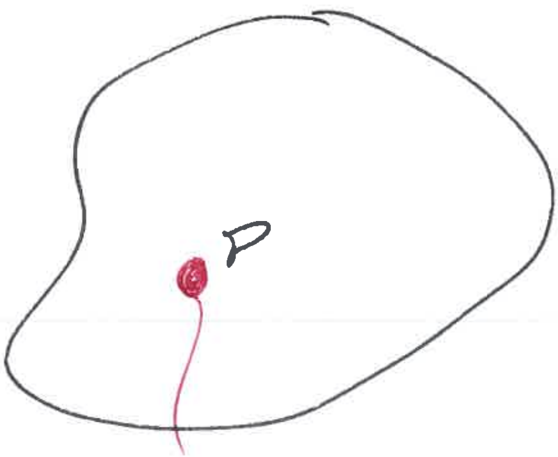
9. Mai 2012

Stress tensor

SPANNUNG

It was the concept of stress in mechanics that originally led to the invention of tensors

stress < tenseur / tension > tensor



rigid body M

infinitesimal body region around point P

We assume the body is acted upon by external forces but is in static equilibrium.

Two types of <u>external forces</u>

<u>body forces</u>

=

forces whose magnitudes are proportional to the <u>volume</u>/mass of the region; for instance:

— gravity

— centrifugal force

<u>surface forces</u>

=

forces whose magnitudes are proportional to the <u>area</u> of the region

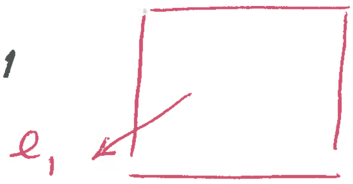
<u>stress</u> = surface force per unit of area

We will concentrate on <u>homogeneous stress</u> i.e. stress which does not depend on location of element in the body (but depends on the orientation of the surface/plane).

Choose an o.n. basis e_1, e_2, e_3

Consider a plane through P parallel to the $e_2 e_3$ coordinate plane.

That plane has normal e_1



Let ΔA_1 be the area of the slice of the region cut by that plane and let ΔF be the force acting on that slice.

$$\Delta F = \underbrace{\Delta F^1}\, e_1 + \underbrace{\Delta F^2}\, e_2 + \underbrace{\Delta F^3}\, e_3$$

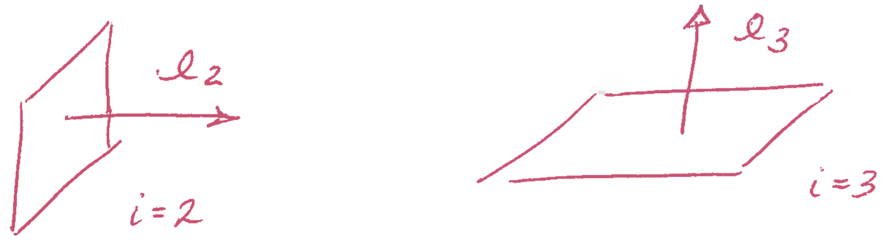
components of the force vector

We define the numbers

$$\sigma^{1j} = \lim_{\Delta A_1 \to 0} \frac{\Delta F^j}{\Delta A_1} \qquad \text{force/area}$$

$j = 1, 2, 3$

Similarly by considering planes parallel to the other coordinate planes.



We define
$$\sigma^{ij} = \lim_{\Delta A_i \to 0} \frac{\Delta F^j}{\Delta A_i}$$

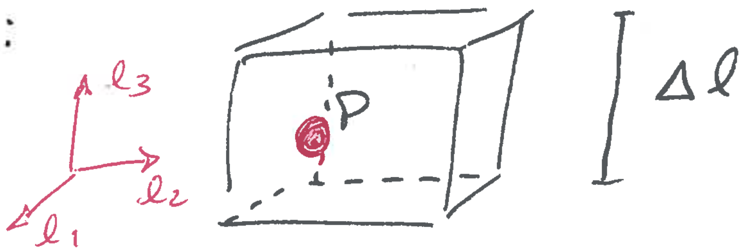
It turns out that the resulting nine numbers σ^{ij} form a contravariant 2-tensor called the <u>stress tensor</u>.

