DS-GA 3001.03: Homework Problem Set 6

Optimization and Computational Linear Algebra for Data Science

(Fall 2016)

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This homework problem set is due on November 8, before class at the homework dropbox in the CDS reception.

If you have any questions about the homework post them on the piazza page for the course, contact Vlad Kobzar at vkobzar@cims.nyu.edu **or myself at** bandeira@cims.nyu.edu**, or stop by our office hours.**

Try not to look up the answers, you'll learn much more if you try to think about the problems without looking up the solutions. If you need hints, feel free to email me or Vlad.

You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list, on your submission, the students you work with for the homework (this will not affect your grade).

If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n to be 2, e.g.), state explicitly that you have done so. Solutions where extra conditions were assume, or where only special cases where treated, will also be graded (probably scored as a partial answer).

Problems with a $(*)$ are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 (four) extra credit questions successfully answered you get a homework "bye": your lowest homework score (or one you did not hand in) gets replaced by a perfect score.

Problem 6.1 *Given a graph* $G = (V, E,)$ *define the graph Laplacian matrix as*

$$L_G = \sum_{i<j \text{ such that } (i,j)\in E} (e_i - e_j)(e_i - e_j)^T.$$

Show

- *(a)* L_G *is symmetric and all eigenvalues of* L_G *are non-negative.*
- *(b)* L_G *has a non-zero vector in its kernel.*
- *(c) The second smallest eigenvalue of* L_G *is non-zero if and only* G *is connected.*

Problem 6.2 *Let* $x, y \in \mathbb{R}$ *if we constraint* $x + y = 10$ *what is the maximum possible value for* xy*?*

Problem 6.3 *Consider*

$$p(x) = x^4 - 7x^2 - 4x + 20.$$

- *(a) What is the minimum of* $p(x)$*?*
- *(b) What is the maximum of* $p(x)$*?*
- *(c) Is there any point* x *for which every point* u *very close to* x *has a smaller value (*$\exists\epsilon > 0$ *s.t.* $p(x) > p(u)$ *for all* u *satisfying* $|u-x| < \epsilon$*)? If so find all such points. Are they all maximizers?*
- *(d) Is there any point* x *for which every point* u *very close to* x *has a larger value (*$\exists\epsilon > 0$ *s.t.* $p(x) < p(u)$ *for all* u *satisfying* $|u - x| < \epsilon$*)? If so find all such points. Are they all minimizers?*

Problem 6.4 *Consider*

$$q(x, y) = 1 + x^2 - 2y - 2xy + y^2$$

What is the minimum of $q(x, y)$*?* $\left(\min\limits_{x,y\in\mathbb{R}} q(x, y)\right)$

$(*)$ **Problem 6.5 (For Extra Credit)** *The ring graph on* n *nodes is a graph where node* $1 < k < n$ *is connected to node* $k - 1$ *and* $k + 1$ *and node* 1 *is connected to node* n*. Derive the two-dimensional diffusion map embedding for the ring graph (if the eigenvectors are complex valued, try creating real valued ones using multiplicity of the eigenvalues). Is it a reasonable embedding of this graph in two dimensions?*