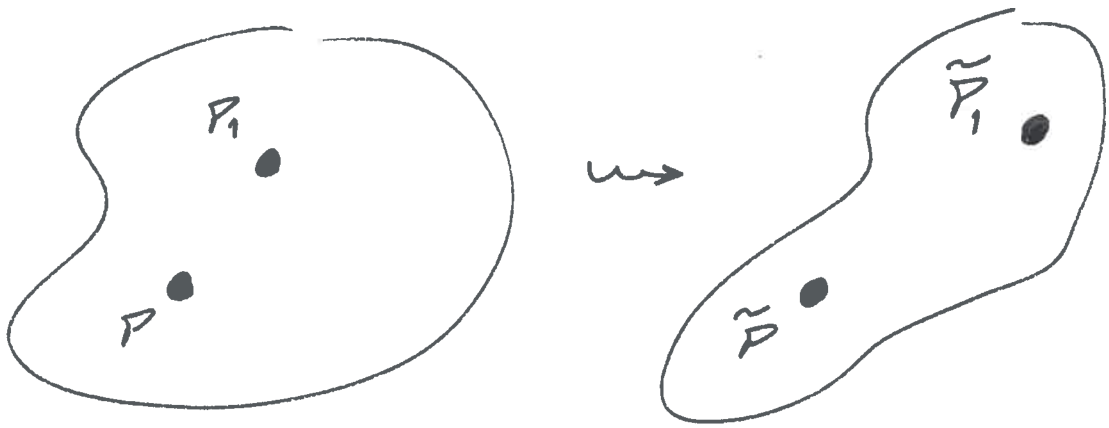
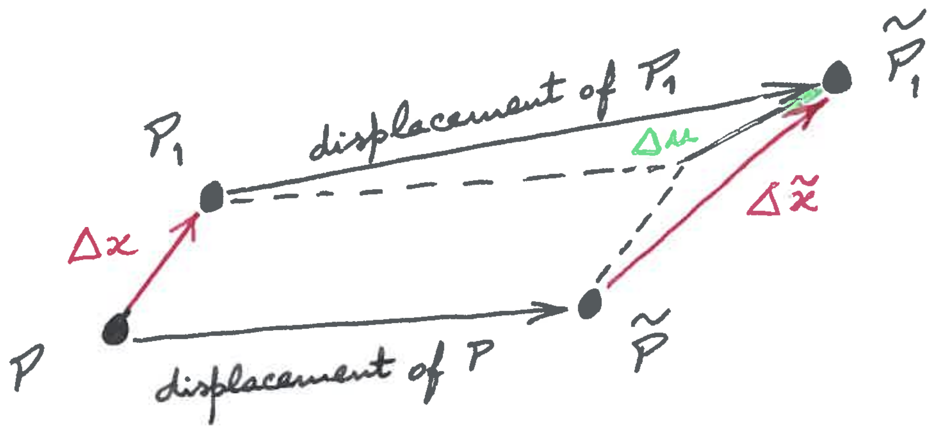
16. Mai 2012

Strain tensor VERZERRUNG

Consider a body being slightly deformed



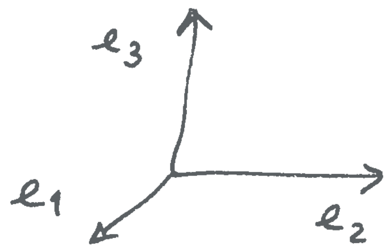


$$\underbrace{\Delta\tilde{x}}_{\substack{\text{new}\\\text{relative}\\\text{position}}} = \underbrace{\Delta x}_{\substack{\text{old}\\\text{relative}\\\text{position}}} + \underbrace{\Delta u}_{\substack{\text{difference of displacements}\\\text{measures the deformation}}}$$

Assume that we have a small <u>homogeneous deformation</u>

$$\Delta u = \underbrace{f}_{\substack{\text{small linear function} \\ \text{independent of point } P}}(\Delta x)$$

