Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Family name:	**Department:**
First name:	**ETH ID No.:**

For the grading:

	1K	**2K**	**Points**	**Comments:**
1				
2				
3				
4				
5				
6				
7				
8				
9-24				
Total				

MATHEMATICS I AND II EXAM

for students of Agricultural Science, Earth Sciences, Environmental Sciences, and Food Science

Important:

- Please fill the header on the cover page and lay your ETH-card visible on the table.
- Please write neatly with a non erasable blue or black pen, in particular not with a pencil. Beware that something that is too hard to read could be ignored.
- Please leave some empty space on the margins for the correction.
- This exam has 24 questions and lasts for 180 minutes.

For questions 1-8:

- Please write down all intermediate steps of your calculations and solutions.
- Write your name and ETH ID / Legi-Nr. on each additional sheet.
- The maximal score of each exercise part is given in the right margin.

For questions 9-24:

- Mark your answers clearly.
- There is always only one correct answer and 2 points per question.

Permitted aids:

- Written notes up to 40 A4-Pages, one English dictionary,
- **no** calculator, **no** mobile phone, **no** laptop.
- Please switch off your mobile phone and stow it away.

Good Luck!

1. Consider the function

$$f(x) = \frac{1}{1 - \tan(x)}.$$

a) Determine the derivative $f'(x)$. 2 points

b) Determine the linearization of $f(x)$ in $x_0 = 0$. 2 points

c) Determine the range of $\tan(x)$ for $-\frac{\pi}{2} < x < \frac{\pi}{4}$. 2 points

d) Determine the range of $f(x)$ for $-\frac{\pi}{2} < x < \frac{\pi}{4}$. 3 points

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2. Determine the general solution of the the following differential equations:

a) $y'' = 4y' - 4y$. 4 points

b) $3xy' - y = x + 1$ for $x > 0$. 4 points

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3. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{pmatrix}.$$

a) Determine the rank of the matrix A. 3 points

b) Determine a basis for the solution set of the matrix equation $A\vec{x} = \vec{0}$. 3 points

c) Let $\vec{b}$ be the sum of all four columns of A. Determine the general solution of the system $A\vec{x} = \vec{b}$. 2 points

4. Consider the following system of differential equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} .$$

a) Determine the eigenvalues and the corresponding eigenvectors of the coefficient matrix A of the system. 3 points

b) Determine the solution of the system with the initial value

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} .$$

3 points

c) Find all values k such that every solution of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ k & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

is bounded for all $t \in \mathbb{R}$. 3 points

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5. Consider the function

$$f(x, y) = \ln(1 + x^2 - y^2) \ .$$

a) Determine the gradient of f in terms of (x, y). 2 points

b) In which direction does f grow the fastest in the point $(x, y) = (1, 1)$? 2 points

c) Find an equation of the tangent plane to the graph of the function at the point $(x, y, f(x, y)) = (1, 1, 0)$. 2 points

d) Classify the critical point $(0, 0)$ as local maximum, local minimum or saddle point. 3 points

6. Consider the vector field

$$\vec{F}(x,y,z) = \begin{pmatrix} yz^2 \\ xz^2 \\ 2xyz \end{pmatrix}.$$

a) Is $\vec{F}$ conservative? 2 points

b) Determine the work of $\vec{F}$ along a straight line from the point $(1,1,1)$ to the point (x,y,z). Write your answer in terms of (x,y,z). 3 points

c) For which points on the coordinate plane $z=0$ is the divergence of $\vec{F}$ positive? Sketch this set of points. 3 points