For homogeneous stress the stress tensor σ^{ij} does not depend on the point P.
However, when we flip the orientation of the normal, the stress flips sign.

The stress tensor σ^{ij} is a symmetric tensor

Why?

Consider an infinitesimal cube of side Δl surrounding P:



The forces acting on each of the six faces of the cube are: (assume that the faces are parallel to the coordinate planes)

$\sigma^{1j} \Delta A_1 \, l_j$ and $-\sigma^{1j} \Delta A_1 \, l_j$ for the front and back faces, respec.

$\sigma^{2j} \Delta A_2 \, l_j$ and $-\sigma^{2j} \Delta A_2 \, l_j$ for the right and left faces, respectively

$\sigma^{3j} \Delta A_3 \, l_j$ and $-\sigma^{3j} \Delta A_3 \, l_j$ for the top and bottom faces, respectively

where $\Delta A_1 = \Delta A_2 = \Delta A_3 = \Delta s = (\Delta l)^2$ is the common face area.

Compute the total <u>moment</u>

tendency of a force to "twist" or rotate an object

assuming forces are at the center of the faces (whose distance is $\frac{\Delta l}{2}$ to the center point P):

$$
\begin{aligned}
L = {} & \frac{\Delta l}{2} e_1 \times \sigma^{1j} \Delta s \, e_j + \left(-\frac{\Delta l}{2} e_1\right) \times \left(-\sigma^{1j} \Delta s \, e_j\right) \\
& + \frac{\Delta l}{2} e_2 \times \sigma^{2j} \Delta s \, e_j + \left(-\frac{\Delta l}{2} e_2\right) \times \left(-\sigma^{2j} \Delta s \, e_j\right) \\
& + \frac{\Delta l}{2} e_3 \times \sigma^{3j} \Delta s \, e_j + \left(-\frac{\Delta l}{2} e_3\right) \times \left(-\sigma^{3j} \Delta s \, e_j\right) \\
= {} & \Delta l \, \Delta s \left(\left(\sigma^{23} - \sigma^{32}\right) e_1 + \left(\sigma^{31} - \sigma^{13}\right) e_2 + \left(\sigma^{12} - e^{21}\right) e_3 \right)
\end{aligned}
$$

Static equilibrium $\Longrightarrow$ $L = 0$

$$\Longrightarrow \boxed{\sigma^{ij} = \sigma^{ji}}$$

$\square$

$$\sigma = \begin{pmatrix} \sigma^{11} & \sigma^{12} & \sigma^{13} \\ \sigma^{12} & \sigma^{22} & \sigma^{23} \\ \sigma^{13} & \sigma^{23} & \sigma^{33} \end{pmatrix}$$

<u>normal components</u>

~ components of force acting perpendicularly to coord. planes

<u>shear components</u>

~ components of force acting parallel to coord. planes

Since it is a <u>symmetric</u> 2-tensor σ can be orthogonally diagonalized:

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

<u>principal stresses</u> = eigenvalues of σ

The eigenspaces of σ are the <u>principal directions</u>

(Shear components disappear for <u>principal planes</u>.)

Special forms of the stress tensor *

- Uniaxial stress

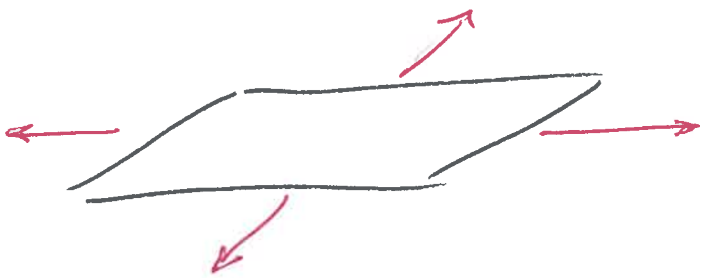
$$\begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>Ex</u> stress tensor in a long vertical rod loaded by hanging a weight on the end

