Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Last Name:	**Department:**
First Name:	**ETH ID Legi-Nr.:**

For the grading:

	1K	**2K**	**points**	**comments:**
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
Total				

ETH Zurich
Dr. Ana Cannas

Summer 2015

Mathematics I and II Exam

for students of Earth Sciences, Food Science, Agricultural Science and Environmental Sciences

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Important:

- Lay your ETH-Card visible on the table.
- Fill in the heading of the front page.
- Note all intermediate results and approaches to the solution.
- Write your name on each additional sheet.
- After each exercise there is its maximally reachable number of points.
- Use a blue or a black pen.

Permitted aid material:

- written notes (40 A4-pages)
- **no** calculator
- **no** mobile phone

Good Luck!

1. Let

$$A = \begin{pmatrix} 1 & 2 & -1 & -3 \\ -2 & -4 & 2 & 6 \\ 0 & 0 & -1 & 4 \end{pmatrix}.$$

a) Is the system

$$A\,\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

solvable?

3 points

b) Solve the system

$$A\,\vec{x} = \vec{0}.$$

3 points

c) What is the dimension of the space of solutions of $A\,\vec{x} = \vec{0}$?

2 points

2. We consider the family of matrices

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & a & 1 \\ 0 & 0 & b \end{pmatrix},$$

where a and b are real parameters.

a) Compute the inverse A^{-1}, when $a = b = 1$.

3 points

b) For which values a and b is A invertible?

You do **not** have to determine the inverses.

3 points

c) What is the rank of A, when $a = 2$ and $b = 1$?

2 points

3. We consider the system of differential equations

$$\begin{cases} \dot{x}(t) &= -x(t) - 2y(t) \\ \dot{y}(t) &= \ \ 3x(t) + 4y(t) \end{cases}$$

a) Determine the eigenvalues and the corresponding eigenvectors of the coefficient matrix of this system.

4 points

b) Determine the general solution of this system.

2 points

c) Is the origin a stable or an unstable equilibrium point? Do not forget to justify your answer.

2 points

4. We consider differential equations of the form

$$y'' + 4y' + 13y = f(x),$$

where $y(x)$ is an unknown real function and $f(x)$ is a given real function.

a) Determine the zeros of the characteristic polynomial.

2 points

b) Determine the general solution of the differential equation, when $f(x) = -40\,\sin x$.

4 points

c) Can a nontrivial solution of the homogeneous differential equation (that is with $f(x) \equiv 0$) be periodic? Do not forget to justify your answer.

2 points

5. Let $f(x)$ be the function defined for $x > 0$ with

$$f'(x) = x \ln x \quad \text{and} \quad f(1) = 0.$$

a) What is the limit

$$\lim_{x \to 1} \frac{x \ln x}{\sin(x-1)} \quad ?$$

2 points

b) What is the smallest value of $f(x)$?

3 points

c) Determine $f(x)$.

3 points

6. The position of a particle at time t is given by

$$\vec{r}(t) = \begin{pmatrix} 2\cos t \\ 2\sin t \\ t \end{pmatrix}, \quad t \geq 0.$$

a) Give a parametrisation of the tangent line to this path at time $t = \dfrac{\pi}{2}$.

2 points

b) What is the length of the part of the curve between the points

$$A = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -2 \\ 0 \\ \pi \end{pmatrix}?$$

3 points

c) This particle moves in the gradient vector field

$$\vec{F}(x, y, z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}.$$

Give a potential function of $\vec{F}$.

What is the work of $\vec{F}$ from point A to point B from part (b) along this curve?

3 points

7. Consider the function

$$f(x, y) = x^2 + y^4 - 2y^2 + 1.$$

a) Determine all critical points of f.

2 points

b) Classify the critical points of f (as saddle point, local minimum, local maximum).

3 points

c) Determine the maximal and the minimal value, that f achieves on the line segment given by

$$y = x, \quad 0 \leq x \leq 1.$$

3 points

8. Let V be the solid body above the plane

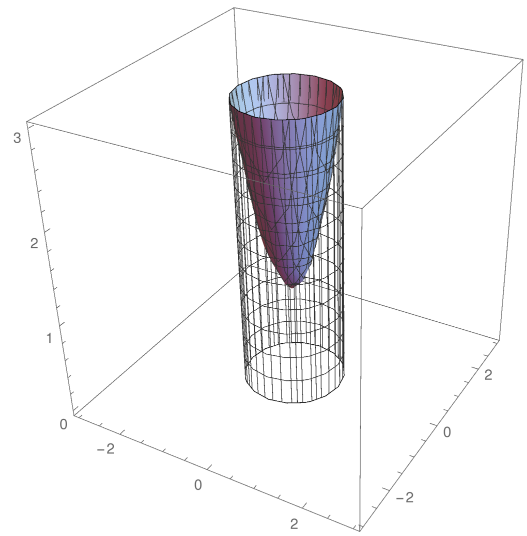
$$z = 0,$$

below the paraboloid

$$z = 1 + 2x^2 + 2y^2,$$

and within the cylinder

$$x^2 + y^2 = 1.$$



a) Compute the volume of V.