Write the components of Δu and Δx w.r.t. an o.n. basis



The function f will than be represented by a matrix with entries called e_{ij} :

$$\Delta u^i = e_{ij} \, \Delta x^j$$

Write the matrix (ℓ_{ij}) as a sum of a symmetric and an antisymmetric matrix:

$$\boxed{\ell_{ij} = \underbrace{\varepsilon_{ij}}_{\text{symmetric}} + \underbrace{\omega_{ij}}_{\text{antisymmetric}}}$$

where

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\ell_{ij} + \ell_{ji}\right)}$$

is called the <u>strain tensor</u> (or deformation tensor)

$$\boxed{\omega_{ij} = \frac{1}{2}\left(\ell_{ij} - \ell_{ji}\right)}$$

is called the <u>rotation tensor</u>

Why these names : <u>strain</u> and <u>rotation</u> ?

- First see how a (small) antisymmetric 3×3 matrix represents a (small) rotation in 3-space.

Let $\omega = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a fixed vector

The rule associating to each vector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ the cross product $\omega \times v$ is a linear function, hence representable by a <u>matrix</u> :

$$\omega \times v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ a & b & c \\ x & y & z \end{pmatrix}$$

$$= \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}}_{\text{antisymmetric matrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

For the matrix
$$\omega_{ij} = \begin{pmatrix} 0 & \omega_{12} & -\omega_{31} \\ -\omega_{12} & 0 & \omega_{23} \\ \omega_{31} & -\omega_{23} & 0 \end{pmatrix}$$

the corresponding vector is
$$\omega = \begin{pmatrix} -\omega_{23} \\ -\omega_{31} \\ -\omega_{12} \end{pmatrix}.$$

Suppose that the matrix (e_{ij}) was already antisymmetric so that
$$\omega_{ij} = e_{ij} \quad \text{and} \quad \varepsilon_{ij} = 0.$$

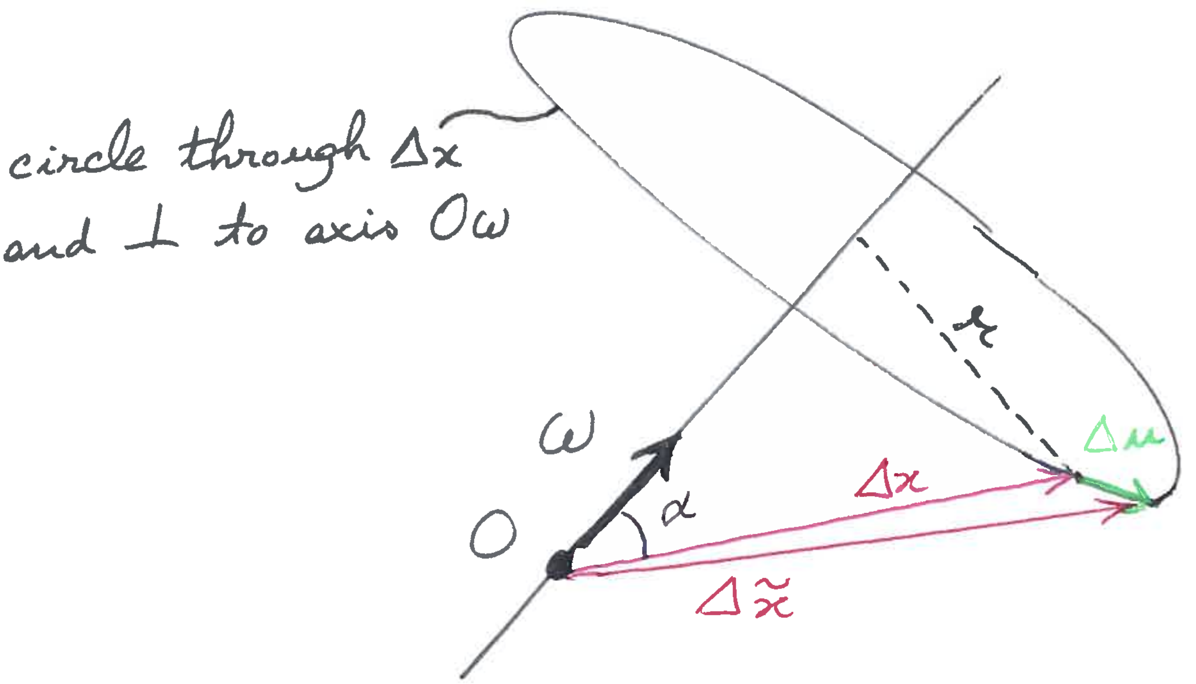
The relation $\Delta u^i = e_{ij}\, \Delta x^j$

