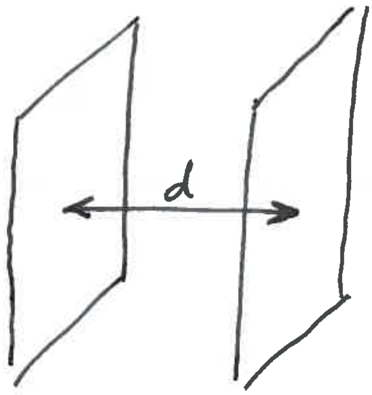
21. Nov. 2011

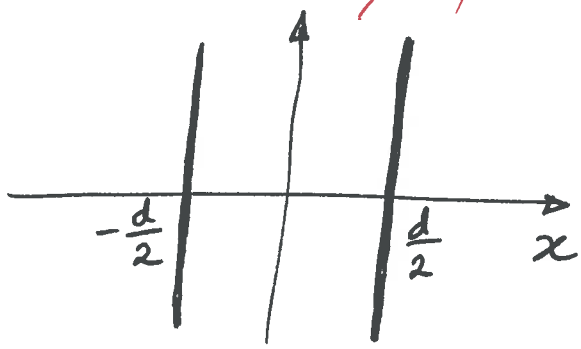
Laplace's eq in 2D (or 3D with translation invariance) $\longrightarrow$ Complex Analysis

Example 1 Find potential between two parallel infinite plates at distance d and kept at constant potential u_1 & u_2.



$u(x,\cancel{y},\cancel{z}) = u(x)$ potential only depends on x

$\Delta u = u''$



$$\begin{cases} u'' = 0 \\ u\left(-\frac{d}{2}\right) = u_1 \\ u\left(\frac{d}{2}\right) = u_2 \end{cases}$$

Easy problem!

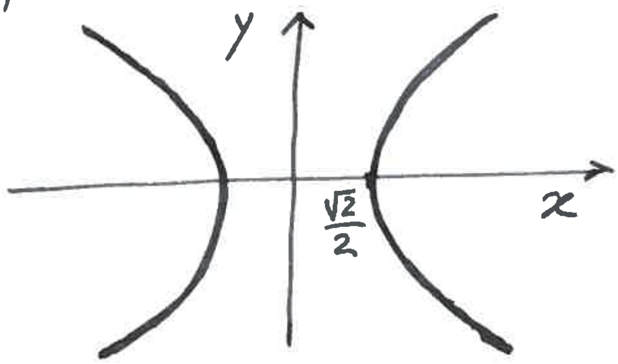
Solution : $u(x) = ax + b$ with

a & b determined from BC

$$\begin{cases} a\left(-\frac{d}{2}\right) + b = u_1 \\ a\frac{d}{2} + b = u_2 \end{cases}$$

$$a = \frac{1}{d}(u_2 - u_1) \quad \& \quad b = \frac{1}{2}(u_1 + u_2)$$

<u>Example 2</u> Same as Example 1 but plates are bent along hyperbola $x^2 - y^2 = \frac{1}{2}$ (any z).



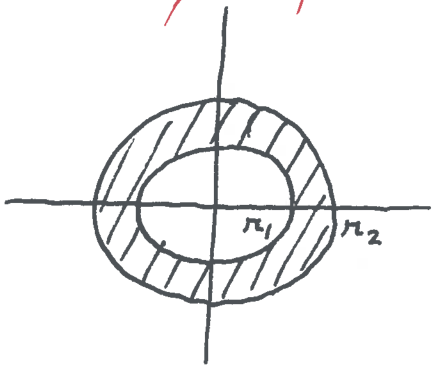
How to handle this ?!

