Wallpaper Stamps

imprinting Mathematics

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Plan:

Have you ever been to Lisbon?

Restauradores

Rossio

Belém

Chiado

Stepping on Mathematics

?

?

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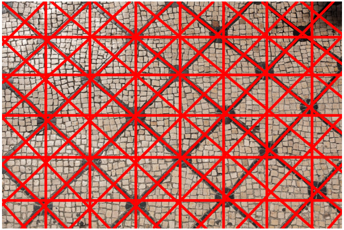
?

A closer look at the square grid



The *fishing net* pattern is preserved under certain mirror reflections.

There are *three types* of mirror crossings:

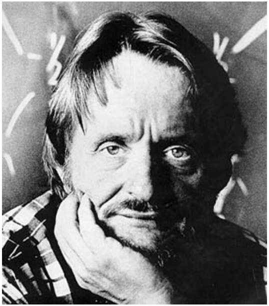
- 4 mirrors cross at the center of each white square;
- 4 mirrors cross at each black node;
- 2 mirrors cross at the center of each black edge.

New Talents in Mathematics – a program for college students created in 2000 by the *Gulbenkian Foundation* – involving math projects and themed summer schools

A new look into the symmetries of things

Murray MacBeath

Bill Thurston



John Horton Conway

Conway is the most avid prophet of the new geometric perspective on symmetry and devised the terminology and signature notation. He was a guest lecturer of *New Talents in Math* in Lisbon in 2004.

The four fundamental features of symmetry

kaleidoscopes, **gyrations**, **miracles** and **wonders**

gyrations and wonders are tr**ue** to orientation – **blue**

kaleidoscopes and miracles **re**flect – **red**

and their symbols: $*$, **digits or** ∞, **X** and **O**

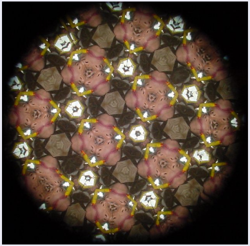
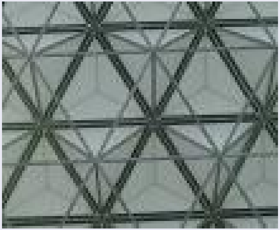
each symmetry type corresponds to a **symbolic signature**
such as $*$ **4 4 2** or **2 2** $*$ or **4 4 2** or **X X**

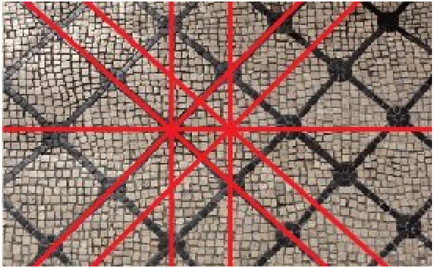
KALEIDOSCOPES

$* \; \mathbf{a} \; \mathbf{b} \; \mathbf{c}$ symbolizes mirrors crossing with angles

$$\frac{180^\circ}{a}, \frac{180^\circ}{b}, \frac{180^\circ}{c}.$$

Examples $* \; 3 \; 3 \; 3$

The three representative mirror crossings have angles

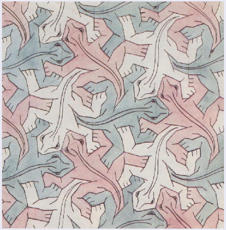
$$\frac{180^\circ}{4}, \frac{180^\circ}{4}, \frac{180^\circ}{2}$$

hence the *pattern signature* $*\ 4\ 4\ 2$.

GYRATIONS

a b c symbolizes **gyration points** by angles $\frac{360^\circ}{a}$, $\frac{360^\circ}{b}$, $\frac{360^\circ}{c}$, which do not lie on *any mirror line*.

