Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Last Name:	**Department:**
First Name:	**ETH ID Legi-Nr.:**

For the correction:

	1K	**2K**	**Punkte**	**Bemerkungen:**
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
Total				

ETH Zurich
Dr. Ana Cannas

Winter 2015

Mathematics I and II Exam

**for students of Earth Sciences, Food Science,
Agricultural Science and Environmental Sciences**

Important:

- Lay your ETH-Card visible on the table.
- Fill in the heading of the front page.
- Note all intermediate results and approaches to the solution.
- Write your name on each additional sheet.
- After each exercise there is its maximally reachable number of points.
- Use a blue or a black pen.

Permitted aid material:

- written notes (40 A4-pages)
- **no** calculator
- **no** mobile phone

Good Luck!

1. We consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \\ 4 & 8 & 12 \end{pmatrix}$$

a) Solve the system $A\,\vec{x} = \vec{0}$.

4 points

b) Solve the system $A\,\vec{x} = \vec{b}$, where $\vec{b} = \begin{pmatrix} 6 \\ 9 \\ 15 \\ 24 \end{pmatrix}$ is the sum of the three columns of A.

2 points

c) For which values of $k \in \mathbb{R}$ is the system $A\,\vec{x} = \begin{pmatrix} 2 \\ 2 \\ 4 \\ k \end{pmatrix}$ solvable?

2 points

2. We consider the function

$$f(x) = x\ln(x).$$

a) Find the Taylor polynomial of order 2 generated by f at $x_0 = 2$.

3 points

b) Compute the limit

$$\lim_{x\to 1} \frac{x\ln(x)}{1-x^3}.$$

2 points

c) Is the equation $x\ln(x) = 1$ solvable?
Remember to justify your answer.

3 points

3. Solve the following initial value problem

$$\begin{cases} y' = x^2(1+y^2) \\ y(0) = 0. \end{cases}$$

What is the domain of the solution?

8 points

4. We consider the system of differential equations

$$\begin{cases} \dot{x}(t) = 3x(t) - 5y(t) \\ \dot{y}(t) = 2x(t) - 3y(t). \end{cases}$$

a) Compute the eigenvalues and the corresponding eigenvectors of the coefficient matrix of this system.

3 points

b) Compute the general **real** solution of this system.

3 points

c) Compute the solution of this system with intial values

$$\begin{cases} x(0) = 5 \\ y(0) = 1. \end{cases}$$

3 points

5. We consider the differential equation

$$y'' + 2y' = 6.$$

a) Compute the general real solution of the corresponding homogeneous differential equation.

4 points

b) Compute a particular solution of the nonhomogeneous differential equation.

3 points

6. We consider the function

$$f(x, y) = x^4 + 2y^3 - 2x^2y.$$

a) Determine the equation of the tangent plane of the graph of f at the point $(1, 0, 1)$.

3 points

b) The level curve $f(x, y) = 1$ near the point $(1, 0)$ is the graph of a function $y = y(x)$. What is the value of $y'(1)$?

3 points

7. Determine the greatest and the smallest values that the function

$$f(x, y) = 4x - 3y$$

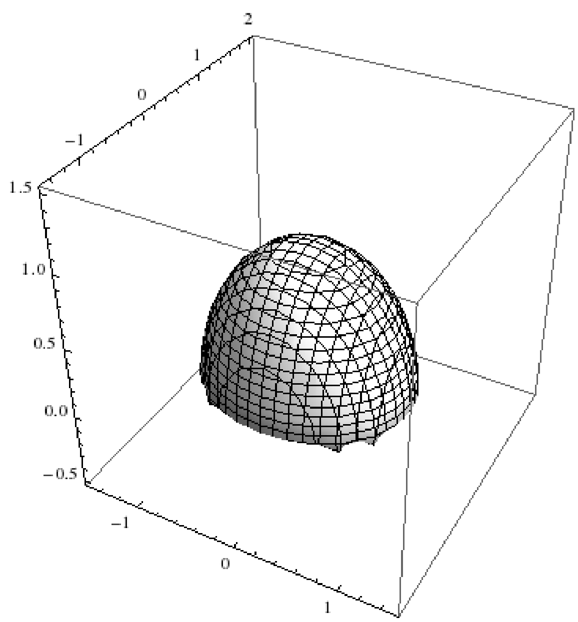
takes on the circle $x^2 + y^2 = 1$.

7 points

8. Using **cylindrical** coordinates, compute the integral

$$\iiint_V (1 - x^2 - y^2)\, dV,$$

where V is the half ball $x^2 + y^2 + z^2 \leq 1, \quad z \geq 0.$



8 points

9. We consider the surface in space

$$D = \left\{(x, y, z) | \; z = x^2 + y^2, \; x^2 + y^2 \leq 1\right\}.$$

a) Compute the area of D.

5 points

b) Compute the flux of $\nabla \times F$ across D from bottom to top, where

$$\vec{F}(x, y, z) = \begin{pmatrix} -2yz^2 \\ x(z + z^3) \\ -e^{x+y+z} \end{pmatrix}.$$

5 points

10. Solve the following problem for the heat equation:

$$\begin{cases} u_t = u_{xx} \\ u(x+2\pi, t) = u(x,t) \\ u(x,0) = 2\cos(3x) - 5\sin(4x) \end{cases}$$

for $x \in \mathbb{R}$ and $t \geq 0$.

9 points

11. Instructions: Mark the correct answers. There is always only one answer correct and 2 points per question. Wrong or multiple crosses are valuated with 0 points.

