ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Family name:	**Department:**
First name:	**ETH ID No.:**

For the grading:

	1K	2K	Points	Comments:
1				
2				
3				
4				
5				
6				
7				
8				
9-24				
Total				

MATHEMATICS I AND II EXAM

for students of Agricultural Science, Earth Sciences, Environmental Sciences, and Food Science

Important:

- Please fill the header on the cover page and lay your ETH-card visible on the table.
- Please write neatly with a non erasable blue or black pen, in particular not with a pencil. Beware that something that is too hard to read could be ignored.
- Please leave some empty space on the margins for the correction.
- This exam has 24 questions and lasts for 180 minutes.

For questions 1-8:

- Please write down all intermediate steps of your calculations and solutions.
- Write your name and ETH ID / Legi-Nr. on each additional sheet.
- The maximal score of each exercise part is given in the right margin.

For questions 9-24:

- Mark your answers clearly.
- There is always only one correct answer and 2 points per question.

Permitted aids:

- Written notes up to 40 A4-Pages, one English dictionary,
- **no** calculator, **no** mobile phone, **no** laptop.
- Please switch off your mobile phone and stow it away.

Good Luck!

1. Consider the function

$$f(x) = \frac{2x}{x^2+1} \quad \text{with } x \in \mathbb{R}.$$

a) Find and classify all local extrema of $f(x)$. 3 points

b) Determine the range of $f(x)$. 3 points

c) Solve the following initial value problem:

$$\begin{cases} F'(x) = f(x) \\ F(0) = 3. \end{cases}$$

3 points

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2. Determine the general solution of the following differential equations:

a) $y'' = 4y' - 5y$. 4 points

b) $xy' = x^2 + 3y \quad$ for $x > 0$. 4 points

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3. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 & 5 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 4 & 8 \end{pmatrix}.$$

a) Is the following matrix equation solvable?

$$A\vec{x} = \begin{pmatrix} 0 \\ 6 \\ -4 \end{pmatrix}$$

2 points

b) Determine a basis for the solution set of the matrix equation $A\vec{x} = \vec{0}$. 4 points

c) Determine a basis for the space of all vectors $\vec{v}$ for which the matrix equation $A\vec{x} = \vec{v}$ is solvable. 3 points

4. Consider the following matrix which is depending on a parameter k:

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & k & 0 \\ 1 & k+2 & k \end{pmatrix} .$$

a) Find all k such that A is invertible. 3 points

b) Determine the inverse of A when $k = 1$. 3 points

c) Find all k such that the system of linear differential equations

$$\dot{\vec{x}} = A\vec{x}$$

has a solution of the form

$$\vec{x}(t) = e^{2t}\vec{x}_0$$

for some nonzero vector $\vec{x}_0$. 2 points

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5. Consider the function

$$f(x, y) = 2x^3 + 3y^4 - 6xy^2 \ .$$

a) Determine the gradient of f in terms of (x, y). 2 points

b) Consider the direction $\vec{u} = \binom{1}{0}$ and the points $(x_1, y_1) = (1, 0)$ and $(x_2, y_2) = (0, 1)$. In which of these two points does f grow faster in the direction of $\vec{u}$? 2 points

c) Determine the critical point of f with $x > 0$ and $y > 0$. Classify this point (as local maximum/ local minimum / saddle point). 3 points

d) The equation

$$f(x, y) = 3$$

defines a differentiable function $y = y(x)$ in a neighbourhood of the point $(x, y) = (0, 1)$. Calculate $y'(0)$. 2 points

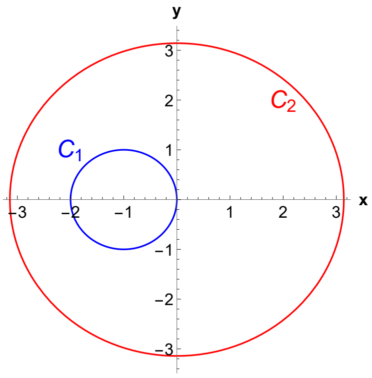
6. Consider the vector field

$$\vec{F}(x,y) = \begin{pmatrix} \dfrac{2y}{(x+1)^2+y^2} \\ \dfrac{-2(x+1)}{(x+1)^2+y^2} \end{pmatrix}.$$

a) Is $\vec{F}$ a gradient field in the first quadrant? 2 points

b) What is the line integral of $\vec{F}$ along the unit circle around $(-1,0)$ in counterclockwise direction? 3 points

c) Calculate the line integral of $\vec{F}$ along the circle C_2 with radius π around the origin in counterclockwise direction. 3 points



