ETH Zürich
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Summer 2016

# Mathematics I and II Exam

**for students of Earth Sciences, Food Science, Agricultural Science and Environmental Sciences**

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**Important:**

- Lay your ETH-Card visible on the table.
- Fill in the heading of the front page.
- Note all intermediate results and approaches to the solutions of exercises 1-4.
- Write your name on each additional sheet.
- After each exercise there is its maximally reachable number of points.
- Use a blue or a black pen.

**Permitted aid material:**

- written notes (40 A4 pages)
- **no** calculator
- **no** mobile phone

Good Luck!

**1.** Consider the function

$$f(x) = \ln(2x - x^2).$$

**a)** What is the (maximal) domain where $f(x)$ is defined? 2 points

**b)** Find the points $x$ where $f(x)$ attains local extrema. 2 points

**c)** What is the range of the function $f(x)$? 3 points

**d)** Determine the value of 3 points

$$\lim_{x \to 0^+} x f(x).$$

**2.** Consider the ordinary differential equation

$$y'' - 4y' + 5y = e^{2x}.$$

**a)** Determine the zeros of the characteristic polynomial. 2 points

**b)** Determine the general real solution of the associated homogeneous differential equation. 2 points

**c)** Find a particular solution of the inhomogeneous equation. 3 points

**d)** Find the solution of the differential equation satisfying the conditions 3 points

$$y(0) = 1 \text{ und } y'(0) = 3.$$

**3.** Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 4 & 3 \\ 3 & 6 & 3 & 9 & 6 \end{pmatrix}.$$

**a)** Is the system

$$A\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

solvable? Either solve it or show its unsolvability. 4 points

**b)** For which real numbers $k$ is the system

$$A\vec{x} = \begin{pmatrix} k \\ k \\ k \end{pmatrix}$$

solvable? 2 points

**c)** Determine the dimension of the null space of $A$, $\dim(N(A))$. 2 points

**4.** Consider the system of ordinary differential equations

$$\begin{cases} \dot{x}(t) &= 3x(t) - 4y(t) \\ \dot{y}(t) &= 2x(t) - 3y(t). \end{cases}$$

**a)** Find the eigenvalues and corresponding eigenvectors of the coefficient matrix of this system. 4 points

**b)** Determine the general solution of this system. 2 points

**c)** Find the solution to the system satisfying the initial condition

$$\begin{cases} x(0) &= 1, \\ y(0) &= 0. \end{cases}$$

2 points

**d)** For which initial conditions

$$\begin{cases} x(0) &= x_0 \\ y(0) &= y_0 \end{cases}$$

does the solution to the system stay bounded for $t \geq 0$? 2 points

**5.** Consider the function

$$f(x, y) = 2xy + y^2 - 2x.$$

**a)** Determine the gradient of $f$ as a function of $(x, y)$. 2 points

**b)** Find the critical points of $f$ and classify them. 3 points

**c)** Find the maximal value $f(x, y)$ reaches on the triangle $D$ given by the conditions

$$x \geq 0,\ y \geq 0,\ x + y \leq 1.$$

3 points

**d)** Provide an expression involving iterated integrals for the integral of $f$ over the triangle $D$ from part c).
You **need not** compute the integral. 2 points

**6.** The position of a particle at time $t$ is given in polar coordinates by

$$\theta = t,\ r = e^{-t},\ 0 \leq t \leq \pi.$$

**a)** Give a parametrization

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

of the particle trajectory. 2 points

**b)** At which point is the tangent line to this path horizontal? 3 points

**c)** Suppose the particle moves in the gradient field

$$\vec{F}(x,y) = \left( \frac{2x}{x^2+y^2} \, , \, \frac{2y}{x^2+y^2} + 1 \right)^T , \; (x,y) \neq (0,0).$$

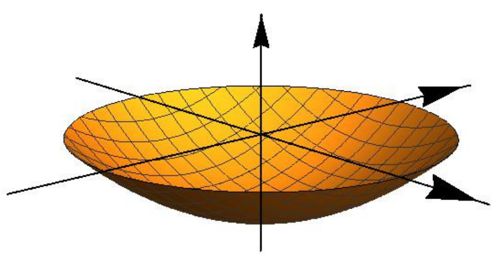
Give a potential function for $\vec{F}$. 3 points

**d)** What is the line integral of the vector field $\vec{F}$ from part c) from point $(1,0)$ to point $(-e^{-\pi}, 0)$ along this path? 2 points

**7.** Let $S$ be the part of a paraboloid given by

$$z = x^2 + y^2 - 1, \; z \leq 0$$

as depicted in the figure below.



