19. Dec. 2011

Poisson equation $\boxed{\Delta u = f}$

given function

In Electrostatics this is one of Maxwell's equations:

u = electrostatic potential

$\vec{F} = \nabla u$ electric field (= gradient of u)

f = electric charge density

Gauss' law: $\nabla \cdot \vec{F} = f$

that is $\Delta u = f$

because $\Delta = \nabla \cdot \nabla$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$\boxed{\Delta u = f}$ is a nonhomogeneous PDE

$\underbrace{f}$: this term does not contain u nor any of its derivatives

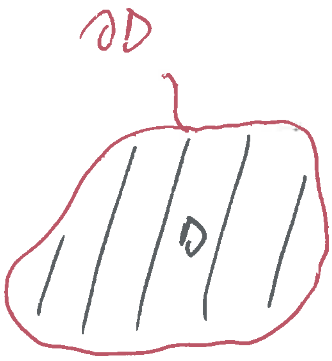
Hence
- superposition principle doesn't apply
- our earlier techniques do not work
- we need Green's functions to solve it

Green's functions are especially suitable to solve nonhomogeneous PDEs.

Last time: The solution of the Dirichlet problem for the Poisson eq

$$\boxed{\begin{array}{l} \Delta u = f \text{ on } D \\ u = g \text{ on } \partial D \end{array}}$$

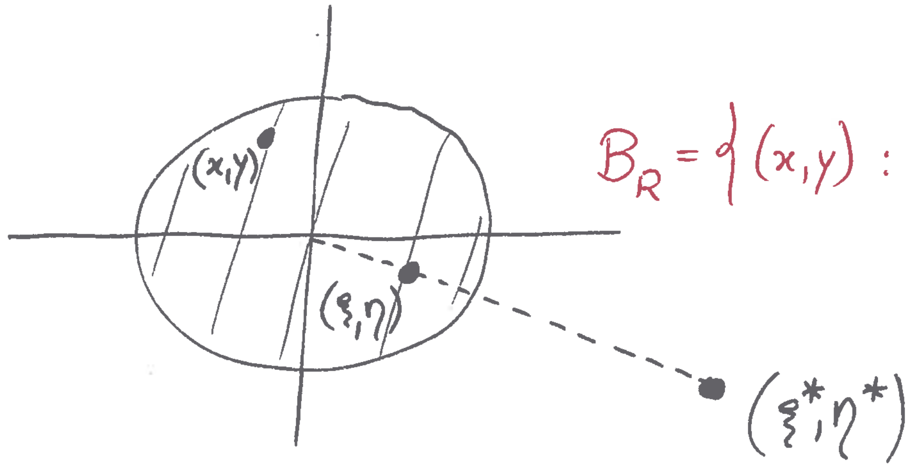
is given by an integral formula involving Green's function (for any continuous functions f on $\bar{D}$ & g on ∂D).



$$u(\xi) = \int_D \cdots + \int_{\partial D} \cdots$$

See formula on page 12.11 of notes.

Green's function for Δ on the disk B_R



$B_R = \{(x,y) : x^2 + y^2 \leq R^2\}$

By the reflection principle $\exists\, (\xi^*, \eta^*)$ s.t.

$$G_{B_R}(x,y;\xi,\eta) = \underbrace{\Gamma(x,y;\xi,\eta)}_{\text{fundamental solution}} - \Gamma(x,y;\xi^*,\eta^*) + \underbrace{c}_{\text{constant}}$$

is a Green's function for Δ on B_R.

We can find the suitable (ξ^*, η^*) by imposing the properties of a Green's function. See details in notes from last time.

We find that $(\xi^*, \eta^*) = \left(\frac{R^2}{r^2}\xi, \frac{R^2}{r^2}\eta\right)$

where $r = \sqrt{\xi^2 + \eta^2}$

and that $c = \frac{1}{4\pi} \ln \frac{R^2}{r^2}$.

Equivalently, if (ξ, η) has polar coords (r, φ) then (ξ^*, η^*) will have polar coords. $\left(\frac{R^2}{r}, \varphi\right)$.

Note that ∂B_R is fixed: if $r = R$, then $(\xi^*, \eta^*) = (\xi, \eta)$.

