21. März . 2012

Bilinear forms

V vector sp.

$\varphi : V \times V \longrightarrow \mathbb{R}$ bilinear

means that φ is linear in each entry

Exercise Which φ is linear
and which is bilinear ?

$\varphi(x,y) = 2x - y$ is linear
in $(x,y) \in \mathbb{R}^2$

$\varphi(x,y) = 3xy$ is bilinear

linear in $x \in \mathbb{R}$ and linear in $y \in \mathbb{R}$

$b_1, \dots, b_n$ basis of V

$\beta^1, \dots, \beta^n$ dual basis of V^*

Last time:

- $\beta^i \otimes \beta^j$ are bilinear forms on V defined by $(\beta^i \otimes \beta^j)(v, w) = \beta^i(v)\beta^j(w)$

  tensor product of linear forms

- any bilinear form $\varphi : V \times V \to \mathbb{R}$ can be uniquely written as

$$\boxed{\varphi = \underbrace{B_{ij}\, \beta^i \otimes \beta^j}}$$

Einstein notation for double sum

$$\sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij}\, \beta^i \otimes \beta^j$$

where $\boxed{B_{ij} = \varphi(b_i, b_j)}$ ← real numbers

Example $V = \mathbb{R}^3$

Fix vector $a \in \mathbb{R}^3$

$$\varphi_a(v, w) = a \cdot (v \times w)$$

scalar triple product

is a bilinear form

$\varphi_a(v,w)$ = signed volume of parallelepiped spanned by a, v, w

sign depends on orientation of a, v, w

Find components B_{ij} of φ_a w.r.t. standard basis of $\mathbb{R}^3$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $a = \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}$ be the given vector.

Recall the cross product in $\mathbb{R}^3$:

$$e_i \times e_j = \begin{cases} 0 & \text{when } i = j \\ e_k & \text{when } (i,j,k) \text{ is a } \underline{\text{cyclic}} \text{ permutation of } (1,2,3) \\ -e_k & \text{when } (i,j,k) \text{ is a } \underline{\text{non-cyclic}} \text{ permutation of } (1,2,3) \end{cases}$$

$$\left.\begin{aligned} e_1 \times e_2 &= e_3 \\ e_2 \times e_3 &= e_1 \\ e_3 \times e_1 &= e_2 \end{aligned}\right] \text{cyclic} \qquad \left.\begin{aligned} e_2 \times e_1 &= -e_3 \\ e_3 \times e_2 &= -e_1 \\ e_1 \times e_3 &= -e_2 \end{aligned}\right] \text{non-cyclic}$$

$$a \bullet e_k = a^k$$

Therefore $B_{ij} = \mathcal{E}_a(e_i, e_j) = a \bullet (e_i \times e_j)$

$$= \begin{cases} 0 & \text{when } i = j \\ a^k & \text{when } (i,j,k) \text{ is a cyclic permutation of } (1,2,3) \\ -a^k & \text{when } (i,j,k) \text{ is a non-cyclic permutation of } (1,2,3) \end{cases}$$

Conclusion : $B_{12} = a^3 = -B_{21}$

$B_{23} = a^1 = -B_{32}$

$B_{31} = a^2 = -B_{13}$

$B_{II} = 0$ ← diagonal components are zero

Can write in a matrix

$$B = \begin{pmatrix} 0 & a^3 & -a^2 \\ -a^3 & 0 & a^1 \\ a^2 & -a^1 & 0 \end{pmatrix} \qquad \square$$

<u>Matrix representation</u>

$\varphi : V \times V \longrightarrow \mathbb{R}$ bilinear

$\mathcal{B} = (b_1, \dots, b_n)$ basis of V

$B_{ij} = \varphi(b_i, b_j)$ components of φ w.r.t. $\mathcal{B}$

Define $B = \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \vdots \\ B_{n1} & \cdots & B_{nn} \end{pmatrix}$ B_{ij} row column

matrix of φ w.r.t. basis $\mathcal{B}$

How do the components of φ change when we change basis?

OLD $\mathcal{B} = (b_1, \dots, b_n)$
NEW $\tilde{\mathcal{B}} = (\tilde{b}_1, \dots, \tilde{b}_n)$ $\Big\}$ two bases of V

$\mathcal{B}^* = (\beta^1, \dots, \beta^n)$
$\tilde{\mathcal{B}}^* = (\tilde{\beta}^1, \dots, \tilde{\beta}^n)$ $\Big\}$ corresponding dual bases of V^*

$\beta^i \otimes \beta^j$, $i, j = 1, \dots, n$
$\tilde{\beta}^i \otimes \tilde{\beta}^j$, $i, j = 1, \dots, n$ $\Big\}$ corresponding bases of $\underbrace{V^* \otimes V^*}_{\overset{\|}{\mathrm{Bil}(V \times V, \mathbb{R})}}$

Let $\underbrace{\varphi : V \times V \longrightarrow \mathbb{R} \text{ bilinear}}_{\varphi \in V^* \otimes V^*}$

Components of φ w.r.t. $\mathcal{B}$ are

$$B_{ij} = \varphi(b_i, b_j) \quad \text{OLD}$$

Components of φ w.r.t. $\tilde{\mathcal{B}}$ are

$$\tilde{B}_{ij} = \varphi(\tilde{b}_i, \tilde{b}_j) \quad \text{NEW}$$

What is the relation between the B_{ij}'s and the $\tilde{B}_{ij}$'s?

