ETH Zurich
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Winter 2018

MATHEMATICS I EXAM

for students of Agricultural Science, Earth Sciences, Food Science and Environmental Sciences

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Important:

- Lay your ETH-Card visible on the table.
- Fill in the heading of the front page.
- Note all intermediate results and approaches to the solutions of exercises 1-4.
- Write your name on each additional sheet.
- After each exercise there is its maximally reachable number of points.
- Use a blue or a black pen.
- This exam contains 12 questions and lasts 90 minutes.

Permitted aid material:

- written notes (20 A4 pages)
- **no** calculator
- **no** mobile phone
- **no** laptop

Good Luck!

1. Consider the function

$$f(x) = \frac{x}{\ln x}.$$

a) What is the (maximal) domain where f is defined? 2 points

b) In what (maximal) interval is f strictly increasing? 2 points

c) Find the local maxima and minima of f? 2 points

d) What is the range of the function f? 3 points

2. Determine the general solution of each of the following differential equations:

a) $y' = e^y$. 4 points

b) $y'' = 6y' - 9y$. 4 points

3. Consider the system of ordinary differential equations

$$\begin{cases} x + y + z & = 1 \\ 2x + 7y + 7z & = 2 \\ 4x + 9y + k^2 z & = k + 7 \end{cases}$$

where k is a real parameter.

a) Determine the general solution of this system when $k = 0$. 3 points

For which values of k ...

b) ... does the coefficient matrix of the system have rank 2 ? 2 points

c) ... does the system have no solution? 2 points

d) ... does the system have infinitely many solutions? 2 points

4. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 6 & -2 & -4 \\ -2 & 2 & 2 \end{pmatrix}.$$

a) Is $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ an eigenvector of A? 2 points

b) Is the matrix $A + I$ invertible, where I denotes the 3×3 identity matrix? 2 points

c) Determine the eigenvalues of A and their corresponding multiplicities. 2 points

d) Does the system of linear differential equations

$$\dot{\vec{x}} = A\vec{x}$$

admit unbounded solutions? 2 points

For questions 5-12:
Mark the correct answer. There is always exactly one correct answer and 2 points per question. Wrong or multiple answers will be valuated with 0 points.

5. For what values of k is the function

$$f(x) = \begin{cases} x^3 + k & \text{für } x < -1, \\ 2x + 2 & \text{für } x \geq -1, \end{cases}$$

continuous for all $x \in \mathbb{R}$?

☐ (a) 1

☐ (b) 0

☐ (c) -1

☐ (d) -2

6. Which of the following polynomials has both $1 + i$ and $1 - 2i$ as zeroes?

☐ (a) $x^4 + 3x^2 + 6x + 10$.

☐ (b) $x^4 + 3x^2 - 6x + 10$.

☐ (c) $x^4 + 4x^3 + 11x^2 + 14x + 10$.

☐ (d) $x^4 - 4x^3 + 11x^2 - 14x + 10$.

7. Which expression is equal to the area of a disk of radius r?

☐ (a) $\int_0^r (r^2 - x^2)\,\mathrm{d}x$.

☐ (b) $4\int_0^r \sqrt{r^2 - x^2}\,\mathrm{d}x$.

☐ (c) $\int_0^{2\pi} \sqrt{r^2 - x^2}\,\mathrm{d}x$.

☐ (d) $\int_{-r}^r \sqrt{r^2 - x^2}\,\mathrm{d}x$.

8. The integral $\int_0^{\sqrt{2}} \frac{3x^2}{\sqrt{2-x^2}}\mathrm{d}x$ equals

☐ (a) 3.

☐ (b) $\frac{3}{2}\pi$.

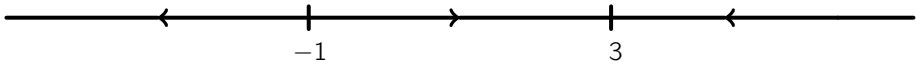
☐ (c) $3(1 - \sqrt{2})$.

☐ (d) $3\sqrt{2}$.