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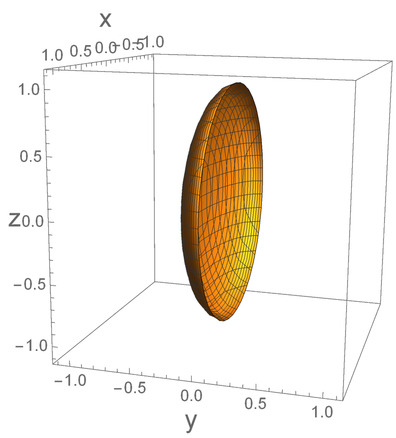
7. Consider the vector field

$$\vec{G}(x,y,z) = \begin{pmatrix} z\sin(y) \\ ye^x \\ x+z \end{pmatrix}.$$

and the half ellipsoid A given by

$$x^2 + 9y^2 + z^2 = 1 \text{ and } y \leq 0 .$$



a) Parametrize A. 3 points

b) Parametrize the boundary curve of A (in an arbitrary direction). 2 points

c) Determine the flux of $\operatorname{rot}\vec{G}$ through A from the left to the right (i.e. the y-component of the normal vector is positive),

$$\iint_A \left(\operatorname{rot}\vec{G}\right)\cdot\vec{n}\,dA .$$

3 points

8. We consider problems of the form

$$\begin{cases} u_{xx} = u_t + 2u \\ u_x(0,t) = u_x(\pi,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

for $0 \leq x \leq \pi$ and $t \geq 0$.

a) Determine the fundamental solutions $u_n(x,t)$. 6 points

b) Determine the solution $u(x,t)$ of the problem when

$$f(x) = 3\cos(4x) - \cos(5x) \ .$$

3 points

For exercises 9-24: Each question gives 2 points. Wrong or multiple answers give 0 points. Mark your answers on this exam.

9. The path of a moving mass point can be described by the following initial value problem:

$$\begin{cases} \dfrac{d\vec{r}}{dt} = \begin{pmatrix} 3e^t \\ 4t^3 \end{pmatrix}, \\ \vec{r}(0) = \begin{pmatrix} 3-e \\ 2 \end{pmatrix}. \end{cases}$$

What is the position vector of the mass point at time $t = 1$?

(a) $\vec{r}(1) = \begin{pmatrix} 2e \\ 3 \end{pmatrix}$.

(b) $\vec{r}(1) = \begin{pmatrix} 3e \\ 1 \end{pmatrix}$.

(c) $\vec{r}(1) = \begin{pmatrix} 4e \\ 3 \end{pmatrix}$.

(d) $\vec{r}(1) = \begin{pmatrix} 5e \\ 1 \end{pmatrix}$.

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10. Which of the following equations does a curve with the following parametrization satisfy?

$$\vec{r}(t) = \begin{pmatrix} \cos t - \sin t \\ \sin(2t) \end{pmatrix}, \quad \text{für } 0 \le t \le 2\pi$$

(a) $x^2 + y^2 - y = 0$.

(b) $x^2 + y^2 + y = 0$.

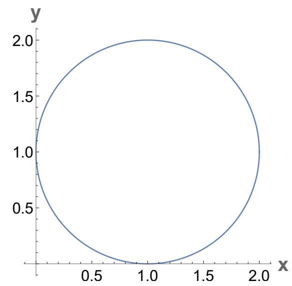
(c) $x^2 + y - 1 = 0$.

(d) $x^2 + y + 1 = 0$.

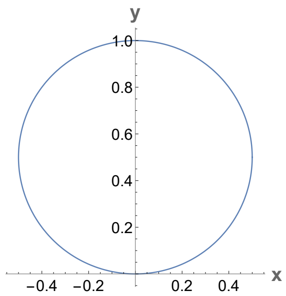
11. Which picture shows the curve with the parametrization

$$\vec{r}(t) = \begin{pmatrix} t + \sin t \\ \cos t \end{pmatrix}, \quad \text{for } 0 \leq t \leq 2\pi \text{ ?}$$

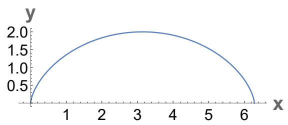
(a)



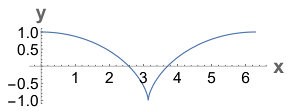
(c)