Let L = change of basis matrix i.e. $\tilde{b}_j = L^i_j \, b_i$

$$\tilde{B}_{ij} \underset{\substack{\uparrow \\ \text{defn of } \tilde{B}_{ij}}}{=} \varphi(\tilde{b}_i, \tilde{b}_j) \underset{\substack{\uparrow \\ \text{change names of dummy indices} \\ \text{as needed}}}{=} \varphi(L^k_i b_k, L^l_j b_l)$$

$$\underset{\substack{\uparrow \\ \text{bilinearity of } \varphi}}{=} L^k_i L^l_j \, \varphi(b_k, b_l) \underset{\substack{\uparrow \\ \text{defn of } B_{kl}}}{=} L^k_i L^l_j B_{kl}$$

We conclude that $\boxed{\tilde{B}_{ij} = L^k_i L^l_j B_{kl}}$

NEW OLD

and hence say that a bilinear form φ is a covariant 2-tensor.

Exercise In terms of matrices $\boxed{\tilde{B} = L^T B L}$

Example For scalar triple product φ_a find components $\tilde{B}_{ij}$ w.r.t basis $\tilde{\mathcal{B}} = (\underbrace{e_2}_{\tilde{b}_1}, \underbrace{e_1+e_3}_{\tilde{b}_2}, \underbrace{e_3}_{\tilde{b}_3})$

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\tilde{B} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{L^T} \underbrace{\underbrace{\begin{bmatrix} 0 & a^3 & -a^2 \\ -a^3 & 0 & a^1 \\ a^2 & -a^1 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{L}}_{}$$

$$BL = \begin{bmatrix} a^3 & -a^2 & -a^2 \\ 0 & a^1-a^3 & a^1 \\ -a^1 & a^2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a^1-a^3 & a^1 \\ a^3-a^1 & 0 & -a^2 \\ -a^1 & a^2 & 0 \end{bmatrix}$$

components $\tilde{B}_{ij}$ are the entries of ← this matrix

Reality check :

- $\tilde{B}$ is antisymmetric like B ✓
- $\tilde{B}_{12} = \varphi_a(\tilde{b}_1, \tilde{b}_2) = a \cdot \underbrace{(e_2 \times (e_1+e_3))}_{-e\ +e} = a^1 - a^3$ ✓
- $\tilde{B}_{13} = \ldots$
- $\tilde{B}_{23} = \ldots$

$\square$

Similarly for <u>trilinear</u> and

<u>multilinear</u> forms

$T : V \times V \times V \longrightarrow \mathbb{R}$ is trilinear

when T is linear in each of its 3 variables.

The space of all trilinear forms is

denoted $V^* \otimes V^* \otimes V^*$, has dimension n^3

and has basis $\beta^i \otimes \beta^j \otimes \beta^k$, $i, j, k = 1, \ldots, n$.

A trilinear form $T : V \times V \times V \longrightarrow \mathbb{R}$

is a <u>covariant 3-tensor</u> because

its components T_{ijk} w.r.t. a basis

change by $\widetilde{T}_{ijk} = L^l_i \, L^p_j \, L^q_k \, T_{lpq}$

when the basis changes by L.

Example of trilinear form

$$V = \mathbb{R}^3$$

$$\varphi : V \times V \times V \longrightarrow \mathbb{R}$$

$$\varphi(u, v, w) = \underbrace{u \bullet (v \times w)}_{\text{scalar triple product}}$$

$$= \det \begin{bmatrix} - & u & - \\ - & v & - \\ - & w & - \end{bmatrix}$$

check this equality in coordinates

recall Leibniz formula for determinants

case of 3×3 matrices :

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= \sum_{\sigma \in S_3} \underbrace{\operatorname{sign}\sigma}_{\pm}\, a_{1\sigma(1)}\, a_{2\sigma(2)}\, a_{3\sigma(3)}$$

notation

$$\sigma = (\sigma(1), \sigma(2), \sigma(3))$$

$$S_3 = \{\text{permutations of 3 elements}\} = \{(1,2,3), (1,3,2), (2,3,1), (2,1,3), \underbrace{(3,1,2)}_{+}, \underbrace{(3,2,1)}_{-}\}$$

$T : \underbrace{V \times \dots \times V}_{k \text{ factors}} \longrightarrow \mathbb{R}$ is <u>multilinear</u> or k-linear

when T is linear in each of its k variables.

The space $\underbrace{V^* \otimes \dots \otimes V^*}_{k \text{ factors}}$ of all k-linear

forms has dimension n^k and has basis

$\beta^{i_1} \otimes \beta^{i_2} \otimes \dots \otimes \beta^{i_k}$, $i_1, i_2, \dots, i_k = 1, \dots, n$.

A k-linear form T is a <u>covariant k-tensor</u>.