Examples **3 3 3**



A closer look at the *ropes* pattern

is preserved by rotation about three points by angles

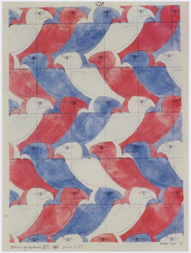
$$\frac{360^\circ}{4}, \frac{360^\circ}{4}, \frac{360^\circ}{2}$$

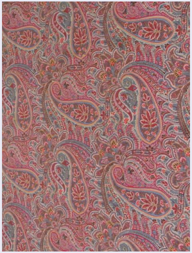
$\Longrightarrow$ signature **4 4 2**

MIRACLES

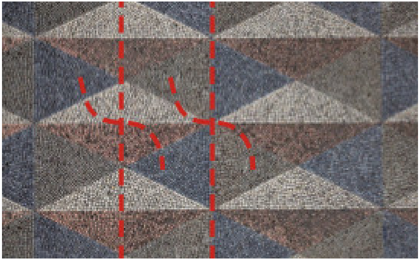
X represents a *mirrorless crossing*, i.e, a **path from a point to a reflected copy of itself without ever touching a mirror line**.

Examples X X and * X





A closer look at the four-colour cobblestone pattern



has two miracles

$\Longrightarrow$ signature XX

WONDERS

O (*Oh!* or *zero*) represents a pattern
without kaleidoscopes, **nor** gyrations, **nor** miracles.

Examples **O**

How to find a signature of a pattern

1. Mark one kaleidoscope corner of each type with a $*$ and the number of mirrors through it.
2. Mark one gyration point of each type with a ● and its order.
3. Can you walk from some point to a reflected copy of itself without ever touching a mirror line? If so, a miracle has occurred. Mark that path with a broken red line and a **X**.
4. If you've found none of the above, then there is a wonder. Mark it with **O**.

Menu: cost of the symmetry features

Symbol	Cost (Fr.)	Symbol	Cost (Fr.)
0	2	$\ast$ or **X**	1
2	$\frac{1}{2}$	2	$\frac{1}{4}$
3	$\frac{2}{3}$	3	$\frac{1}{3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
number n	$\frac{n-1}{n}$	number n	$\frac{n-1}{2n}$
∞	1	∞	$\frac{1}{2}$

For instance, **3** costs $\frac{2}{3}$ and $\ast$ **3** costs $1 + \frac{1}{3} = \frac{4}{3}$.

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The *meal* **2 2** ∞ costs $\frac{1}{2} + \frac{1}{2} + 1 =$ **Fr. 2**.

The *meal* $*$ **4 4 2** also costs $1 + \frac{3}{8} + \frac{3}{8} + \frac{1}{4} =$ **Fr. 2**.

Budget

the types of patterns and friezes are exactly those with signatures (meals) with total cost 2

17 types of plane patterns

(signatures not involving ∞)

7 types of friezes

(signatures involving ∞)

*632	632	*442	442	*333	*22∞	22∞	2*∞
333	*2222	2222	4*2	3*3	2*22	*∞∞	∞∞
22*	**	*X	XX	22X	O	∞*	∞X

You just have to know how to add fractions!

A longer walk

4 4 2

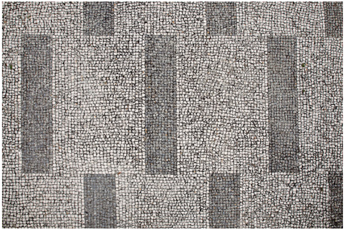
2 2 *

X X

* 4 4 2

2 2 2 2

3 * 3



2 * 2 2

* *

∗ 6 3 2

∗ X

∗ 2 2 2 2

∗ 4 4 2

And a few friezes

2 * ∞

2 2 ∞

* ∞ ∞

∞∞

Friezes are analysed analogously

$$\underbrace{\text{gyration}}_{}^{2}\ \underbrace{\text{kaleidoscope}}^{*\infty}$$

