23. Mai 2012

Conductivity tensors

Consider a homogeneous continuous crystal.

Its properties can be divided into two classes:

- properties that do *not* depend on a direction | and are hence described by *scalars*

  Exs density, heat capacity

- properties that *depend* on a direction | and are hence described by *tensors*

  Exs elasticity, electrical conductivity, heat conductivity

We say the crystal is *anisotropic* when it has such "tensorial" properties.

Electrical conductivity

Let | E electric field
| J electric current density

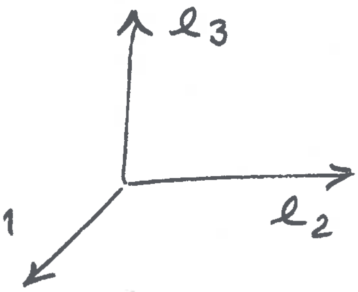
We assume that these are constant
i.e. are the same at all points of
the crystal.

At each point of the crystal:

- E gives the <u>electric force</u> (in volts per meter) that would be exerted on a positive test charge (of 1 coulomb) placed at that point;
- J (in amperes per square meter) gives the direction the charge carriers move and the <u>rate of electric current</u> across an infinitesimal surface perpendicular to that direction.

J is a function of E : $J = f(E)$

Consider a small increment in J ΔJ
caused by a " " " E ΔE
and write these increments in terms
of their components w.r.t. a chosen
o.n. basis



$$\Delta J = \Delta J^i e_i \qquad \Delta E = \Delta E^i e_i$$

The increments are related by

$$\Delta J^i = \underbrace{\frac{\partial f^i}{\partial E^j}} \Delta E^j + \text{higher order terms}$$

(in $(\Delta E^j)^2, (\Delta E^j)^3, \ldots$ recall Taylor expansion)

assume this is
independent of
the point of
the crystal

$$\boxed{\frac{\partial f^i}{\partial E^j} = K^i_j}$$

We obtain the relation

$$\boxed{\Delta J^i = k^i_j \, \Delta E^j}$$

or simply $\qquad \Delta J = k \, \Delta E$

where k is the electrical conductivity tensor. This is a $(1,1)$-tensor and may depend on the initial value of E, i.e., electrical conductivity may be different for small and large electric forces.

When initially we have $E = 0$ (and still consider small increments ΔE and ΔJ), the relation

$$J = k \, E$$

is called the generalized Ohm's law.

The <u>electrical resistivity tensor</u> is

$$r = K^{-1}$$

i.e., is the (1,1)-tensor such that

$$r_i^j \, K_j^l = \delta_i^l$$

For an <u>isotropic</u> crystal all directions are equivalent and these tensors are spherical:

$$K_i^j = \underset{\uparrow \atop \text{scalar}}{K} \, \delta_i^j \, , \quad r_i^j = \frac{1}{K} \, \delta_i^j$$

$$\begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix} , \quad \begin{bmatrix} \frac{1}{K} & 0 & 0 \\ 0 & \frac{1}{K} & 0 \\ 0 & 0 & \frac{1}{K} \end{bmatrix}$$

In general, K_i^j is neither symmetric nor antisymmetric (actually symmetry does not even make sense for a (1,1)-tensor unless a metric is fixed).

Heat conductivity

Let | T temperature
| H heat flux vector

For a homogeneous crystal and constant H and constant gradient of T, Fourier's heat conduction law says that

$$\boxed{H = -K \operatorname{grad} T}$$
(*)

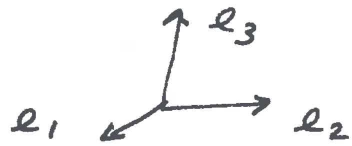
At each point of the crystal:

- grad T points the highest ascent of the temperature and measures the rate of increase of T in that direction;
- H measures the amount of heat passing per unit area perpendicular to its direction per unit time

(*) hence the minus sign: heat flows opposite to increasing temperature

K is the <u>heat conductivity tensor</u> (or <u>thermal conductivity tensor</u>)

In terms of components w.r.t. a chosen o.n. basis



$$H^i = -K^{ij} (\mathrm{grad}\, T)_j$$

<u>Note</u> The gradient of a real function is a covariant 1-tensor; see exercise 3 of series 7.

