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Analysis III

How to solve some of the most important types of $\underbrace{\text{PDEs}}_{\text{partial differential equations}}$

Main examples:

- Wave equation ← today & next week
- Heat equation
- Laplace equation

References:

- Felder
- Kreyszig

} basic for lectures *

- Hungerbühler
- Pinchover - Rubinstein

} more in depth **

* look up PDF at coordination webpage or ask assistant

** available at Polybuchhandlung

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Exercise series and informal lecture notes will be posted throughout the semester at

http://www.math.ethz.ch/education/bachelor/lectures/hs2011/other/analysis3_itet

ODE = ordinary differential equation
= equation involving functions of <u>one</u> independent variable and one or more of their derivatives

<u>Example</u>

$$\boxed{m \frac{d^2 x(t)}{dt^2} = F(x(t))}$$

Newton's second law

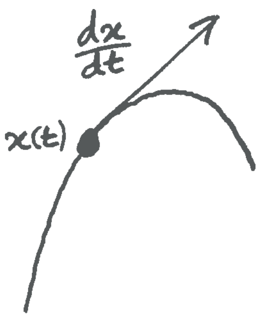
Unknown function
$x(t) = (x_1(t), x_2(t), x_3(t))$
represents position of particle at time t

$\frac{dx(t)}{dt}$ = velocity vector

$\frac{d^2x(t)}{dt^2}$ = acceleration

m = mass (constant)

$F(x)$ = force field



PDE = partial differential equation

= equation involving one or more partial derivatives of an unknown function of <u>two or more</u> independent variables

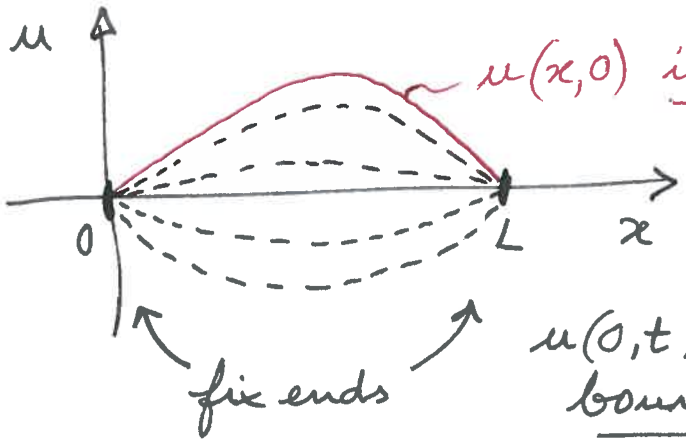
<u>Example</u>

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

Wave equation (for a string)

Unknown function $u(x,t)$ represents deflection at point $x \in [0,L]$ and at time $t \in [0,+\infty)$

