4. April 2012

Reciprocal basis and covariant coordinates

Setup | V vector sp. of $\dim = n$
$g: V \times V \to \mathbb{R}$ inner product (or "metric")
$\mathcal{B} = (b_1, \dots, b_n)$ basis of V

Last time we defined another basis of V called the reciprocal basis $\mathcal{B}^g = (b^1, \dots, b^n)$ by

$$\boxed{g(b^i, b_j) = \delta^i_j}$$

Two bases of V: $\mathcal{B}$ and $\mathcal{B}^g$

Let G be the $n \times n$ matrix with entries

$$\boxed{g_{ij} \stackrel{\text{defn}}{=} g(b_i, b_j)}$$

The entries of G^{-1} are denoted g^{ij}

upper indices

so that we have the property

$$\boxed{g^{ik} g_{kj} = \delta^i_j}$$

says $G^{-1}G = I$

Last time we saw that G^{-1} is the change of basis matrix from $\mathcal{B}$ to the reciprocal basis $\mathcal{B}^g$, i.e.

$$\boxed{b^i = g^{ij} b_j}$$

Let us now compute $g(b^i, b^j)$:

$$g(b^i, b^j) = g\left(g^{ik} b_k ,\, g^{jl} b_l\right)$$

reciprocal basis vectors b^i
written in terms of basis vectors b_k

$$= g^{ik} g^{jl}\, g(b_k, b_l)$$

bilinearity of g

$$= g^{ik} g^{jl} g_{kl}$$

definition of $g_{kl} = g(b_k, b_l)$

$$= g^{jl} \delta^i_l$$

g^{ij} & g_{ij} are the entries of inverse matrices

$$= g^{ji} = g^{ij}$$

symmetry

Conclusion : $\boxed{g^{ij} = g(b^i, b^j)}$

<u>Claim</u> The reciprocal basis is contravariant.

<u>Proof</u> Let $\mathcal{B}$, $\tilde{\mathcal{B}}$ be two bases of V, L the corresponding change of basis matrix and $\Lambda = L^{-1}$ its inverse.

$$\boxed{\tilde{b}_i = L_i^j \, b_j}$$

We have to check that, if $\mathcal{B}^g = (b^1, \ldots, b^n)$ is a reciprocal basis for $\mathcal{B}$ then a reciprocal basis for $\tilde{\mathcal{B}}$ is $\tilde{\mathcal{B}}^g = (\tilde{b}^1, \ldots, \tilde{b}^n)$ where

$$\boxed{\tilde{b}^i = \Lambda_k^i \, b^k}$$

We need to check if we have the property of reciprocal basis

$$g(\tilde{b}^i, \tilde{b}_j) = \delta^i_j$$

when $\tilde{b}^i$ is defined as in the previous page.

Let us compute :

$$g(\tilde{b}^i, \tilde{b}_j) = g(\Lambda^i_k b^k, L^l_j b_l)$$

$\tilde{b}^i$ & $\tilde{b}_j$ written in terms of b^k & b_l

$$= \Lambda^i_k L^l_j \underbrace{g(b^k, b_l)}_{= \delta^k_l}$$

bilinearity of g

by property of reciprocal basis for $\mathcal{B}^g$

$$= \Lambda^i_k L^k_j$$

$L^l_j \delta^k_l = L^k_j$

$$= \delta^i_j$$

$\Lambda = L^{-1}$

$\square$

$v \in V$

Two ways of writing vector v in coordinates

$$\boxed{v = \begin{cases} v^i b_i & \text{w.r.t. } \mathcal{B} \\ v_i b^i & \text{w.r.t. } \mathcal{B}^g \end{cases}}$$

Recall: v^i (ordinary) coordinates of v w.r.t. $\mathcal{B}$ are <u>contravariant</u>

change by L^{-1} when basis changes by L

<u>Claim</u> Vector coordinates w.r.t. reciprocal basis $\mathcal{B}^g$ are <u>covariant</u>

change by L when basis $\mathcal{B}$ changes by L

See proof on page $\boxed{6.19}$ or see later today.

Coordinates v_i of v w.r.t. reciprocal basis $\mathcal{B}^g$ are called <u>covariant coordinates</u> of v.

Relation between contravariant and covariant coordinates

$$v = \begin{cases} v^i b_i \\ v_i b^i \end{cases}$$

contravariant coords. v^i

covariant coords. v_i

$$\boxed{v_j} b^j = v = v^i b_i = \boxed{v^i g_{ij}} b^j$$

must be equal because coords w.r.t. B^g are unique

$$\boxed{v_j = v^i g_{ij}}$$

$$\boxed{v^i} b_i = v = v_j b^j = \boxed{v_j g^{ji}} b_i$$

must be equal because coords w.r.t. B are unique

$$\boxed{v^i = v_j g^{ji}}$$

Example

$V = \mathbb{R}[x]_1$ polys of degree ≤ 1
(linear polynomials $a + bx$)

$$g(p,q) = \int_0^1 p(x)\, q(x)\, dx$$

$\mathcal{B} = (\underbrace{1}_{b_1}, \underbrace{x}_{b_2})$ basis of V

Find a) G

b) G^{-1}

c) reciprocal basis $\mathcal{B}^g$

d) (contravariant) coords of $p(x) = 6x$

e) covariant coords of $p(x) = 6x$

a) G has entries

$$g_{11} = g(b_1, b_1) = \int_0^1 1.1\, dx = 1$$

$$g_{12} = g(b_1, b_2) = \int_0^1 1.x\, dx = \frac{1}{2} = g_{21}$$

$$g_{22} = g(b_2, b_2) = \int_0^1 x.x\, dx = \frac{1}{3}$$

$$G = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

b) $\det G = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$$G^{-1} = \begin{pmatrix} 4 & -6 \\ -6 & 12 \end{pmatrix}$$

