,q_0}(s,t)$, and we thus still obtain the same bound in (4.6). If we further allow the pair of times s, t to include the times σ^m_k for $m \leq n$ and $k \in \mathbb{N} \cup \{0\}$, then this at most doubles the total number of partition points N, so we can again obtain the same bound. Since the points $\rho^m_{k,i}$ lie inside the interval $[\sigma^m_k, \varsigma^m_k]$ on which X is constant, it is clear for instance that $X_{\sigma^m_k, \rho^m_{k,i}} = 0$ and $\int_{\sigma^m_k}^{\rho^m_{k,i}} X^n_{u-} \otimes \mathrm{d}X_u = 0$ for every $i \in \mathbb{N}$, and it follows that these terms will not contribute anything to the bound above.

Thus, the bound in (4.6) actually holds for all $0 \leq s < t \leq T$ with $s, t \in \mathcal{Q}^n_X$, with X^n defined relative to the partition $\mathcal{Q}^n_X$, as in (4.1).

Step 3. We first consider the case that $w_{X,q_0}(s,t)^{\frac{2}{p(1-\varepsilon)}} \leq 2^{-n}$. In this case it follows from (4.3) and (4.6) that there exists a constant $C_1 = C_1(\omega, q_0, \varepsilon)$ such that

$$\left| \int_s^t X^n_{u-} \otimes \mathrm{d}X_u - X_s \otimes X_{s,t} \right|^{\frac{p}{2}} \leq C_1 w_{X,q_0}(s,t). \tag{4.7}$$

Now we consider the case that $w_{X,q_0}(s,t)^{\frac{2}{p(1-\varepsilon)}} \geq 2^{-n}$. Let $\mathbb{X}_{s,t} := \int_s^t X_{u-} \otimes \mathrm{d}X_u - X_s \otimes X_{s,t}$ be the second level component of the Itô lift of X. By [LP18, Proposition 3.4], we know that $\mathbb{X}$ possesses finite $\frac{p}{2}$-variation; that is, there exists a control function $w_{\mathbb{X},\frac{p}{2}}$ such that

$$\sup_{(s,t) \in \Delta_{[0,T]}} \frac{|\mathbb{X}_{s,t}|^{\frac{p}{2}}}{w_{\mathbb{X},\frac{p}{2}}(s,t)} \leq 1.$$

Then, in view of (4.2), we obtain

$$\begin{aligned}
\left| \int_s^t X^n_{u-} \otimes \mathrm{d}X_u - X_s \otimes X_{s,t} \right| &\leq 2 \left\| \int_0^\cdot X^n_{u-} \otimes \mathrm{d}X_u - \int_0^\cdot X_{u-} \otimes \mathrm{d}X_u \right\|_{\infty,[0,T]} + |\mathbb{X}_{s,t}| \\
&\leq C_2 \Big(2^{-n(1-\varepsilon)} + w_{\mathbb{X},\frac{p}{2}}(s,t)^{\frac{2}{p}} \Big) \\
&\leq C_2 \Big(w_{X,q_0}(s,t)^{\frac{2}{p}} + w_{\mathbb{X},\frac{p}{2}}(s,t)^{\frac{2}{p}} \Big)
\end{aligned}$$

for some constant $C_2 = C_2(\varepsilon, \omega)$, and hence

$$\left| \int_s^t X^n_{u-} \otimes \mathrm{d}X_u - X_s \otimes X_{s,t} \right|^{\frac{p}{2}} \leq C_3 \Big(w_{X,q_0}(s,t) + w_{\mathbb{X},\frac{p}{2}}(s,t) \Big), \tag{4.8}$$

where $C_3 = (2C_2)^{\frac{p}{2}}$. Letting $\tilde{w}_{X,p}(s,t) := 2(1 + C_1 + C_3)(w_{X,q_0}(s,t) + w_{\mathbb{X},\frac{p}{2}}(s,t))$, and combining (4.5), (4.7) and (4.8), we conclude that for every $\omega \in \Omega'$,

$$\sup_{(s,t) \in \Delta_{[0,T]}} \frac{|X_{s,t}|^p}{\tilde{w}_{X,p}(s,t)} + \sup_{n \in \mathbb{N}} \sup_{\substack{(s,t) \in \Delta_{[0,T]} \\ s,t \in \mathcal{Q}^n_X}} \frac{|\int_s^t X^n_{u-} \otimes \mathrm{d}X_u - X_s \otimes X_{s,t}|^{\frac{p}{2}}}{\tilde{w}_{X,p}(s,t)} \leq 1,$$

from which Property (RIE) follows. □

Remark 4.2. *Proposition 4.1 holds true even if the semimartingale X takes values in an (infinite dimensional) Hilbert space E, as long as the norm on $E \otimes E$ is admissible in the sense of* [LCL07, Definition 1.25]. *In particular, an extension of Proposition 4.1 to so-called piecewise semimartingales, which were introduced in* [Str14, Definition 2.2] *as generalized semimartingales with an image dimension evolving randomly in time, appears to be straightforward to implement. As discussed in* [KK20, Remark 6.2 and Section 7]*, piecewise semimartingales provide a realistic framework to model so-called open markets, which are financial markets with an evolving number of traded assets.*