The heat conductivity tensor is a contravariant 2-tensor and experiments show that it is <u>symmetric</u>

can be orthogonally diagonalized

The <u>heat resistivity tensor</u> is

$$r = K^{-1}$$

also symmetric.

W.r.t. an o.n. eigenbasis, K is represented by

$$\begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix}$$

the eigenvalues of K are called the <u>principal coefficients</u> of heat conductivity

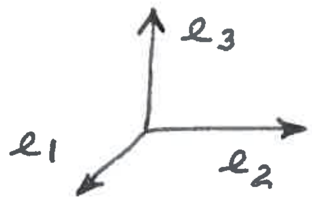
Physical considerations (heat flows towards decreasing temperature always) show that the eigenvalues are positive:

$$K_i > 0$$

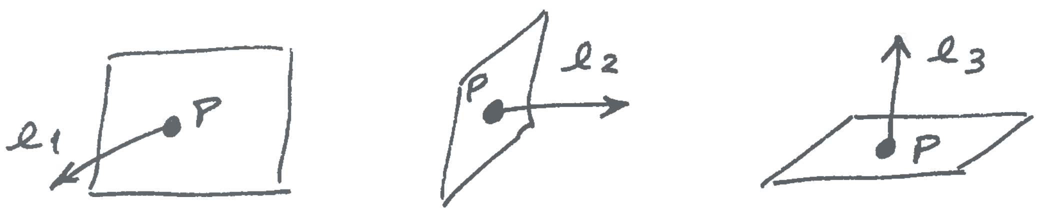
The eigenspaces of K are called the <u>principal directions</u> of heat conductivity.

Back to the <u>stress tensor</u>

Fix an o.n. basis



Consider <u>slices</u> through point P of a rigid body and parallel to coord. planes.



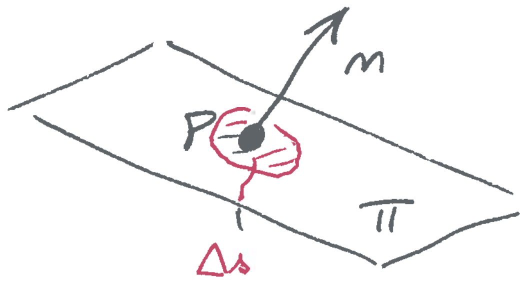
The <u>stress tensor</u> has components

$$\boxed{\sigma^{ij} = \lim_{\Delta A_i \to 0} \frac{\Delta F^j}{\Delta A_i}}$$

where ΔA_i is the area of a small element/slice of the plane $\perp e_i$ and containing point P

ΔF^j is the j-th component of the force acting on that slice

What about the stress across <u>other</u> slices (i.e. planes with other normal vectors) through the point P ?



Let

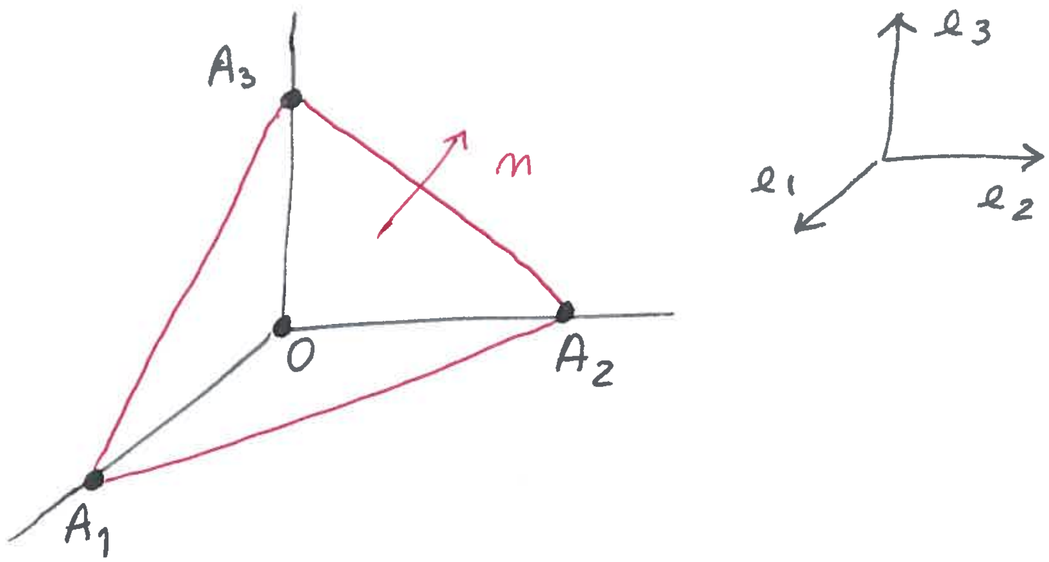
Π	plane passing through P
n	unit vector $\perp$ to plane Π
Δs	area of small element of plane Π containing P
ΔF	force acting on that element