Example 3 Find steady-state* temperature between two infinite coaxial cylinders of radii $r_1 < r_2$ kept at constant temperature u_1 & u_2 respectively.
Assume thermal diffusity is 1.**

Heat equation $\underbrace{\dfrac{\partial u}{\partial t}}_{0^{*}} = \underbrace{c^2}_{1^{**}} \Delta u$

$\underbrace{u(r, \cancel{\theta}, \cancel{z})}_{\text{cylindrical coords}} = u(r)$ temperature only depends on r

$\Delta u = u'' + \frac{1}{r} u'$



$$\begin{cases} u'' + \frac{1}{r} u' = 0 \\ u(r_1) = u_1 \\ u(r_2) = u_2 \end{cases}$$

Easy problem!

$$\frac{u''}{u'} = -\frac{1}{r}$$

$$\ln|u'| = -\ln r + \text{constant}$$

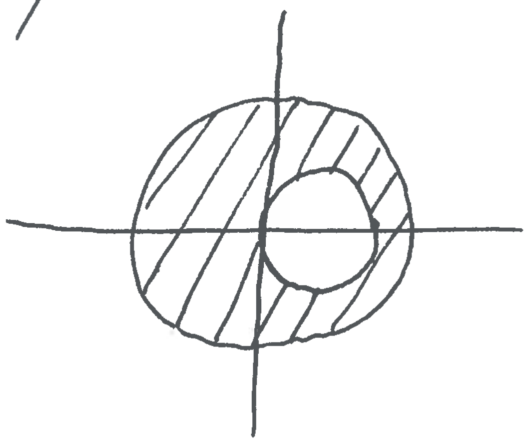
$$u' = \frac{a}{r}$$

Solution: $u(r) = a\ln r + b$ with a & b determined from BC

$$\begin{cases} a\ln r_1 + b = u_1 \\ a\ln r_2 + b = u_2 \end{cases}$$

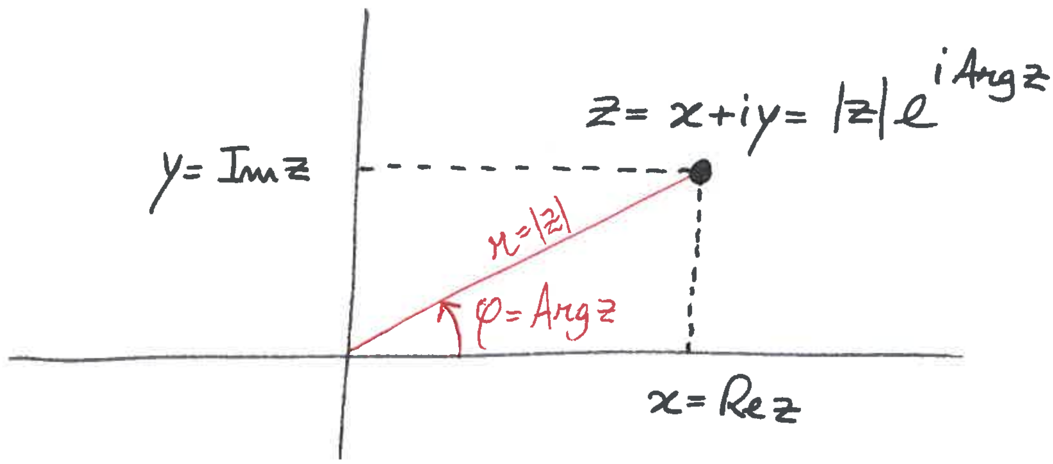
$$a = \frac{u_1 - u_2}{\ln r_1 - \ln r_2} \quad \& \quad b = \frac{u_1 \ln r_2 - u_2 \ln r_1}{\ln r_2 - \ln r_1}$$

Example 4 Same as Example 3 but cylinders are not coaxial.



How to handle ?!

Complex plane $\mathbb{C}$



$$f(z) = \underbrace{u(x,y)}_{\substack{\operatorname{Re} f \\ \text{real part}}} + i\, \underbrace{v(x,y)}_{\substack{\operatorname{Im} f \\ \text{imaginary part}}} \quad \text{complex function}$$

|| Here assume real functions are twice continuously differentiable

f is analytic $\iff$ u & v satisfy Cauchy-Riemann equations

$$\text{CR} \quad \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \underset{\text{CR}}{=} \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \underset{\substack{\text{mixed partials} \\ \text{are equal}}}{=} 0$$

Similarly $\Delta v = 0$

When $\Delta u = 0$ we say u is <u>harmonic</u>

Conclusion : f analytic $\Rightarrow$ $\mathrm{Re} f$ & $\mathrm{Im} f$ are harmonic

Moreover, given u harmonic, can always find v (up to a constant) such that $f = u + iv$ is analytic. We say v is then a <u>conjugate harmonic</u> for u.