11.1 The level surface $f = 0$ of the function

$$f(x, y, z) = x^2 - y^2 + z^2$$

is

- ☐ (a) a sphere.
- ☐ (b) a paraboloid.
- ☐ (c) a cone.
- ☐ (d) a hyperboloid.

11.2 Consider the function

$$f(x, y) = y^3 - x^3 + 3x^2 - 3y.$$

Which statement is **false**?

- ☐ (a) $(0, -1)$ is a saddle point of f.
- ☐ (b) $(0, 1)$ is a local minimum of f.
- ☐ (c) $(2, -1)$ is a local minimum of f.
- ☐ (d) $(2, 1)$ is a saddle point of f.

11.3 The curve in the xy-plane given by the equation

$$e^{(x-1)^3}(y^2-1)=0$$

can be represented as

- ☐ (a) the graph of a differentiable function of x near the point $(1,0)$.

- ☐ (b) the graph of a differentiable function of y near the point $(1,0)$.

- ☐ (c) the graph of a differentiable function of x near the point $(0,1)$.

- ☐ (d) the graph of a differentiable function of y near the point $(0,1)$.

11.4 Which function is a solution of the following problem?

$$\begin{cases} u_{xx}+u_{yy}=0 \\ u(0,y)=u(\pi,y)=0 \\ u(x,0)=0 \end{cases}$$

- ☐ (a) $u(x,y)=(\sinh x)\,(\sin y)$.

- ☐ (b) $u(x,y)=\left(\sinh\frac{x}{\pi}\right)(\sin y)$.

- ☐ (c) $u(x,y)=(\sin x)\,(\sinh y)$.

- ☐ (d) $u(x,y)=\left(\sin\frac{x}{\pi}\right)(\sinh y)$.

11.5 Consider the integral

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r^2 \sin\theta \, dr \, d\theta \ .$$

Which one of the following integrals is **not** equal to I?

☐ (a) $\displaystyle\int_0^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} y \, dx \, dy.$

☐ (b) $\displaystyle\int_0^{1} \int_x^{\sqrt{2-x^2}} y \, dy \, dx.$

☐ (c) $\displaystyle\int_0^{\sqrt{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^2 \sin\theta \, d\theta \, dr.$

☐ (d) $\displaystyle\left(\int_0^{\sqrt{2}} r^2 \, dr\right) \cdot \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\theta \, d\theta\right).$

11.6 Consider the surface in space defined by

$$x^2 + y^2 + z^2 = 4 \quad \text{und} \quad z \le \sqrt{2}.$$

Which of the following expressions describes this surface in spherical coordinates?

☐ (a) $\varrho = 2$ und $0 \le \phi \le \frac{\pi}{4}.$

☐ (b) $\varrho = 2$ und $\frac{\pi}{4} \le \phi \le \pi.$

☐ (c) $\varrho = 4$ und $0 \le \phi \le \frac{\pi}{4}.$

☐ (d) $\varrho = 4$ und $\frac{\pi}{4} \le \phi \le \pi.$

11.7 Which of the following is a parametrization of the curve

$$(x-1)^2 + 4y^2 = 16 \quad ?$$

☐ (a) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t - 1 \\ 2\sin t \end{pmatrix}, \ t \in [0, 2\pi].$

☐ (b) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{\cos t - 1}{4} \\ \frac{\sin t}{2} \end{pmatrix}, \ t \in [0, 2\pi].$

☐ (c) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 + 4\cos t \\ 2\sin t \end{pmatrix}, \ t \in [0, 2\pi].$

☐ (d) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \cos t \\ \frac{1}{2}\sin t \end{pmatrix}, \ t \in [0, 2\pi].$

11.8 Which statement about the following vector field is correct?

$$\vec{F}(x, y, z) = \begin{pmatrix} 2\,x\,y\,z \\ x^2 z \\ x^2 y \end{pmatrix}$$

☐ (a) $\vec{F}$ is conservative and $\operatorname{div} \vec{F} = 0$.

☐ (b) $\vec{F}$ is conservative but $\operatorname{div} \vec{F} \neq 0$.

☐ (c) $\vec{F}$ is not conservative but $\operatorname{div} \vec{F} = 0$.

☐ (d) $\vec{F}$ is neither conservative nor $\operatorname{div} \vec{F} = 0$.

11.9 The theorem of Green establishes a relation between the flux of the rotation of a vector field $\vec{F}$ through a planar area A and

- [ ] (a) the flux of $\vec{F}$ through the boundary of A.
- [ ] (b) the length of the boundary curve of A.
- [ ] (c) the circulation of $\vec{F}$ around the boundary of A.
- [ ] (d) the area of A.

11.10 The Fourier series of the (2π)-periodic function

$$f(x) = |x|, \quad -\pi \leq x \leq \pi$$

is

- [ ] (a) $\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} \cos((2n+1)x).$
- [ ] (b) $\frac{\pi}{2} - (4\pi) \sum_{n=0}^{+\infty} \frac{1}{n^2} \sin(nx).$
- [ ] (c) $\frac{\pi}{2} - \sum_{n=0}^{+\infty} \left(\frac{4\pi}{n^2} \sin(nx) + \frac{4}{\pi(2n+1)^2} \cos((2n+1)x) \right).$
- [ ] (d) $\frac{\pi}{2} - \sum_{n=0}^{+\infty} \left(\frac{4}{\pi(2n+1)^2} \sin((2n+1)x) + \frac{4\pi}{n^2} \cos(nx) \right).$