DS-GA 1014: Homework Problem Set 9

Optimization and Computational Linear Algebra for Data Science (Fall 2018)

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Due on Friday November 30, 2018

This homework problem set is due on November 30 on NYU Classes.

If you have questions about the homework feel free to contact Brett Bernstein (brett.bernstein@nyu.edu**) or myself, or stop by our office hours.**

Unless otherwise stated all answers must be mathematically justified, even questions that ask for a computation.

Try not to look up the answers. You'll learn much more if you try to think about the problems without looking up the solutions. If you need hints, feel free to ask on Piazza.

You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list, on your submission, the students you work with for the homework (this will not affect your grade).

Late submissions will be graded with a penalty of 10% per day late. Weekend days do not count — from Friday to Monday counts as only 1 day.

If you need to impose extra conditions on a problem to make it easier (or consider specific cases of the question, like taking n to be 2, e.g.), state explicitly that you have done so. Solutions where extra conditions were assumed, or where only special cases where treated, will also be graded (probably scored as a partial answer).

Problems with a $(*)$ are extra credit — they will not (directly) contribute to your score for this homework. However, for every four extra credit questions successfully answered you get a homework "bye": your lowest homework score (or one you did not hand in) gets replaced by a perfect score.

For problems 1 and 2, assume we are working with a graph that has vertices labeled $1, \ldots, n$. Some pairs of the vertices are connected by (undirected) edges. The corresponding adjacency matrix $A \in \mathbb{R}^{n \times n}$ is defined so that $A_{ij} = 1$ if there is an edge between vertices i and j, and $A_{ij} = 0$ otherwise. Thus A is a symmetric matrix with zero diagonal. The degree $\deg(i)$ of vertex i is defined to be the number of 1's in the ith row of A (i.e., the number of edges that have vertex i as an endpoint). We assume that $\deg(i) > 0$ for $i = 1, \ldots, n$. Define $D \in \mathbb{R}^{n \times n}$ to be the diagonal matrix with $D_{ii} = \deg(i)$ for $i = 1, \ldots, n$. Finally, define the transition matrix $M \in \mathbb{R}^{n \times n}$ to be $M = D^{-1}A$.

Problem 9.1 *Suppose we can partition the nodes into two non-empty sets $S, T \subseteq \{1, \ldots, n\}$ such that there are no edges between S and T. More formally, suppose that the following hold:*

1. $S \cup T = \{1, \ldots, n\}$, $S \neq \emptyset$, $T \neq \emptyset$, $S \cap T = \emptyset$.
2. *If $i \in S$ and $j \in T$ then $A_{ij} = A_{ji} = 0$ where A is the adjacency matrix.*

Prove that the transition matrix M has at least two linearly independent eigenvectors with eigenvalue 1.

Problem 9.2 *Suppose we can partition the nodes into two non-empty sets $S, T \subseteq \{1, \ldots, n\}$ such that there are only edges between S and T. More formally, suppose that the following hold:*

1. $S \cup T = \{1, \ldots, n\}$, $S \neq \emptyset$, $T \neq \emptyset$, $S \cap T = \emptyset$.
2. *If $i, j \in S$ then $A_{ij} = A_{ji} = 0$ where A is the adjacency matrix.*
3. *If $i, j \in T$ then $A_{ij} = A_{ji} = 0$ where A is the adjacency matrix.*

Prove that the transition matrix M has -1 as an eigenvalue.

Problem 9.3 *Let $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$ with A symmetric. A quadratic function $q : \mathbb{R}^n \to \mathbb{R}$ is defined by*

$$q(x) = x^T A x + b^T x + c.$$

1. *Compute the gradient $\nabla q(x)$ and show your work.*
2. *Compute the Hessian $\nabla^2 q(x)$ and show your work.*

Problem 9.4 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Let $x \in \mathbb{R}^n$ and suppose $\nabla f(x) \neq 0$. Show there exists $s > 0$ such that*

$$f\left(x - t\nabla f(x)\right) < f(x)$$

for all $0 < t < s$.

($*$) **Problem 9.5 (For Extra Credit)** *Let $A, B \subseteq \mathbb{R}^n$ be convex sets, and let C denote the union of all line segments connecting A to B:*

$$C = \{ta + (1-t)b : t \in [0,1],\ a \in A,\ b \in B\}.$$

Prove that C is convex.