DEDEKIND SUMS VIA ATIYAH-BOTT-LEFSCHETZ

ANA CANNAS DA SILVA

Abstract. This paper, written for differential geometers, shows how to deduce the reciprocity laws of Dedekind and Rademacher, as well as n-dimensional generalizations of these, from the Atiyah-Bott-Lefschetz formula, by applying this formula to appropriate elliptic complexes on weighted projective spaces.

1. Introduction

Dedekind was entrusted with Riemann's papers after his death, including some notes related to elliptic modular functions which Dedekind edited. Dedekind then published a famous addendum to those notes where, among other useful comments, he devotes consideration to a finite sum, which was essentially what became known as a *Dedekind sum.*

Let p and q be positive integers which are relatively prime. The Dedekind sum for the pair (p, q) is

$$s(p,q) = \sum_{k=1}^{q} \left(\!\!\left(\frac{pk}{q}\right)\!\!\right) \left(\!\!\left(\frac{k}{q}\right)\!\!\right) ,$$

where $((\cdot))$ is the function defined by

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

with $\lfloor x \rfloor$ the greatest integer not exceeding x, known as the *floor* of x. This is a sawtooth function of period 1 with graph

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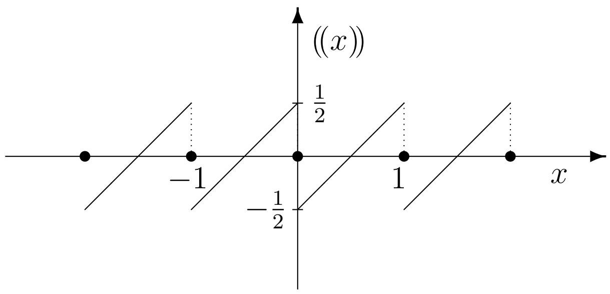

Dedekind discovered a reciprocity law obeyed by these sums, which Rademacher generalized. The precise statements of these laws are recalled in §4. Mordell related Dedekind sums to the number of lattice[1] points in the tetrahedron

$$0 \leq x < a \ , \ 0 \leq y < b \ , \ 0 \leq z < c \ , \ 0 < \frac{x}{a} + \frac{y}{b} + \frac{z}{c} < 1 \ ,$$

where a, b and c are relatively prime positive integers. The statement and a proof of Mordell's result can be found in [RG, p.40-43].

There is by now a vast literature on Dedekind sums and their generalizations of different sorts. They appear naturally in the theory of elliptic functions, modular transformations, lattice points, etc. For a survey, see [RG].

Hirzebruch [Hi] was probably the first to tackle generalized Dedekind sums from topological considerations. In particular, he arrived at reciprocity laws and Mordell's theorem using his signature theorem and results of Atiyah, Bott and Singer for group actions on 4-dimensional manifolds. Afterwards, Zagier [Z] studied connections between topology and number theory for higher-dimensional manifolds. Their book [HZ] develops topological approaches to number theory.

Numerous authors have obtained results on enumeration of lattice points in convex lattice polytopes by looking at toric varieties, such as Brion [B] applying a Lefschetz-Riemann-Roch formula and Morelli [Mo] studying the Todd class. Ishida [I] reproved and generalized Brion's results by a more elementary approach using contractibility of convex sets. Pommersheim [P] obtained a formula for the number of lattice points in an arbitrary lattice tetrahedron generalizing Mordell's 1951 formula, as well as a n-term generalization of Rademacher's three-term reciprocity formula for Dedekind sums, by using a formula for the Todd class of a toric variety. Sardo Infirri [SI] obtained variants of Brion's results using Brion's and Ishida's methods. Brion and Vergne [BV]

[1]Throughout, we consider the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$.

extended Pommersheim's work. The theme of number-theoretic applications from topological analysis of symplectic toric orbifolds may be found also among Guillemin's research interests; see, for instance, [G].

In this paper, we show how to deduce the reciprocity laws of Dedekind and Rademacher, as well as n-dimensional generalizations of these formulas and expressions for numbers of lattice points inside polytopes, by applying the enormously fruitful Atiyah-Bott-Lefschetz fixed-point formula to appropriate elliptic complexes on weighted projective spaces.