4 points

b) Determine the divergence of

$$\vec{G}(x,y,z) = \begin{pmatrix} \frac{z}{y^2+1} + 2xyz \\ \sin(xz) - y^2 z \\ x^2 + y^2 + z \end{pmatrix}.$$

Can $\vec{G}$ be the rotation of another vector field $\vec{F}$? Do not forget to justify your answer.

2 points

c) Compute the outwards flux of the above vector field $\vec{G}$ through the surface of V.

2 points

9. We consider the initial-boundary value problem

$$\begin{cases} u_{xx} = u_t + 2u \\ u_x(0,t) = u_x(2\pi,t) = 0 \\ u(x,0) = f(x), \end{cases}$$

where $0 \leq x \leq 2\pi,\ t \geq 0$.

a) After applying separation of variables $u(x,t) = X(x)T(t)$ this partial differential equation splits into a system of ordinary differential equations for $X(x)$ and $T(t)$ depending on a parameter $k \in \mathbb{R}$.

What is the corresponding system?

3 points

b) Determine a series of eigenfunctions

$$u_n(x,t) = X_n(x)T_n(t), \quad n = 0, 1, 2, \cdots$$

of the homogeneous part of this problem.

3 points

c) Solve the initial-boundary value problem for

$$f(x) = \begin{cases} x, & 0 \leq x \leq \pi \\ 2\pi - x, & \pi < x \leq 2\pi \end{cases},$$

where

$$\mathcal{F}_f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos((2k+1)x)$$

is the cosine-series of $f(x)$.

2 points

10. We consider the temperature distribution $u(x,t)$ in a bar, which is modelled by the following partial differential equation:

$$u_t = 4u_{xx}.$$

a) Let $L = 2$ be the length of the bar and let the ends be kept at constant temperature -1 and 1. Solve the problem

$$\begin{cases} u_t = 4u_{xx} \\ u(-1,t) = -1, \quad u(1,t) = 1 \\ u(x,0) = x + 7\sin(3\pi x). \end{cases}$$

Hint: You may use eigenfunctions of the corresponding homogeneous problem without deriving them.

3 points

b) Determine the Fourier-integral of

$$f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

3 points

c) For modelling purposes let the bar be infinitely long. Determine an integral expression of the bounded solution $u(x,t)$, $x \in \mathbb{R}$, $t \ge 0$ of

$$\begin{cases} u_t = 4u_{xx} \\ u(x,0) = f(x), \end{cases}$$

where $f(x)$ is the function from part (b).

2 points

11. Instructions: Mark the correct answer. There is always only one answer correct and 2 points per question. Wrong or multiple crosses are valuated with 0 points.

11.1 Which is a parametrisation of the ellipse

$$(x-1)^2 + (2y+2)^2 = 4 \quad ?$$

☐ (a) $x = 2 + \cos t, \quad y = 1 - \sin t, \quad 0 \le t < 2\pi.$

☐ (b) $x = 1 + \cos t, \quad y = -1 + \frac{1}{2}\sin t, \quad 0 \le t < 2\pi.$

☐ (c) $x = 2 - 2\cos t, \quad y = -\frac{1}{2} - \sin t, \quad 0 \le t < 2\pi.$

☐ (d) $x = 1 + 2\cos t, \quad y = -1 + \sin t, \quad 0 \le t < 2\pi.$

11.2 Which pair of conic sections does **not** intersect?

☐ (a) $x^2 - y^2 = 1$ and $x^2 + y^2 = 4.$

☐ (b) $x^2 - y^2 = 4$ and $x^2 + y^2 = 1.$

☐ (c) $x^2 + 4y^2 = 4$ and $4x^2 + y^2 = 4.$

☐ (d) $x^2 + y^2 = 1$ and $16x^2 + y^2 = 4.$

11.3 Let $f(x, y)$ be four times continuously differentiable and with

$$\frac{\partial^2 f}{\partial x \partial y} = e^{2x} - y.$$

Which statement is **wrong**?

- ☐ (a) $\frac{\partial^2 f}{\partial y \partial x} = e^{2y} - x.$
- ☐ (c) $\frac{\partial^3 f}{\partial x \partial y^2} = -1.$
- ☐ (b) $\frac{\partial^3 f}{\partial y \partial x^2} = 2e^{2x}.$
- ☐ (d) $\frac{\partial^4 f}{\partial x^2 \partial y^2} = 0.$

11.4 The level curve

$$x^2 + y^4 - 2y^2 = 1$$

is in a neighborhood of the point $(1, \sqrt{2})$ the graph of a function $x = x(y)$. What is the value of $x'(\sqrt{2})$?