How : Given u solve CR to find v

$$\begin{cases} \dfrac{\partial v}{\partial x} = \dots \\[2ex] \dfrac{\partial v}{\partial y} = \dots \end{cases} \iff \begin{cases} v = \dots \\[2ex] v = \dots \end{cases}$$

Apply to case where u is a potential function

$\downarrow$

$\Delta u = 0$

Find conjugate harmonic v.

$f = u + iv$ analytic function

called the complex potential

$\downarrow$

up to a constant

Example 1 $u = ax + b$

Find $f(z)$ analytic with $\mathrm{Re} f = u$

Solve CR to find $v = \mathrm{Im} f$

$$\begin{cases} \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = 0 \\[2ex] \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = a \end{cases} \iff \begin{cases} v \text{ does not depend on } x \\[2ex] v = ay + c(x) \end{cases}$$

Choose $v = ay$

$$f(z) = \underbrace{ax+b}_{u} + i\underbrace{ay}_{v} = a(x+iy) + b = \underbrace{az+b}_{\text{Answer}}$$

<u>Example 3</u> $u = a \ln \underbrace{r} + b$

$$= a \ln \underbrace{\sqrt{x^2+y^2}}_{|z|} + b$$

Find the complex potential $f(z)$

Strategy 1 : Solve CR $\begin{cases} \frac{\partial v}{\partial x} = \cdots \\ \frac{\partial v}{\partial y} = \cdots \end{cases}$

Strategy 2 : Recall <u>complex logarithm</u>

$$z = |z| e^{i \operatorname{Arg} z}$$

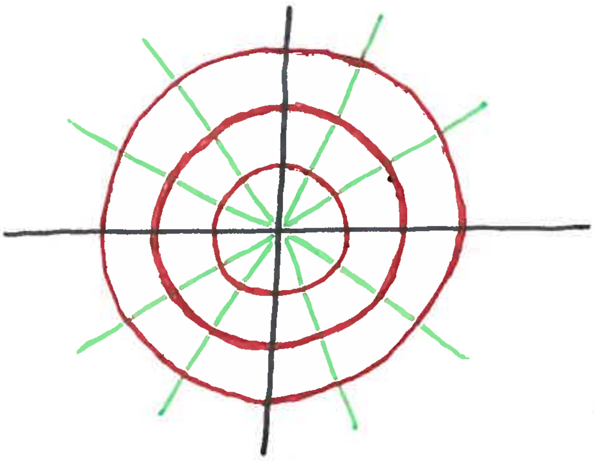
$$\operatorname{Log} z = \underbrace{\ln |z|}_{\substack{\text{real} \\ \text{part}}} + i \underbrace{\operatorname{Arg} z}_{\substack{\text{imaginary} \\ \text{part}}}$$

Need $\operatorname{Arg} z$ <u>chosen</u> in some interval of length 2π, for instance $(-\pi, \pi)$

