DS-GA 1014: Homework Problem Set 4

Optimization and Computational Linear Algebra for Data Science (Fall 2018)

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Due on Friday September 28, 2018

This homework problem set is due on September 28 on NYU Classes.

If you have questions about the homework feel free to contact Brett Bernstein (brett.bernstein@nyu.edu**) or myself, or stop by our office hours.**

Unless otherwise stated all answer must be mathematically justified, even questions that ask for a computation.

Try not to look up the answers, you'll learn much more if you try to think about the problems without looking up the solutions. If you need hints, feel free to email me or Vlad.

You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list, on your submission, the students you work with for the homework (this will not affect your grade).

Late submissions will be graded with a penalty of 10% **per day late. Weekend days do not count, from Friday to Monday counts only 1 day.**

If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n **to be** 2**, e.g.), state explicitly that you have done so. Solutions where extra conditions were assume, or where only special cases where treated, will also be graded (probably scored as a partial answer).**

Problems with a $(*)$ **are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 (four) extra credit questions successfully answered you get a homework "bye": your lowest homework score (or one you did not hand in) gets replaced by a perfect score.**

Problem 4.1 *Let $v_1, \ldots, v_m \in \mathbb{R}^n$ with $v_i \neq 0$ for $i = 1, \ldots, m$ and $\langle v_i, v_j \rangle = 0$ for all $i \neq j$. Prove that $v_1, \ldots, v_m$ are linearly independent.*

Problem 4.2 *Suppose we define an inner product $\langle \cdot, \cdot \rangle_A$ by $\langle x, y \rangle_A = x^T A y$ for some $A \in \mathbb{R}^{n \times n}$. Prove that in order for $\langle x, y \rangle_A$ to be an inner product, the following are necessary conditions:*

1. *A is symmetric.*
2. *$A_{ii} > 0$ for $i = 1, \ldots, n$.*
3. *A is invertible.*

Problem 4.3 *In each of the following questions, it is intended that you solve the problem using the numpy/scipy libraries in Python, but you can use any programming language. Please include just the answer in your writeup, and include the code files in your submission (i.e., do not include the code in your writeup). A good idea is to test your code on some small simple instances you create by hand to make sure it is correct. Note that calculations on the computer do not use exact arithmetic, so you should allow for some margin of error in your code.*

1. *The file `HW4matrix1.txt` contains a 7×5 matrix M, where each row is on a separate line, and the values on each line are space-delimited. Which of the vectors $e_1, e_2, \ldots, e_7$ are in the column space of M? Here e_i denotes the ith standard basis vector containing 1 in the ith position and zeros elsewhere.*

2. *The file `HW4matrix2.txt` contains a 3×6 matrix A, where each row is on a separate line, and the values on each line are space-delimited. Find a vector $x \in \mathbb{R}^6$ of the form*

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad \text{such that } Ax = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

   Give the values of x_1, x_2, and x_3 (only include 2 digits after the decimal place).

The files are available at `https://piazza.com/nyu/fall2018/dsga1014/resources`.

($*$) **Problem 4.4 (For Extra Credit)** *Let $n > 1$. Compute*

$$\min\{|x^T z| : x, y, z \in \mathbb{R}^n,\ \|x\| = \|y\| = \|z\| = 1,\ x^T y \geq 0.9,\ y^T z \geq 0.8\}$$

and prove that your calculation is correct.

Note that the answer may generally depend on n, but it also may be that it coincides for different values of n (it is often useful to start with the cases $n = 2$ or $n = 3$, and indeed partial credit will be given for those cases).