4.2. **Generalized semimartingales.** It is a well observed fact in the empirical literature, see e.g. [Lo91], that price processes appear regularly in financial markets which are not semimartingales. Motivated by this fact, many researchers have proposed and investigated financial models based on fractional Brownian motions; see for instance [JPS09, Che03, Ben12]. One example of such models are the so-called mixed Black–Scholes models. In these models the (one-dimensional) price process $S = (S_t)_{t\in[0,\infty)}$ is usually given by

$$S_t := s_0 \exp(\sigma W_t + \eta Y_t + \nu t + \mu t^{2H}), \qquad t \in [0,\infty), \tag{4.9}$$

for constants $s_0, \sigma, \eta > 0$ and $\nu, \mu \in \mathbb{R}$, where $W = (W_t)_{t\in[0,\infty)}$ is standard Brownian motion and $Y = (Y_t)_{t\in[0,\infty)}$ is a fractional Brownian motion with Hurst index $H \in (0,1)$. Multi-dimensional versions of the mixed Black–Scholes model (4.9) can be obtained by standard modifications. Notice that, while the price process S as defined in (4.9) is not a semimartingale if $H \neq 1/2$, the mixed Black–Scholes model (4.9) is still arbitrage-free when restricting the admissible trading strategies to classes of trading strategies which, roughly speaking, exclude continuous rebalancing of the positions in the underlying market, cf. [JPS09, Che03, Ben12]. In particular, the mixed Black–Scholes model (4.9) is free of simple arbitrage opportunities if $H > 1/2$, as proven in [Ben12, Section 4.1].

In order to demonstrate that Property (RIE) is satisfied by various financial models based on fractional Brownian motion, we consider the following class of generalized semimartingales. On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ with a complete right-continuous filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$, let Z be a d-dimensional process admitting the decomposition

$$Z_t = X_t + Y_t, \qquad t \in [0,\infty),$$

where X is a semimartingale, and Y is a càdlàg adapted process with finite q-variation for some $q \in [1,2)$. Processes Z of this form are sometimes called *Young semimartingales*, and belong to the class of càdlàg Dirichlet processes in the sense of [Föl81b].

We introduce stopping times (τ_k^n), such that, for each $n \in \mathbb{N}$, $\tau_0^n = 0$, and

$$\tau_k^n := \inf\{t > \tau_{k-1}^n : |X_t - X_{\tau_{k-1}^n}| \geq 2^{-n} \ \text{ or } \ |Y_t - Y_{\tau_{k-1}^n}| \geq 2^{-n}\}$$

for $k \in \mathbb{N}$, and set

$$\mathcal{P}_Z^n := \{\tau_k^m : m \leq n,\, k \in \mathbb{N} \cup \{0\}\}.$$

As in the previous section, since we insist that the sequence of partitions in Property (RIE) has vanishing mesh size, we also define

$$\mathcal{Q}_Z^n := \mathcal{P}_Z^n \cup \{\sigma_k^m : m \leq n,\, k \in \mathbb{N} \cup \{0\}\} \cup \{\rho_{k,i}^m : m \leq n,\, k \in \mathbb{N} \cup \{0\},\, i \in \mathbb{N}\},$$

where the times σ_k^m and $\rho_{k,i}^m$ are defined analogously as in Section 4.1.

Proposition 4.3. *Let $Z = X + Y$ be a d-dimensional process such that X is a càdlàg semimartingale, and Y is a càdlàg process with finite q-variation for some $q \in [1,2)$. Then, for any $p \in (2,3)$ such that $1/p + 1/q > 1$, almost every sample path of Z is a price path in the sense of Definition 3.1.*

Proof. Step 1. It is sufficient to prove that almost all sample paths of Z satisfy Property (RIE) along $(\mathcal{Q}^n_Z([0,T]))_{n\in\mathbb{N}}$ for an arbitrary $T > 0$. To this end, for each $n \in \mathbb{N}$ we set

$$Z^n_t := \sum_{[u,v]\in\mathcal{Q}^n_Z([0,T])} Z_u \mathbf{1}_{[u,v)}(t), \qquad t \in [0,T],$$

and define X^n and Y^n in the same way with respect to the partition $\mathcal{Q}^n_Z([0,T])$.