A weighted (or twisted) projective space, X, is obtained by taking the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the action

$$\rho(\omega)(z_0, \ldots, z_n) = (\omega^{q_0} z_0, \ldots, \omega^{q_n} z_n) \ , \quad \omega \in \mathbb{C}^* \ ,$$

where the q_i's are relatively prime positive integers (this is also known as the case of a *well-formed* weighted projective space). This is a nice type of orbifold, namely a quotient of a smooth manifold by a finite group action that is free except at isolated points (see §3.4).

We equip this weighted projective space X with the holomorphic line bundle associated with the representation with weight $-\ell \cdot q_0 \cdots q_n$ for some nonnegative integer ℓ, so that the dimension of its space of holomorphic sections be the number, $N_n(q_0, \ldots, q_n, \ell)$, of integer lattice points $(m_0, \ldots, m_n)$ with $m_0, \ldots, m_n \geq 0$ satisfying

$$q_0 m_0 + \ldots + q_n m_n = \ell q_0 \ldots q_n \ .$$

Our formulas are then of the form

$$N_n(q_0, q_1, \ldots, q_n, \ell) = A_n(q_0, q_1, \ldots, q_n) + B_n(q_0, q_1, \ldots, q_n, \ell) \ ,$$

where the $A_n(q_0, q_1, \ldots, q_n)$ are related to generalized Dedekind sums and the $B_n(q_0, q_1, \ldots, q_n, \ell)$ are evaluated using Laurent series; the exact definitions of these numbers are given in §3.5.

Since $A_n(q_0, \ldots, q_n)$ is independent of ℓ and $N_n(q_0, \ldots, q_n, 0) = 1$, we end up with generalized reciprocity formulas of the form

$$A_n(q_0, \ldots, q_n) = 1 - B_n(q_0, \ldots, q_n, 0) \ ,$$

as well as formulas for the numbers of lattice points

$$N_n(q_0, \ldots, q_n, \ell) = 1 - B_n(q_0, \ldots, q_n, 0) + B_n(q_0, \ldots, q_n, \ell) \ .$$

The version of the Atiyah-Bott-Lefschetz formula needed for our purposes is reviewed in §2, the application to weighted projective spaces is described in §3, and the number-theoretic consequences are discussed in §4.

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2. Atiyah-Bott-Lefschetz Formulas for Orbifolds

2.1. **Case of Good Orbifolds.** We will derive the needed theorems for *good orbifolds*, i.e., orbifolds that are global quotients of a compact manifold by an action of a finite group.

Let M be a compact complex (smooth) manifold of (complex) dimension n, acted upon by a finite group G in a holomorphic fashion. We denote by $\psi_g : M \to M$ the holomorphic diffeomorphism of M corresponding to the element $g \in G$. The quotient space, $X = M/G$, is a *good complex orbifold.*

We denote the standard splitting induced by local holomorphic coordinates on M as

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0} \oplus T^{0,1} \ , \tag{2.1}$$

where $T^{1,0}$ is spanned by the dz's and $T^{0,1}$ by the $d\overline{z}$'s. Then the corresponding bigraded wedge powers are

$$\bigwedge\nolimits^{a,b} := \bigwedge\nolimits^{a}(T^{1,0}) \otimes_{\mathbb{C}} \bigwedge\nolimits^{b}(T^{0,1}) \ .$$

We will consider $a = 0$ and $b = 0, 1, \ldots, n$.

The Dolbeault cohomology groups of X, denoted $H^k(X)$ (or $H^{0,k}_{\overline{\partial}}(X)$), are the subgroups of G-invariant elements in the Dolbeault cohomology groups of M, i.e.,

$$H^k(X) := H^k_G(M), \quad k = 0, 1, \ldots, n,$$

where $H^k(M)$ are the homology groups of the elliptic complex

$$0 \longrightarrow \Gamma(\bigwedge\nolimits^{0,0}) \xrightarrow{\overline{\partial}} \Gamma(\bigwedge\nolimits^{0,1}) \xrightarrow{\overline{\partial}} \Gamma(\bigwedge\nolimits^{0,2}) \xrightarrow{\overline{\partial}} \ldots \Gamma(\bigwedge\nolimits^{0,n}) \longrightarrow 0$$

acted upon by G via pullback.