Can choose $v = a \operatorname{Arg} z$

Answer : $f(z) = a \operatorname{Log} z + b$

$$u + iv = a \operatorname{Log} z + b$$



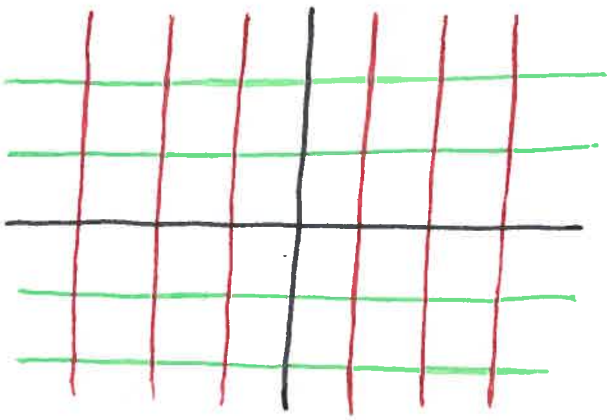
u constant $\Leftrightarrow$ r const

equipotential lines

v constant $\Leftrightarrow$ φ const

lines of force

Example 1 $\quad u + iv = az + b$



u constant $\Leftrightarrow$ x const

equipotential lines

v constant $\Leftrightarrow$ y const

lines of force

Electric particles move along lines of force

In general, $f = \underbrace{\boxed{u}}_{\text{potential}} + iv$ complex potential

$\vec{F} = \text{grad}\, u$ force

$\text{grad}\, u \perp$ equipotential lines

lines where $u = \text{const}$

$$\text{grad}\, u = \Big(\underbrace{\frac{\partial u}{\partial x}}_{A}, \underbrace{\frac{\partial u}{\partial y}}_{B}\Big)$$

$$\text{grad}\, v = \Big(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\Big) \overset{CR}{=} \Big(\underbrace{-\frac{\partial u}{\partial y}}_{-B}, \underbrace{\frac{\partial u}{\partial x}}_{A}\Big)$$

$$(A,B)\bullet(-B,A) = 0$$

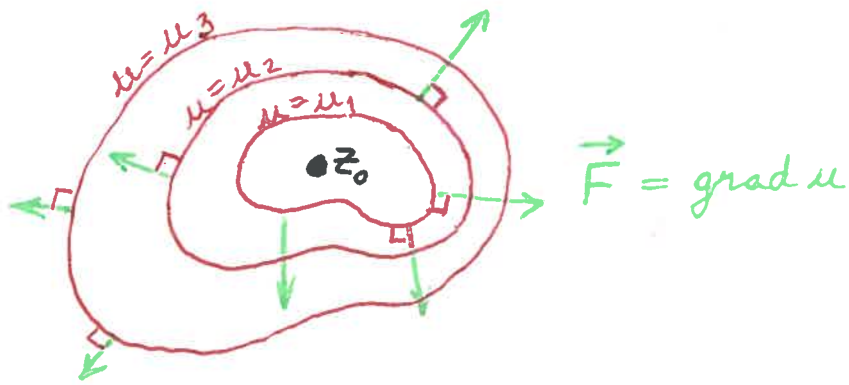
so $\text{grad}\, u \perp \text{grad}\, v$

$\text{grad}\, u$ is parallel to lines of force

lines where $v = \text{const}$

except at points z_0 where $\text{grad}\, u = \text{grad}\, v = 0$

i.e. $f'(z_0) = 0$ i.e. z_0 critical point



Change coords by analytic functions

$$\text{domain } D \text{ in } \mathbb{C} \xrightarrow{\boxed{g(z)=w}} \text{domain } D^* = g(D) \text{ in } \mathbb{C}$$

$z = x+iy$ $\qquad\qquad$ $w = s+it$

$$g(z) = s(x,y) + i\,t(x,y)$$

When g is analytic and has no critical point in D we say that g is a <u>conformal mapping</u> from D to D^*.

$\downarrow$

g <u>preserves angles</u>

We usually restrict D so that g is invertible

$$D \underset{z = g^{-1}(w)}{\overset{g(z)=w}{\rightleftarrows}} D^* = g(D)$$

Then :

$$f(z) = f^*(g(z)) \text{ analytic} \iff f^*(w) = f(g^{-1}(w)) \text{ analytic}$$

Proof that g preserves angles

A curve C parametrized by $z(\tau)$ gets transformed by g into a curve C^*

$$w(\tau) = g(z(\tau))$$

Tangent to C is given by vector $z'(\tau)$

″ ″ C^* ″ ″ ″ ″

$$w'(\tau) = g'(z(\tau)) \cdot z'(\tau)$$

Angle of C^* at point $w_0 = w(\tau_0)$ is

$$\mathrm{Arg}(w'(\tau_0)) = \mathrm{Arg}\big(g'(\underbrace{z(\tau_0)}_{z_0})\big) + \underbrace{\mathrm{Arg}\,(z'(\tau_0))}_{\text{angle of } C \text{ at } z_0 = z(\tau_0)}$$

Therefore, at the point $w_0 = g(z_0)$, the transformation g rotates curves always by the same angle $\mathrm{Arg}\,(g'(z_0))$.