Let $X = M + A$ be a decomposition of the semimartingale X such that M is a locally square integrable martingale and A is of bounded variation. By setting $B := A + Y$, we can write $Z = M + B$, where B has finite q-variation. We define the Itô integral of Z with respect to itself by

$$\int_0^t Z_{u-} \otimes \mathrm{d}Z_u := \int_0^t Z_{u-} \otimes \mathrm{d}M_u + \int_0^t Z_{u-} \otimes \mathrm{d}B_u,$$

where the first integral on the right-hand side is an Itô integral and the second one is interpreted as a Young integral, which exists since Z has finite p-variation and $1/p + 1/q > 1$; see e.g. [FZ18]. Then, since $\|Z^n_{\cdot-} - Z_{\cdot-}\|_{\infty,[0,T]} \leq 2^{1-n}$, by the Burkholder–Davis–Gundy inequality and the Borel–Cantelli lemma, we deduce that for almost all ω and for every $\varepsilon \in (0,1)$, there exists a constant $C = C(\omega,\varepsilon)$ such that, for all $n \in \mathbb{N}$,

$$\left\| \int_0^\cdot Z^n_{u-} \otimes \mathrm{d}M_u - \int_0^\cdot Z_{u-} \otimes \mathrm{d}M_u \right\|_{\infty,[0,T]} \leq C 2^{-n(1-\varepsilon)}, \tag{4.10}$$

cf. the proof of [LP18, Proposition 3.4]. By a standard bound for Young integrals (e.g. [FZ18, Proposition 2.4]), for every $t \in [0,T]$, we also have (noting that $Z^n_0 = Z_0$ for all n) that

$$\left| \int_0^t (Z^n_{u-} - Z_{u-}) \otimes \mathrm{d}B_u \right| \leq C \|Z^n_{\cdot-} - Z_{\cdot-}\|_{p,[0,t]} \|B\|_{q,[0,T]},$$

for some constant $C = C(p,q)$. Since $\|Z^n_{\cdot-}\|_{p_0,[0,t]} \leq \|Z\|_{p_0,[0,T]}$ holds for every n and every $p_0 \in (2,p)$, a routine interpolation argument shows that, for each $n \in \mathbb{N}$,

$$\|Z^n_{\cdot-} - Z_{\cdot-}\|_{p,[0,T]} \leq C \|Z^n_{\cdot-} - Z_{\cdot-}\|^{1-\frac{p_0}{p}}_{\infty,[0,T]} \|Z\|^{\frac{p_0}{p}}_{p_0,[0,T]}$$

for some constant $C = C(p,p_0)$. Hence, since $\|Z^n_{\cdot-} - Z_{\cdot-}\|_{\infty,[0,T]} \leq 2^{1-n}$ for all n, we have

$$\lim_{n\to\infty} \left\| \int_0^\cdot Z^n_{u-} \otimes \mathrm{d}B_u - \int_0^\cdot Z_{u-} \otimes \mathrm{d}B_u \right\|_{\infty,[0,T]} = 0. \tag{4.11}$$

Combining (4.10) with (4.11), we conclude that, almost surely, the integral $\int_0^\cdot Z^n_{u-} \otimes \mathrm{d}Z_u$ converges uniformly to $\int_0^\cdot Z_{u-} \otimes \mathrm{d}Z_u$.

Step 2. For every $n \in \mathbb{N}$ we set

$$\begin{aligned}
\mathbb{Z}^n_{s,t} &:= \int_s^t Z^n_{s,u-} \otimes \mathrm{d}Z_u \\
&= \int_s^t X^n_{s,u-} \otimes \mathrm{d}X_u + \int_s^t Y^n_{s,u-} \otimes \mathrm{d}Y_u + \int_s^t X^n_{s,u-} \otimes \mathrm{d}Y_u + \int_s^t Y^n_{s,u-} \otimes \mathrm{d}X_u,
\end{aligned}$$

for $(s,t) \in \Delta_{[0,T]}$. Moreover, we define

$$\mathbb{X}^n_{s,t} := \int_s^t X^n_{s,u-} \otimes \mathrm{d}X_u, \qquad \mathbb{Y}^n_{s,t} := \int_s^t Y^n_{s,u-} \otimes \mathrm{d}Y_u,$$

and

$$\mathbb{XY}^n_{s,t} := \int_s^t X^n_{s,u-} \otimes \mathrm{d}Y_u, \qquad \mathbb{YX}^n_{s,t} := \int_s^t Y^n_{s,u-} \otimes \mathrm{d}X_u,$$

for $(s,t) \in \Delta_{[0,T]}$. We seek a control function c such that

$$\sup_{n \in \mathbb{N}} \sup_{\substack{(s,t) \in \Delta_{[0,T]} \\ s,t \in \mathcal{Q}^n_Z([0,T])}} \frac{|\mathbb{Z}^n_{s,t}|^{\frac{p}{2}}}{c(s,t)} \leq 1.$$

