10. Oct. 2011

Heat equation

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

Unknown fn: $u(x,t)$ = temperature at point x and time t

$x \in [0, L]$, $t \geq 0$

c^2 = "thermal diffusity"

rod with perfect lateral insulation

BC boundary condition:
keep temperature constant $= 0$ at two ends

IC initial condition:
know temperature at each point at time $t = 0$

Problem

$$\begin{cases} \text{PDE} & \text{1-dim'l heat equation} \\ \text{BC} & u(0,t) = u(L,t) = 0 \quad \text{for all } t \\ \text{IC} & u(x,0) = f(x) \end{cases}$$

only this IC because PDE only has $\frac{\partial}{\partial t}$

(need $f(0) = f(L) = 0$)

$$\left.\begin{array}{l}\text{PDE} \\ \text{BC}\end{array}\right\} \text{ are linear homogeneous}$$

$\Longrightarrow$ attempt <u>separation of variables</u>

Step 1 | Find solutions of PDE of form
$u(x,t) = X(x)T(t)$

$$\frac{\partial u}{\partial t} = X(x)\,T'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t)$$

heat eqn : $X(x)T'(t) = c^2 X''(x)T(t)$

$$\underbrace{\frac{T'(t)}{c^2 T(t)}}_{\text{only depends on } t} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{only depends on } x} = \underbrace{k}_{\text{constant}}$$

Get two ODEs :

$$\boxed{X'' - kX = 0}$$

$$\boxed{T' - kc^2 T = 0}$$

Step 2 Impose BC

$$\begin{cases} u(0,t) = X(0)\,T(t) = 0 \\ u(L,t) = X(L)\,T(t) = 0 \end{cases} \quad \text{for all } t$$

$$\Longleftrightarrow \quad \underbrace{\begin{matrix} T(t)=0 \\ \text{for all } t \end{matrix}}_{\text{trivial solution}} \quad \underline{\underline{\text{or}}} \quad \underbrace{\begin{cases} X(0)=0 \\ X(L)=0 \end{cases}}_{\text{interesting}}$$

Solve
$$\boxed{\begin{matrix} X'' - kX = 0 \\ X(0) = 0 \\ X(L) = 0 \end{matrix}}$$

Cases $k \geq 0$: get only trivial solution (Exercise !)

Case $k = -p^2 < 0$:

$$\begin{cases} X(x) = A\cos px + B\sin px \\ X(0) = 0 \\ X(L) = 0 \end{cases} \longrightarrow \begin{array}{|l} A=0 \\ B\sin pL = 0 \end{array}$$

interesting when $pL = m\pi$

$$p = \frac{m\pi}{L}$$

Get solutions $\quad X_m = \sin\frac{m\pi x}{L}, \quad m = 1, 2, \dots$

Now solve $\boxed{T' + \lambda_m^2 T = 0}$

$$\lambda_m^2 = p^2 c^2 \qquad p = \frac{m\pi}{L}$$

$$\boxed{\lambda_m = \frac{m\pi c}{L}}$$ eigenvalues

General solution $T_m(t) = B_m e^{-\lambda_m^2 t}$, $m = 1, 2, \ldots$

arbitrary constant

Summary Got solutions of heat equation satisfying BC of the form

$$\boxed{\begin{aligned} u_m(x,t) &= X_m(x)\, T_m(t) \\ &= B_m e^{-\lambda_m^2 t} \sin \frac{m\pi x}{L} \end{aligned}}$$

$m = 1, 2, \ldots$

eigenfunctions

Step 3 Impose IC

Consider series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} B_n \underbrace{e^{-\lambda_n^2 t}}_{\downarrow \text{ as } t \to +\infty \atop 0} \sin \frac{n\pi x}{L}$$

$$u(x,0) = f(x)$$

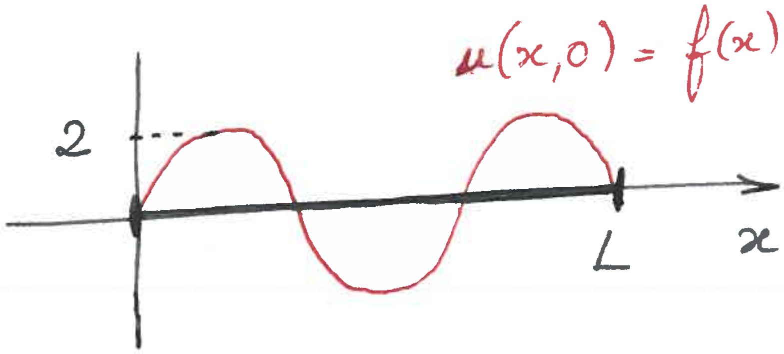
$$\Longleftrightarrow \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x)$$