<u>Claim</u> The stress at P across surface perpendicular to n is

$$\boxed{\sigma(n) \underset{\text{definition}}{=} \lim_{\Delta s \to 0} \frac{\Delta F}{\Delta s} \underset{\text{claim}}{=} \sigma^{ij} (n \cdot e_i)\, e_j}$$

Proof of claim

Consider the tetrahedron $OA_1A_2A_3$ formed by the triangular slice on the plane Π having area Δs and three triangles on planes parallel to coord. planes.



Consider all forces acting on this tetrahedron as a volume element of the rigid body.

There can be two types of forces:

- body force $= \underbrace{f}_{\text{force per unit volume}} \cdot \underbrace{\Delta v}_{\text{volume of tetrahedron}}$

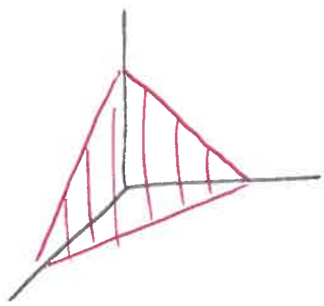
  We don't know f but it won't matter (see below).

- surface force = sum of forces on each of the four sides of the tetrahedron

Assess each of these four surface contributions:

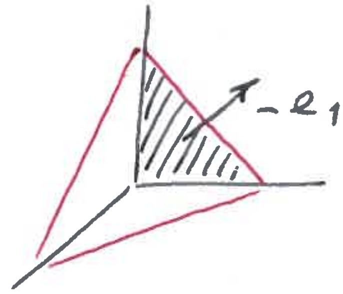
If Δs is the area of the slice on plane Π, the contribution of that slice is

$$\sigma(n)\,\Delta s$$



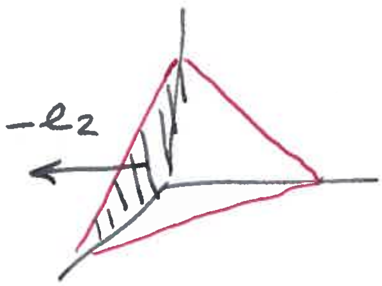
If Δs_1 is the area of the slice on the plane with normal $-e_1$, the contribution of that slice is

$$-\sigma^{1j} e_j \, \Delta s_1$$



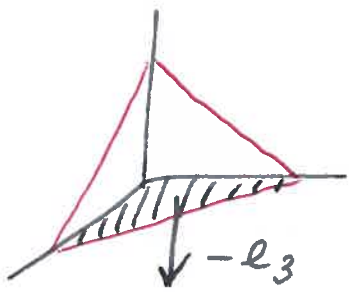
Similarly, the contributions of the other two slices are

$$-\sigma^{2j} e_j \, \Delta s_2$$



and

$$-\sigma^{3j} e_j \, \Delta s_3$$



Note we use outside pointing normals everywhere hence the minus signs.

So the total surface force is

$$\sigma(n)\,\Delta s - \sigma^{1j} e_j\,\Delta s_1 - \sigma^{2j} e_j\,\Delta s_2 - \sigma^{3j} e_j\,\Delta s_3$$

Because we have a ~~static~~ equilibrium the sum of all (body & surface) forces must be zero :

$$\underbrace{f\,\Delta v}_{\substack{\text{term of higher order} \\ (\Delta v \to 0 \text{ faster than } \Delta s \to 0) \\ \text{can be neglected when } \Delta s \text{ small}}} + \sigma(n)\,\Delta s - \sigma^{ij} e_j\,\Delta s_i = 0$$

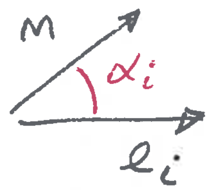
We conclude that

$$\sigma(n)\,\Delta s = \sigma^{ij} e_j\,\Delta s_i$$

It remains to relate Δs to $\Delta s_1, \Delta s_2, \Delta s_3$

The side with area Δs_i is the orthogonal projection of the side with area Δs onto the plane with normal e_i.

