

DS-GA 3001.03: Homework Problem Set 7

Optimization and Computational Linear Algebra for Data Science

(Fall 2016)

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Due on November 15, 2016

This homework problem set is due on November 15, before class at the homework dropbox in the CDS reception.

If you have any questions about the homework post them on the piazza page for the course, contact Vlad Kobzar at vkobzar@cims.nyu.edu or myself at bandeira@cims.nyu.edu, or stop by our office hours.

Try not to look up the answers, you'll learn much more if you try to think about the problems without looking up the solutions. If you need hints, feel free to email me or Vlad.

You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list, on your submission, the students you work with for the homework (this will not affect your grade).

If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n to be 2, e.g.), state explicitly that you have done so. Solutions where extra conditions were assume, or where only special cases were treated, will also be graded (probably scored as a partial answer).

Problems with a (*) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 (four) extra credit questions successfully answered you get a homework "bye": your lowest homework score (or one you did not hand in) gets replaced by a perfect score.

Problem 6.1 Which of the following sets are convex? (Justify your answer)

- (a) $\{x \in \mathbb{R}^n | x_k \geq 0 \text{ for all } k \text{ and } \sum_{k=1}^n x_k = 1\}$
- (b) $\{x \in \mathbb{R}^n | 1 \leq \|x\|_2 \leq 2\}$
- (c) $\{x \in \mathbb{R} | x^3 \geq 1\}$
- (d) $\{x \in \mathbb{R} | x^2 \geq 1\}$

Problem 6.2 • (a) Show that if a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex then its sublevel sets

$$C_\gamma = \{x \in \mathbb{R}^n | f(x) \leq \gamma\}$$

are convex.

- (b) Can you find an example of a function that has convex sublevel sets but is not convex?

Problem 6.3 Show that the ℓ_2 norm is convex. In other words show that the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $f(x) = \|x\|_2$ is convex.

Problem 6.4 Which of the following functions ($f : \mathbb{R} \rightarrow \mathbb{R}$) are convex? (Justify your answer)

- (a) $f(x) = 1$
- (b) $f(x) = x$
- (c) $f(x) = x^2$
- (d) $f(x) = x^3$
- (e) $f(x) = x^4$

(*) **Problem 6.5 (For Extra Credit)** Given a set of points $x_1, \dots, x_n \in \mathbb{R}^p$ and a partition of them in k clusters S_1, \dots, S_k recall the k -means objective

$$\min_{S_1, \dots, S_k} \min_{\mu_1, \dots, \mu_k} \sum_{l=1}^k \sum_{i \in S_l} \|x_i - \mu_l\|^2.$$

Show that this is equivalent to

$$\min_{S_1, \dots, S_k} \sum_{l=1}^k \frac{1}{|S_l|} \sum_{i, j \in S_l} \|x_i - x_j\|^2.$$