Choose B_n's as coefficients of Fourier sine series of $f(x)$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

Example 1

$$f(x) = \underline{\underline{2}} \sin \frac{3\pi x}{L}$$



$$B_3 = \underline{\underline{2}}$$

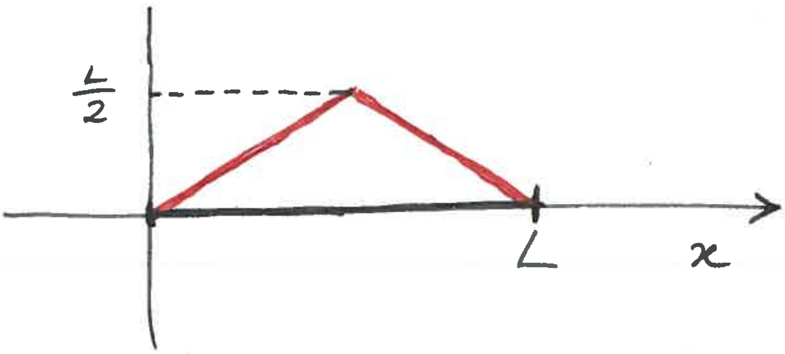
$$B_n = 0 \, , \quad n \neq 3$$

Solution is

$$u(x,t) = \underline{\underline{2}} \, e^{-\left(\frac{3\pi c}{L}\right)^2 t} \sin \frac{3\pi x}{L}$$

Example 2

$$f(x) = \begin{cases} x & 0 \le x \le \frac{L}{2} \\ L-x & \frac{L}{2} \le x \le L \end{cases}$$



$$B_m = \frac{2}{L}\left(\int_0^{\frac{L}{2}} x \sin \frac{m\pi x}{L}\, dx + \int_{\frac{L}{2}}^{L} (L-x) \sin \frac{m\pi x}{L}\, dx \right)$$

$$= \dots = \begin{cases} 0 & m \text{ even} \\ \frac{4L}{m^2\pi^2} & m = 4k+1 \\ -\frac{4L}{m^2\pi^2} & m = 4k+3 \end{cases}$$

integration by parts

Solution is

$$u(x,t) = \sum_{m \text{ odd}} (-1)^{\frac{m-1}{2}} \frac{4L}{m^2\pi^2} e^{-\lambda_m^2 t} \sin \frac{m\pi x}{L}$$

Other types of boundary conditions

- nonzero boundary conditions

$$u(0,t) = U_0 \qquad u(L,t) = U_L$$

(need $f(0) = U_0$, $f(L) = U_L$ in IC)

- adiabatic conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$$

(ends are perfectly insulated)

- no boundary conditions

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infinite bar

$\Big\}$ next time

Nonzero boundary conditions

First : find solution $u_p(x)$ of

$$\begin{cases} \dfrac{d^2 u_p}{dx^2} = 0 \quad \rightsquigarrow u_p(x) = ax + b \\ u_p(0) = U_0 \\ u_p(L) = U_L \end{cases}$$

Second: find solution $u_H(x,t)$ of

$$\begin{cases} \dfrac{\partial u_H}{\partial t} = c^2 \dfrac{\partial^2 u_H}{\partial x^2} \\ u_H(0,t) = 0 \, , \quad u_H(L,t) = 0 \\ u_H(x,0) = f(x) - u_p(x) \end{cases}$$

Then $\boxed{u(x,t) = u_p(x) + u_H(x,t)}$ is the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = U_0 \, , \quad u(L,t) = U_L \\ u(x,0) = f(x) \end{cases}$$

Adiabatic conditions

... separation of variables

Step 1 same

Step 2
$$\begin{cases} \frac{\partial u}{\partial x}(0,t) = X'(0)\,T(t) = 0 \\ \frac{\partial u}{\partial x}(L,t) = X'(L)\,T(t) = 0 \end{cases} \quad \text{for all } t$$

$$\Longleftrightarrow \quad \begin{matrix} T(t) = 0 \\ \text{for all } t \end{matrix} \quad \underline{\underline{\text{or}}} \quad \underbrace{\begin{cases} X'(0) = 0 \\ X'(L) = 0 \end{cases}}$$

Solve
$$\begin{cases} X'' - kX = 0 \\ X'(0) = 0 \\ X'(L) = 0 \end{cases}$$

Case $k = -p^2 < 0$:
$$\begin{cases} X(x) = A\cos px + B\sin px \\ X'(0) = 0 \longrightarrow B = 0 \\ X'(L) = 0 \end{cases}$$

$pL = m\pi$

Get solutions $X_m = \cos \frac{m\pi x}{L}$

$m = \underline{\underline{0}}, 1, 2, \ldots$

Step 3 $$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2 t} \cos \frac{n\pi x}{L}$$

$$\lambda_n = \frac{n\pi c}{L}$$

IC: $$u(x,0) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} = f(x)$$

Choose A_n's as coefficients of Fourier cosine series of $f(x)$

Note $$u(x,t) \longrightarrow A_0 \quad \text{as } t \to +\infty$$

where

$$A_0 = \frac{1}{L} \int_0^L f(x)\, dx = \underline{\text{average of } f(x)}$$

makes sense!