Towards this aim, we first construct a control function $c_{\mathbb{X}}$ such that the above bound holds for $\mathbb{X}^n$. Since $\|X^n_{\cdot-} - X_{\cdot-}\|_{\infty,[0,T]} \leq 2^{1-n}$, we still have the bound in (4.2). We also choose $q_0 \in (2,3)$ and $\varepsilon \in (0,1)$ as in (4.3), and let w_{X,q_0} and $w_{Y,q}$ be control functions dominating the q_0-variation and q-variation for X and Y respectively. From the definition of the stopping times $\tau^m_k \in \mathcal{P}^n_Z$ and the fact that $q < 2 < q_0$, it is easy to check that, for all $s < t$ with $s,t \in \mathcal{P}^n_Z$, the number N of partition points in $\mathcal{P}^n_Z$ between s and t can be bounded by

$$N \leq \sum_{m=1}^{n} \Big(2^{m q_0} w_{X,q_0}(s,t) + 2^{mq} w_{Y,q}(s,t) \Big) \lesssim 2^{n q_0} w_{q_0,q}(s,t),$$

where $w_{q_0,q}(s,t) := w_{X,q_0}(s,t) + w_{Y,q}(s,t)$. By the same argument as in Step 2 of the proof of Proposition 4.1, we deduce that, for all $n \in N$ and all $s < t$ with $s,t \in \mathcal{Q}^n_Z$,

$$\left| \int_s^t X^n_{u-} \otimes \mathrm{d}X_u - X_s \otimes X_{s,t} \right| \lesssim 2^{n(q_0-2)} w_{q_0,q}(s,t),$$

which, as in Step 3 of the proof of Proposition 4.1, allows us to conclude the existence of a control function $c_{\mathbb{X}}$ such that

$$\sup_{n \in \mathbb{N}} \sup_{\substack{(s,t) \in \Delta_{[0,T]} \\ s,t \in \mathcal{Q}^n_Z([0,T])}} \frac{|\mathbb{X}^n_{s,t}|^{\frac{p}{2}}}{c_{\mathbb{X}}(s,t)} \leq 1.$$

Next we use the local estimates of Young integration to show that there exists a control $c_{\mathbb{XY}}$ such that the above bound holds for $\mathbb{XY}^n$ and $c_{\mathbb{XY}}$. Indeed, by [FZ18, Proposition 2.4], for all $s < t$ in $[0,T]$, we have

$$\begin{aligned} \left| \int_s^t X^n_{s,u-} \otimes \mathrm{d}Y_u \right|^{p/2} &\leq C_{p,q} \|Y\|^{p/2}_{q,[s,t]|} \|X^n\|^{p/2}_{p,[s,t]} \\ &\leq C_{p,q} \|Y\|^{p/2}_{q,[s,t]} \|X\|^{p/2}_{p,[s,t]} \\ &\leq C_{p,q} w_{Y,q}(s,t)^{p/2q} w_{X,p}(s,t)^{1/2}, \end{aligned}$$

for some constant $C_{p,q}$ depending only on p and q. Since $p \in (2,3)$ and $q \in [1,2)$, we have that $p/2q > 1/2$, and thus $p/2q + 1/2 > 1$, which implies that the map $(s,t) \mapsto w_{Y,q}(s,t)^{p/2q} w_{X,p}(s,t)^{1/2}$ is superadditive, and hence is itself a control function. Thus, the control function $c_{\mathbb{XY}} := C_{p,q} w_{Y,q}(s,t)^{p/2q} w_{X,p}(s,t)^{1/2}$ gives the desired bound. Similarly, we can find control functions $c_{\mathbb{YX}}$ and $c_{\mathbb{Y}}$ for $\mathbb{YX}^n$ and $\mathbb{Y}^n$, respectively. Hence, our claim follows by noting that

$$|\mathbb{Z}^n| \leq |\mathbb{X}^n| + |\mathbb{Y}^n| + |\mathbb{XY}^n| + |\mathbb{YX}^n|.$$

All together, we deduce that the sample paths of Z almost surely satisfy Property (RIE) along $(\mathcal{Q}^n_Z([0,T]))_{n\in\mathbb{N}}$. □

4.3. **Typical price paths.** The notion of "typical price paths" was introduced by Vovk, who introduced a model-free hedging-based approach to mathematical finance allowing to investigate the sample path properties of such "typical price paths" based on arbitrage considerations; see for instance [Vov08, Vov12, PP16]. Let us briefly recall the basic setting and definitions of Vovk's approach.

