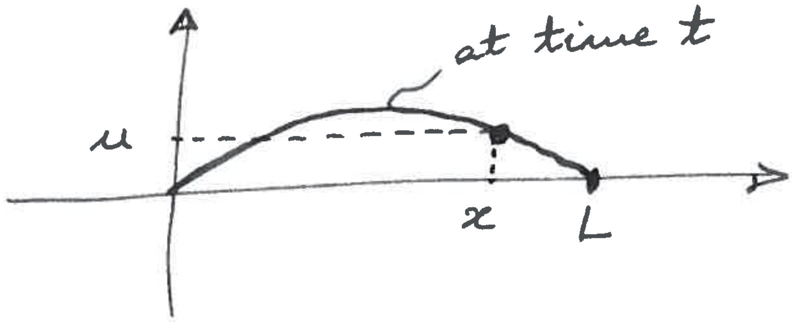
3. Oct. 2011

Wave equation

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

$u(x,t)$ unknown function

$0 \leq x \leq L \quad , \quad t \geq 0$



The wave equation is a linear 2nd order PDE like most equations that interest us.

Terminology

- The <u>order</u> of the equation is the order of the highest derivative in it.

- An equation is <u>linear</u> if it is linear in the unknown function and its partial derivatives.

  - A linear eqn is <u>homogeneous</u> if each term contains either the unknown function or one of its derivatives.

  - Otherwise <u>nonhomogeneous</u>.

Examples

$$t\,\frac{\partial^2 u}{\partial x^2} + 2e^{xt}\,\frac{\partial^2 u}{\partial t^2} + u = \underbrace{1}$$

linear nonhomogeneous

$$t\,\frac{\partial^3 u}{\partial x^2 \partial t} + 2e^{xt}\,\frac{\partial^2 u}{\partial t^2} + u = \underbrace{0}$$

linear homogeneous

$$\frac{\partial^2 u}{\partial x^2} + \underbrace{x u \frac{\partial u}{\partial t}}_{\Downarrow} = \sin x$$

not linear

$$\underbrace{\left(\frac{\partial u}{\partial x}\right)^2}_{\Downarrow} + \sin x \,\frac{\partial u}{\partial t} = 1$$

not linear

Superposition principle

If u_1, u_2 solutions of a homogeneous linear PDE

then $u = \underbrace{c_1 u_1 + c_2 u_2}$ is also a soln

linear combination of u_1, u_2

for any constants c_1, c_2

$$u \longrightarrow \boxed{\mathcal{L}} \xrightarrow{\mathcal{L}[u]}$$

linear operator

$$\mathcal{L}[c_1 u_1 + c_2 u_2] = c_1 \mathcal{L}[u_1] + c_2 \mathcal{L}[u_2]$$

Examples

- $\mathcal{L}[u] = \dfrac{\partial u}{\partial t}$
- $\mathcal{L}[u] = e^{xt} \dfrac{\partial^2 u}{\partial x \partial t}$
- $\mathcal{L}[u] = \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2}$ ← for wave eq
- $\mathcal{L}[u] = \displaystyle\int_0^x u(s,t)\, ds$

Solve

	PDE	Wave equation
	BC	$u(0,t) = u(L,t) = 0$ for all t boundary conditions
	IC	$u(x,0) = f(x)$ $\frac{\partial u}{\partial t}(x,0) = g(x)$ initial conditions

When PDE & BC are linear & homogeneous attempt method of

<u>Separation of variables</u>

3 steps

Last week: steps 1 & 2

Got <u>eigenfunctions</u> $u_n(x,t)$
$\parallel$
nontrivial solutions of PDE & BC

$$u_n(x,t) = X_n(x)\, T_n(t)$$
$$= \sin\frac{n\pi x}{L}\left(C_n\cos\lambda_n t + D_n\sin\lambda_n t\right)$$
$$n = 1, 2, \dots$$

where $\lambda_n = \dfrac{cn\pi}{L}$ "eigenvalues"

$u_n(x,t)$ represents <u>harmonic motion</u> called <u>normal mode</u> with <u>period</u> $\dfrac{2L}{cn}$, <u>frequency</u> $\dfrac{cn}{2L}$

