2. Mai 2012

Tensor product

Recall from lecture 8 (page 8.9 of notes):

A tensor of type (p,q) or (p,q)-tensor

contravariant rank p covariant rank q

is a $(p+q)$-linear function

$$\underbrace{V^* \times \dots \times V^*}_{p \text{ factors}} \times \underbrace{V \times \dots \times V}_{q \text{ factors}} \longrightarrow \mathbb{R}$$

And recall from lecture 5 (page 5.11 of notes)

The <u>tensor product</u> of a $(0,q)$-tensor T

$$T : \underbrace{V \times \dots \times V}_{q} \longrightarrow \mathbb{R}$$

with a $(0,\ell)$-tensor U is the $(0,q+\ell)$-tensor

$$U : \underbrace{V \times \dots \times V}_{\ell} \longrightarrow \mathbb{R}$$

$$T \otimes U : \underbrace{V \times \dots \times V}_{q+\ell \text{ factors}} \longrightarrow \mathbb{R}$$ defined by

$$(T \otimes U)(v_1, \dots, v_{q+\ell}) = T\big(\underbrace{v_1, \dots, v_q}_{\text{first } q \text{ vectors}}\big)\, U\big(\underbrace{v_{q+1}, \dots, v_{q+\ell}}_{\text{last } \ell \text{ vectors}}\big)$$

More generally:

Defn The <u>tensor product</u> of a (p,q)-tensor T

$$T: \underbrace{V^* \times \dots \times V^*}_{p} \times \underbrace{V \times \dots \times V}_{q} \to \mathbb{R}$$

with a (k,ℓ)-tensor U is the

$$U: \underbrace{V^* \times \dots \times V^*}_{k} \times \underbrace{V \times \dots \times V}_{\ell} \to \mathbb{R}$$

$(p+k, q+\ell)$-tensor $T \otimes U: \underbrace{V^* \times \dots \times V^*}_{p+k} \times \underbrace{V \times \dots \times V}_{q+\ell} \to \mathbb{R}$

defined by

$$(T \otimes U)(\alpha_1, \dots, \alpha_{p+k}, v_1, \dots, v_{q+\ell})$$

$$= T(\alpha_1, \dots, \alpha_p, v_1, \dots, v_q)\, U(\alpha_{p+1}, \dots, \alpha_{p+k}, v_{q+1}, \dots, v_{q+\ell})$$

$T \otimes U$ and $U \otimes T$ are tensors of the same type, but in general

$$T \otimes U \neq U \otimes T$$

The set of all tensors of type (p,q) on V is a vector space denoted

$$\mathcal{T}^{p}_{q}(V) = \{\text{all } (p,q)\text{-tensors on } V\}$$

$$= \underbrace{V \otimes \ldots \otimes V}_{p} \otimes \underbrace{V^{*} \otimes \ldots \otimes V^{*}}_{q}$$

This last notation will be justified later (see page 10.6).

Choose $\mathcal{B} = (b_1, \dots, b_n)$ basis of V

Let $\mathcal{B}^* = (\beta^1, \dots, \beta^n)$ corresponding dual basis of V^*

($\mathcal{B}^*$ depends on $\mathcal{B}$)

By taking appropriate tensor products of elements of $\mathcal{B}$ and elements of $\mathcal{B}^*$ we obtain a basis for $\mathcal{T}_q^p(V)$:

$$b_{i_1} \otimes b_{i_2} \otimes \dots \otimes b_{i_p} \otimes \beta^{j_1} \otimes \beta^{j_2} \otimes \dots \otimes \beta^{j_q}$$

where the indices $i_1, \dots, i_p, j_1, \dots, j_q$ take all values between 1 and n.

(We checked this earlier for tensors of order 2. The general case is similar.)

There are $\underbrace{n \times \dots \times n}_{p} \times \underbrace{n \times \dots \times n}_{q}$ elements in this basis, hence $\dim \mathcal{T}_q^p(V) = n^{p+q}$.

Let V and W be two finite-dimensional vector spaces.

$$\dim V = n \ , \quad \dim W = m$$

Choose $(b_1, \dots, b_n)$ basis of V
$(a_1, \dots, a_m)$ basis of W.

<u>Defn</u> $V \otimes W$ is the $n \cdot m$ - dimensional vector space with basis

$$b_i \otimes a_j \qquad \begin{array}{l} i = 1, \dots, n \\ j = 1, \dots, m \end{array}$$

$V \otimes W$ is called the <u>tensor product</u> of vector spaces V and W.