We conclude that the angle between two curves C_1 & C_2 is the same as the angle between their transformed curves C_1^* & C_2^*. $\square$

Example $\boxed{g(z) = \sin z}$

$$= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

definition of sin

$$= \frac{1}{2i}\left(e^{ix}e^{-y} - e^{-ix}e^{y}\right)$$

$z = x + iy$

$$= \sin x \, \frac{e^{-y} + e^{y}}{2} + i \cos x \, \frac{-e^{-y} + e^{y}}{2}$$

$e^{ix} = \cos x + i \sin x$

$$= \underbrace{\sin x \cosh y}_{\substack{\text{real part} \\ s(x,y)}} + i \underbrace{\cos x \sinh y}_{\substack{\text{imaginary part} \\ t(x,y)}}$$

definition of cosh & sinh

$$W = g(z) = \underbrace{s(x,y)}_{} + i \underbrace{t(x,y)}_{}$$

Consider curve C given by $x = \underbrace{x_0}_{\text{constant}}$

What is the transformed curve C^* ?

$$C^* : \begin{cases} s = \sin x_0 \cosh y \\ t = \cos x_0 \sinh y \end{cases}$$

$$\frac{s^2}{\sin^2 x_0} - \frac{t^2}{\cos^2 x_0} = \cosh^2 y - \sinh^2 y = 1$$

$\sin^2 x_0$ const, $\cos^2 x_0$ const

for $x_0 \neq k\frac{\pi}{2}$

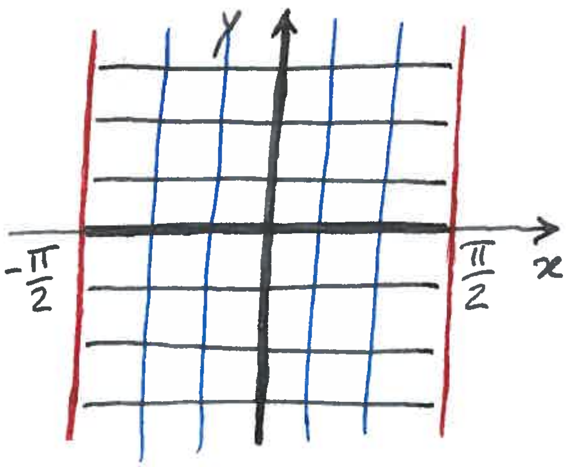
$$\frac{s^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{hyperbola}$$

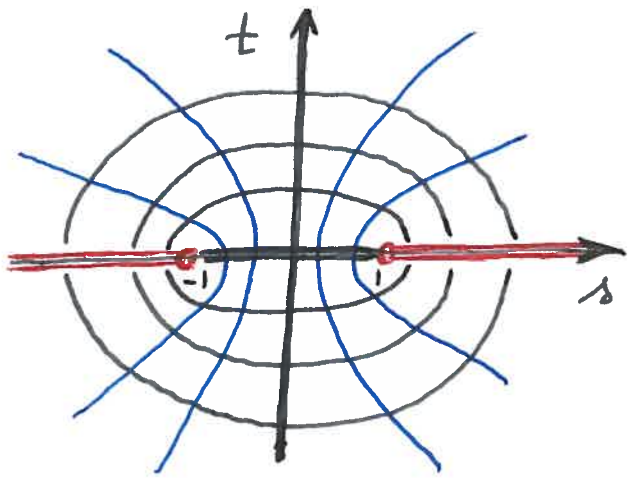
So C^* is a (branch of a) hyperbola

<u>Exercise</u> Study cases $x_0 = k\frac{\pi}{2}$, $k \in \mathbb{Z}$

<u>Exercise</u> What about the curve $y = y_0$?