- ☐ (a) $-2\sqrt{2}.$
- ☐ (c) $\frac{1}{2\sqrt{2}}.$
- ☐ (b) $-\frac{1}{2\sqrt{2}}.$
- ☐ (d) $2\sqrt{2}.$

11.5 We consider the function

$$f(x,y,z) = \frac{x}{y} + \sqrt{\sin z}.$$

In which direction $\vec{u}$ is the directional derivative of f at the point $P = \left(1, 1, \frac{\pi}{2}\right)$ maximal and how big is this maximal value?

- [ ] (a) $\vec{u} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T, \quad D_{\vec{u}}f(P) = 2.$

- [ ] (b) $\vec{u} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T, \quad D_{\vec{u}}f(P) = -2$

- [ ] (c) $\vec{u} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)^T, \quad D_{\vec{u}}f(P) = \sqrt{2}.$

- [ ] (d) $\vec{u} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)^T, \quad D_{\vec{u}}f(P) = -\sqrt{2}.$

11.6 Which is the quadratic Taylor-polynomial of

$$f(x,y) = \frac{e^x}{1+y}$$

at the origin $(x,y) = (0,0)$?

- [ ] (a) $1 + x + y + \frac{1}{2}x^2 + xy + y^2.$

- [ ] (b) $1 + x - y + \frac{1}{2}x^2 - xy + y^2.$

- [ ] (c) $1 + x + y + x^2 + 2xy + 2y^2.$

- [ ] (d) $1 + x - y + x^2 - 2xy + 2y^2.$

11.7 After changing the order of integration the integral

$$\int_0^2 \int_0^1 f(x,y)\,dy\,dx + \int_{-2}^0 \int_{-\frac{1}{2}x}^1 f(x,y)\,dy\,dx$$

is equal to

☐ (a) $\displaystyle\int_{-2}^1 \int_{2y}^2 f(x,y)\,dx\,dy.$

☐ (b) $\displaystyle\int_0^1 \int_{-2y}^2 f(x,y)\,dx\,dy.$

☐ (c) $\displaystyle\int_0^2 \int_0^1 f(x,y)\,dx\,dy + \int_{-2}^0 \int_{-\frac{1}{2}y}^1 f(x,y)\,dx\,dy.$

☐ (d) $\displaystyle\int_0^1 \int_0^2 f(x,y)\,dx\,dy + \int_0^1 \int_0^{2y} f(x,y)\,dx\,dy.$

11.8 We consider the region defined by

$$1 \le x^2 + y^2 + z^2 \le 4 \quad \text{and} \quad z = \sqrt{x^2 + y^2}.$$

Which description in spherical coordinates corresponds to this region?

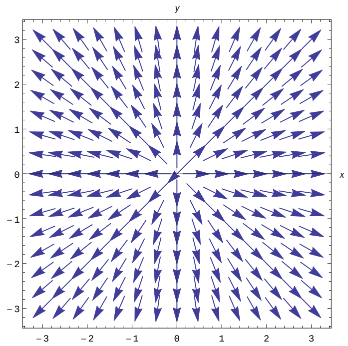
☐ (a) $1 \le R \le 2$ and $\varphi = \dfrac{\pi}{4}$.

☐ (b) $1 \le R \le 2$ and $\varphi = \dfrac{\pi}{2}$.

☐ (c) $1 \le R \le 4$ and $\varphi = \dfrac{\pi}{4}$.

☐ (d) $1 \le R \le 4$ and $\varphi = \dfrac{\pi}{2}$.

11.9 Which vector field matches this sketch?



☐ (a) $\vec{F}(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right).$

☐ (b) $\vec{F}(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}\right).$

☐ (c) $\vec{F}(x,y) = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}\right).$

☐ (d) $\vec{F}(x,y) = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right).$

11.10 Let $\vec{H} = (H_1, H_2)$ be a vector field in $\mathbb{R}^2 \setminus \{(0,0)\}$, so that $\frac{\partial H_2}{\partial x} = \frac{\partial H_1}{\partial y}$ and the circulation of $\vec{H}$ along the unit circle $C_1 : x^2 + y^2 = 1$ in positive direction is $\oint_{C_1} H_1\,dx + H_2\,dy = 1$.

What is the circulation of $\vec{H}$ along the circle $C_2 : x^2 + y^2 = 4$ in positive direction?

☐ (a) 1.

☐ (b) 4.

☐ (c) −1.

☐ (d) −4.