Let $\Omega_+ := D([0,\infty);\mathbb{R}^d_+)$ be the space of all non-negative càdlàg functions $\omega\colon [0,\infty) \to \mathbb{R}^d_+$. For each $t \in [0,\infty)$, $\mathcal{F}^\circ_t$ is defined to be the smallest σ-algebra on Ω_+ that makes all functions $\omega \mapsto \omega(s)$, $s \in [0,t]$, measurable and $\mathcal{F}_t$ is defined to be the universal completion of $\mathcal{F}^\circ_t$. Stopping times $\tau\colon \Omega_+ \to [0,\infty) \cup \{\infty\}$ with respect to the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual. The coordinate process on Ω_+ is denoted by S, i.e. $S_t(\omega) := \omega(t)$ for $t \in [0,\infty)$.

A process $H\colon \Omega_+ \times [0,\infty) \to \mathbb{R}^d$ is a *simple (trading) strategy* if there exists a sequence of stopping times $0 = \sigma_0 < \sigma_1 < \sigma_2 < \dots$ such that for every $\omega \in \Omega_+$ there exists an $N(\omega) \in \mathbb{N}$ such that $\sigma_n(\omega) = \sigma_{n+1}(\omega)$ for all $n \geq N(\omega)$, and a sequence of $\mathcal{F}_{\sigma_n}$-measurable bounded functions $h_n\colon \Omega_+ \to \mathbb{R}^d$, such that $H_t(\omega) = \sum_{n=0}^\infty h_n(\omega)\mathbf{1}_{(\sigma_n(\omega),\sigma_{n+1}(\omega)]}(t)$ for $t \in [0,\infty)$. For a simple strategy H, the corresponding integral process

$$(H \cdot S)_t(\omega) := \sum_{n=0}^{\infty} h_n(\omega) S_{\sigma_n\wedge t, \sigma_{n+1}\wedge t}(\omega)$$

is well-defined for all $(t,\omega) \in [0,\infty) \times \Omega_+$. For $\lambda > 0$, we write $\mathcal{H}_\lambda$ for the set of all simple strategies H such that $(H \cdot S)_t(\omega) \geq -\lambda$ for all $(t,\omega) \in [0,\infty) \times \Omega_+$.

Definition 4.4. Vovk's outer measure $\overline{P}$ *of a set* $A \subseteq \Omega_+$ *is defined as the minimal superhedging price for* $\mathbf{1}_A$, *that is*

$$\overline{P}(A) := \inf\Big\{\lambda > 0 \,:\, \exists (H^n)_{n\in\mathbb{N}} \subset \mathcal{H}_\lambda \text{ s.t. } \forall \omega \in \Omega_+,\ \liminf_{n\to\infty}(\lambda + (H^n \cdot S)_T(\omega)) \geq \mathbf{1}_A(\omega)\Big\}.$$

A given set $A \subseteq \Omega_+$ *is called a* null set *if it has outer measure zero. A property (P) holds for* typical price paths *if the set* A *where (P) is violated is a null set.*

Remark 4.5. *Loosely speaking, the outer measure* $\overline{P}$ *corresponds to the (model-free) notion of "no unbounded profit with bounded risk", see* [PP16, Section 2.2] *for a more detailed discussion in this direction. Furthermore, the outer measure* $\overline{P}$ *dominates all local martingale measures on the space* Ω_+, *see* [ŁPP18, Lemma 2.3 and Proposition 2.5]. *As a consequence, all results proven for typical price paths hold simultaneously under all martingale measures (or in other words quasi-surely with respect to all martingale measures).*

Let us recall that typical price paths are of finite p-variation for every $p > 2$ (see [Vov11, Theorem 1]) and Vovk's model-free framework allows for setting up a model-free Itô integration, see e.g. [ŁPP18].

Lemma 4.6. *Typical price paths are price paths in the sense of Definition 3.1, with any* $p \in (2,3)$.

The proof of Lemma 4.6 works verbatim as that of Proposition 4.1 keeping in mind [LP18, Proposition 3.10] and [ŁPP18, Corollary 4.9], and is therefore omitted for brevity.

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