$(\xi^*, \eta^*) = \frac{R^2}{r^2}(\xi, \eta)$ is called the <u>inverse point</u> of (ξ, η) relative to the circle ∂B_R

We conclude that the Green's function for Δ on the disk B_R is

$$\boxed{G_{B_R}(x, y; \xi, \eta) = \frac{1}{4\pi} \ln\left(\frac{R^2}{r^2} \cdot \frac{(x-\xi)^2 + (y-\eta)^2}{\left(x - \frac{R^2}{r^2}\xi\right)^2 + \left(y - \frac{R^2}{r^2}\eta\right)^2} \right)}$$

where $r = \sqrt{\xi^2 + \eta^2}$

or, in polar coordinates, is

$$\boxed{G_{B_R}(\rho, \theta; r, \varphi) = \frac{1}{4\pi} \ln\left(\frac{R^2}{r^2} \cdot \frac{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}{\rho^2 + \frac{R^4}{r^2} - 2\rho \frac{R^2}{r} \cos(\theta - \varphi)} \right)}$$

Using this Green's function we obtain the integral formula for the solution of the Dirichlet problem for the Poisson eq. on a disk B_R

$$\begin{cases} \Delta u = f & \text{on } B_R \\ u = g & \text{on } \partial B_R \end{cases}$$

$$\boxed{\begin{aligned} u(r,\varphi) &= \int_0^{2\pi}\int_0^R G_{B_R}(\rho,\theta;r,\varphi)\, f(\rho,\theta)\, \underline{\underline{\rho}}\, d\rho\, d\theta \\ &+ \int_0^{2\pi} g(R,\theta)\, \frac{\partial}{\partial\rho} G_{B_R}(\rho,\theta;r,\varphi)\Big|_{\rho=R}\, \underline{\underline{R}}\, d\theta \end{aligned}}$$

where we integrate using polar coordinates ($\underline{\underline{\rho}}$ is the jacobian and $\underline{\underline{R}}$ comes from the parametrization for the line integral) and we use the fact that for the circle the normal derivative is given by $\frac{\partial}{\partial\rho}$.

Back to the case of the Laplace eq:

$$\begin{cases} \Delta u = 0 & \text{on } B_R \\ u = g & \text{on } \partial B_R \end{cases}$$

which is the case when $f \equiv 0$.

The solution given by Green's function is then

$$u(r,\varphi) = \iint \underbrace{\qquad\qquad}_{\substack{0 \\ \text{because } f\equiv 0}} d\rho\, d\theta$$

$$+ \int_0^{2\pi} g(R,\theta) \frac{\partial}{\partial\rho} G_{B_R}(\rho,\theta; r,\varphi)\Big|_{\rho=R} R\, d\theta$$

Compare with the solution given by Poisson's integral formula on page $\boxed{11.1}$.

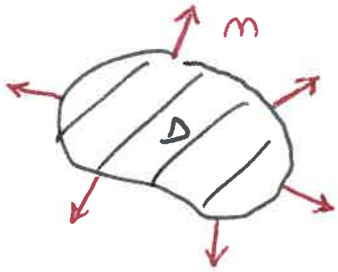
Since they must coincide we conclude that

$$\boxed{\underbrace{\frac{\partial}{\partial\rho} G_{B_R}(\rho,\theta;r,\varphi)}_{\text{Green's function}}\Big|_{\rho=R} = \frac{1}{2\pi R} \underbrace{K(r,\varphi;R,\theta)}_{\text{Poisson's kernel}}}$$

This can be checked directly with the formulas for G_{B_R} and for K (exercise).

Neumann problem for the Poisson eq

$$\begin{cases} \Delta u = f & \text{on } D \\ \dfrac{\partial u}{\partial n} = g & \text{on } \partial D \end{cases}$$



Uniqueness of solution?

Suppose u_1 & u_2 were two solutions.

Their difference $u_1 - u_2$ satisfies

$$\begin{cases} \Delta (u_1 - u_2) = f - f = 0 & \text{on } D \\ \dfrac{\partial (u_1 - u_2)}{\partial n} = g - g = 0 & \text{on } \partial D \end{cases}$$

(homogeneous Neumann problem for the Laplace eq)

and hence can be any constant:

$$u_1 = u_2 + c, \quad c \in \mathbb{R}$$

Hence, whereas the solution is unique for the Dirichlet problem, the solution is **not** unique for the Neumann problem.

Existence of solution ?