Hint : $\sin^2 x + \cos^2 x = 1$





<u>Exercise</u> What are the pictures for the conformal mappings a) $w = \cos z$

b) $w = \sinh z$

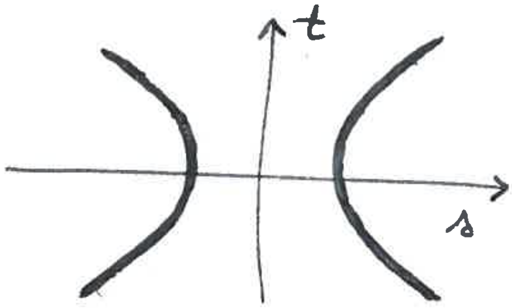
c) $w = \cosh z$

Hints : $\cos z = \sin\left(z + \frac{1}{2}\pi\right)$

$\sinh z = -i \sin(iz)$

$\cosh z = \cos iz$

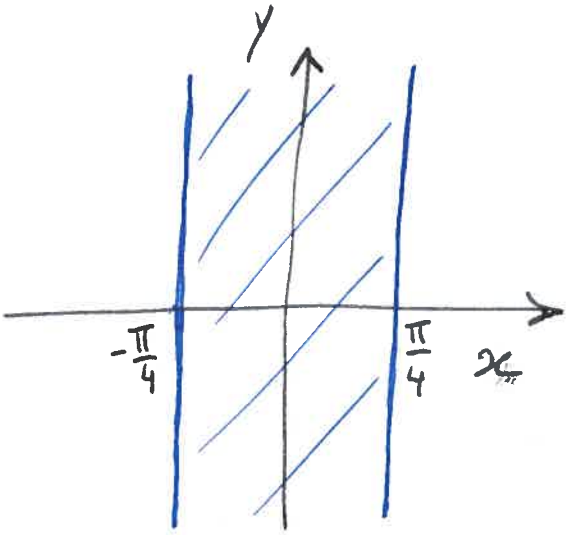
Back to Example 2



$$s^2 - t^2 = \frac{1}{2}$$

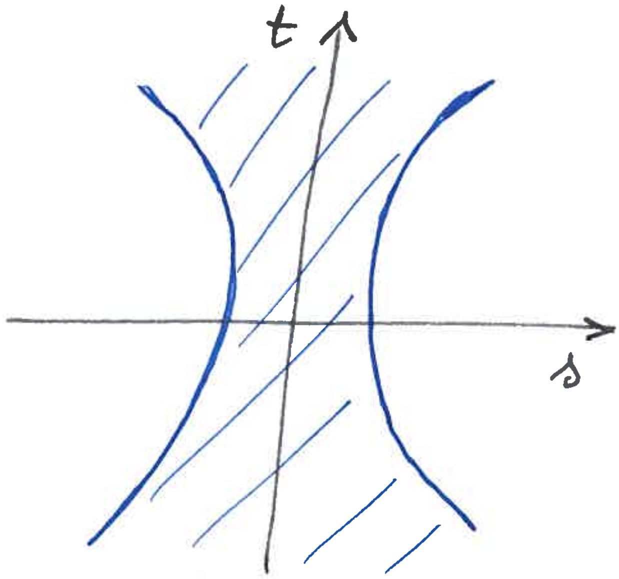
Want $a^2 = b^2 = \frac{1}{2}$

Choose $x_0 = \pm \frac{\pi}{4}$



D

$-\frac{\pi}{4} < x < \frac{\pi}{4}$



D^*

region between branches of hyperbola

$\sin Z = W$

We had solved the problem in D

$$u(x,y) = ax+b = \operatorname{Re} f(z)$$

$$f(z) = az+b \quad \text{complex potential}$$

where $d = \frac{\pi}{2}$ distance

$$a = \frac{2}{\pi}(u_2-u_1) \;\&\; b = \frac{1}{2}(u_1+u_2)$$

We want now to solve it in D^*

$$u^*(s,t) = ? = \operatorname{Re} f^*(w) \quad \text{solution}$$

$$f^*(w) = ? \quad \text{complex potential}$$

Since $g(z) = \sin z$ transforms the problem in D into the problem in D^*

it must be $\quad f(z) = f^*(\sin z)$

that is $\quad f(\arcsin w) = f^*(w)$

<u>Answer</u> $\quad u^*(s,t) = \operatorname{Re} f(\arcsin w)$

$$= a \operatorname{Re}(\arcsin w) + b$$

Example $\boxed{g(z) = \dfrac{az+b}{cz+d}}$ where $a,b,c,d \in \mathbb{C}$, $ad-bc \neq 0$