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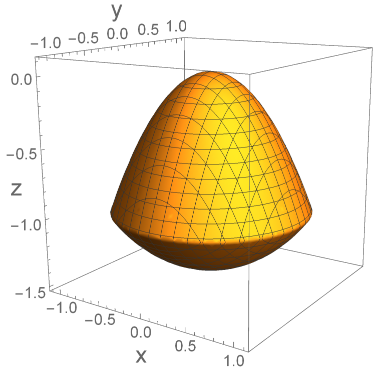
7. Consider the vector field

$$\vec{G}(x,y,z) = \begin{pmatrix} e^y + z \\ e^z + y \\ e^x \end{pmatrix}.$$

and the solid V above the sphere $x^2+y^2+z^2=2$ and below the paraboloid $x^2+y^2=-z$.



a) Represent the solid V as a multiple integral in cylindrical coordinates. 4 points

b) Find the flux of the vector field $\overrightarrow{G}$ outward through the surface S of V. 3 points

c) Find the flux of the vector field $\operatorname{rot}\overrightarrow{G}$ outward through the surface of V. 2 points

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8. We consider problems of the form

$$\begin{cases} u_t = u_{xx} \\ u_x(0,t) = u_x(\pi,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

where $0 \le x \le \pi$ and $t \ge 0$.

a) Determine the cosine series of the function

$$f(x) = x \ , \ \text{for } 0 \le x \le \pi \ .$$

5 points

b) Determine the solution $u(x,t)$ for the problem with $f(x) = x$.

(You may use relevant eigenfunctions without deriving them.)

If you didn't solve part (a), you can write the solution $u(x,t)$ of the problem in terms of the coefficients a_n of the cosine series of $f(x)$:

$$\sum_{n=0}^{\infty} a_n \cos(nx) \ .$$

3 points

For exercises 9-24: Each question gives 2 points. Wrong or multiple answers give 0 points. Mark your answers on this exam.

9. What is the value of the solution of the following initial value problem at $t = 1$?

$$\dot{\vec{r}}(t) = \begin{pmatrix} 3t^2 - 1 \\ 4e^{2t} \end{pmatrix}, \quad \vec{r}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(a) $\vec{r}(1) = \begin{pmatrix} 0 \\ e^2 \end{pmatrix}$

(b) $\vec{r}(1) = \begin{pmatrix} 1 \\ 2e^2 \end{pmatrix}$

(c) $\vec{r}(1) = \begin{pmatrix} 2 \\ 8e^2 \end{pmatrix}$

(d) $\vec{r}(1) = \begin{pmatrix} 4 \\ 4e^2 \end{pmatrix}$

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10. Which equation represents a circle with center $(-1, 1)$ and radius 2?

(a) $x^2 + 2x + y^2 - 2y = 0.$

(b) $x^2 - 2x + y^2 + 2y = 0.$

(c) $x^2 - 2x + y^2 + 2y = 2.$

(d) $x^2 + 2x + y^2 - 2y = 2.$

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11. Which is the equation in polar coordinates of the curve

$$x^2 = y^3,\ x > 0\,?$$

(a) $r = \cos^2\theta - \sin^3\theta,\ 0 < \theta < \frac{\pi}{2}.$

(b) $r = \sin^3\theta - \cos^2\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$

(c) $r = \frac{\sin^3\theta}{\cos^2\theta},\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$

(d) $r = \frac{\cos^2\theta}{\sin^3\theta},\ 0 < \theta < \frac{\pi}{2}.$

12. The domain D of the function

$$f(x,y) = \frac{1}{x^2+y^2-1}$$

is

(a) open and bounded.

(b) open and unbounded.

(c) closed and bounded.

(d) closed and unbounded.

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13. Consider the composition of the function

$$f(x,y) = e^x \sin y$$

with differentiable functions $x(t)$ and $y(t)$. What is the derivative

$$\frac{d}{dt} f\Big(x(t), y(t)\Big) \ ?$$

(a) $\dot{x}(t)e^{x(t)} + e^{x(t)} \cos\Big(y(t)\Big)$.

(b) $\dot{x}(t)e^{x(t)} - e^{x(t)} \sin\Big(y(t)\Big)$.