$n = 1$: <u>fundamental</u> or <u>first harmonic</u> mode
(other frequencies are multiples of it)

adjust pitch $\longleftrightarrow$ adjust frequency $\longleftrightarrow$ adjust L, $c = \sqrt{\dfrac{T}{\rho}}$

By superposition principle

$$u(x,t) = \sum_{\text{finite}} u_n(x,t)$$

is a solution of PDE & BC

Step 3 Impose IC by considering formal solution of form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left(C_n \cos \lambda_n t + D_n \sin \lambda_n t \right) \sin \frac{n\pi x}{L}$$

2

convergence!

$$u(x,0) = f(x)$$

$$\Longleftrightarrow \quad \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} = f(x)$$

Choose C_n's as coefficients of <u>Fourier sine series</u> of $f(x)$

$$\boxed{C_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

$$\Longleftrightarrow \left.\sum_{n=1}^{\infty}\left(-C_n\lambda_n \sin\lambda_n t + D_n\lambda_n \cos\lambda_n t\right)\sin\frac{n\pi x}{L}\right|_{t=0} = g(x)$$

2 convergence

$$\Longleftrightarrow \sum_{n=1}^{\infty} D_n\lambda_n \sin\frac{n\pi x}{L} = g(x)$$

Choose $D_n\lambda_n$'s as coefficients of <u>Fourier sine series</u> of $g(x)$

$$D_n\lambda_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$$\boxed{D_n = \frac{2}{c\,n\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx}$$

So
$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos \lambda_n t + D_n \sin \lambda_n t \right) \sin \frac{n\pi x}{L}}$$

where $\lambda_n = \dfrac{c n \pi}{L}$

C_n & D_n are as above
(from Fourier sine series of $f(x)$, $g(x)$)

is a <u>formal solution</u> of our problem

ignore convergence issues
work term by term

When is this an honest <u>solution</u>?

(Depends on $f(x)$, $g(x)$.)

Recall <u>Fourier series</u>

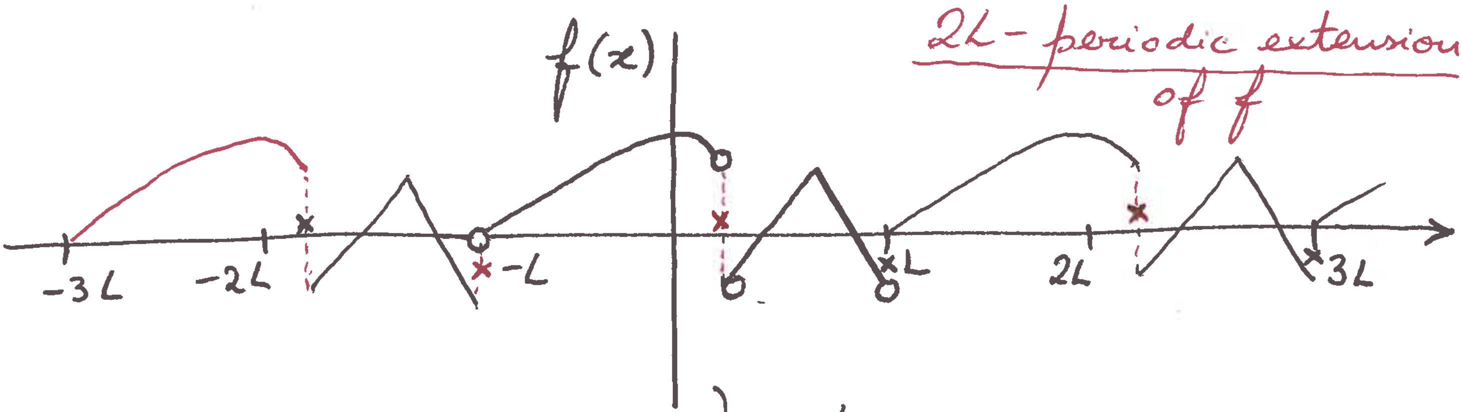
$$\underbrace{f(x)}_{\text{given function}} \sim \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}}_{\text{its Fourier Series F.S.}}$$