We can take the <u>tensor product</u> of a covariant k-tensor T with a covariant l-tensor U to obtain a covariant $(k+l)$-tensor

$$T \otimes U : \underbrace{V \times \dots \times V}_{k+l \text{ factors}} \longrightarrow \mathbb{R}$$

$$(T \otimes U)(v_1, \dots, v_{k+l}) = T(\underbrace{v_1, \dots, v_k}_{\text{first } k \text{ vectors}})\, U(\underbrace{v_{k+1}, \dots, v_{k+l}}_{\text{last } l \text{ vectors}})$$

Inner products

as a special case of bilinear forms

An inner product on a vector space V is a bilinear form on V which is

- symmetric and
- positive definite.

$$g : V \times V \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto g(v, w)$$

bilinear
↓
linear in v
& linear in w

- symmetric $\quad g(v, w) = g(w, v) \quad \forall v, w \in V$

- positive definite $\quad g(v, v) > 0 \quad \forall v \in V, v \neq 0$

Examples $V = \mathbb{R}^3$

Are the following bilinear forms inner products?

Check
- symmetry and
- positive definiteness

a) $\varphi(v, w) = v \bullet w$

$\uparrow$ standard inner product

(scalar product
dot product)

$v = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}$ $w = \begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix}$ $v \bullet w = v^i w^i$

b) $\varphi(v, w) = - v \bullet w$

This φ is negative definite $\varphi(v,v) < 0 \quad \forall v \neq 0$

c) $\varphi(v, w) = v \bullet w + 2 v^1 w^2$

This φ is not symmetric

d) $\varphi(v, w) = v \bullet 3w$

Answers: a) Yes b) No c) No d) Yes

Examples $V = \mathbb{R}[x]_2$ polys. degree ≤ 2

Are the following bilinear forms inner products ?

Check
- symmetry and
- positive definiteness

a) $\varphi(p,q) = \int_0^1 p(x)\,q(x)\,dx$

- $\int_0^1 p(x)q(x)\,dx = \int_0^1 q(x)p(x)\,dx$ because $p(x)q(x) = q(x)p(x)$
- $\int_0^1 (p(x))^2\,dx \geq 0$ always because $(p(x))^2 \geq 0$
  $= 0$ only when $p(x) = 0 \;\forall x \in [0,1]$
  which is only when $p \equiv 0$

b) $\varphi(p,q) = \int_0^1 p'(x)\,q'(x)\,dx$

$\int_0^1 (p'(x))^2\,dx = 0 \Rightarrow p'(x) = 0 \;\forall x \in [0,1] \not\Rightarrow p \equiv 0$

c) $\varphi(p,q) = \int_3^{\pi} e^x p(x)\,q(x)\,dx$

d) $\varphi(p,q) = p(1)\,q(1) + p(2)\,q(2)$

Is there $p \in \mathbb{R}[x]_2$, $p \neq 0$ s.t. $(p(1))^2 + (p(2))^2 = 0$?

e) $\varphi(p,q) = p(1)q(1) + p(2)q(2) + p(3)q(3)$

Is there a nonzero polynomial of degree $\underline{2}$ with $\underline{3}$ distinct zeros ?

Answers: a) Yes b) No c) Yes d) No e) Yes

$g : V \times V \longrightarrow \mathbb{R}$ inner product

$v, w \in V$

<u>norm</u> of v $= \|v\| \overset{\text{defn}}{=} \underbrace{\sqrt{g(v,v)}}_{\geq 0 \text{ so ok}}$

$= |v|$

<u>unit</u> vector : $\|v\| = 1$

<u>orthogonal</u> vectors : $v \perp w \overset{\text{defn}}{\Longleftrightarrow} g(v,w) = 0$

= perpendicular

<u>orthonormal</u> vectors : $v \perp w$ <u>and</u> $\|v\| = \|w\| = 1$

= o.n.

$\mathcal{B} = (b_1, \dots, b_n)$ basis of V

<u>orthonormal basis</u> : $b_1, \dots, b_n$ are o.n.

= o.n. basis

$$\boxed{g(b_i, b_j) = \delta_{ij}}$$

$$= \begin{cases} 1 & \text{when } i = j \leftarrow \|b_i\| = 1 \text{ says} \\ 0 & \text{when } i \neq j \leftarrow b_i \perp b_j \end{cases}$$

<u>Examples</u> a) $V = \mathbb{R}^n$

g = standard inner product

$\mathcal{B} = (e_1, \dots, e_n)$

standard basis is an o.n. basis for the standard inner product

b) $V = \mathbb{R}[x]_2$

$$g(p,q) = \int_{-1}^{1} p(x)\, q(x)\, dx$$

$\mathcal{B} = (p_1, p_2, p_3)$ where

$$p_1(x) = \frac{1}{\sqrt{2}}$$

$$p_2(x) = \sqrt{\frac{3}{2}}\, x$$

$$p_3(x) = \sqrt{\frac{5}{8}}\,(3x^2 - 1)$$

Check that $\mathcal{B}$ is an o.n. basis for the inner product above.

p_1, p_2, p_3 are the first three <u>Legendre polynomials</u> up to scaling