Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Last Name:	**Department:**
First Name:	**ETH ID Legi-Nr.:**

For the grading:

	1K	**2K**	**points**	**comments:**
1				
2				
3				
4				
5				
Total				

ETH Zürich
Dr. Ana Cannas

Summer 2016

Mathematics II Exam

for students of Agricultural Sciences, Earth Sciences, Food Science and Environmental Sciences

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Important:

- Lay your ETH-Card visible on the table.
- Fill in the heading of the front page.
- Note all intermediate results and approaches to the solutions of exercises 1-4.
- Write your name on each additional sheet.
- After each exercise there is its maximally reachable number of points.
- Use a blue or a black pen.

Permitted aid material:

- written notes (20 A4 pages)
- **no** calculator
- **no** mobile phone

Good Luck!

1. Consider the function

$$f(x, y) = 2xy + y^2 - 2x.$$

a) Determine the gradient of f as a function of (x, y). 2 points

b) Find the critical points of f and classify them. 3 points

c) Find the maximal value $f(x, y)$ reaches on the triangle D given by the conditions

$$x \geq 0,\ y \geq 0,\ x + y \leq 1.$$

3 points

d) Provide an expression involving iterated integrals for the integral of f over the triangle D from part c).
You **need not** compute the integral. 2 points

2. The position of a particle at time t is given in polar coordinates by

$$\theta = t,\ r = e^{-t},\ 0 \leq t \leq \pi.$$

a) Give a parametrization

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

of the particle trajectory. 2 points

b) At which point is the tangent line to this path horizontal? 3 points

c) Suppose the particle moves in the gradient field

$$\vec{F}(x, y) = \left(\frac{2x}{x^2 + y^2}\ ,\ \frac{2y}{x^2 + y^2} + 1 \right)^T,\ (x, y) \neq (0, 0).$$

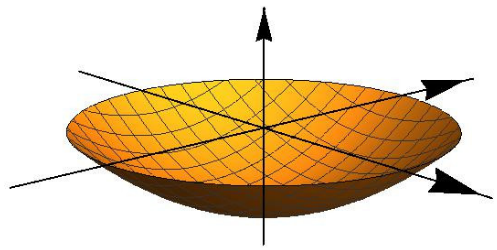
Give a potential function for $\vec{F}$. 3 points

d) What is the line integral of the vector field $\vec{F}$ from part c) from point $(1, 0)$ to point $(-e^{-\pi}, 0)$ along this path? 2 points

3. Let S be the part of a paraboloid given by

$$z = x^2 + y^2 - 1,\ z \leq 0$$

as depicted in the figure below.



surface S

a) Parametrize S using cylindrical or polar coordinates. 2 points

b) Calculate the area of S. 4 points

c) Determine the upward flux of the curl of $\vec{F}$, where

$$\vec{F}(x,y,z) = \begin{pmatrix} y - z \\ e^z - x \\ x - e^y \end{pmatrix},$$

through S *using Stokes' theorem*. 4 points

4. We consider the initial-boundary value problem

$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < 1,\ t > 0, \\ u(0,t) = 0, & t \geq 0, \\ u(1,t) = 1, & t \geq 0, \\ u(x,0) = f(x), & 0 < x < 1, \\ u_t(x,0) = g(x), & 0 < x < 1, \end{cases} \tag{1}$$

where $f(x)$ and $g(x)$ denote real functions.

a) Find a stationary solution $u^*(x)$ to the wave equation $u_{tt} = u_{xx}$ on the interval $(0,1)$ satisfying the same boundary conditions as in (1). 2 points

b) We choose the ansatz

$$u(x,t) = u^*(x) + v(x,t)$$

for the solution of the full problem. Formulate the corresponding problem with the appropriate conditions for the function $v(x,t)$. 3 points

c) Solve problem (1) for

$$f(x) = x - \sin(3\pi x) \text{ and } g(x) = 0.$$

You may use relevant eigenfunctions without deriving them. 3 points

5. Instructions: Mark the correct answer. There is always exactly one correct answer! There are 2 points for each question. Wrong or multiple answers will be valuated with 0 points.

5.1 Which one of the following integrals represents the graph of the function

$$f(x) = x^2, \quad 0 \le x \le 1 \quad ?$$

- ☐ (a) $\int_0^1 (1 + 2x)\, dx.$
- ☐ (b) $\int_0^1 (2x - 1)\, dx.$
- ☐ (c) $\int_0^1 \sqrt{1 + x^2}\, dx.$
- ☐ (d) $\int_0^1 \sqrt{1 + 4x^2}\, dx.$

5.2 What is the quadratic Taylor polynomial of

$$f(x, y) = x \tan(y)$$

at the point $(x, y) = (0, \frac{\pi}{4})$?

- ☐ (a) $x + xy.$
- ☐ (b) $x + 2xy.$
- ☐ (c) $x - \pi x + 2xy.$
- ☐ (d) $x - \frac{\pi}{2}x + 2xy.$

5.3 The level set

$$\ln(x+y) + x^2 y + x + y = 1$$

can in a neighborhood of the point $(x, y) = (1, 0)$ be written as graph of a function $x = x(y)$. What is the value of $x'(0)$?

☐ (a) $-\frac{3}{2}$.

☐ (b) $-\frac{2}{3}$.

☐ (c) $\frac{2}{3}$.

☐ (d) $\frac{3}{2}$.

5.4 We consider the spatial domain given by

$$1 \leq x^2 + y^2 + z^2 \leq 3 \text{ and } z < -\sqrt{x^2 + y^2}.$$

Which is the corresponding description in spherical coordinates?

- ☐ (a) $1 \leq \varrho \leq \sqrt{3}$ and $0 \leq \varphi < \dfrac{3\pi}{4}$.

- ☐ (b) $1 \leq \varrho \leq \sqrt{3}$ and $\dfrac{3\pi}{4} < \varphi \leq \pi$.

- ☐ (c) $1 \leq \varrho \leq 3$ and $0 \leq \varphi < \dfrac{3\pi}{4}$.

- ☐ (d) $1 \leq \varrho \leq 3$ and $\dfrac{3\pi}{4} < \varphi \leq \pi$.

5.5 On which of the following domains does the vector field

$$\vec{F}(x,y,z) = \left(\frac{-z}{x^2+z^2}, y^2, \frac{x}{x^2+z^2}\right)^T$$

not constitute a gradient field?

☐ (a) $A = \{(x,y,z) |\ x > z > 0\}$.

☐ (b) $B = \{(x,y,z) |\ x > 0 \text{ and } z > 0\}$.

☐ (c) $C = \{(x,y,z) |\ 1 < x^2 + z^2 < 4\}$.

☐ (d) $D = \{(x,y,z) |\ x^2 + z^2 > 1 \text{ and } z > 0\}$.

5.6 What is the value of the π-periodic Fourier series of

$$f(x) = x^2 - \pi x,\ 0 \leq x \leq \pi$$

at the point $x = \frac{3\pi}{2}$?

☐ (a) $-\dfrac{3\pi^2}{4}$.

☐ (b) $-\dfrac{\pi^2}{4}$.

☐ (c) $\dfrac{\pi^2}{4}$.

☐ (d) $\dfrac{3\pi^2}{4}$.