ETH

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Last Name:	**Department:**
First Name:	**ETH ID Legi-Nr.:**

For the grading:

	1K	**2K**	**points**	**comments:**
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
Total				

ETH Zurich
Dr. Ana Cannas

Winter 2016

MATHEMATICS I AND II EXAM

for students of Earth Sciences, Food Science, Agricultural Science and Environmental Sciences

---

Important:

- Lay your ETH-Card visible on the table.
- Fill in the heading of the front page.
- Note all intermediate results and approaches to the solutions of exercises 1-10.
- Write your name on each additional sheet.
- After each exercise there is its maximally reachable number of points.
- Use a blue or a black pen.

Permitted aid material:

- written notes (40 A4-pages)
- **no** calculator
- **no** mobile phone

Good Luck!

1. We consider the system of the form

$$\begin{cases} x + 2y + z = 2 \\ \quad\quad ky + 2z = -2 \\ 2x + 4y + kz = 6 \end{cases}$$

where k is a real parameter.

a) Solve the system for $k = 0$.

3 points

b) Determine the inverse of the coefficient matrix, when $k = 1$.

3 points

c) For which value(s) of k is the system **not** solvable?

2 points

2. We consider the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}.$$

a) Is the vector

$$\vec{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Do not forget to justify your answer.

3 points

b) Find a vector $\vec{c} \neq \vec{0}$ in $\mathbb{R}^3$, that is perpendicular to both vectors $\vec{v}_1$ and $\vec{v}_2$.

3 points

c) What is the dimension of the space of all vectors, that are perpendicular to $\vec{v}_1$ and $\vec{v}_2$?

2 points

3. We consider the differential equation

$$y'' - 2y' + 5y = 8e^x - 5.$$

a) Determine the zeros of the characteristic polynomial.

2 points

b) Determine a particular solution of this differential equation with the ansatz

$$y(x) = Ae^x + B \quad (A, B \in \mathbb{R}).$$

3 points

c) Determine the general **real** solution of the differential equation.

3 points

4. We consider the system of differential equations

$$\begin{cases} \dot{x}(t) &= \quad x(t) + 4y(t) \\ \dot{y}(t) &= -x(t) - 3y(t). \end{cases}$$

a) Determine the eigenvalues and corresponding eigenvectors of the coefficient matrix of this system.

3 points

b) Determine the general solution of this system.

3 points

c) Under which initial conditions

$$\begin{cases} x(0) &= x_0 \\ y(0) &= y_0, \end{cases}$$

does the solution of the system remain bounded for $t \geq 0$?

2 points

5. Let $f(x)$ be the function defined for $x > 0$ with

$$f'(x) = \frac{-x+1}{x^2+x} \quad \text{and} \quad f(1) = 0.$$

a) In what interval is f strictly increasing?

In what interval is f strictly decreasing?

3 points

b) Determine the quadratic Taylor-polynomial of $f(x)$ at $x = 1$.

2 points

c) Determine $f(x)$.

3 points

6. We consider the function

$$f(x, y) = 6x^2 - 2x^3 + 3y^2 - 6xy.$$

a) Determine all critical points of f.

2 points

b) Classify the critical points of f (as saddle point, local minimum, local maximum).

3 points

c) Is the level curve $f(x, y) = 5$ in a neighborhood of the point $(-1, -1)$ the graph of a differentiable function $y = y(x)$?

3 points

7. We consider the function

$$g(x, y, z) = xy + z^2 - xz.$$

a) What is the linearisation of g at the point $P = (2, 1, 2)$?

3 points

b) In what direction does g have the steepest descent at the point $P = (2, 1, 2)$? What is the corresponding value of the directional derivative?

2 points

c) Give an integral expression for the integral of $g(x, y, z)$ over the solid

$$x^2 + y^2 \leq 4, \quad -1 \leq z \leq 3.$$

using cylindrical coordinates.

You do **not** have to compute the integral.

3 points

8. Let S be the part of the sphere

$$x^2 + y^2 + z^2 = 9$$

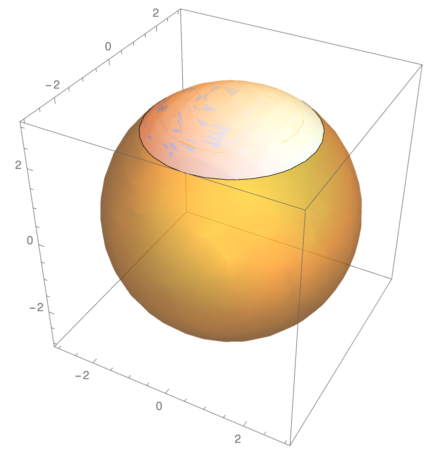
above the plane

$$z = \sqrt{5}$$

and let

$$\vec{F}(x, y, z) = \begin{pmatrix} (x^2 + y^2 + 1)y \\ -5x + z \\ xyz \end{pmatrix}$$

$$\vec{G} = \operatorname{rot} \vec{F}.$$



a) Determine the upwards flux of $\vec{G}$ through S.