surface S

**a)** Parametrize $S$ using cylindrical or polar coordinates. 2 points

**b)** Calculate the area of $S$. 4 points

**c)** Determine the upward flux of the curl of $\vec{F}$, where

$$\vec{F}(x,y,z) = \begin{pmatrix} y-z \\ e^z - x \\ x - e^y \end{pmatrix},$$

through $S$ *using Stokes' theorem.* 4 points

**8.** We consider the initial-boundary value problem

$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < 1,\ t > 0, \\ u(0,t) = 0, & t \geq 0, \\ u(1,t) = 1, & t \geq 0, \\ u(x,0) = f(x), & 0 < x < 1, \\ u_t(x,0) = g(x), & 0 < x < 1, \end{cases} \tag{1}$$

where $f(x)$ and $g(x)$ denote real functions.

**a)** Find a stationary solution $u^*(x)$ to the wave equation $u_{tt} = u_{xx}$ on the interval $(0,1)$ satisfying the same boundary conditions as in (1). 2 points

**b)** We choose the ansatz

$$u(x,t) = u^*(x) + v(x,t)$$

for the solution of the full problem. Formulate the corresponding problem with the appropriate conditions for the function $v(x,t)$. 3 points

**c)** Solve problem (1) for

$$f(x) = x - \sin(3\pi x) \text{ and } g(x) = 0.$$

You may use relevant eigenfunctions without deriving them. 3 points

**9. Instructions:** Mark the correct answer. There is always exactly one correct answer and 2 points per question. Wrong or multiple answers will be valuated with 0 points.

9.1 Which of the following is a parametrization of the ellipse

$$(x+1)^2 + 16(y-2)^2 = 16$$

for $0 \leq t < 2\pi$?

☐ (a) $x = 4\cos(t) - 4, \quad y = \sin(t) - 2.$

☐ (b) $x = 4\cos(t) - 1, \quad y = \sin(t) + 2.$

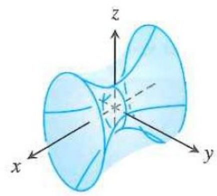
☐ (c) $x = \dfrac{\cos(t) - 4}{4}, \quad y = \sin(t) - 2.$

☐ (d) $x = \dfrac{\cos(t) - 1}{4}, \quad y = \sin(t) + 2.$

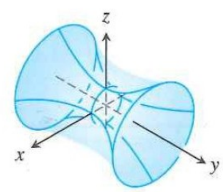
9.2 Which of the following depicts the surface given by

$$x = y^2 - z^2 \quad ?$$

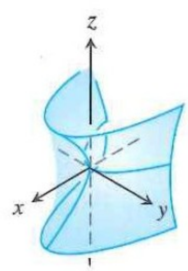
☐ (a)



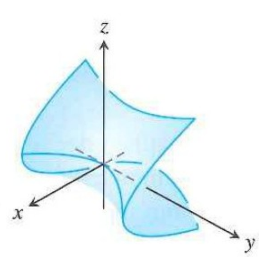

☐ (b)




☐ (c)




☐ (d)



9.3 Which integral represents the arc length of the graph of the function

$$f(x) = x^2, \quad 0 \leq x \leq 1 \quad ?$$

☐ (a) $\int_0^1 (1 + 2x)\, dx.$

☐ (b) $\int_0^1 (2x - 1)\, dx.$

☐ (c) $\int_0^1 \sqrt{1 + x^2}\, dx.$

☐ (d) $\int_0^1 \sqrt{1 + 4x^2}\, dx.$

9.4 Suppose $f(x, y, z)$ is twice continuously differentiable. The second order partial derivatives of $f$ are at most

☐ (a) 3 different functions.

☐ (b) 4 different functions.

☐ (c) 6 different functions.

☐ (d) 9 different functions.

9.5 Which polynomial is the quadratic Taylor polynomial of the function

$$f(x, y) = x\tan(y)$$

at the point $(x, y) = (0, \frac{\pi}{4})$?

☐ (a) $x + xy$.

☐ (b) $x + 2xy$.

☐ (c) $x - \pi x + 2xy$.

☐ (d) $x - \frac{\pi}{2}x + 2xy$.

9.6 The level curve

$$\ln(x + y) + x^2y + x + y = 1$$

is in a neighborhood of the point $(x, y) = (1, 0)$ the graph of a function $x = x(y)$. What is the value of $x'(0)$?