$u(x,0)$ <u>initial condition</u>

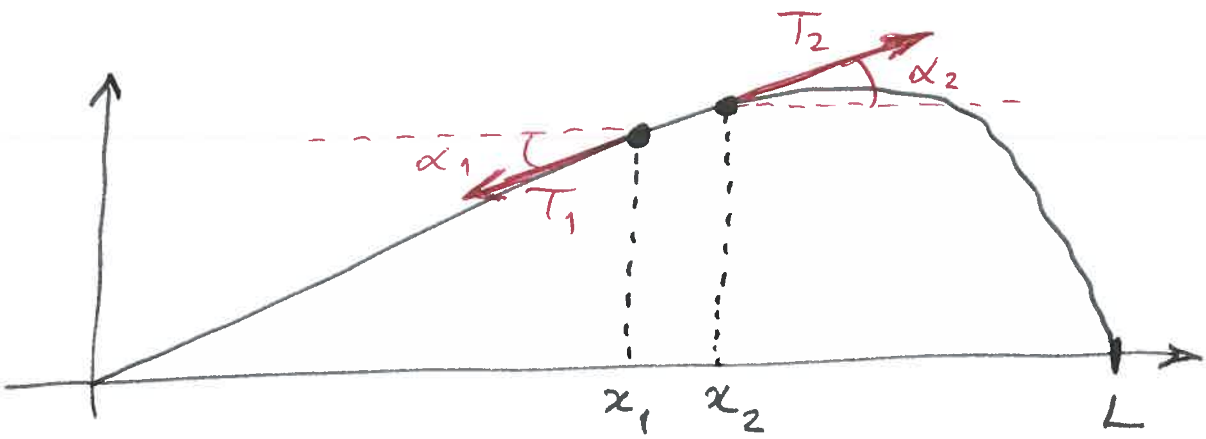


$u(0,t) = u(L,t) = 0$ for all t

<u>boundary conditions</u>

Wave equation

for an "ideal string"
similar enough to a real string



T_1, T_2 tension at two nearby points

horizontal components

no horizontal motion $\Rightarrow T_1 \cos\alpha_1 = T_2 \cos\alpha_2 = T$
constant

vertical components

$$\text{force} = T_2 \sin\alpha_2 - T_1 \sin\alpha_1 = m \frac{\partial^2 u}{\partial t^2}$$

Newton

$$\text{mass} = m = \rho(x_2 - x_1)$$

mass per unit length
(for undeflected string)

Derivation of the wave equation

Divide force eqn. $T_2 \sin\alpha_2 - T_1 \sin\alpha_1 = m \frac{\partial^2 u}{\partial t^2}$
by the constant $T_2 \cos\alpha_2 = T_1 \cos\alpha_1 = T$

$$\underbrace{\frac{T_2 \sin\alpha_2}{T_2 \cos\alpha_2}}_{\substack{\tan\alpha_2 \\ \parallel \\ \left.\frac{\partial u}{\partial x}\right|_{x_2} \\ \text{slope at } x_2}} - \underbrace{\frac{T_1 \sin\alpha_1}{T_1 \cos\alpha_1}}_{\substack{\tan\alpha_1 \\ \parallel \\ \left.\frac{\partial u}{\partial x}\right|_{x_1} \\ \text{slope at } x_1}} = \frac{\rho(x_2 - x_1)}{T} \frac{\partial^2 u}{\partial t^2}$$

Divide by $x_2 - x_1$

$$\frac{\left.\frac{\partial u}{\partial x}\right|_{x_2} - \left.\frac{\partial u}{\partial x}\right|_{x_1}}{x_2 - x_1} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

Let $x_2, x_1 \longrightarrow x$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \underset{\substack{\uparrow \\ \text{positive constant}}}{\frac{\rho}{T}} \frac{\partial^2 u}{\partial t^2}}$$

1-dimensional wave equation

$c^2 = \frac{T}{\rho}$ positive constant

$$\boxed{\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\[2ex] u(0,t) = u(L,t) = 0 \quad \text{for all } t \\[2ex] u(x,0) = f(x) \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) \end{array}}$$

PDE

BC <u>boundary conditions</u>

IC <u>initial conditions</u>

turns out to be a <u>well-posed</u> mathematical problem

i.e. <u>has</u> a <u>unique</u> solution which is <u>continuous</u> with respect to the data

Analysis III

Problems	Tools
from different types of PDEs	separation of variables
	Fourier series
	Fourier integrals
	Fourier transform
	Laplace transform
	spherical harmonics
	characteristics
	⋮

Today: use method of separation of variables to solve "initial boundary value" problem with wave equation

Separation of variables

Step 1 Find solutions of PDE of the form

$$\boxed{u(x,t) = \underbrace{X(x)}_{\substack{\text{only}\\ \text{depends}\\ \text{on } x}}\ \underbrace{T(t)}_{\substack{\text{only}\\ \text{depends}\\ \text{on } t}}} \quad \star$$

$$\frac{\partial^2 u}{\partial t^2} \overset{\star}{=} X(x)\, T''(t)$$

$$\frac{\partial^2 u}{\partial x^2} \overset{\star}{=} X''(x)\, T(t)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

becomes $X(x)T''(t) = c^2 X''(x)\, T(t)$

Step 1, cont.

