18. April 2012

Tensors

Let V be a vector space.
Up to now we had:

tensors	comps.	behavior under change of basis	type
vectors in V	v^i	contravariant	$(1,0)$
linear transfs. $V \to V$	A^i_j	mixed: contravar. & covariant	$(1,1)$
linear forms $V \to \mathbb{R}$	α_j	covariant	$(0,1)$
bilinear forms $V \times V \to \mathbb{R}$ ⊛	B_{ij}	covariant 2-tensor	$(0,2)$
k-linear forms $\underbrace{V \times \ldots \times V}_{k} \to \mathbb{R}$	$F_{i_1 i_2 \ldots i_k}$	covariant k-tensor	$(0,k)$

⊛ includes inner products (= symmetric positive definite bilinear forms)

What should a (2,0)-tensor be like?

$$V^* = \{\text{linear } V \to \mathbb{R}\} = \{(0,1)\text{-tensors}\}$$

dual space

$$\{\text{bilinear } V \times V \to \mathbb{R}\} = \{(0,2)\text{-tensors}\}$$

Notice A vector $v \in V$ may be regarded as a linear form on V^*:

$$\begin{aligned} v : V^* &\longrightarrow \mathbb{R} \\ \alpha &\longmapsto \alpha(v) \end{aligned}$$

and vice-versa:
any linear form on V^* is given by a vector in V.

Why?
Just count dimensions and see that the map above taking vectors v to linear forms on V^* is injective.

$$\dim (V^*)^* = \dim V^* = \dim V.$$

Therefore :

$V = \{ \text{linear } V^* \to \mathbb{R} \} = \{ (1,0)\text{-tensors} \}$

$$\boxed{\{ \text{bilinear } V^* \times V^* \to \mathbb{R} \} = \{ (2,0)\text{-tensors} \}}$$

Defn A tensor of type (2,0) or (2,0)-tensor is a bilinear form on V^*, that is, is a bilinear function $\sigma : V^* \times V^* \to \mathbb{R}$.

Let $Bil(V^* \times V^*, \mathbb{R}) = \{ \text{all bilinear forms } \sigma : V^* \times V^* \to \mathbb{R} \}$

$= \{ \text{all } (2,0)\text{-tensors} \}$

Exercise Check that $Bil(V^* \times V^*, \mathbb{R})$ is a vector space.

Hint : It is enough to check that if $\sigma, \tau \in Bil(V^* \times V^*, \mathbb{R})$, $\lambda, \mu \in \mathbb{R}$ then $\underbrace{\lambda\sigma + \mu\tau \in Bil(V^* \times V^*, \mathbb{R})}_{\text{linear combinations of (2,0)-tensors are (2,0)-tensors}}$.

Construction of (2,0) - tensors by tensor product of (1,0) - tensors

Take two vectors $\underbrace{v, w}_{\text{two (1,0)-tensors}} \in V$.

Define $\sigma : V^* \times V^* \longrightarrow \mathbb{R}$

by $\boxed{\sigma(\alpha, \beta) = \alpha(v)\, \beta(w)}$

linear in α linear in β

Then σ is bilinear and called

$\boxed{\sigma = v \otimes w}$ tensor product of v and w

tensor product

$\sigma = v \otimes w$ is hence a (2,0) - tensor.

Note In general $v \otimes w \neq w \otimes v$

because there can be linear forms α, β

s.t. $\alpha(v)\,\beta(w) \neq \alpha(w)\,\beta(v)$.

Basis for the space of (2,0)-tensors

$\mathcal{B} = (b_1, \dots, b_n)$ basis of V

$b_i \otimes b_j$ tensor product of b_i and b_j is a (2,0)-tensor

Claim $\boxed{\text{The } b_i \otimes b_j \,,\ i,j = 1,\dots,n \text{ form a basis of } \mathrm{Bil}(V^* \times V^*, \mathbb{R})}$

Consequence: $\dim \mathrm{Bil}(V^* \times V^*, \mathbb{R}) = n^2$

Notation $\mathrm{Bil}(V^* \times V^*, \mathbb{R}) = V \otimes V$

tensor product of vector spaces V and V

Feels déjà vu?
Compare with Lecture 4 (pages 4.13–4.16).
The claim here is proved in an analogous way to the proof in that lecture.