would be equivalent to
$$\Delta u = \omega \times \Delta x$$

so that
$$\Delta \tilde{x} = \Delta x + \Delta u = \Delta x + \omega \times \Delta x$$

For ω small (infinitesimal) this represents a small (infinitesimal) rotation through the small angle (infinitesimal) $\|\omega\|$ about the axis $O\omega$.

All this can be made precise using differential calculus.



$\Delta u = \omega \times \Delta x$ is tangent to the circle and has magnitude

$$\underbrace{\|\omega\|}_{\sim \text{ angle in circumference}} \; \underbrace{\|\Delta x\| \sin\alpha}_{r}$$

- The opposite extreme case is when the matrix (e_{ij}) was already <u>symmetric</u> so that

$$\varepsilon_{ij} = e_{ij} \quad \text{and} \quad \omega_{ij} = 0$$

Since $\Delta\tilde{x} = \Delta x + \Delta u$
the relation $\Delta u^i = \varepsilon_{ij}\,\Delta x^j$ gives

$$\begin{aligned} \Delta\tilde{x}^i &= \underbrace{\Delta x^i} + \underbrace{\Delta u^i} \\ &= \underbrace{\delta_{ij}\,\Delta x^j} + \underbrace{\varepsilon_{ij}\,\Delta x^j} \\ &= \Big(\delta_{ij} + \underbrace{\varepsilon_{ij}}\Big)\,\Delta x^j \end{aligned}$$

we always assume this is small
meaning infinitesimal
(differential calculus makes this precise)

We will see that it is ε_{ij} that encodes the changes in the distances :

$$\|\Delta\tilde{x}\|^2 = \Delta\tilde{x}\cdot\Delta\tilde{x}$$

$$= (\Delta x + \Delta u)\cdot(\Delta x + \Delta u)$$

$$= \Delta x\cdot\Delta x + 2\,\Delta x\cdot\Delta u + \underbrace{\Delta u\cdot\Delta u}_{\sim 0}$$

neglect $\|\Delta u\|^2$ when $\Delta u \to 0$ (compared to Δu)

$$\sim \|\Delta x\|^2 + 2\,\varepsilon_{ij}\,\Delta x^i\,\Delta x^j$$

Note For an antisymmetric ω_{ij} the term $2\,\omega_{ij}\,\Delta x^i\,\Delta x^j = 0$ so that it is always the symmetric part that is relevant for the distortion of distances.

Recall metric tensor or inner product encoding distances among points.