(c) $\dot{x}(t)e^{x(t)} \sin\Big(y(t)\Big) + e^{x(t)}\dot{y}(t) \cos\Big(y(t)\Big)$.

(d) $\dot{x}(t)e^{x(t)} \cos\Big(\dot{y}(t)\Big) - e^{x(t)}\dot{y}(t) \sin\Big(y(t)\Big)$.

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14. Which of the following partial differential equations does the function

$$u(x,y) = e^{x+cy^2}$$

satisfy?

(a) $u_{yy} = 2cy + u_{xy}$.

(b) $u_{yy} = 2cu + 2cyu_y$.

(c) $u_{yy} = 2cu_y + u_{xy}$.

(d) $u_{yy} = 2cyu + 2cu_y$.

15. What is the Taylor polynomial of

$$f(x, y) = \ln(1 + xy)$$

around $(x_0, y_0) = (0, 0)$?

(a) $\frac{1}{2}xy$. (b) $1 + \frac{1}{2}xy$. (c) xy. (d) $1 + xy$.

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16. What is the value of the integral of the function $f(x, y) = e^{4-x^2-y^2}$ over the disc with radius 2 and center in the origin?

(a) $\pi(e^4 - 1)$. (b) $2\pi(e^4 - 1)$. (c) $\pi(1 - e^{-4})$. (d) $2\pi(1 - e^{-4})$.

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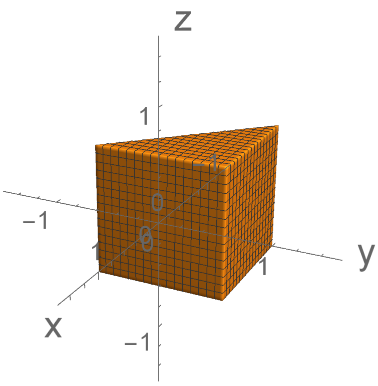
17. Consider an integral of the form

$$\int_0^1 \int_{1-x}^1 \int_0^1 f(x, y, z)\, dz\, dy\, dx.$$

What is the corresponding domain of integration?

(a)


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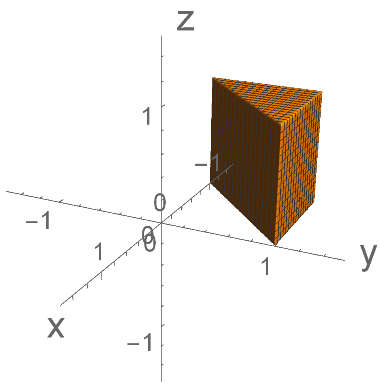

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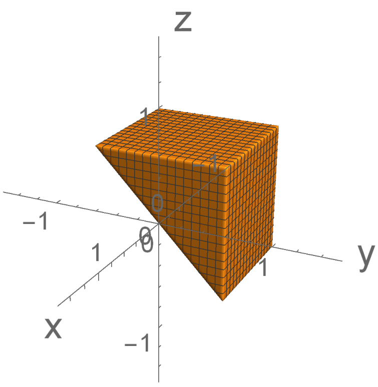

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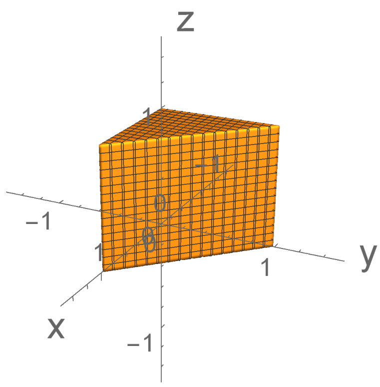

18. In spherical coordinates, a 3D-object V can be described by the following inequalities:

$$1 \leq \rho \leq 2, \quad \frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{4}, \quad 0 \leq \theta \leq 2\pi.$$

Which of the following inequalities describe V in cartesian coordinates?

(a) $1 \leq x^2 + y^2 + z^2 \leq 2, \quad -\sqrt{x^2+y^2} \leq z \leq 0.$

(b) $1 \leq x^2 + y^2 + z^2 \leq 2, \quad -2 \leq z \leq -\sqrt{x^2+y^2}.$

(c) $1 \leq x^2 + y^2 + z^2 \leq 4, \quad -\sqrt{x^2+y^2} \leq z \leq 0.$

(d) $1 \leq x^2 + y^2 + z^2 \leq 4, \quad -2 \leq z \leq -\sqrt{x^2+y^2}.$

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19. What is the mass of a wire of the form

$$x^2 + y^2 = 1, \quad x > 0,$$

with density function

$$f(x, y) = 3x\sqrt{2 + 2y} \ ?$$

(a) 2. (b) 3. (c) 4. (d) 8.