- Plane stressed state / biaxial stress

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>Ex</u>



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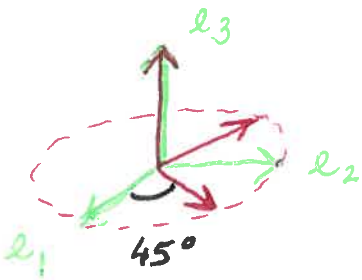
* written with respect to an o.n. eigenbasis or another special basis

- Pure shear

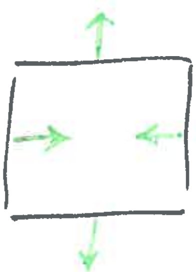
$$\underbrace{\begin{pmatrix} -\sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{stress tensor written w.r.t. o.n. eigenbasis}} \quad \text{or} \quad \underbrace{\begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{stress tensor written w.r.t. o.n. basis obtained by rotating eigenbasis through } 45^\circ \text{ about 3rd axis}}$$

$$\begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T} \begin{pmatrix} -\sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{L}$$



columns of L give new basis in terms of old eigenbasis

Ex



or

- Shear deformation

$$\begin{pmatrix} 0 & \sigma^{12} & \sigma^{13} \\ \sigma^{12} & 0 & \sigma^{23} \\ \sigma^{13} & \sigma^{23} & 0 \end{pmatrix}$$

w.r.t. some o.n. basis

<u>Fact</u> σ is a shear deformation <u>iff</u> its trace is zero

<u>Ex</u> $\begin{pmatrix} 2 & -4 & 0 \\ -4 & 0 & 4 \\ 0 & 4 & -2 \end{pmatrix}$ represents a shear deformation

Check:

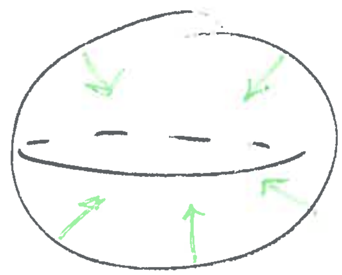
$$\underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}}_{L^T} \begin{pmatrix} 2 & -4 & 0 \\ -4 & 0 & 4 \\ 0 & 4 & -2 \end{pmatrix} \underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}}_{L} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 4\sqrt{2} \\ -2 & 4\sqrt{2} & 0 \end{pmatrix}$$

- Hydrostatic pressure $\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$

$$p = \text{pressure} \neq 0$$

(all eigenvalues are equal to $-p$)

<u>Ex</u> pressure of fluid on a bubble



<u>Exercise</u> Any stress tensor can be written as the sum of a hydrostatic pressure and a shear deformation.

Hint : look at the trace.

Invariants

Let A be a 3×3 matrix with entries a^{ij}

The characteristic polynomial of A $p_A(\lambda)$ is invariant under change of basis.

$$p_A(\lambda) = \det(A-\lambda I) = \det\begin{pmatrix} a^{11}-\lambda & a^{12} & a^{13} \\ a^{21} & a^{22}-\lambda & a^{23} \\ a^{31} & a^{32} & a^{33}-\lambda \end{pmatrix}$$

$$= -\lambda^3 + \operatorname{tr} A\,\lambda^2 - \underbrace{\ldots}_{\substack{\text{quadratic expression}\\ \text{in entries of } A}}\lambda + \det A$$

Applying this to the stress tensor $(A=\sigma)$ we obtain stress invariants:

$$I_1 = \operatorname{tr}\sigma = \sigma^{11}+\sigma^{22}+\sigma^{33}$$

$$I_2 = (\sigma^{12})^2+(\sigma^{23})^2+(\sigma^{13})^2-\sigma^{11}\sigma^{22}-\sigma^{22}\sigma^{33}-\sigma^{33}\sigma^{11}$$

$$I_3 = \det\sigma$$

I_1, I_2, I_3 remain unchanged under o.n. basis change