3 points

b) What is the downwards flux of $\vec{G}$ through the circular disk

$$x^2 + y^2 \leq 4, \quad z = \sqrt{5} \quad ?$$

Hint: Compare with a).

3 points

c) What is $\operatorname{div} \vec{G}$?

2 points

9. We consider the boundary problem

$$\begin{cases} u_{xx} + u_{yy} = \pi^2 u \\ u(0,y) = u(1,y) = u(x,0) = 0 \\ u(x,1) = f(x), \end{cases}$$

where $0 \le x \le 1,\ 0 \le y \le 1$ and $f(x)$ is a real function.

a) After applying separation of variables $u(x,t) = X(x)T(t)$ this partial differential equation splits into a system of ordinary differential equations for $X(x)$ and $T(t)$ depending on a parameter $k \in \mathbb{R}$.

What is the corresponding system?

3 points

b) Determine a series of eigenfunctions

$$u_n(x,y) = X_n(x)Y_n(y), \quad n = 1, 2, 3, \cdots$$

of the homogeneous part of this problem.

3 points

c) Solve the boundary problem for

$$f(x) = \begin{cases} -2x, & 0 \le x \le \frac{1}{2} \\ 2(x-1), & \frac{1}{2} < x \le 1, \end{cases}$$

where

$$\mathcal{F}_f(x) = -\frac{8}{\pi^2} \sum_{k \ge 0} \frac{(-1)^k}{(2k+1)^2} \sin((2k+1)\pi x)$$

is the sine-series of $f(x)$.

2 points

10. We consider an oscillating string, which is modelled by the following differential equation:

$$u_{tt} = 9u_{xx}.$$

a) Let $L = \pi$ be the length of the string and let its ends be kept at constant deflection 0 and π. Solve the problem

$$\begin{cases} u_{tt} = 9u_{xx} \\ u(0,t) = 0, \quad u(\pi,t) = \pi \\ u(x,0) = x \\ u_t(x,0) = 12\sin(2x). \end{cases}$$

Hint: You may use eigenfunctions of the corresponding homogeneous problem without deriving them.

3 points

b) Determine the Fourier-integral of

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$$

3 points

c) For modelling purposes let the string be infinitely long. Determine an integral expression of the bounded solution $u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$ of

$$\begin{cases} u_{tt} = 9u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = 0, \end{cases}$$

where $f(x)$ is the function form part (b).

Hint: The initial velocity simplifies the integral expression.

2 points

11. Instructions: Mark the correct answer. There is always only one answer correct and 2 points per question. Wrong or multiple crosses are valuated with 0 points.

11.1 Three of the following parametrisations describe the same curve. Which parametrisation describes *another* curve?

☐ (a) $\vec{r}(t) = \begin{pmatrix} t^2 \\ 3t+3 \end{pmatrix}, \ t \in [0,1].$

☐ (b) $\vec{r}(t) = \begin{pmatrix} (t-1)^2 \\ 3t \end{pmatrix}, \ t \in [1,2].$

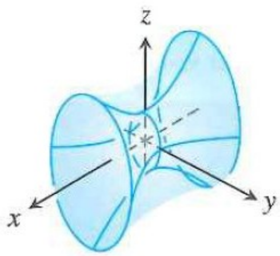
☐ (c) $\vec{r}(t) = \begin{pmatrix} \frac{1}{9}t^2 \\ t+1 \end{pmatrix}, \ t \in [0,3].$

☐ (d) $\vec{r}(t) = \begin{pmatrix} \frac{1}{9}(t-3)^2 \\ t \end{pmatrix}, \ t \in [3,6].$

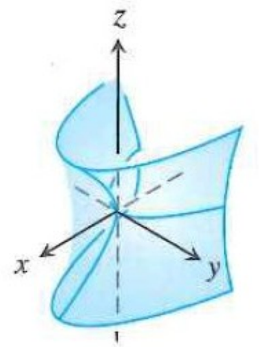
11.2 Which is the picture of the surface with the equation

$$x^2 - y^2 + z^2 = 1 \quad ?$$

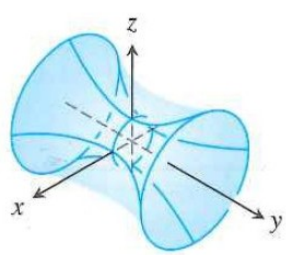
☐ (a)



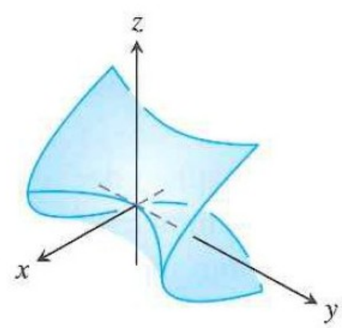
☐ (c)



☐ (b)



☐ (d)



11.3 Which inequality describes the domain of definition of the function

$$f(x, y) = \ln\left(1 + \frac{y}{x^2 + 4}\right) \quad ?$$

☐ (a) $y < x^2 + 4$.

