DS-GA 1014: Homework Problem Set 3

Optimization and Computational Linear Algebra for Data Science

(Fall 2018)

Afonso S. Bandeira
`bandeira@cims.nyu.edu`
`http://www.cims.nyu.edu/~bandeira`

Brett D. Bernstein
`brettb@`
`~brettb`

Due on Friday September 21, 2018

This homework problem set is due on September 21 on NYU Classes.

If you have questions about the homework feel free to contact Brett Bernstein (`brett.bernstein@nyu.edu`**) or myself, or stop by our office hours.**

Unless otherwise stated all answer must be mathematically justified, even questions that ask for a computation.

Try not to look up the answers, you'll learn much more if you try to think about the problems without looking up the solutions. If you need hints, feel free to email me or Vlad.

You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list, on your submission, the students you work with for the homework (this will not affect your grade).

Late submissions will be graded with a penalty of 10% **per day late. Weekend days do not count, from Friday to Monday counts only 1 day.**

If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n **to be** 2**, e.g.), state explicitly that you have done so. Solutions where extra conditions were assume, or where only special cases where treated, will also be graded (probably scored as a partial answer).**

Problems with a $(*)$ **are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 (four) extra credit questions successfully answered you get a homework "bye": your lowest homework score (or one you did not hand in) gets replaced by a perfect score.**

Problem 3.1 *Let $B \in \mathbb{R}^{4\times 3}$ be an arbitrary matrix with entries*

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \\ B_{4,1} & B_{4,2} & B_{4,3} \end{bmatrix}.$$

Find matrices A, C such that

$$ABC = \begin{bmatrix} B_{1,2} & B_{1,1} & B_{1,3} & B_{1,2} \\ B_{2,2} + B_{3,2} & B_{2,1} + B_{3,1} & B_{2,3} + B_{3,3} & B_{2,2} + B_{3,2} \\ B_{4,2} & B_{4,1} & B_{4,3} & B_{4,2} \end{bmatrix}$$

holds for any B as defined above.

Problem 3.2 *Let $A \in \mathbb{R}^{3\times 4}$ be defined by*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}.$$

1. *Compute a basis for* $\ker(A)$.
2. *What is* $\mathrm{Rank}(A)$*?*
3. *Compute the set*
$$\left\{ x \in \mathbb{R}^4 : Ax = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}.$$

Problem 3.3 *Given a matrix $L \in \mathbb{R}^{n\times m}$ (meaning that L is a linear transformation $L : \mathbb{R}^m \to \mathbb{R}^n$) show that*

$$\ker(L) = \ker(L^T L).$$

Problem 3.4 *Use the result from the proof above (and the fundamental theorem of linear algebra) to show that*

$$\mathrm{Rank}(L) = \mathrm{Rank}(L^T)$$

Hint: Start by showing that for any two matrices A, B we have $\mathrm{Rank}(A) \geq \mathrm{Rank}(AB)$.

($*$) **Problem 3.5 (For Extra Credit)** *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{k\times n}$. Prove that there is a matrix C with $A = CB$ if and only if* $\ker(B)$ *is a subspace of* $\ker(A)$.