Components of a (2,0) - tensor

$\sigma : V^* \times V^* \longrightarrow \mathbb{R}$ bilinear

i.e. a (2,0) - tensor

After choosing a basis $b_1, \dots, b_n$ of V

$\Downarrow$

basis $\beta^1, \dots, \beta^n$ of V^*

basis $b_i \otimes b_j$ of $\{(2,0)\text{-tensors}\}$

the (2,0) - tensor σ is represented by its components

$$\boxed{S^{ij} = \sigma(\beta^i, \beta^j)}$$

means

$\sigma = S^{ij} b_i \otimes b_j$

values of σ on pairs of elements of dual basis

which can be arranged in a matrix

$$S = \begin{pmatrix} S^{11} & \cdots & S^{1n} \\ \vdots & & \vdots \\ S^{n1} & \cdots & S^{nn} \end{pmatrix}$$

S^{ij}

row column

called the matrix of the (2,0) - tensor w.r.t. the chosen basis of V.

How do the components of a (2,0)-tensor change when we change basis ?

OLD $\mathcal{B} = (b_1, \dots, b_n)$
NEW $\tilde{\mathcal{B}} = (\tilde{b}_1, \dots, \tilde{b}_n)$ } two bases of V

$\mathcal{B}^* = (\beta^1, \dots, \beta^n)$
$\tilde{\mathcal{B}}^* = (\tilde{\beta}^1, \dots, \tilde{\beta}^n)$ } corresponding dual bases of V^*

Let $\underbrace{\sigma : V^* \times V^* \longrightarrow \mathbb{R} \quad \text{bilinear}}_{(2,0)\text{-tensor}}$

Components of σ w.r.t. $\mathcal{B}$ are

$$S^{ij} = \sigma(\beta^i, \beta^j) \qquad \text{OLD}$$

Components of σ w.r.t. $\tilde{\mathcal{B}}$ are

$$\tilde{S}^{ij} = \sigma(\tilde{\beta}^i, \tilde{\beta}^j) \qquad \text{NEW}$$

What is the relation between the S^{ij}'s and the $\tilde{S}^{ij}$'s ?

Let L = change of basis matrix

i.e. $\tilde{b}_j = L^i_j \, b_i$

Then $\tilde{\beta}^i = \Lambda^i_j \, \beta^j$ where $\Lambda = L^{-1}$.

$$\tilde{S}^{ij} \underset{\substack{\uparrow \\ \text{defn of } \tilde{S}^{ij}}}{=} \sigma(\tilde{\beta}^i, \tilde{\beta}^j) \underset{\substack{\uparrow \\ \text{change of basis}}}{=} \sigma(\Lambda^i_k \beta^k, \Lambda^j_\ell \beta^\ell)$$

$$\underset{\substack{\uparrow \\ \text{bilinearity of } \sigma}}{=} \Lambda^i_k \Lambda^j_\ell \, \sigma(\beta^k, \beta^\ell) \underset{\substack{\uparrow \\ \text{defn of } S^{k\ell}}}{=} \Lambda^i_k \Lambda^j_\ell \, S^{k\ell}$$

We conclude that $\boxed{\tilde{S}^{ij} = \Lambda^i_k \Lambda^j_\ell \, S^{k\ell}}$

NEW OLD

and hence say that σ is a <u>contravariant 2-tensor</u>.

<u>Exercise</u> In terms of matrices $\boxed{\tilde{S} = \Lambda^T S \Lambda}$

Compare with Lecture 5, pages 5.6-5.7.

