31. Oct. 2011

2nd order linear PDEs

Case of two variables:

$u(x,t)$ unknown function

$$\boxed{\underbrace{A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x\partial t} + C\frac{\partial^2 u}{\partial t^2}}_{\substack{\text{principal part} \\ (A,B,C \text{ functions of } x,t)}} = \underbrace{F\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right)}_{\substack{\text{linear in} \\ u,\ \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial t}}}}$$

<u>Example</u> $\frac{1}{5}x^3\frac{\partial^2 u}{\partial x\partial t} + \sin xt\,\frac{\partial^2 u}{\partial t^2} = \pi u - t^2\frac{\partial u}{\partial x} + e^{xt}$

<u>Principal part</u> = 2nd order term

$$\boxed{\text{determines many fundamental properties of solutions}}$$

3 types :

<u>elliptic</u> if $AC - B^2 > 0$

<u>parabolic</u> if $AC - B^2 = 0$

<u>hyperbolic</u> if $\underbrace{AC - B^2}_{\text{discriminant}} < 0$

<u>Recall conics</u> $\underbrace{AX^2 + BXY + CY^2}_{} = DX + EY + F$

$$\begin{bmatrix} X & Y \end{bmatrix} \underbrace{\begin{bmatrix} A & B \\ B & C \end{bmatrix}}_{\text{symmetric matrix } M} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$\det M = AC - B^2$

<u>ellipse</u> when $AC - B^2 > 0$

<u>parabola</u> $= 0$

<u>hyperbola</u> < 0 $)($

The fundamental equations of mathematical physics

Wave equation

$$c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

$A = c^2$ $\quad C = -1 \quad B = 0 \quad F \equiv 0$

is *hyperbolic*

heat equation

$$c^2 \frac{\partial^2 u}{\partial x^2} = \underbrace{\frac{\partial u}{\partial t}}_{F}$$

$A = c^2 \quad B = C = 0$

is *parabolic*

Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$A = C = 1 \quad B = 0$

is *elliptic*

Remarks

- Type may vary from point to point.
- Type is invariant under change of coordinates.
- Can always change coordinates so that the principal part becomes equal to the principal part of a fundamental equation of mathematical physics.

$\Downarrow$

$$\left.\begin{array}{l}\text{Wave eq}\\ \text{heat eq}\\ \text{Laplace eq}\end{array}\right\} \text{ are prototypes}$$

(Solutions of a hyberbolic equation are wave-like, etc)

Types of boundary conditions:

- Dirichlet conditions or BC of first kind

  $u(0,t) = \ldots \quad \& \quad u(L,t) = \ldots \quad , \quad t \geq 0$

  (u is prescribed on boundary)

- Neumann conditions or BC of second kind

  $\frac{\partial u}{\partial x}(0,t) = \ldots \quad \& \quad \frac{\partial u}{\partial x}(L,t) = \ldots \quad , \quad t \geq 0$

  (for higher dimensions, normal derivative is prescribed on boundary)

- Robin conditions or BC of third kind

  $$\alpha u + \beta \frac{\partial u}{\partial x} = 0 \quad \text{at } x = 0 \, , \; t \geq 0$$
  $$\gamma u + \delta \frac{\partial u}{\partial x} = 0 \quad \text{at } x = L \, , \; t \geq 0$$

- mixed conditions

  for instance, Dirichlet at one end
  Neumann at other end

- periodic conditions

  $u(0,t) = u(L,t) \quad \& \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t)$

- Cauchy conditions

  $u(0,t) = \ldots \quad \& \quad \frac{\partial u}{\partial x}(0,t) = \ldots$

- Etc

Laplace's equation

$$\boxed{\Delta u = 0}$$

Δ laplacian

2D $\quad \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

3D $\quad \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

Occurrences :

- steady state heat equation

  steady state $\downarrow$ u independent of t

  $\dfrac{\partial u}{\partial t} = 0$ $\qquad \cancel{\dfrac{\partial u}{\partial t}} = c^2 \Delta u$

- steady state wave equation

  $\cancel{\dfrac{\partial^2 u}{\partial t^2}} = c^2 \Delta u$

- electromagnetism $\rightsquigarrow$ u = electric potential
- astronomy $\rightsquigarrow$ u = gravitational potential
- fluid dynamics $\rightsquigarrow$ u = fluid potential