<u>A deformation changes the metric tensor.</u>

Let g be the metric before the deformation
$\tilde{g}$ ... after ...

$$\|\Delta\tilde{x}\|^2 \overset{\text{defn}}{\equiv} \tilde{g}(\Delta\tilde{x}, \Delta\tilde{x})$$
$$= \tilde{g}_{ij}\,\Delta\tilde{x}^i\,\Delta\tilde{x}^j = \tilde{g}_{ij}(\Delta x^i + \Delta u^i)(\Delta x^j + \Delta u^j)$$

$$\|\Delta x\|^2 \overset{\text{defn}}{\equiv} g(\Delta x, \Delta x)$$
$$= g_{ij}\,\Delta x^i\,\Delta x^j$$

For infinitesimal deformations $(\Delta u \sim 0)$
we obtain from the formula on page 12.8 that

$$\tilde{g}_{ij}\,\Delta x^i\,\Delta x^j \sim g_{ij}\,\Delta x^i\,\Delta x^j + 2\,\varepsilon_{ij}\,\Delta x^i\,\Delta x^j$$

and hence

$$\varepsilon_{ij} \sim \frac{1}{2}\left(\tilde{g}_{ij} - g_{ij}\right)$$

<u>ε_{ij} measures the change in the metric.</u>

By definition, the strain tensor ε_{ij} is a symmetric tensor

$$\varepsilon = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix}$$

determine the relative elongation/contraction of the body along coordinate directions e_1, e_2, e_3

shear components of the strain tensor

Since it is a symmetric 2-tensor ε can be orthogonally diagonalized:

$$\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

principal coefficients of the deformation = eigenvalues of ε

The eigenspaces are the principal directions of the deformation.

Special forms of the strain tensor

- Shear deformation when $\mathcal{E}$ is traceless:

$$\operatorname{tr} \mathcal{E} = \mathcal{E}_{11} + \mathcal{E}_{22} + \mathcal{E}_{33} = 0$$

- Uniform compression when the principal coefficients of $\mathcal{E}$ are equal (and nonzero):

$$\mathcal{E} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

Exercise Any strain tensor can be written as the sum of a uniform compression and a shear deformation.

Elasticity tensor

The stress tensor represents an external exertion on the material.

The strain tensor represents the material reaction to that exertion.

These are called field tensors in crystallography because they represent imposed conditions, as opposed to matter tensors which represent material properties.

Hooke's law says that, for small deformations strain is related to stress by a matter tensor called elasticity tensor (or stiffness tensor):

$$\boxed{\sigma^{ij} = E^{ijkl}\, \varepsilon_{kl}}$$

the elasticity tensor has rank 4

An arbitrary 4-tensor in 3-dim'l space has $3^4 = \underline{\underline{81}}$ components!

Fortunately symmetry reduces the number of independent entries for E^{ijkl}.

<u>Minor symmetries</u>:

The symmetry of stress & strain tensors

$$\sigma^{ij} = \sigma^{ji} \qquad \& \qquad \varepsilon_{kl} = \varepsilon_{lk}$$

implies that

$$\underbrace{E^{ijkl} = E^{jikl}}_{\text{for each } k,l} \qquad \& \qquad \underbrace{E^{ijkl} = E^{ijlk}}_{\text{for each } i,j}$$

the i,j "matrix" is symmetric
only 6 instead of 9 indep. entries

the k,l "matrix" is symm.
only 6 instead of 9 ...

so E^{ijkl} has at most $6^2 = \underline{\underline{36}}$ independent components

major symmetries:

It follows from the existence of a "strain energy density functional U" satisfying

$$\frac{\partial^2 U}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = E^{ijkl}$$

that

$$\underbrace{E^{ijkl} = E^{klij}}$$

the matrix with rows labelled by $\underbrace{(i,j)}_{6 \text{ possibilities}}$

and columns labelled by $\underbrace{(k,l)}_{6 \text{ possibilities}}$

is symmetric

$$\begin{bmatrix} \ast & \ast & \ast & \ast & \ast & \ast \\ & \ast & \ast & \ast & \ast & \ast \\ & & \ast & \ast & \ast & \ast \\ & & & \ast & \ast & \ast \\ & & & & \ast & \ast \\ & & & & & \ast \end{bmatrix}$$

lower part is determined by upper part

so E^{ijkl} in fact only has

$6 + 5 + 4 + 3 + 2 + 1 = \underline{\underline{21}}$ independent components