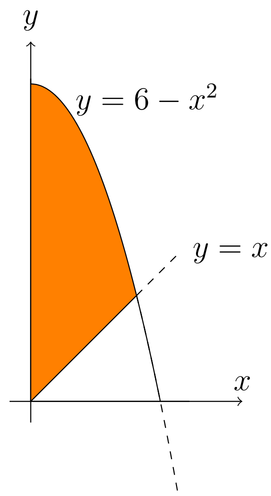
☐ (a) $-\dfrac{3}{2}$.

☐ (b) $-\dfrac{2}{3}$.

☐ (c) $\dfrac{2}{3}$.

☐ (d) $\dfrac{3}{2}$.

9.7 What is the area of the region bounded by the lines $x = 0$, $y = x$ and the parabola $y = 6 - x^2$?




- [ ] (a) $\frac{31}{6}$.
- [ ] (b) $\frac{37}{6}$.
- [ ] (c) $\frac{22}{3}$.
- [ ] (d) $\frac{34}{3}$.

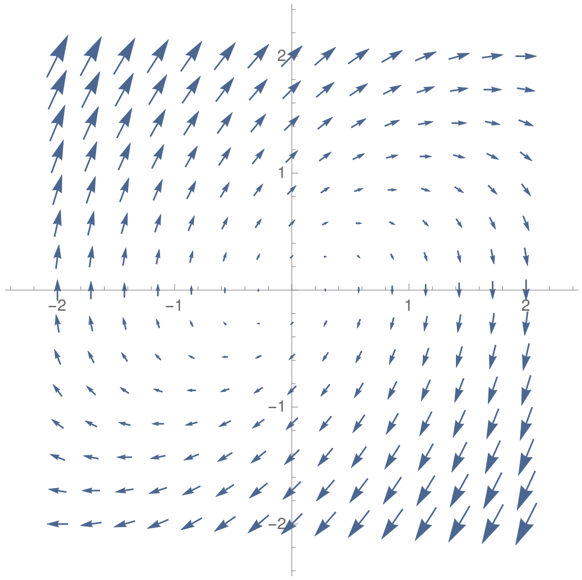
9.8 We consider the solid region given by

$$1 \leq x^2 + y^2 + z^2 \leq 3 \text{ and } z < -\sqrt{x^2 + y^2}.$$

Which one of the following descriptions in spherical coordinates corresponds to this region?

- [ ] (a) $1 \leq \varrho \leq \sqrt{3}$ and $0 \leq \varphi < \frac{3\pi}{4}$.
- [ ] (b) $1 \leq \varrho \leq \sqrt{3}$ and $\frac{3\pi}{4} < \varphi \leq \pi$.
- [ ] (c) $1 \leq \varrho \leq 3$ and $0 \leq \varphi < \frac{3\pi}{4}$.
- [ ] (d) $1 \leq \varrho \leq 3$ and $\frac{3\pi}{4} < \varphi \leq \pi$.

9.9 Which vector field matches this sketch?




- [ ] (a) $(y^2, 0)$.
- [ ] (b) $(y, y - x)$.
- [ ] (c) $(x, -2y)$.
- [ ] (d) $(x, y^2)$.

9.10 On which of the following domains does the vector field

$$\vec{F}(x, y, z) = \left( \frac{-z}{x^2 + z^2}, y^2, \frac{x}{x^2 + z^2} \right)^T$$

**not** constitute a gradient vector field?

- [ ] (a) $A = \{(x, y, z) | \ x > z > 0\}$.
- [ ] (b) $B = \{(x, y, z) | \ x > 0 \text{ and } z > 0\}$.
- [ ] (c) $C = \{(x, y, z) | \ 1 < x^2 + z^2 < 4\}$.
- [ ] (d) $D = \{(x, y, z) | \ x^2 + z^2 > 1 \text{ and } z > 0\}$.

9.11 What is the value of the $\pi$-periodic Fourier series of

$$f(x) = x^2 - \pi x,\ 0 \leq x \leq \pi$$

at the point $x = \frac{3\pi}{2}$ ?

- [ ] (a) $-\dfrac{3\pi^2}{4}$.
- [ ] (b) $-\dfrac{\pi^2}{4}$.
- [ ] (c) $\dfrac{\pi^2}{4}$.
- [ ] (d) $\dfrac{3\pi^2}{4}$.

9.12 The partial differential equation

$$u_{tt} = (1 + x^2 + 2t^2)u_{xx}$$

is

- [ ] (a) elliptic.
- [ ] (b) hyperbolic.
- [ ] (c) linear, but not homogeneous.
- [ ] (d) not linear.