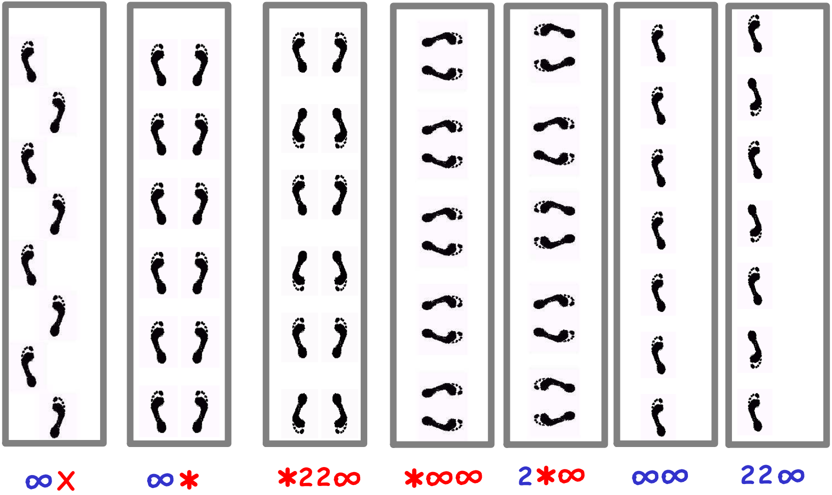
gyration kaleidoscope

2 $*\infty$

infinite parallel mirrors *cross at infinity*

gyration point by angle $\frac{360°}{2}$

Frieze fitness



Summary:

- There are **4 features of symmetry**:
  kaleidoscopes, gyrations, miracles and wonders.
- The **magic theorem** implies that there are:
  17 types of plane patterns and 7 types of friezes.

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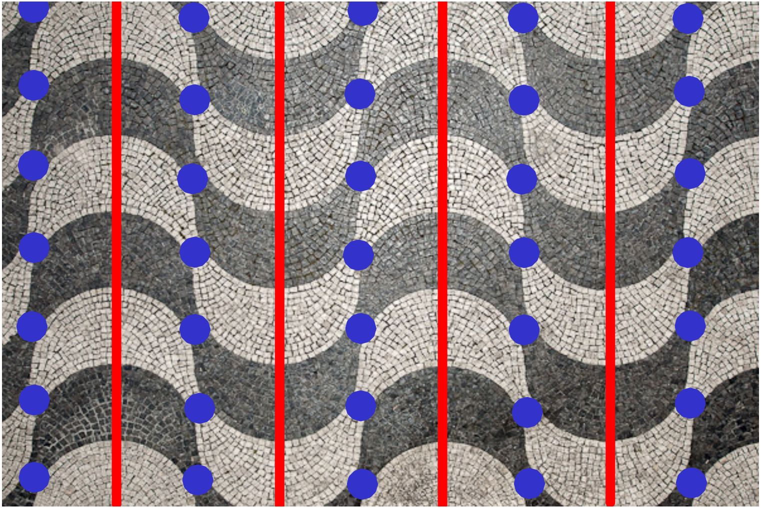
Where does this come from?!