(b)



(d)



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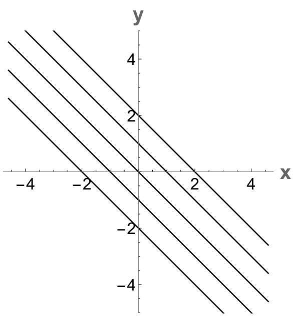
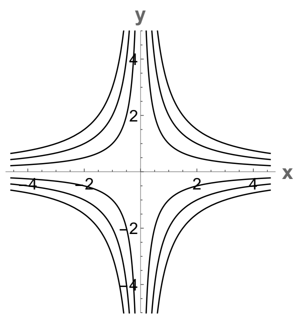
12. Which picture shows the level curves of the function

$$f(x, y) = (x - y)^2 \text{ ?}$$

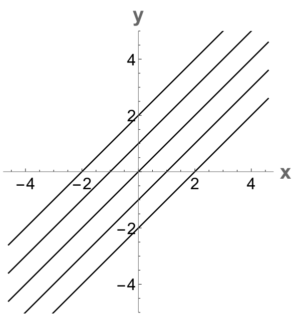
(a)



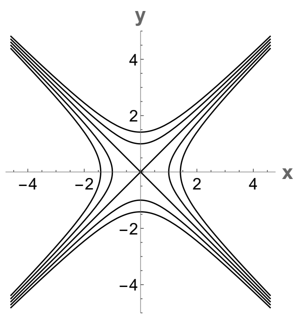
(c)



(b)



(d)



13. The partial derivative of

$$f(x,y) = e^{x+y^2}$$

with respect to y is given by

(a) e^{x+y^2}.

(b) $2ye^{x+y^2}$.

(c) $(1+2y)e^{x+y^2}$.

(d) $(x+y^2)e^{x+y^2}$.

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14. Consider the function

$$f(x,y) = 3xy^3 - x^2 - 9xy \ .$$

How many critical points does f have in the plane?

(a) 1. (b) 3. (c) 5. (d) 7.

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15. What is the slope of the curve given by

$$x^5 - 2x^2y + xy^3 = 0$$

at the point $(x,y) = (1,1)$?

(a) -2. (b) $-\frac{1}{2}$. (c) $\frac{1}{2}$. (d) 2.

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16. Which integral is generally equal to

$$\int_0^2 \int_{-\sqrt{4-x^2}}^0 f(x^2+y^2)\, dy\, dx \ ?$$

Look carefully at the integrand.

(a) $\int_0^{\frac{\pi}{2}} \int_0^2 f(r)\, dr\, d\theta$.

(b) $\int_{-\frac{\pi}{2}}^0 \int_0^2 r f(r)\, dr\, d\theta$.

(c) $\int_{-\frac{\pi}{2}}^0 \int_0^2 f(r^2)\, dr\, d\theta$.

(d) $\int_0^{\frac{\pi}{2}} \int_0^2 r f(r^2)\, dr\, d\theta$.

17. The value of the integral

$$\iint_A \sin(x+y)\,dx\,dy\ ,$$

where A is the region bounded by $y=0$, $x=\frac{\pi}{2}$ and $y=x$, is equal to

(a) 0.

(b) $\frac{1}{2}$.

(c) 1.

(d) 2.

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18. In cartesian coordinates, a solid $V \subseteq \mathbb{R}^3$ can be described by the following inequalities:

$$1 \leq x^2+y^2 \leq 4 \qquad \text{and} \qquad y \geq 0.$$

Which of the following inequalities describe V in spherical coordinates?

(a) $1 \leq \rho \sin\varphi \leq 2$ and $0 \leq \theta \leq \dfrac{\pi}{2}$.

(b) $1 \leq \rho \sin\varphi \leq 2$ and $0 \leq \theta \leq \pi$.