It follows from Gauss' divergence thm (*) that

$$\int_D \Delta u = \int_{\partial D} \frac{\partial u}{\partial n}$$

Therefore, a necessary condition for the existence of solution for the Neumann problem is

$$\boxed{\int_D \underbrace{f}_{\Delta u} = \int_{\partial D} \underbrace{g}_{\frac{\partial u}{\partial n}}}$$

Integral formula for solution ?

for the Neumann problem

(*) $\int_D \nabla \psi = \int_{\partial D} \psi \cdot n$ ← Gauss' thm

Take $\psi = \nabla u$

Consider Green's <u>second identity</u>

(from which we derived an integral formula for the solution of the Dirichlet problem)

$$\int_D (v\,\Delta u - u\,\Delta v) = \int_{\partial D} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right)$$

When $v = G_D$ <u>Green's fn</u> of Δ relative to D

$$\int_D (G_D\,\Delta u - u\,\underbrace{\Delta G_D}_{\delta_\xi}) = \int_{\partial D} \left(\underbrace{G_D}_{0} \frac{\partial u}{\partial n} - u \frac{\partial G_D}{\partial n}\right)$$

<u>good</u> for a <u>Dirichlet</u> problem since $\frac{\partial u}{\partial n}$ is <u>not</u> given there

For a <u>Neumann</u> problem, instead of a Green's function, use another type of function:

$$v = N_D \quad \text{s.t.} \quad \begin{cases} \Delta N_D = \delta_\xi & \text{on } D \\ \dfrac{\partial N_D}{\partial n} = \ldots & \text{on } \partial D \end{cases}$$

<u>Defn</u> A <u>Neumann's function</u> for the laplacian Δ in a region D is a family of functions parametrized by $(\xi,\eta)\in D$ of the form

$$N_D(x,y;\xi,\eta) = \Gamma(x,y;\xi,\eta) + h(x,y;\xi,\eta)$$

defined for $(x,y)\neq(\xi,\eta)$

where
- $\Gamma(x,y;\xi,\eta) = \frac{1}{4\pi}\ln\left((x-\xi)^2+(y-\eta)^2\right)$ is our fundamental solution
- h satisfies $\begin{cases} \Delta h = 0 & \text{on } D \\ \frac{\partial h}{\partial n} = -\frac{\partial \Gamma}{\partial n} + \frac{1}{L} & \text{on } \partial D \end{cases}$
- L = length of ∂D

Therefore, a Neumann's function satisfies

$$\begin{cases} \boxed{\Delta N_D = \Delta T + \Delta h = \delta_{\vec{\xi}}} & \text{on } D \\ \boxed{\dfrac{\partial N_D}{\partial n} = \dfrac{\partial T}{\partial n} + \dfrac{\partial h}{\partial n} = \dfrac{1}{L}} & \text{on } \partial D \end{cases}$$

whereas a Green's function satisfies

$$\begin{cases} \Delta G_D = \delta_{\vec{\xi}} & \text{on } D \\ G_D = 0 & \text{on } \partial D \end{cases}$$

WARNING

In some Physics texts these functions have opposite signs.
In particular, there

$$\Delta G_D = - \delta_{\vec{\xi}}$$

The use of a Neumann's function :

<u>Thm</u> Let N be a Neumann's function of Δ relative to the region D.

Then the solutions of the Neumann problem for the Poisson eq

$$\boxed{\begin{cases} \Delta u = f & \text{on } D \\ \dfrac{\partial u}{\partial n} = g & \text{on } \partial D \end{cases}}$$

are given by

$$\boxed{u(\vec{\xi}) = \int_D N(\vec{x};\vec{\xi})\, f(\vec{x})\, d\vec{x} - \int_{\partial D} N(\vec{x};\vec{\xi})\, g(\vec{x})\, ds + \text{constant}}$$

Compare this with the statement for the Dirichlet problem on page $\boxed{12.11}$.