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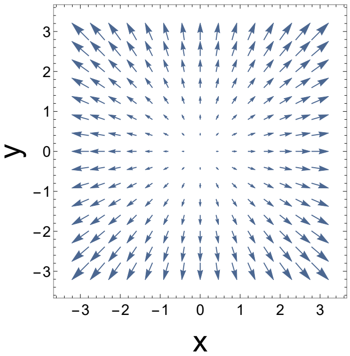
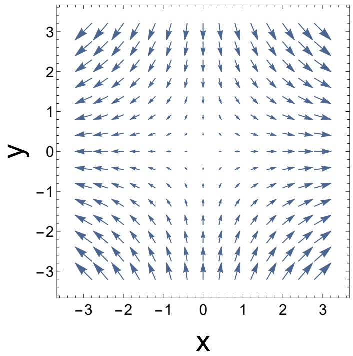
20. Which picture shows the gradient field with the scalar potential

$$f(x, y) = xy\ ?$$

(a)


(c)


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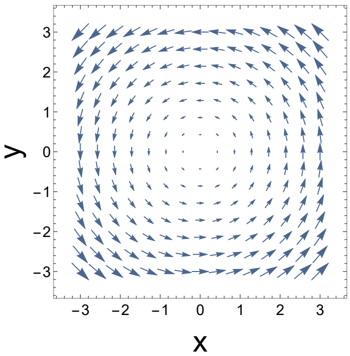

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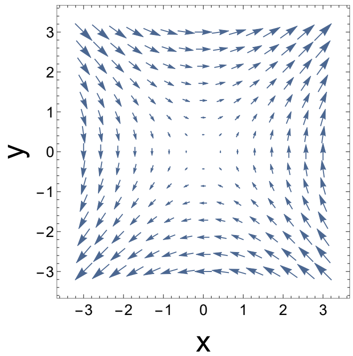

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21. Which is a parametrization of the intersection curve of the ellipsoid

$$9(x-1)^2 + 4y^2 + (z+2)^2 = 40$$

with the plane

$$y = 1\,?$$

(a) $\vec{r}(t) = \begin{pmatrix} 1+2\cos t \\ 1 \\ -2+6\sin t \end{pmatrix}, \ t \in [0, 2\pi].$

(c) $\vec{r}(t) = \begin{pmatrix} 1+3\cos t \\ 1 \\ -2+6\sin t \end{pmatrix}, \ t \in [0, 2\pi].$

(b) $\vec{r}(t) = \begin{pmatrix} 1+2\cos t \\ 4 \\ -2+\sin t \end{pmatrix}, \ t \in [0, 2\pi].$

(d) $\vec{r}(t) = \begin{pmatrix} 1+3\cos t \\ 4 \\ -2+\sin t \end{pmatrix}, \ t \in [0, 2\pi].$

22. What is the circulation of the vector field

$$\vec{F} = \begin{pmatrix} xy^2 + x^2 \\ x^2y + x \end{pmatrix}$$

along the unit circle with center in the origin in clockwise direction?

(a) -2π. (b) $-\pi$. (c) π. (d) 2π.

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23. Determine the value of the 4-periodic continuation of

$$f(x) = x^3, \quad 0 \leq x \leq 2$$

at $x = 3$.

(a) -27. (b) -1. (c) 1. (d) 27.

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24. Consider the following partial differential equation:

$$u_{xx} = u_{tt} - 3u \ .$$

With the ansatz $u(x,t) = X(x)T(t)$ the PDE can be separated into a system of ODE's for$X(x)$ and $T(t)$ which are dependent of a parameter $k \in \mathbb{R}$. Into which system?

(a) $X'' - kX = 0$ und $T'' - (k-3)T = 0$.

(b) $X'' - kX = 0$ und $T'' + (k-3)T = 0$.

(c) $X'' - kX = 0$ und $T'' - (k+3)T = 0$.

(d) $X'' - kX = 0$ und $T'' + (k+3)T = 0$.