Tensors of type (p,q)

contravariant rank p — covariant rank q

Defn A tensor of type (p,q) or (p,q)-tensor is a multilinear form on the space

$$\underbrace{V^* \times \dots \times V^*}_{p} \times \underbrace{V \times \dots \times V}_{q}$$

i.e. is a $(p+q)$-linear function

$$\underbrace{V^* \times \dots \times V^*}_{p} \times \underbrace{V \times \dots \times V}_{q} \longrightarrow \mathbb{R} .$$

What we have done earlier extends to (p,q)-tensors.

In particular ...

Let T be a (p,q) - tensor

$\mathcal{B} = (b_1, \ldots, b_n)$ basis of V

$\mathcal{B}^* = (\beta^1, \ldots, \beta^n)$ corresp. dual basis of V^*.

The components of T w.r.t. these bases are

$$\boxed{T^{i_1, \ldots, i_p}_{j_1, \ldots, j_q} = T(\beta^{i_1}, \ldots, \beta^{i_p}, b_{j_1}, \ldots, b_{j_q})}$$

If $\tilde{\mathcal{B}} = (\tilde{b}_1, \ldots, \tilde{b}_n)$ another basis of V

$\tilde{\mathcal{B}}^* = (\tilde{\beta}_1, \ldots, \tilde{\beta}_n)$ corresp. dual basis

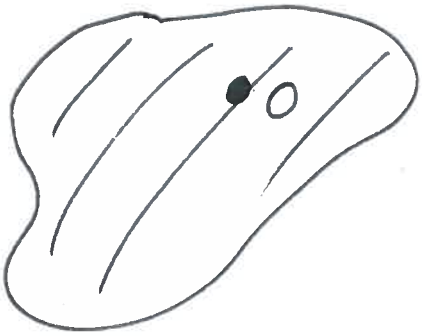
L = change of basis matrix

$\Lambda = L^{-1}$

the components of T w.r.t. these new bases are

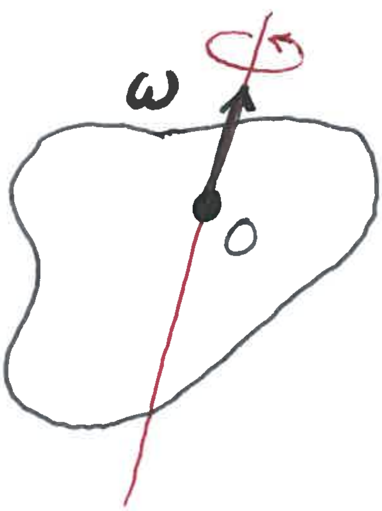
$$\boxed{\tilde{T}^{i_1, \ldots, i_p}_{j_1, \ldots, j_q} = \Lambda^{i_1}_{k_1} \cdots \Lambda^{i_p}_{k_p} L^{l_1}_{j_1} \cdots L^{l_q}_{j_q} T^{k_1, \ldots, k_p}_{l_1, \ldots, l_q}}$$

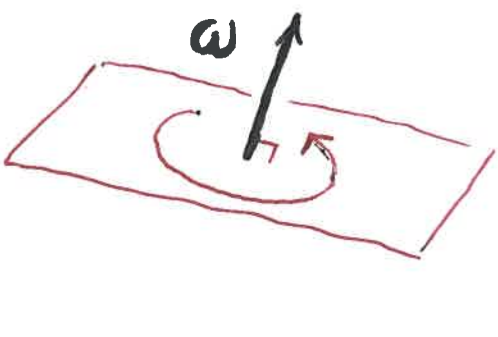
Inertia tensor



rigid body M
fixed at a point O

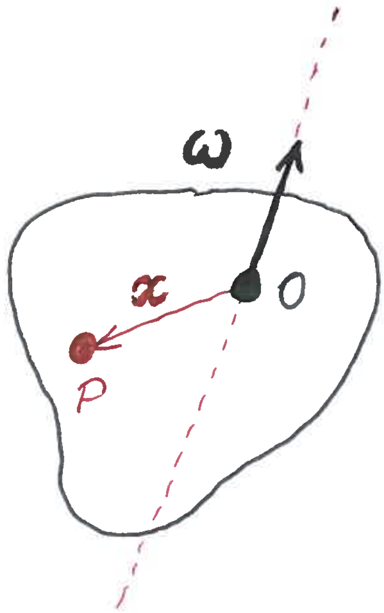
The motion of this body at time t is rotation with angular velocity $\boldsymbol{\omega}$ about some axis through O.