The scaling factor for area under projection is $\cos \alpha_i$ where α_i is the angle between the plane normal vectors.



convex angle

$$\frac{\Delta s_i}{\Delta s} = \cos \alpha_i = n \bullet e_i$$

$$n \bullet e_i = \underbrace{\|n\|}_{1} \, \underbrace{\|e_i\|}_{1} \cos \alpha_i$$

Therefore

$$\sigma(n) \cancel{\Delta s} = \sigma^{ij} e_j \, (n \bullet e_i) \cancel{\Delta s}$$

or, equivalently,

$$\boxed{\sigma(n) = \sigma^{ij} (n \bullet e_i) \, e_j} .$$

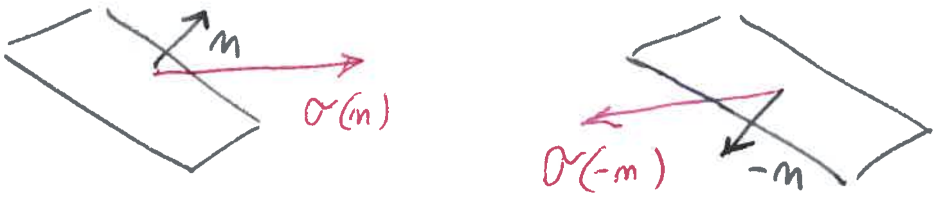
$\square$

Comments

1) $\sigma(n)$ = stress across surface with normal n

$$\boxed{\sigma(-n) = -\sigma(n)}$$

(Stress considers orientation as if the forces on each side of the surface have to balance each other in static equilibrium.



2) $$\sigma(n) = \sigma^{ij}\,\underbrace{(n \cdot e_i)}\,e_j$$

these are the coordinates of n w.r.t. o.n. basis e_1, e_2, e_3

$$n = \underbrace{(n \cdot e_1)}_{n^1} e_1 + \underbrace{(n \cdot e_2)}_{n^2} e_2 + \underbrace{(n \cdot e_3)}_{n^3} e_3$$

Contravariance of stress

$\mathcal{B} = (e_1, e_2, e_3)$

$\tilde{\mathcal{B}} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ another basis

Let $\tilde{e}_i = L_i^j e_j$ where $L =$ change of basis matrix

$e_i = \Lambda_i^j \tilde{e}_j \qquad \Lambda = L^{-1}$

Let n be a given unit vector

S be the stress across surface perpendicular to n

Then S can be expressed in two ways :

$$S = \sigma^{ij}(n \cdot e_i)\, e_j \quad \text{w.r.t. basis } \mathcal{B}$$

$$S = \tilde{\sigma}^{ij}(n \cdot \tilde{e}_i)\, \tilde{e}_j \quad \text{" " } \tilde{\mathcal{B}}$$

The goal is to relate $\tilde{\sigma}^{ij}$ to σ^{ij}.

Start with the first expression for S and rename the indices for later convenience.

$$S = \sigma^{km} (n \cdot e_k)\, e_m$$

$$= \sigma^{km} (n \cdot \Lambda^i_k \tilde{e}_i)(\Lambda^j_m \tilde{e}_j)$$

$$= \sigma^{km} \Lambda^i_k \Lambda^j_m (n \cdot \tilde{e}_i)\, \tilde{e}_j$$

using linearity twice

expressing e_k, e_m in terms of $\tilde{e}_i, \tilde{e}_j$

Comparing the last expression with the expression on the previous page we get

$$\Longrightarrow \quad \tilde{\sigma}^{ij} = \sigma^{km} \Lambda^i_k \Lambda^j_m$$

showing that σ is a <u>contravariant 2-tensor</u>.