☐ (b) $y > x^2 + 4$.

☐ (c) $y < -x^2 - 4$.

☐ (d) $y > -x^2 - 4$.

11.4 Let $f(x, y, z) = x\sqrt{y + z}$. Then $\dfrac{\partial^3 f}{\partial x \partial y \partial z}$ is equal to

☐ (a) $-\frac{1}{4}(y + z)^{-\frac{3}{2}}$.

☐ (b) $\frac{1}{2}(y + z)^{-\frac{1}{2}}$.

☐ (c) $(y + z)^{-\frac{1}{2}} + (y + z)^{\frac{1}{2}}$.

☐ (d) $(y + z)^{-\frac{1}{2}} + \dfrac{x}{y + z}$.

11.5 What is the area of the bounded region enclosed by the curves

$$y = x^2 \quad \text{and} \quad y = x \quad ?$$

☐ (a) $\frac{1}{6}$.

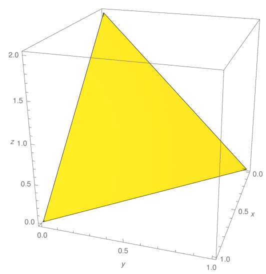
☐ (b) $\frac{1}{4}$.

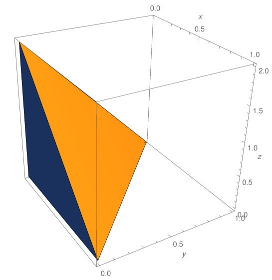
☐ (c) $\frac{1}{3}$.

☐ (d) $\frac{1}{2}$.

11.6 What is the volume of the region defined by

$$x \geq 0, \quad y \geq 0, \quad x + y \leq 1, \quad 0 \leq z \leq 2 - 2x - 2y \quad ?$$





☐ (a) $\frac{1}{6}$.

☐ (b) $\frac{1}{4}$.

☐ (c) $\frac{1}{3}$.

☐ (d) $\frac{1}{2}$.

11.7 We consider the region defined by

$$x^2 + y^2 + z^2 \geq 2 \quad \text{and} \quad z^2 = x^2 + y^2.$$

Which description in spherical coordinates corresponds to this region?

☐ (a) $R \geq \sqrt{2}$ and $\varphi = \frac{\pi}{2}$.

☐ (b) $R \geq \sqrt{2}$ and $\varphi = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

☐ (c) $R \geq 2$ and $\varphi = \frac{\pi}{4}$.

☐ (d) $R \geq 2$ and $\varphi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

11.8 What is the mass of the filament with the shape of the curve

$$y = e^x, \quad 0 \leq x \leq 1$$

and with density function

$$f(x, y) = 3y^2 \quad ?$$

- [ ] (a) $\sqrt{e^2 + 1} - 2\sqrt{2}$.
- [ ] (b) $2\sqrt{e^2 + 1} - 4\sqrt{2}$.
- [ ] (c) $(e^2 + 1)^{\frac{3}{2}} - 2\sqrt{2}$.
- [ ] (d) $2(e^2 + 1)^{\frac{3}{2}} - 4\sqrt{2}$.

11.9 The outward flux of the vector field $\vec{F}(x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ through the surface of a sphere

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| \; x^2 + y^2 + z^2 = 1 \right\}$$

is equal to

- [ ] (a) 0.
- [ ] (b) 1.
- [ ] (c) Minus the area of S.
- [ ] (d) Three times the volume of the unit sphere.

11.10 Let $\vec{F}$ be a vector field in the plane defined for $(x, y) \neq (0, 0)$, whose components satisfy the equation

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

and whose circulation along the unit circle C around the origin vanishes:

$$\oint_C \vec{F} \cdot d\vec{r} = 0.$$

Which statement is correct?

☐ (a) $\vec{F}$ is certainly a gradient field.

☐ (b) $\vec{F}$ is certainly not a gradient field.

☐ (c) It cannot be decided whether $\vec{F}$ is a gradient field or not.

☐ (d) Such an $\vec{F}$ does not exist.