When is $\underbrace{\text{formal solution}}_{\text{given by series}}$ an $\underbrace{\text{honest solution}}_{\text{of } \left\{\begin{array}{l}\text{PDE}\\ \text{BC}\\ \text{IC}\end{array}\right.}$?

Answer 1 : In physical applications so far ...] Analysis III

Answer 2 : This is a delicate problem depending on regularity of initial conditions

$$f(x) \sim \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos \ldots + \sum_{n=1}^{\infty} b_n \sin \ldots}_{\text{series}}$$

series "represents" function

series

- converges ?
- converges to $f(x)$?
- continuity ?
- differentiability ?
- $\frac{d}{dx} \sum_n f_n(x) = \sum_n \frac{d}{dx} f_n(x)$?
- ...

] Fourier Thm Etc

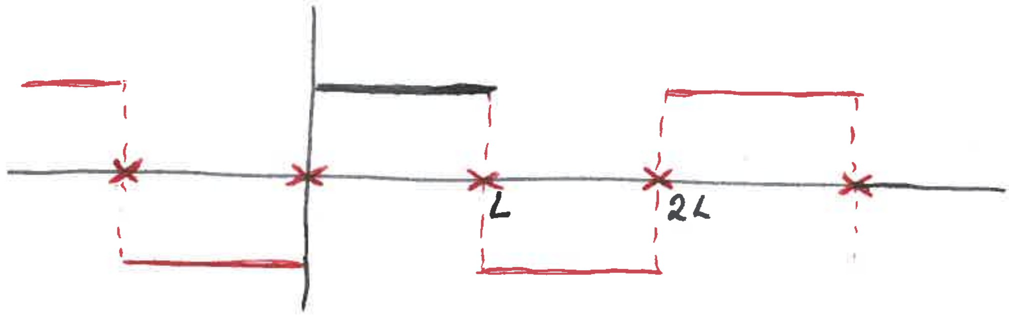
Watch out!

Example

$$1 \sim \sum_{n \text{ odd}} \frac{4}{n\pi} \sin \frac{n\pi x}{L}$$

$0 \le x \le L$

(see last time)



$$\sum_{n \text{ odd}} \frac{4}{n\pi} \sin \frac{n\pi x}{L} = \begin{cases} 1 & \cdots \\ 0 & x = 0, \pm L, \pm 2L \ldots \\ -1 & \cdots \end{cases}$$

Fourier Thm

Derivative?

$$\frac{d}{dx} 1 \overset{?}{\sim} \sum_{n \text{ odd}} \frac{d}{dx} \left(\frac{4}{n\pi} \sin \frac{n\pi x}{L} \right)$$

$$0 \cancel{\overset{?}{\sim}} \underbrace{\sum_{n \text{ odd}} \frac{4}{L} \cos \frac{n\pi x}{L}}_{\text{diverges}}$$

Answer 3 : Sometimes we can <u>sum the series</u>

Example (last week)

$$\begin{cases} \text{wave equation} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x), \quad \dfrac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

choose $g(x) \equiv 0$
to keep things shorter

Solution is

$$u(x,t) = \sum_{m=1}^{\infty} C_m \cos \underbrace{\frac{cm\pi t}{L}}_{\beta} \sin \underbrace{\frac{m\pi x}{L}}_{\alpha}$$