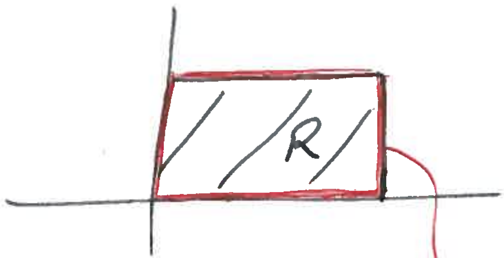
Dirichlet problem for Laplace's equation on a rectangle

$$\begin{cases} \Delta u = 0 & \text{PDE} \\ u(a,y) = f(y) & \\ u(0,y) = u(x,0) = u(x,b) = 0 & \text{BC} \end{cases}$$

$(x,y) \in$ rectangle R

$0 \leq x \leq a$

$0 \leq y \leq b$



∂R boundary of R

Solve by separation of variables

Step 1

$$u(x,y) = X(x)\,Y(y)$$

$$X''Y + XY'' = 0 \qquad \text{PDE}$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

constant

$$\boxed{X'' - kX = 0} \quad \& \quad \boxed{Y'' + kY = 0}$$

Step 2

Impose homogeneous parts of BC

$$u(0,y) = u(x,0) = u(x,b) = 0$$

for all x, y

$$\Downarrow$$

$$X(0) = Y(0) = Y(b) = 0$$

Solve
$$\begin{cases} Y'' + kY = 0 \\ Y(0) = Y(b) = 0 \end{cases}$$

Interesting for $k > 0$

$$Y(y) = A\cos\sqrt{k}\,y + B\sin\sqrt{k}\,y\ ,\quad A, B \in \mathbb{R}$$

Basic solutions are

$$Y_m(y) = \sin\underbrace{\left(\frac{m\pi}{b}\right)}_{=\sqrt{k}} y\ ,\quad m = 1, 2, \dots$$

Solve
$$\begin{cases} X'' - kX = 0 \\ X(0) = 0 \end{cases} \qquad \text{where } k = \left(\frac{m\pi}{b}\right)^2$$

$$X(x) = A^* e^{\sqrt{k}x} + B^* e^{-\sqrt{k}x}\ ,\quad A^*, B^* \in \mathbb{R}$$

Basic solutions are

$$X_m(x) = \sinh\frac{m\pi}{b}x\ ,\quad m = 1, 2, \dots$$

where $\sinh\alpha = \dfrac{e^{\alpha} - e^{-\alpha}}{2}$

Step 3 Impose rest of BC

Consider the series

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \underbrace{\sinh \frac{n\pi}{b} x}_{X_n(x)} \underbrace{\sin \frac{n\pi}{b} y}_{Y_n(y)}} \quad (\star)$$

$$u(a,y) = f(y) \quad \Longleftrightarrow$$

$$\sum_{n=1}^{\infty} A_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi}{b} y = f(y)$$

must be the coefficients of the Fourier sine series of f

$$A_n \sinh \frac{n\pi a}{b} = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi}{b} y \, dy$$

$$\Rightarrow \quad A_n = \frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b f(y) \sin \frac{n\pi}{b} y \, dy \quad (\star\star)$$

Solution of problem is $(\star)$ with coefficients A_n given by $(\star\star)$

Exercise Dirichlet problem for Laplace's eq on a 3D box

$u(x,y,z)$

$$0 \le x \le a$$
$$0 \le y \le b$$
$$0 \le z \le c$$

$$\begin{cases} \Delta u = 0 & \text{PDE} \\ u(a,y,z) = f(y,z) & \\ u(0,y,z) = u(x,0,z) = u(x,b,z) & \\ \quad = u(x,y,0) = u(x,y,c) = 0 & \text{BC} \end{cases}$$

Separation of variables :

$$u(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$\frac{X''}{X} = \underbrace{-\frac{Y''}{Y} - \frac{Z''}{Z}}_{} = k_1$$

$$-\frac{Y''}{Y} = \frac{Z''}{Z} + k_1 = k_2$$

Fill in details!

$$\begin{cases} Y'' + k_2 Y = 0 \\ Y(0) = Y(b) = 0 \end{cases}$$

Need $k_2 > 0$

$$Y_m = \sin\left(\underbrace{\frac{m\pi}{b}}_{=\sqrt{k_2}} y\right), \quad m = 1, 2, \dots$$

$$\begin{cases} Z'' + (k_1 - k_2) Z = 0 \\ Z(0) = Z(c) = 0 \end{cases}$$

Need $k_1 - k_2 > 0$

$$Z_n = \sin\left(\underbrace{\frac{n\pi}{c}}_{=\sqrt{k_1 - k_2}} z\right), \quad n = 1, 2, \dots$$

$$k_1 = \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2 > 0$$

$$k_2 = \left(\frac{m\pi}{b}\right)^2$$

$$\begin{cases} X'' - k_1 X = 0 \\ X(0) = 0 \end{cases}$$

$$X_{mn} = \sinh \underbrace{\pi\sqrt{\left(\frac{m}{b}\right)^2+\left(\frac{n}{c}\right)^2}}_{=\sqrt{k_1}}\, x$$