Assume $c^2 X(x)\,T(t) \neq 0$
and divide by it.

$$\underbrace{\frac{T''(t)}{c^2 T(t)}}_{\text{only depends on } t} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{only depends on } x} = k$$

must be constant

Get two ODEs!

$$\boxed{X'' - kX = 0}$$

&

$$\boxed{T'' - kc^2 T = 0}$$

Step 2 | Find solutions X & T of ODEs above so that product XT satisfies BC

BC $\begin{cases} u(0,t) = X(0)\,T(t) = 0 \\ u(L,t) = X(L)\,T(t) = 0 \end{cases}$ for all t

$\Longleftrightarrow$ $\underbrace{T(t) = 0 \text{ for all } t}$ or $\begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases}$

$\Downarrow$

$u \equiv 0$

not interesting

Solve
$$\boxed{\begin{aligned} &X'' - kX = 0 \\ &X(0) = 0 \\ &X(L) = 0 \end{aligned}}$$

Recall linear ODEs

Three cases:

$k = 0$ $\qquad X(x) = Ax + B$

$k = \mu^2 > 0$ $\qquad X(x) = Ae^{\mu x} + Be^{-\mu x}$

$k = -p^2 < 0$ $\qquad X(x) = A\cos px + B\sin px$

$k = 0$ $\quad \begin{cases} X(x) = Ax + B \\ X(0) = 0 \\ X(L) = 0 \end{cases} \longrightarrow \left| \begin{array}{l} B = 0 \\ A = 0 \end{array} \right.$

$\Rightarrow u \equiv 0$

not interesting

$k = \mu^2 > 0$ $\quad \begin{cases} X(x) = Ae^{\mu x} + Be^{-\mu x} \\ X(0) = 0 \\ X(L) = 0 \end{cases} \longrightarrow \left| \begin{array}{l} A + B = 0 \\ Ae^{\mu L} + Be^{-\mu L} = 0 \end{array} \right. \left| \begin{array}{l} A = 0 \\ B = 0 \end{array} \right.$

$\Rightarrow u \equiv 0$

not interesting

$k = -p^2 < 0$ $\quad \begin{cases} X(x) = A\cos px + B\sin px \\ X(0) = 0 \\ X(L) = 0 \end{cases} \longrightarrow \left| \begin{array}{l} A = 0 \\ B\sin pL = 0 \end{array} \right.$

interesting when $pL = m\pi$

(B can be $\neq 0$)

$$p = \frac{m\pi}{L}$$

$$\boxed{X_m(x) = \sin \frac{m\pi}{L} x} \qquad m = 1, 2, \dots$$

are elementary solutions of

$$\begin{cases} X'' - kX = 0 \\ X(0) = 0 \\ X(L) = 0 \end{cases} \qquad \text{for } \boxed{k = -p^2 = -\left(\frac{m\pi}{L}\right)^2}$$

Note : $\quad X_{-m}(x) = -X_m(x)$

Now solve $$\boxed{T'' + \left(\frac{m\pi c}{L}\right)^2 T = 0}$$

use $k = -\left(\frac{m\pi}{L}\right)^2$

Let $\lambda_m = \dfrac{m\pi c}{L}$

$$\boxed{T'' + (\lambda_m)^2 T = 0}$$

General solution for each m :

$$T_m(t) = C_m \cos \lambda_m t + D_m \sin \lambda_m t$$

Summary

Got solutions of wave equation satisfying BC of the form

$$u_m(x,t) = X_m(x)\, T_m(t)$$

$$= \sin \frac{m\pi x}{L} \left(C_m \cos \lambda_m t + D_m \sin \lambda_m t \right)$$

$m = 1, 2, \ldots$

called eigenfunctions or characteristic functions

$\lambda_m = \frac{cm\pi}{L}$ eigenvalues or characteristic values

$\parallel$

values for which there are nontrivial solutions of PDE & BC

Separation of variables

Step 1 | $u(x,t) = X(x)\,T(t)$

PDE $\rightsquigarrow$ 2 ODEs

Step 2 | Impose BC

Get eigenfunctions

Step 3 Impose IC

next week

Eigenfunctions for wave equation represent harmonic motions called <u>normal modes</u>

$u_m \longrightarrow$ frequency $\dfrac{cm}{2L}$ cycles/unit time

$$\left(\frac{cm\pi}{L} t = 2\pi \iff t = \frac{2L}{cm} \text{ period} \right)$$

$\sin \dfrac{m\pi x}{L}$

$m = 1$ fundamental mode

$m = 2$

$m = 3$ overtones

$m = 4$

$\vdots$