<u>Note</u>
- We can have more factors
  $V_1 \otimes V_2 \otimes \dots \otimes V_k$
  (the tensor product is associative).
- $(V \otimes W)^* \simeq V^* \otimes W^*$

We will now see why people say that

> "the tensor product linearizes what was bilinear or multilinear".

Let V, W vector spaces
$\dim V = n$, $\dim W = m$

There are various ways of looking at bilinear maps $V \times W \longrightarrow \mathbb{R}$:

Claim

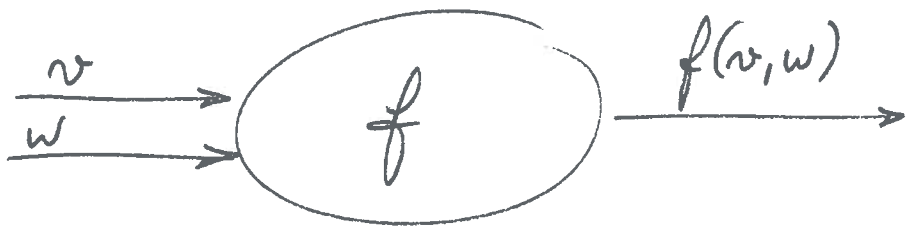
$$\begin{aligned}
\underbrace{\boxed{\mathrm{Bil}(V \times W, \mathbb{R})}}_{\text{cartesian product}} &\simeq \mathrm{Lin}(V, W^*) \\
&\simeq \mathrm{Lin}(W, V^*) \\
&\simeq V^* \otimes W^* \\
&\simeq (V \otimes W)^* \\
&\simeq \underbrace{\boxed{\mathrm{Lin}(V \otimes W, \mathbb{R})}}_{\text{tensor product}}
\end{aligned}$$

$\mathrm{Lin}(V, W^*) = \{$all linear maps $V \to W^*\}$

Here is the idea:

Take $$\underbrace{\boxed{f \in \mathrm{Bil}(V \times W, \mathbb{R})}}_{f : V \times W \longrightarrow \mathbb{R} \ \text{bilinear function}}$$

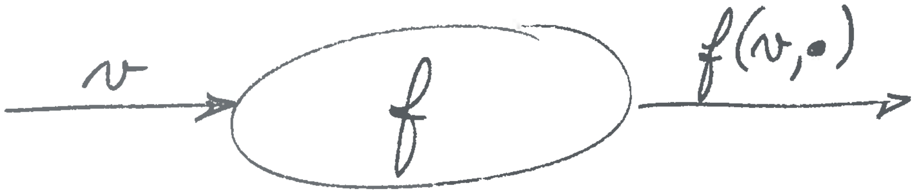
Think of f as a machine taking pairs of vectors and producing numbers out of them.



input: two vectors $v \in V$, $w \in W$

output: real number $f(v,w) \in \mathbb{R}$

If we only feed one vector as input, the machine only gives something that awaits a second vector to produce a no.



input : one vector $v \in V$

output : <u>a linear form</u>!

$$f(v,\cdot) : W \longrightarrow \mathbb{R}$$
$$w \longmapsto f(v,w)$$

so $f(v,\cdot) \in W^*$

Therefore, we can view

$$\boxed{f \in \mathrm{Lin}\,(V, W^*)}$$

Similarly, if the input is only a vector $w \in W$, we can view the output as $f(\cdot, w) \in V^*$.

$$\boxed{f \in \mathrm{Lin}\,(W, V^*)}$$

By our definition of $V^* \otimes W^*$ we have

$$\boxed{\mathrm{Bil}(V \times W, \mathbb{R}) \simeq V^* \otimes W^*}$$

because these spaces both have basis

$$\beta^i \otimes \alpha^j \qquad \begin{matrix} i = 1, \dots, n \\ j = 1, \dots, m \end{matrix}$$

where

$b_1, \dots, b_n$ basis of V
$\beta^1, \dots, \beta^n$ dual basis of V^*
$a_1, \dots, a_m$ basis of W
$\alpha^1, \dots, \alpha^m$ dual basis of V^*

(There is a way to define the tensor product of vector spaces that does not involve bases, but we will not do it here.)

Finally an element $D_{ij}\, \beta^i \otimes \alpha^j \in V^* \otimes W^*$ may be viewed as a linear map $V \otimes W \to \mathbb{R}$, that is, as an element of $(V \otimes W)^*$ by

$$\begin{aligned} V \otimes W &\longrightarrow \mathbb{R} \\ C^{kl} b_k \otimes a_l &\longmapsto D_{ij}\, C^{kl} \underbrace{\beta^i(b_k)}_{\delta^i_k} \underbrace{\alpha^j(a_l)}_{\delta^j_l} \\ &= D_{ij}\, C^{ij} \end{aligned}$$