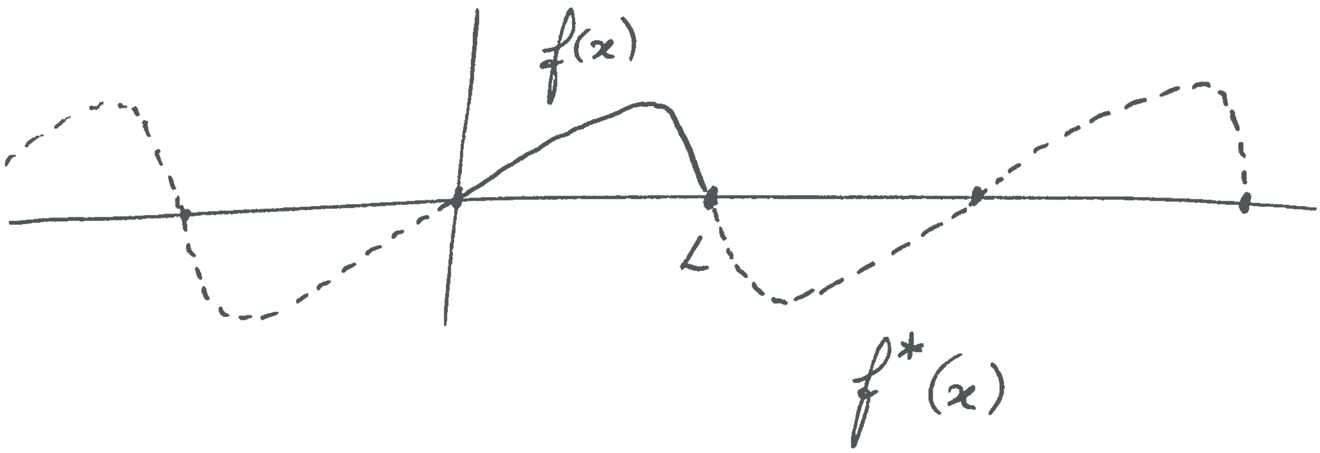
where C_m are the coefficients of the Fourier sine series of $f(x)$

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = \underbrace{f^*(x)}_{\text{odd periodic extension of } f(x)}$$

<u>when</u>

$f(x)$ continuous , $0 \leq x \leq L$

<u>and</u> $f(0) = f(L) = 0$



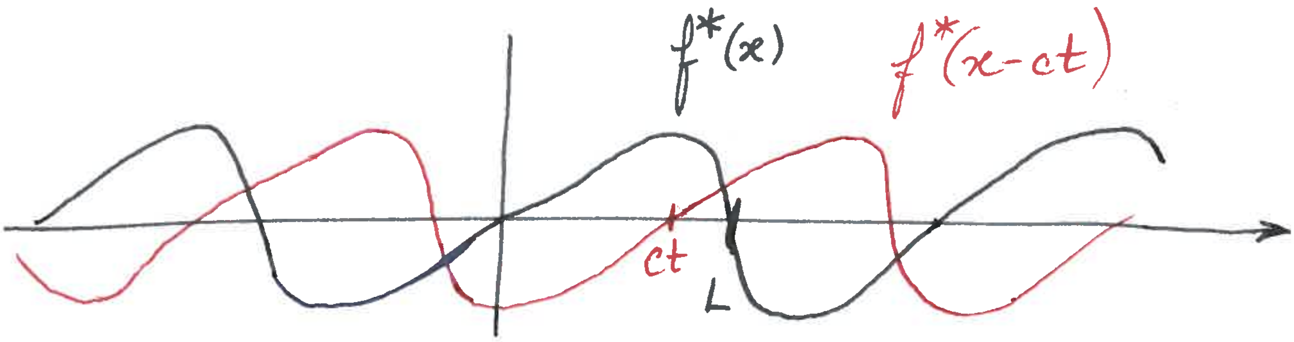
$f^*(x)$ is continuous

(when $f(x)$ continuous and $f(0) = f(L) = 0$)

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$

$$\Rightarrow \quad \underbrace{\cos\beta}\,\underbrace{\sin\alpha} = \frac{1}{2}\Big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\Big)$$

$$u(x,t) = \underbrace{\frac{1}{2}\sum_{m=1}^{\infty} c_m \sin\frac{m\pi}{L}(x+ct)}_{f^*(x+ct)} + \underbrace{\frac{1}{2}\sum_{m=1}^{\infty} c_m \sin\frac{m\pi}{L}(x-ct)}_{f^*(x-ct)}$$



$f^*(x-ct)$ wave traveling to the right (as $t \to +\infty$)

$f^*(x+ct)$ wave traveling to the left

$u(x,t)$ <u>superposition of two waves</u>

$u(x,t)$ is an honest solution

when

$$\boxed{f^*(x) \text{ twice differentiable}}$$

that is, when

$$\boxed{\begin{array}{l} f(x) \text{ twice differentiable}, \ 0 < x < L \\ f(0) = f(L) = 0 \\ f''(0^+) = f''(L^-) = 0 \end{array}}$$

(<u>Exercise</u> What if $g(x) \neq 0$?

Use $\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$

$= \frac{1}{2}\left(\sin\left(\alpha-\beta+\frac{\pi}{2}\right) - \sin\left(\alpha+\beta+\frac{\pi}{2}\right)\right)$

and consider $g^*(x)$ odd periodic extension of $g(x)$)