Waves of the wide ocean around the world

A closer look at the *waves* pattern

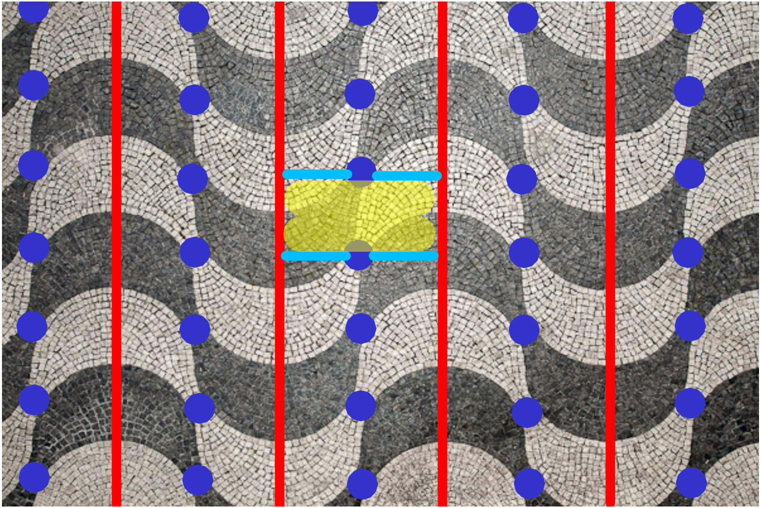
with its mirrors

A closer look at the *waves* pattern



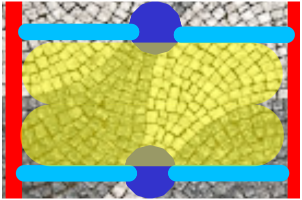
with its mirrors and with its gyrations

A closer look at the *waves* pattern

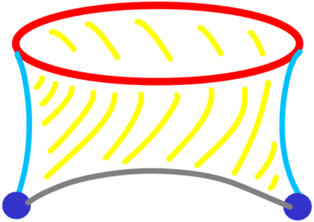


with its mirrors, gyrations and a chosen **fundamental region**

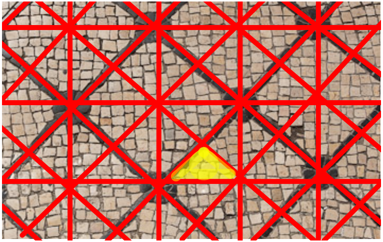
A closer look at the *waves* pattern

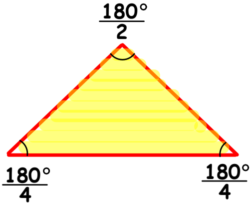


wallpaper stamp $\underbrace{2\ \ 2}_{\text{gyrations}}\ \underbrace{*}_{\text{mirror}}$



A closer look at the square grid





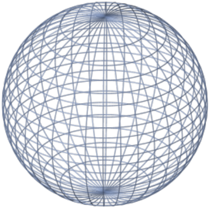
wallpaper stamp $\underbrace{*442}_{\text{kaleidoscope}}$

A wonderful Swiss characteristic

Euler Characteristic of a surface

$$\chi = (\#\text{vertices}) - (\#\text{edges}) + (\#\text{faces})$$

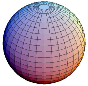
is independent of the chosen polygon subdivision!



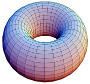
Euler Characteristic of a spherical surface

$$(\#\text{vertices}) - (\#\text{edges}) + (\#\text{faces}) = \mathbf{2}$$

Examples of surfaces $O^m *^n$ or $*^n X^p$

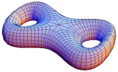


$\chi = 2$



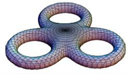
O

$\chi = 0$



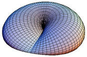
O^2

$\chi = -2$



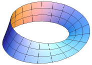
O^3

$\chi = -4$



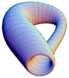
X

$\chi = 1$



$*X$

$\chi = 0$



X^2

$\chi = 0$

...

Classification of surfaces $O^m *^n$ or $*^n X^p$

Any (bounded and connected) surface can be obtained from a sphere by

- $*$ – punching holes (this introduces boundaries);
- O – adding handles and
- X – adding *crosscaps*, i.e. replacing disks by Möbius bands.

These operations affect the Euler Characteristic χ:

- punching a hole or adding a crosscap decrease χ by 1;
- adding a handle decreases χ by 2.

Orbifolds

A 2-dimensional orbifold is a surface endowed with special points:

either cone points of order n or mirror points (along boundary lines) or corner reflectors of order n (intersection of n mirror lines).

The orbifold Euler characteristic is still

$$\chi = (\#\text{vertices}) - (\#\text{edges}) + (\#\text{faces})$$

where now special vertices and edges are weighted as follows:

- if an edge is on a mirror, it contributes only $-\frac{1}{2}$;
- if a vertex is a cone point of order n it contributes $\frac{1}{n}$;
- if a vertex is on a mirror but is not a corner reflector, it contributes $\frac{1}{2}$;
- if a vertex is a corner reflector of order n it contributes $\frac{1}{2n}$;

Ordinary vertices and edges contribute 1 and -1, as before.

Classification of orbifolds

Any (2-dim) orbifold can be obtained from a surface by

- replacing some interior points by cone points and
- replacing some boundary points by corner reflectors.