(c) $\sin\varphi \leq \rho \leq 2\sin\varphi$ and $0 \leq \theta \leq \dfrac{\pi}{2}$.

(d) $\sin\varphi \leq \rho \leq 2\sin\varphi$ and $0 \leq \theta \leq \pi$.

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19. What is the length of the curve with the polar equation

$$r=\cos\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad ?$$

(a) 1.

(b) 2.

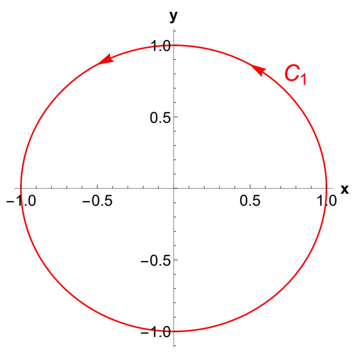
(c) π.

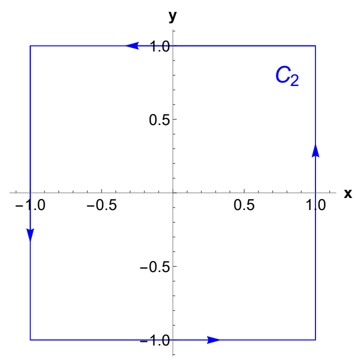
(d) 2π.

20. Consider the vector field

$$\vec{H}(x,y) = \begin{pmatrix} \dfrac{-y}{x^2+y^2} \\ \dfrac{x}{x^2+y^2} \end{pmatrix}$$

and the curves C_1 and C_2 that are shown in the following two pictures:





Which of the following statements is true?

(a) $\oint_{C_1} \vec{H} \cdot d\vec{r} = \oint_{C_2} \vec{H} \cdot d\vec{r} \neq 0.$

(b) $\oint_{C_2} \vec{H} \cdot d\vec{r} = 2 \oint_{C_1} \vec{H} \cdot d\vec{r} \neq 0.$

(c) $\oint_{C_1} \vec{H} \cdot d\vec{r} = \oint_{C_2} \vec{H} \cdot d\vec{r} = 0.$

(d) $\oint_{C_2} \vec{H} \cdot d\vec{r} - 2 \oint_{C_1} \vec{H} \cdot d\vec{r} \neq 0.$

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21. Which is a parametrization of the intersection curve of the sphere

$$x^2 + y^2 + (z-1)^2 = 2$$

with the cone

$$z = 1 + \sqrt{x^2+y^2}\,?$$

(a) $\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 2 \end{pmatrix}, \ t \in [0, 2\pi].$

(b) $\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix}, \ t \in [0, 2\pi].$

(c) $\vec{r}(t) = \begin{pmatrix} \sqrt{2}\cos t \\ \sqrt{2}\sin t \\ 2 \end{pmatrix}, \ t \in [0, 2\pi].$

(d) $\vec{r}(t) = \begin{pmatrix} \sqrt{2}\cos t \\ \sqrt{2}\sin t \\ 1 \end{pmatrix}, \ t \in [0, 2\pi].$

22. What is the circulation of the vector field

$$\vec{F} = \begin{pmatrix} x^2y^3 + y \\ x^3y^2 - x \end{pmatrix}$$

clockwise along the boundary curve of the square $0 \leq x \leq 1, 0 \leq y \leq 1$?

(a) -4. (b) -2. (c) 2. (d) 4.

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23. What is the coefficient of $\sin(2x)$ in the Fourier series of

$$f(x) = \begin{cases} \pi, & 0 \leq x \leq \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < x < 2\pi \end{cases} \quad ?$$

(a) -1. (b) 1. (c) $-\frac{1}{2}$. (d) $\frac{1}{2}$.

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24. What is the type and the order of the following partial differential equation?

$$u_{tt} = x^2 \cdot u_x + u$$

(a) Linear of order 2.

(b) Linear of order 3.

(c) Non-linear of order 2.

(d) Non-linear of order 3.