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sinh \sqrt{k_1}\, x \sin \frac{m\pi}{b} y \sin \frac{n\pi}{c} z$$

with coefficients A_{mn} determined from

$$u(a,y,z) = f(y,z) \quad \Longleftrightarrow$$

$$\sum\sum A_{mn} \sinh \sqrt{k_1}\, a \sin \frac{m\pi}{b} y \sin \frac{n\pi}{c} z = f(y,z)$$

Double Fourier sine series of $f(y,z)$

$$A_{mn} = \frac{1}{\sinh \sqrt{k_1}\, a} \cdot \frac{4}{bc} \int_0^c \int_0^b f(y,z) \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c}\, dy\, dz$$

(formula from last week)

Eigenvalue problems
(Sturm-Liouville)

Suppose we have a problem to which separation of variables applies.

Example (heat flow on some nonhomogeneous rod)

$$\begin{cases} \dfrac{\partial u}{\partial t} - e^{x}(t^2+1)\left(x\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x} + e^{2x}u\right) = 0 & \text{PDE} \\[2ex] \left.\begin{aligned} &u(0,t) - \dfrac{\partial u}{\partial x}(0,t) = 0 \\ &u(L,t) + \dfrac{1}{2}\dfrac{\partial u}{\partial x}(L,t) = 0 \end{aligned}\right\} & \text{BC} \\[2ex] u(x,0) = f(x) & \text{IC} \end{cases}$$

$$0 \leq x \leq L$$

$$t \geq 0$$

Step 1 $\quad u(x,t) = X(x)\,T(t)$

$$XT' - e^{x}(t^2+1)\left(xX''T + X'T + e^{2x}XT\right) = 0$$

PDE

Divide by $(t^2+1)XT$:

$$\frac{T'}{(t^2+1)T} = e^{x}\left(x\frac{X''}{X} + \frac{X'}{X} + e^{2x}\right) = \lambda$$

constant

Get two ODEs :

$$\boxed{\frac{T'}{T} = \lambda(t^2+1)} \qquad \&$$

$$\boxed{\underbrace{xX'' + X' + e^{2x}X}_{\text{is the Sturm-Liouville operator of this problem}} = \lambda \underbrace{e^{-x}}_{W(x)=e^{-x}\ \text{weight function}} X}$$

$$\underbrace{(xX')' + e^{2x}X}$$

Step 2 Impose BC

$$\begin{cases} X(0) - X'(0) = 0 \\ X(L) + \frac{1}{2}X'(L) = 0 \end{cases}$$

Solve
$$\begin{cases} (xX')' + e^{2x}X = \lambda e^{-x}X \\ X(0) - X'(0) = 0 \\ X(L) + \frac{1}{2}X'(L) = 0 \end{cases}$$

only has nontrivial sols for some values of λ

called a Sturm-Liouville eigenvalue problem

A nontrivial solution of it is called an eigenfunction associated with the eigenvalue λ

Linear algebra: $L\vec{v} = \lambda \vec{v}$ eigenvalue problem

$\vec{v} \; (\neq \vec{0})$ eigenvector

λ eigenvalue

<u>Ex</u> $\begin{cases} X'' = \lambda X \\ X(0) = X(L) = 0 \end{cases}$

eigenfunctions $X_n(x) = \sin \frac{n\pi}{L} x$

eigenvalues $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$

$n = 1, 2, \ldots$

Let $X_n(x)$ be a <u>complete orthonormal</u> (*) <u>set of eigenfunctions</u> of our Sturm-Liouville problem with λ_n the corresponding eigenvalues

$n = 1, 2, \ldots$

(*) with respect to <u>inner product</u>

$$\langle g, h \rangle_w = \int_0^L g(x)\, h(x)\, \underbrace{w(x)}_{\text{weight function}}\, dx$$

functions

Solve $\quad \dfrac{T'}{T} = \lambda(t^2+1)$

$$\frac{d}{dt}\ln|T| = \lambda(t^2+1)$$

$$\ln|T| = \lambda\left(\frac{t^3}{3}+t\right) + k$$

$$T(t) = c\, e^{\lambda\left(\frac{t^3}{3}+t\right)}$$

Let $\quad T_n(t) = e^{\lambda_n\left(\frac{t^3}{3}+t\right)}$ for each n

Step 3 Impose IC

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n\, e^{\lambda_n\left(\frac{t^3}{3}+t\right)} X_n(x)}$$

From $u(x,0) = f(x)$, the coefficients c_n will be given by integral formulas

Sturm-Liouville theory generalizes Fourier theory