So any orbifold can be obtained by surgery on a sphere.

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Introducing orbifold features affects the Euler Characteristic χ:

- introducing a cone point of order n decreases χ by $\frac{n-1}{n}$ and
- introducing a corner reflector of order n decreases χ by $\frac{n-1}{2n}$.

Example: the orbifold O225∗34

O225∗34 represents an orbifold obtained from a sphere attaching one handle, three cone points with cone angles $\frac{360^\circ}{2}$, $\frac{360^\circ}{2}$ and $\frac{360^\circ}{5}$, punching one hole $*$ and adding two corner reflectors on the corresponding boundary with angles $\frac{180^\circ}{3}$ and $\frac{180^\circ}{4}$.

O225∗34 has orbifold Euler characteristic
$\chi = 2 - 2 - \frac{1}{2} - \frac{1}{2} - \frac{4}{5} - 1 - \frac{2}{6} - \frac{3}{8} = -\frac{431}{120} = -3.5916666...$

Summary of the proof of the magic theorem

- Any pattern can be *folded-up* into an **orbifold** – the *stamp* of that pattern – by taking all points of the same kind (a so-called orbit) to a single point.

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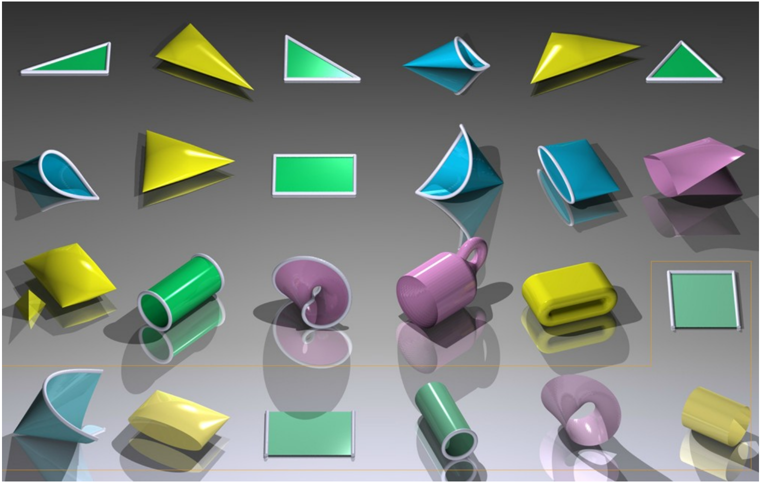
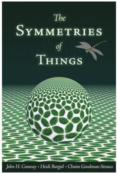
- Every (connected 2-dimensional) orbifold can be obtained by **surgery on a sphere**: adding cone points, holes/mirror lines, corner reflectors, handles and cross-caps.
- The effect of adding such a *feature* to an orbifold has a certain *cost* on its orbifold Euler characteristic (see *menu*). The total cost should be 2, so that $\chi = \mathbf{2} - \mathbf{cost} = \mathbf{0}$.

The 17+7 orbifolds with zero orbifold Euler characteristic

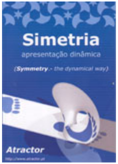


DVD *Symmetry – the dynamical way*

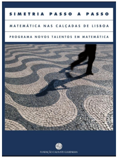
References:



The Symmetries of Things
book by Conway, Burgiel and Goodman-Strauss



Symmetry – the Dynamical Way
DVD by `www.atractor.pt`



Simetria Passo a Passo
Mathematics in the Lisbon sidewalks
`http://www.math.ist.utl.pt/simetria/`

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- Most displayed Lisbon photos were taken by *João Ferrand*.
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- The survey of Lisbon sidewalks started in 2004 with the cooperation of *Bruno Montalto* and *Luís Alexandre Pereira*.
- A team including *José Francisco Rodrigues* and *Leonor Godinho* is pursuing the effort to complete the list of patterns found in Lisbon sidewalks, in cooperation with the *University of Lisbon*, the *Lisbon's Mayor Office* and the *Portuguese Mathematical Society*.