called a <u>fractional linear transf.</u>

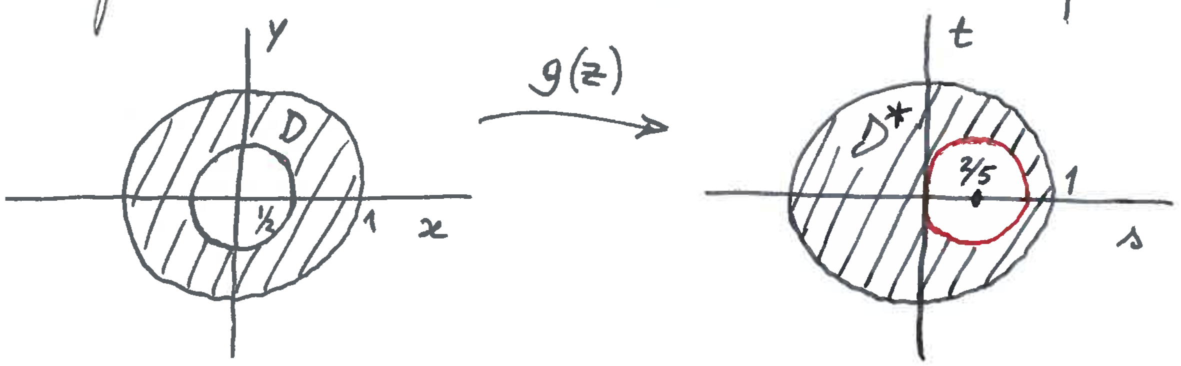
or <u>Möbius transformation</u>

transforms "generalized circles"

into "generalized circles"

A <u>generalized circle</u> is a circle or a straight line.

For instance, $g(z) = \dfrac{2z-1}{z-2}$ takes the unit circle to the unit circle and transforms the circle $|z| = \frac{1}{2}$ into the circle $\left|w - \frac{2}{5}\right| = \frac{2}{5}$.



<u>Exercise</u> Check that

a) if $|z| = 1$ i.e. $z = e^{i\theta}$ then $|g(z)|^2 = 1$

b) if $|z| = \frac{1}{2}$ i.e. $z = \frac{1}{2}e^{i\theta}$ then $\left|g(z) - \frac{2}{5}\right|^2 = \frac{4}{25}$

Back to Example 4

We had solved the problem in D

$$u(x,y) = a \ln\sqrt{x^2+y^2} + b = \mathrm{Re}\, f(z)$$

$$f(z) = a \,\mathrm{Log}\, z + b \quad \text{complex potential}$$

where a & b for $r_1 = \frac{1}{2}$ & $r_2 = 1$

We want now to solve it in D^*

$$u^*(s,t) = \; ? \; = \mathrm{Re}\, f^*(w) \quad \text{solution}$$

$$f^*(w) = \; ? \quad \text{complex potential}$$

Since $g(z) = \dfrac{2z-1}{z-2}$ transforms the problem in D into the problem in D^*

it must be

$$f(z) = f^*\left(\frac{2z-1}{z-2}\right)$$

$$f\underbrace{\left(\frac{2w-1}{w-2}\right)}_{g^{-1}(w) = \frac{2w-1}{w-2}} = f^*(w)$$

coincidence that $g^{-1} = g$!

<u>Answer</u> $\quad u^*(s,t) = \mathrm{Re}\, f\left(\dfrac{2w-1}{w-2}\right)$

$$= a \ln\left|\frac{2w-1}{w-2}\right| + b$$