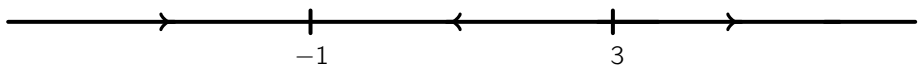
9. Which phase line corresponds to the differential equation

$$y' = \frac{(1+y)(3-y)}{e^y} \; ?$$

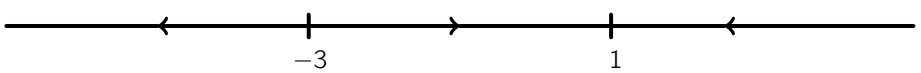
☐ (a)



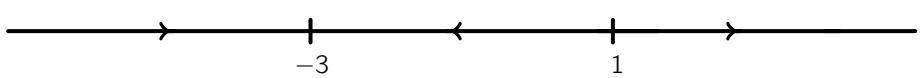
☐ (b)



☐ (c)



☐ (d)



10. Which function is a solution of the differential equation

$$\dot{y} - \frac{1}{t+1} y = 1 \; ?$$

☐ (a) $\ln(t-1) \cdot (t+1)$.

☐ (b) $\ln(t-1) \cdot (t-1)$.

☐ (c) $\ln(t+1) \cdot (t-1)$.

☐ (d) $\ln(t+1) \cdot (t+1)$.

11. Which of the following sets is a subspace of $\mathbb{R}^3$?

☐ (a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$

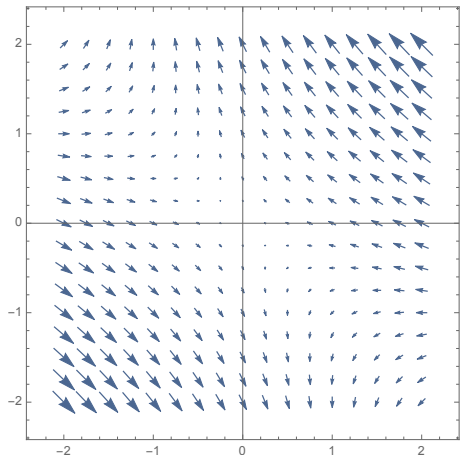
☐ (b) $\left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \middle| x \geq 0 \text{ und } y \geq 0 \right\}$

☐ (c) $\left\{ \begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$

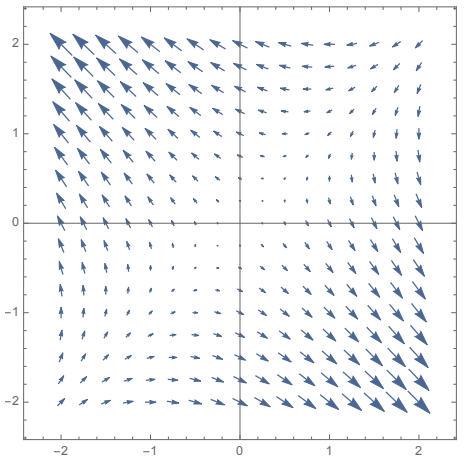
☐ (d) $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| xyz = 0 \right\}$

12. Which is the phase portrait of the system

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \cdot \vec{x} \; ?$$



(a)



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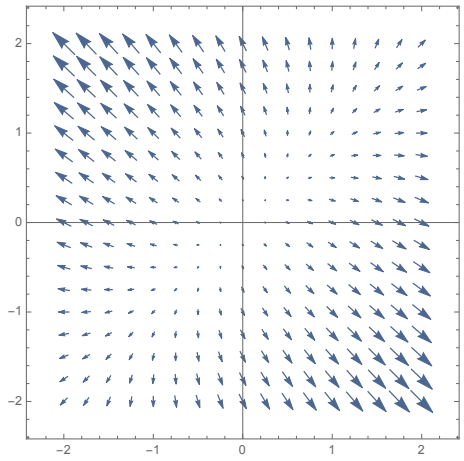

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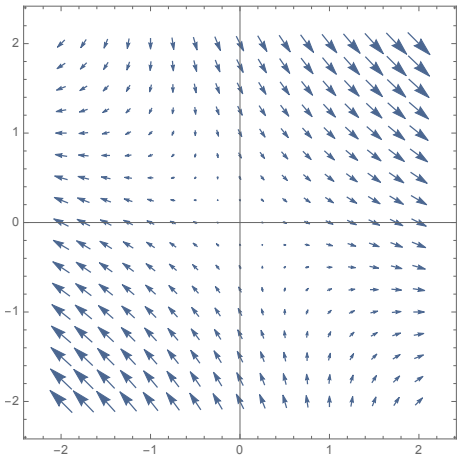

(d)