Sketch of proof

Use Green's second identity with

$$\begin{cases} v = N \\ u = \text{solution of Neumann problem} \end{cases}$$

$$\int_D \Big(N \underbrace{\Delta u}_{f} - u \underbrace{\Delta N}_{\delta_{\vec{\xi}}} \Big) = \int_{\partial D} \Big(N \underbrace{\frac{\partial u}{\partial n}}_{g} - u \underbrace{\frac{\partial N}{\partial n}}_{\frac{1}{L}} \Big)$$

that is

$$\int_D Nf \quad - u(\vec{\xi}) = \int_{\partial D} Ng - \frac{1}{L} \int_{\partial D} u$$

$\square$

The undetermined real constant in the formula is

$$\frac{1}{L} \int_{\partial D} u \, ds = \text{average of } u \text{ on } \partial D.$$

The Neumann's function itself is <u>not</u> unique. In order to force uniqueness, we often impose the <u>normalization</u>

$$\boxed{\int_{\partial D} N \, ds = 0}$$

Note A normalized Neumann's function is also symmetric

$$\boxed{N(\vec{x};\vec{\xi}) = N(\vec{\xi};\vec{x})}$$

Why?

Let $u(\vec{x}) = N(\vec{x};\vec{x}_1)$ & $\vec{v}(\vec{x}) = N(\vec{x};\vec{x}_2)$

u & v satisfy

$$\begin{cases} \Delta u = \delta_{\vec{x}_1} \\ \dfrac{\partial u}{\partial n} = \dfrac{1}{L} \end{cases} \quad \& \quad \begin{cases} \Delta v = \delta_{\vec{x}_2} \\ \dfrac{\partial v}{\partial n} = \dfrac{1}{L} \end{cases}$$

By the identity Green 2

$$\int_D v\underbrace{\Delta u}_{\delta_{\vec{x}_1}} - u\underbrace{\Delta v}_{\delta_{\vec{x}_2}} = \int_{\partial D} v\underbrace{\frac{\partial u}{\partial n}}_{\frac{1}{L}} - u\underbrace{\frac{\partial v}{\partial n}}_{\frac{1}{L}}$$

we get $v(\vec{x}_1) - u(\vec{x}_2) = \dfrac{1}{L}\underbrace{\int_{\partial D}(v-u)}_{0}$

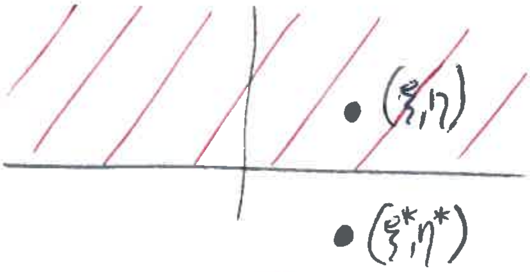
by normalization

and conclude that

$$N(\vec{x}_1;\vec{x}_2) = N(\vec{x}_2;\vec{x}_1).$$

How to find Neumann's functions for Δ

Example Find a Neumann's function for Δ on the upper half plane



$D = \{(x,y) : y > 0\}$

By an "<u>even reflection principle</u>"

take $N_D(x,y;\xi,\eta) = \underbrace{\Gamma(x,y;\xi,\eta)}_{\text{fundamental solution}} + \Gamma(x,y;\xi^*,\eta^*)$

where $(\xi^*,\eta^*) = (\xi,-\eta)$.

Check that properties are satisfied:

(1) $\Delta N_D = \delta(\vec{x}-\vec{\xi}) + \underbrace{\delta(\vec{x}-\vec{\xi}^*)}_{0}$

when $\vec{x},\vec{\xi} \in D$ $(\Rightarrow \vec{\xi}^* \notin D)$

(2) $\left.\dfrac{\partial N_D}{\partial n}\right|_{\partial D} = -\left.\dfrac{\partial N_D}{\partial y}\right|_{y=0} = -\left.\dfrac{\partial \Gamma(\vec{x};\vec{\xi})}{\partial y}\right|_{y=0} - \left.\dfrac{\partial \Gamma(\vec{x};\vec{\xi}^*)}{\partial y}\right|_{y=0}$

$= 0$

exercise (compute $\left.\dfrac{\partial \Gamma}{\partial y}\right|_{y=0}$)

This is good since $L = \infty$ here.

What was $\boxed{\text{not}}$ in this course:

Application of Laplace transform to PDEs

$$f(t) \rightsquigarrow \boxed{F(s) = \int_0^\infty f(t)\, \underbrace{e^{-st}}_{\text{called the kernel of this integral transform}}\, dt}$$

Laplace transform

called the kernel
of this integral transform

This is good for PDEs where one of the variables ranges over the half line $[0,+\infty[$ just as Fourier transform is good for PDEs where variables range over $\mathbb{R}$.

Ref Kreyszig § 11.12

THE END