for $-L \le x \le L$

Where

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{aligned}}$$

Fourier coefficients



f is **piecewise smooth** on $[-L, L]$

only a finite number of finite jumps of f or $\frac{df}{dx}$

elsewhere f and $\frac{df}{dx}$ are continuous

Fourier Thm If $f(x)$, $-L \le x \le L$ is piecewise smooth

then F.S. of $f(x)$ converges to

a) the periodic extension of $f(x)$ where that extension is continuous,

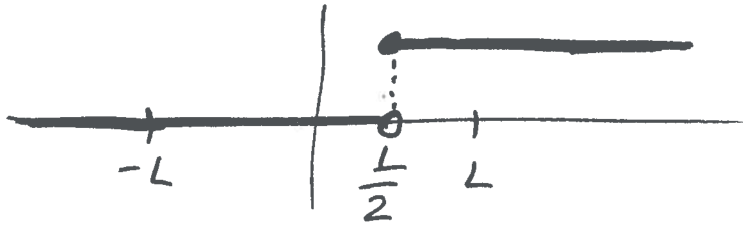
b) the average of the two limits

$$\frac{1}{2}\left[f(x^+) + f(x^-) \right] \quad \times$$

where extension has a jump.

Example

$$f(x) = \begin{cases} 0 & x < \frac{L}{2} \\ 1 & x \geqslant \frac{L}{2} \end{cases}$$

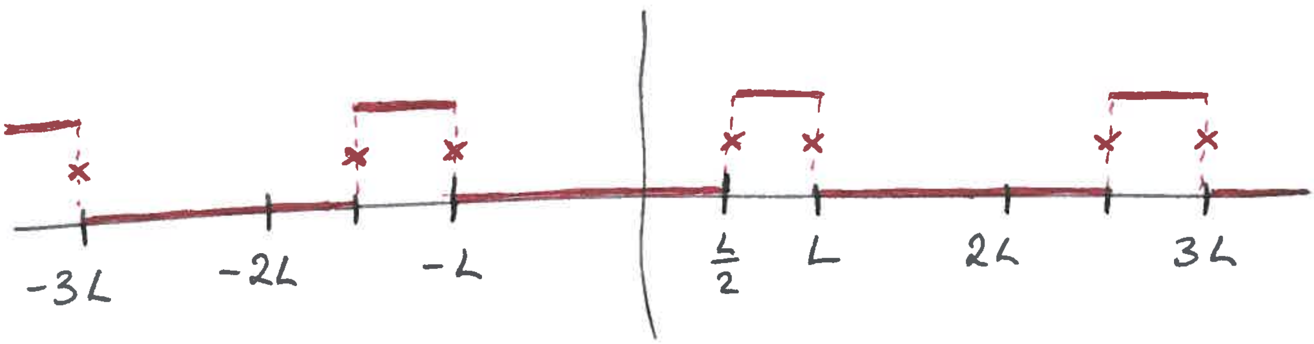


For Fourier series of $f(x)$ on $[-L, L]$:

① What | is it equal to | does it converge to | at each x ?

② What are the Fourier coefficients?

① $f(x)$ is piecewise smooth so can apply Fourier's theorem :



On $[-L, L]$ F.S. converges to

$$\begin{cases} \frac{1}{2} & x = -L \\ 0 & -L < x < \frac{L}{2} \\ \frac{1}{2} & x = \frac{L}{2} \\ 1 & \frac{L}{2} < x < L \\ \frac{1}{2} & x = L \end{cases}$$

(2) $$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx = \frac{1}{2L}\int_{\frac{L}{2}}^{L} dx = \frac{1}{4}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\, dx = \frac{1}{L}\int_{\frac{L}{2}}^{L}\cos\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{L}\left[\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L} = -\frac{1}{n\pi}\sin\frac{n\pi}{2}$$

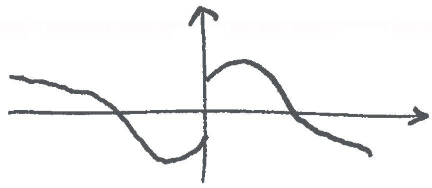
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\, dx = \frac{1}{L}\int_{\frac{L}{2}}^{L}\sin\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{L}\left[-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L} = \frac{1}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

Special cases

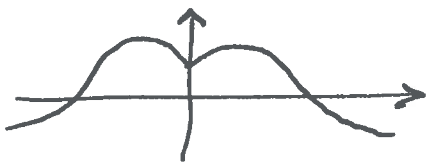
odd functions $\quad f(-x) = -f(x)$

$$\int_{-L}^{L} (\text{odd fn})\, dx = 0 \qquad (*)$$



even functions $\quad f(-x) = f(x)$

$$\int_{-L}^{L} (\text{even fn})\, dx = 2\int_{0}^{L} (\text{even fn})\, dx \qquad (**)$$