c) Reciprocal basis is given by

$$(\underbrace{1}_{b_1} \quad \underbrace{x}_{b_2})\, G^{-1} = (\underbrace{4-6x}_{b^1} \quad \underbrace{-6+12x}_{b^2})$$

$$\mathcal{B}^g = (4-6x\,,\ -6+12x)$$

d) $p(x) = 6x = \underline{\underline{0}} \cdot 1 + \underline{\underline{6}} \cdot x$

has coordinates $v^1 = 0$, $v^2 = 6$

e) $p(x)$ has contravariant coordinates

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = G \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} \underline{\underline{3}} \\ \underline{\underline{2}} \end{pmatrix}$$

Check : $v_1 b^1 + v_2 b^2 = 3 \cdot (4-6x) + 2 \cdot (-6+12x) = 6x$ ✓

Summary

$\mathcal{B} = (b_1, \dots, b_n)$ basis

$\mathcal{B}^g = (b^1, \dots, b^n)$ reciprocal basis

$$\boxed{g(b^i, b_j) = \delta^i_j}$$

$v = v^i b_i$
(contravariant) coords.

$v = v_i b^i$
covariant coords.

$g_{ij} = g(b_i, b_j)$

$g^{ij} = g(b^i, b^j)$

matrices are inverses of each other

$$\boxed{g^{ik} g_{kj} = \delta^i_j}$$

$$\boxed{b_i = g_{ij} b^j}$$

$$\boxed{b^i = g^{ij} b_j}$$

$$\boxed{v^i = g^{ij} v_j}$$

$$\boxed{v_i = g_{ij} v^j}$$

We say that the metric tensor raises or lowers the indices

Recall dual space

V vector sp. $\dim = n$

V^* dual v. sp. $\dim = n$ also

$\dim V = \dim V^*$

$\Longrightarrow$ V and V^* can be identified
if we choose bases for V and V^*

It is enough to choose $\mathcal{B}$ = basis of V

Let $\mathcal{B}^*$ = dual basis of V^*

There is the following correspondence :

vector $v \in V$ $\longleftrightarrow$ linear form $\alpha \in V^*$

exactly when v and α have the same coordinates w.r.t. $\mathcal{B}$ and $\mathcal{B}^*$, respectively

This correspondence depends on choice of basis $\mathcal{B}$, hence is noncanonical.

An inner product (or metric) gives a canonical identification between V and V^*:

Let $g : V \times V \longrightarrow \mathbb{R}$ inner product

Let $v \in V$

Then $g(v, \cdot) : V \longrightarrow \mathbb{R}$ is a linear form!

$$w \longmapsto g(v, w)$$

The canonical identification given by the metric is :

$$\boxed{\begin{array}{ccc} V & \longleftrightarrow & V^* \\ v & \longleftrightarrow & v^* \stackrel{\text{defn}}{=} g(v, \cdot) \end{array}}$$

Note : $g(v, \cdot) \equiv 0 \iff v = 0$

because g is positive definite

Now suppose $\mathcal{B} = (b_1, \dots, b_n)$ basis of V

To what linear forms does the reciprocal basis $\mathcal{B}^g = (b^1, \dots, b^n)$ correspond under the canonical identification of V and V^* given by g ?

$$b^i \longleftrightarrow \underbrace{g(b^i, \bullet)}$$

has property $g(b^i, b_j) = \delta^i_j$ exactly like dual basis !

Conclusion

$$b^i \longleftrightarrow \boxed{\beta^i = g(b^i, \bullet)}$$

reciprocal basis corresponds to dual basis under the canonical identification

This correspondence also explains the covariant character of covariant coords:

$$\boxed{\text{coords. w.r.t. } b^1, \dots, b^n \longleftrightarrow \text{coords. w.r.t. } \beta^1, \dots, \beta^n}$$

by canonical identification between V and V^*

Geometric viewpoint

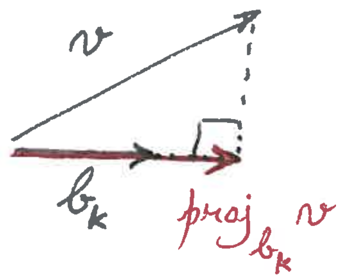
Recall orthogonal projection :

$g : V \times V \longrightarrow \mathbb{R}$ inner product

$\mathcal{B} = (b_1, \dots, b_n)$ basis of V

$v \in V$

$$\boxed{\operatorname{proj}_{b_k} v = \frac{g(v, b_k)}{g(b_k, b_k)} \, b_k}$$



orthogonal projection of v onto b_k

<u>Exercise</u> $v - \operatorname{proj}_{b_k} v \perp b_k$

Now write $v = v_i b^i$ in terms of covariant coords / reciprocal basis.

$$g(v, b_k) = g(v_i b^i, b_k) = v_i \underbrace{g(b^i, b_k)}_{\delta^i_k} = v_k$$

$$\boxed{\operatorname{proj}_{b_k} v = \frac{v_k}{g(b_k, b_k)} \, b_k}$$

Assume $b_1, \dots, b_n$ are unit vectors

Then $\boxed{\mathrm{proj}_{b_k} v = v_k \, b_k}$

‖ covariant coords give
orthogonal projection of v onto $b_1, \dots, b_k$

Write $v = v^i b_i$ in terms of
contravariant coords / usual basis.

‖ contravariant coords give
"parallel projection" of v onto $b_1, \dots, b_k$