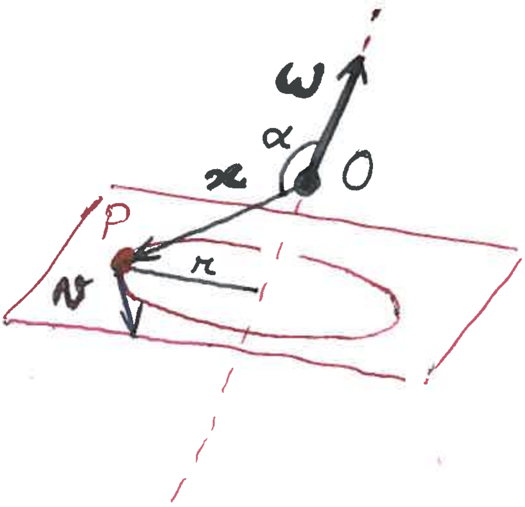
angular velocity $\boldsymbol{\omega}$

has magnitude $|\boldsymbol{\omega}| = \left|\frac{d\theta}{dt}\right|$ and direction giving axis of rotation and orientation by right-hand rule



The <u>position vector</u> of a point P in the body M relative to the origin O is $\mathbf{x} = \overrightarrow{OP}$.

The <u>linear velocity</u> of a point P is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$.



linear velocity $\mathbf{v}$ has magnitude

$$|\mathbf{v}| = \underbrace{|\boldsymbol{\omega}|}_{\left|\frac{d\theta}{dt}\right|} \underbrace{|\mathbf{x}| \sin\alpha}_{r}$$

and direction tangent at P to circle of radius r perpendicular to the axis of rotation

The **kinetic energy** of an infinitesimal region ● of M around P is

$$\underbrace{dE}_{\text{kinetic energy of }\bullet} = \frac{1}{2}\underbrace{v^2}_{v^2 = v\cdot v}\;\underbrace{dm}_{\text{mass of }\bullet}$$

The **total kinetic energy** of M is

$$E = \frac{1}{2}\underbrace{\int_M}_{\substack{\text{integral or sum over } M\\ \text{depending on the type of rigid body}}} v^2\, dm = \frac{1}{2}\int_M (\omega\times x)^2\, dm$$

If M is a solid in 3-d space

$$E = \frac{1}{2} \iiint_M \underbrace{(\omega_p \times x_p)^2 \rho_p}_{\text{function of } P(x^1, x^2, x^3)} dx^1 dx^2 dx^3$$

If M is a flat sheet in 3-d space

$$E = \frac{1}{2} \iint_M (\omega_p \times x_p)^2 \rho_p \, dx^1 dx^2$$

If M is a surface in 3-d space

$$E = \frac{1}{2} \iint_M (\omega_p \times x_p)^2 \rho_p \underbrace{d\sigma}_{\text{for surface integral}}$$

If M is a wire in 3-d space

$$E = \frac{1}{2} \int_M (\omega_p \times x_p)^2 \rho_p \underbrace{ds}_{\text{for line integral}}$$

If M is a finite set of point masses with rigid relative positions

$$E = \frac{1}{2} \sum_{i=1}^{N} (\omega \times x_i)^2 m_i$$

In any case,

we need to work out

$$(\boldsymbol{\omega} \times \boldsymbol{x})^2$$

for vectors $\boldsymbol{\omega}$ and $\boldsymbol{x}$ in 3-d space.

Use Lagrange's identity :

$$(a \times b) \cdot (c \times d) = \det \begin{bmatrix} a \cdot c & a \cdot d \\ b \cdot c & b \cdot d \end{bmatrix}$$

This identity can be proved in coordinates.