$(\text{odd fn}) \cdot (\text{even fn})$ is odd

$(\text{odd fn}) \cdot (\text{odd fn})$ and $(\text{even fn}) \cdot (\text{even fn})$ are even

F.S. of an <u>odd function</u> :

$$a_n = 0 \qquad n = 0, 1, \ldots \qquad \text{by } (*)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \qquad \text{by } (**)$$

$$\boxed{f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}}$$

<u>Fourier sine series</u>

F.S. of an <u>even function</u> :

$$b_n = 0 \qquad \text{by } (*)$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad \text{by } (**)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad \text{—}$$

$$\boxed{f(x) \sim \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}}$$

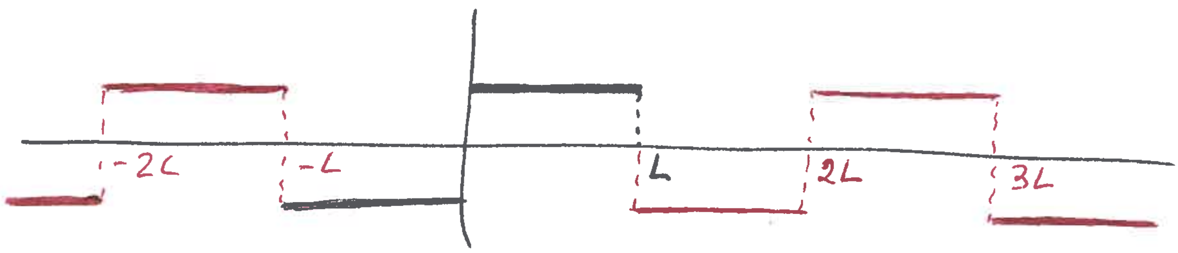
<u>Fourier cosine series</u>

Example $f(x) = 1 \qquad 0 \le x \le L$

① Find <u>Fourier sine series</u>

② Find <u>Fourier cosine series</u>

① Extend f as <u>odd function</u> of period $2L$

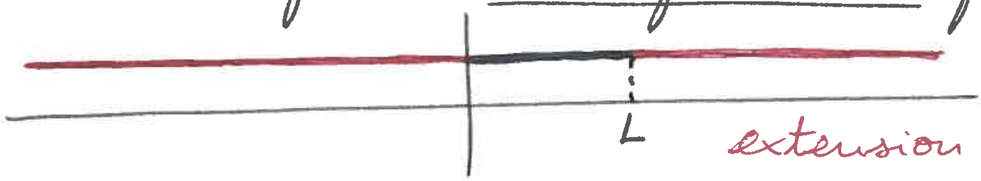


$$b_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\left[-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right]_0^L$$

$$= \frac{2}{n\pi}\Big(1-\underbrace{\cos n\pi}_{(-1)^n}\Big) = \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{n\pi} & n \text{ odd} \end{cases}$$

$$1 \sim \sum_{n \text{ odd}} \frac{4}{n\pi}\sin\frac{n\pi x}{L}$$

② Extend f as <u>even function</u> of period $2L$



extension is always 1

$1 \sim 1 \leftarrow$ Fourier cosine series $\quad \Big|\; a_0 = 1,\; a_n = 0,\ n \ge 1$

Continuity

Consequences of Fourier's theorem for a piecewise smooth $f(x)$:

$$\boxed{\begin{array}{l} \text{F.S. of } f \text{ is continuous for } -L \le x \le L \\ \iff f(x) \text{ is continuous and } f(-L) = f(L) \end{array}}$$

extension of f is continuous

$$\boxed{\begin{array}{l} \text{Fourier cosine series of } f \text{ is continuous for } 0 \le x \le L \\ \iff f(x) \text{ is continuous} \end{array}}$$

even extension is continuous

$$\boxed{\begin{array}{l} \text{Fourier sine series of } f \text{ is continuous for } 0 \le x \le L \\ \iff f(x) \text{ is continuous and } f(0) = f(L) = 0 \end{array}}$$

odd extension is continuous