DS-GA 3001.03: Homework Problem Set 4

Optimization and Computational Linear Algebra for Data Science

(Fall 2016)

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Due on October 11, 2016

This homework problem set is due on October 11, before class at the homework dropbox in the CDS reception.

If you have any questions about the homework post them on the piazza page for the course, contact Vlad Kobzar at `vkobzar@cims.nyu.edu` **or myself at** `bandeira@cims.nyu.edu`**, or stop by our office hours.**

Try not to look up the answers, you'll learn much more if you try to think about the problems without looking up the solutions. If you need hints, feel free to email me or Vlad.

You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list, on your submission, the students you work with for the homework (this will not affect your grade).

If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n to be 2, e.g.), state explicitly that you have done so. Solutions where extra conditions were assume, or where only special cases where treated, will also be graded (probably scored as a partial answer).

Problems with a $(*)$ are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 (four) extra credit questions successfully answered you get a homework "bye": your lowest homework score (or one you did not hand in) gets replaced by a perfect score.

In class we saw:

- (Spectral Decomposition) For every symmetric matrix $A \in \mathbb{R}^{n\times n}$, there exists $V \in \mathbb{R}^{n\times n}$ satisfying $V^T V = I$, and $\Lambda \in \mathbb{R}^{n\times n}$ a diagonal matrix, with diagonal elements $\lambda_1, \ldots, \lambda_n$ (called the eigenvalues of A) such that
$$A = V\Lambda V^T.$$

- (Singular Value Decomposition) For every matrix $B \in \mathbb{R}^{m\times n}$, there exist matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ satisfying $U^T U = I$ and $V^T V = I$, and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ with non-negative diagonal entries $\sigma_1, \ldots, \sigma_{\min\{n,m\}} \geq 0$ called the singular values of B (for a rectangular matrix, diagonal means that for all $i \neq j$, $\Sigma_{ij} = 0$), such that
$$B = U\Sigma V^T.$$

You are welcome (and encouraged) to use these facts in the homework.

Problem 1.1 *Let A be a symmetric $n\times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct). Show that* $\mathrm{Rank}(A)$ *is the number of non-zero eigenvalues of A.*

Problem 1.2 *The trace of a matrix $A \in \mathbb{R}^{n\times n}$,* $\mathrm{Tr}(A)$ *is defined as*

$$\mathrm{Tr}(A) = \sum_{k=1}^{n} A_{kk}.$$

(a) Show that for any matrices $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{n\times m}$ we have $\mathrm{Tr}(BC) = \mathrm{Tr}(CB)$.

(b) Use that to show that for any symmetric matrix $A \in \mathbb{R}^{n\times n}$, its trace is equal to the sum of its eigenvalues.

Problem 1.3 *For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we say that A is Positive Semidefinite ($A \succeq 0$) if all of its eigenvalues are non-negative ($\lambda_k \geq 0$, for $k = 1, \ldots, n$).*

(a) Show that for every symmetric matrix A satisfying $A \succeq 0$, there exists a matrix U such that
$$A = UU^T.$$

(b) Is the converse true? (meaning, if A is of the form $A = UU^T$ for some matrix U, does it need to satisfy $A \succeq 0$?

Problem 1.4 *Let $B \in \mathbb{R}^{m \times n}$ (with, say $m \leq n$). Show that the singular values of B are the square root of the eigenvalues of BB^T. Can you relate the eigenvectors of BB^T with anything in the Singular Value Decomposition of B? What about the eigenvalues and eigenvectors of $B^T B$?*

$(*)$ **Problem 1.5 (For Extra Credit)** *Let $B \in \mathbb{R}^{m \times n}$. Let*

$$\|B\|_2 = \max_{x \neq 0} \frac{\|Bx\|_2}{\|x\|_2},$$

and

$$\gamma(B) = \min_{x \neq 0} \frac{\|Bx\|_2}{\|x\|_2}.$$

Can you relate $\|B\|_2$ and $\gamma(B)$ with singular values of B?