Applying this with $a = c = \boldsymbol{\omega}$, $b = d = \boldsymbol{x}$
we obtain

$$(\boldsymbol{\omega} \times \boldsymbol{x})^2 = (\boldsymbol{\omega} \times \boldsymbol{x}) \cdot (\boldsymbol{\omega} \times \boldsymbol{x})$$

$$= \det \begin{bmatrix} \boldsymbol{\omega} \cdot \boldsymbol{\omega} & \boldsymbol{\omega} \cdot \boldsymbol{x} \\ \boldsymbol{x} \cdot \boldsymbol{\omega} & \boldsymbol{x} \cdot \boldsymbol{x} \end{bmatrix}$$

$$= \boldsymbol{\omega}^2 \, \boldsymbol{x}^2 - (\boldsymbol{\omega} \cdot \boldsymbol{x})^2$$

Let $\mathcal{B} = (e_1, e_2, e_3)$ o.n. basis of $\mathbb{R}^3$

$$\boldsymbol{\omega} = \omega^i e_i$$

$$\boldsymbol{x} = x^i e_i$$

$$\boldsymbol{\omega}^2 = \boldsymbol{\omega} \cdot \boldsymbol{\omega} = \delta_{ij}\, \omega^i \omega^j = \omega^1\omega^1 + \omega^2\omega^2 + \omega^3\omega^3$$

$$\boldsymbol{x}^2 = \boldsymbol{x} \cdot \boldsymbol{x} = \delta_{kl}\, x^k x^l = x^1 x^1 + x^2 x^2 + x^3 x^3$$

$$\boldsymbol{\omega} \cdot \boldsymbol{x} = \delta_{ik}\, \omega^i x^k$$

$$(\boldsymbol{\omega} \times \boldsymbol{x})^2 = \boldsymbol{\omega}^2 \boldsymbol{x}^2 - (\boldsymbol{\omega} \cdot \boldsymbol{x})^2$$

$$= \left(\delta_{ij}\, \omega^i \omega^j\right)\left(\delta_{kl}\, x^k x^l\right) - \left(\delta_{ik}\, \omega^i x^k\right)\left(\delta_{jl}\, \omega^j x^l\right)$$

$$= \left(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}\right) \omega^i \omega^j x^k x^l$$

Therefore, the total kinetic energy is

$$E = \frac{1}{2}\left(\delta_{ij}\,\delta_{kl} - \delta_{ik}\,\delta_{jl}\right)\omega^i\omega^j \int_M x^k x^l \, dm$$

only depends on $\omega^1, \omega^2, \omega^3$

(not on x^1, x^2, x^3 because we integrate over M)

Defn The inertia tensor is the tensor whose components w.r.t. an o.n. basis $\mathcal{B}$ are

$$\boxed{I_{ij} = \left(\delta_{ij}\,\delta_{kl} - \delta_{ik}\,\delta_{jl}\right)\int_M x^k x^l \, dm}$$

Then the kinetic energy of the rotating rigid body is

$$E = \frac{1}{2} I_{ij}\,\omega^i\omega^j \quad \leftarrow \text{Einstein notation}$$

$$= \frac{1}{2}\,\boldsymbol{\omega}\cdot I\boldsymbol{\omega} \quad \leftarrow \text{vector notation}$$

Check :

$$I_{11} = \int_M \left(x^2 x^2 + x^3 x^3 \right) dm$$

$$I_{22} = \int_M \left(x^1 x^1 + x^3 x^3 \right) dm$$

$$I_{33} = \int_M \left(x^1 x^1 + x^2 x^2 \right) dm$$

$$I_{23} = I_{32} = - \int_M x^2 x^3 \, dm$$

$$I_{31} = I_{13} = - \int_M x^1 x^3 \, dm$$

$$I_{12} = I_{21} = - \int_M x^1 x^2 \, dm$$

I_{11}, I_{22}, I_{33} are the <u>moments of inertia</u> of the rigid body M w.r.t. the axes Ox_1, Ox_2, Ox_3 respectively.

I_{12}, I_{23}, I_{31} are the <u>polar moments of inertia</u> or the <u>products of inertia</u> of the rigid body M.