DS-GA 1014: Homework Problem Set 10

Optimization and Computational Linear Algebra for Data Science (Fall 2018)

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This homework problem set is due on December 11 on NYU Classes.

If you have questions about the homework feel free to contact Brett Bernstein (brett.bernstein@nyu.edu**) or myself, or stop by our office hours.**

Unless otherwise stated all answers must be mathematically justified, even questions that ask for a computation.

Try not to look up the answers. You'll learn much more if you try to think about the problems without looking up the solutions. If you need hints, feel free to ask on Piazza.

You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list, on your submission, the students you work with for the homework (this will not affect your grade).

Late submissions will be graded with a penalty of 10% per day late. Weekend days do not count — from Friday to Monday counts as only 1 day.

If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n to be 2, e.g.), state explicitly that you have done so. Solutions where extra conditions were assumed, or where only special cases where treated, will also be graded (probably scored as a partial answer).

Problems with a $(*)$ are extra credit — they will not (directly) contribute to your score for this homework. However, for every four extra credit questions successfully answered you get a homework "bye": your lowest homework score (or one you did not hand in) gets replaced by a perfect score.

Problem 10.1 *Which of the following sets are convex for all $n \geq 1$? In each case either prove the set is convex or show the set is non-convex for some $n \geq 1$.*

1. $S = \{x \in \mathbb{R}^n : x_k \geq 0$ *for all* k *and* $\sum_{k=1}^n x_k = 1\}$.
2. $S = \{x \in \mathbb{R}^n : 1 \leq \|x\| \leq 2\}$.
3. $S = \{x \in \mathbb{R}^n : b^T x \geq a\}$ *where* $a \in \mathbb{R}$ *and* $b \in \mathbb{R}^n$.
4. $S = \{x \in \mathbb{R}^n : x_1 \leq x_2 \leq \cdots \leq x_n\}$ *where* x_i *denotes the* i*th coordinate of* x.

Problem 10.2 *Show that if a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex then its sublevel sets*

$$C_\gamma = \{x \in \mathbb{R}^n : f(x) \leq \gamma\}$$

are convex for all $\gamma \in \mathbb{R}$.

Problem 10.3 *Which of the following functions $f : \mathbb{R}^n \to \mathbb{R}$ are convex? In each case justify your answer.*

1. $f(x) = x^3$ *where* $n = 1$.
2. $f(x) = -\log x$ *where* $n = 1$ *and* $x > 0$.
3. $f(x) = x_1 x_2$ *where* $n = 2$ *and* x_i *denotes the* i*th coordinate of* x.
4. $f(x) = \|x\|$ *for* $n \geq 1$.
5. $f(x) = \sum_{i=1}^k a_i g_i(x)$ *where* $n \geq 1$, $g_i : \mathbb{R}^n \to \mathbb{R}$ *is convex and* $a_i \geq 0$ *for* $i = 1, \ldots, k$.

Problem 10.4 *We say that $x \in \mathbb{R}^n$ is a local minimizer of $f : \mathbb{R}^n \to \mathbb{R}$ if there is a $\delta > 0$ such that any $y \in \mathbb{R}^n$ with $\|x - y\| < \delta$ satisfies $f(y) \geq f(x)$. Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is convex, and x is a local minimizer of f, then x is also a global minimizer of f.*
[Hint: Draw it for $f : \mathbb{R} \to \mathbb{R}$.]

Problem 10.5 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable and let $x \in \mathbb{R}$. Show that for all $\alpha > 0$ that*

$$f\Big(x - \alpha \nabla f(x)\Big) \geq f(x) - \alpha \|\nabla f(x)\|^2.$$

[Hint: Use the fact that the linear approximation to a convex function lies below the function.]

(∗) **Problem 10.6 (For Extra Credit)** *Consider the following optimization problem:*

$$\begin{array}{ll} \textit{minimize}_{x \in \mathbb{R}^n} & c^T x \\ \textit{subject to} & Ax = b \\ & x_i \geq 0 \textit{ for } i = 1 \ldots, n, \end{array}$$

where $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $m < n$. Prove that if there is a minimizer $x^* \in \mathbb{R}^n$, then there exists another (possibly distinct) minimizer $\hat{x} \in \mathbb{R}^n$ with at most m non-zero entries.