A G-equivariant holomorphic map, $\widetilde{f} : M \to M$, induces a G-equivariant endomorphism f^* of the above complex, thus a G-equivariant endomorphism in its homology. Therefore, such a $\widetilde{f} : M \to M$ induces a quotient map, $f : X \to X$, and endomorphisms of complex vector spaces

$$H^k(f) : H^k(X) \longrightarrow H^k(X), \quad k = 0, 1, \ldots, n.$$

By definition, the *Lefschetz number* of f is

$$\mathcal{L}(f) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}\, H^k(f) \ ,$$

where the *trace is taken over* $\mathbb{C}$, i.e., as the trace of an endomorphism of a complex vector space.

We will need the following standard averaging result.

Lemma 2.2. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a finite group G on a finite-dimensional vector space V. If $\widetilde{\ell} : V \to V$ is a G-equivariant linear map, V_G is the subspace of vectors fixed by G, and $\ell : V_G \to V_G$ is the restriction of $\widetilde{\ell}$ to V_G, then*

$$\mathrm{trace}\,(\ell : V_G \to V_G) = \frac{1}{|G|} \sum_{g \in G} \mathrm{trace}\, \Big((\rho_g \circ \widetilde{\ell}) : V \to V \Big) \ .$$

Proof. Consider the projection $p : V \to V_G$ given by

$$p(v) = \frac{1}{|G|} \sum_{g \in G} \rho_g(v) \ .$$

Since $\widetilde{\ell}$ is G-equivariant, we have $p \circ \widetilde{\ell} = \ell \circ p$. Hence, $\widetilde{\ell}$ preserves the splitting $V = V_G \oplus \ker p$, and

$$\mathrm{trace}\,(\ell : V_G \to V_G) = \mathrm{trace}\,(\ell \circ p : V \to V) = \mathrm{trace}\,(p \circ \widetilde{\ell} : V \to V) \ .$$

□

By Lemma 2.2, we have

$$\mathcal{L}(f) = \sum_{k=0}^{n} (-1)^k \frac{1}{|G|} \sum_{g \in G} \mathrm{trace}\, H^k(\psi_g \circ \widetilde{f}) \ .$$

Suppose that $f : X \to X$ has only nondegenerate (hence isolated) fixed points, or, equivalently, that for all $g \in G$ the composition $\psi_g \circ \widetilde{f}$ has only nondegenerate fixed points, i.e.

$$\det \Big(\mathrm{I} - d(\psi_g \circ \widetilde{f})_p \Big) \neq 0 \qquad \text{at a fixed point } p \in M \ .$$

In this holomorphic case, the complexification of the dual of the derivative gives a bundle map

$$d\widetilde{f}^* \ : \ \widetilde{f}^* \left(T^*M \otimes_{\mathbb{R}} \mathbb{C} \right) \longrightarrow T^*M \otimes_{\mathbb{R}} \mathbb{C}$$

preserving the standard splitting (2.1). Hence, at a fixed point $p \in M$, the linear map

$$d(\psi_g \circ \widetilde{f})^*_p \ : \ T^*_pM \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow T^*_pM \otimes_{\mathbb{R}} \mathbb{C}$$

is a direct sum of endomorphisms of $T_p^{1,0}$ and $T_p^{0,1}$ denoted

$$d(\psi_g \circ \widetilde{f})_p^{1,0} \oplus d(\psi_g \circ \widetilde{f})_p^{0,1} \ .$$

Since T_p^*M is (real)-isomorphic to $T_p^{1,0}$, it inherits a complex structure, with respect to which we may view $d(\psi_g \circ \widetilde{f})_p^*$ as a complex endomorphism agreeing with $d(\psi_g \circ \widetilde{f})_p^{1,0}$.

By the Lefschetz fixed point theorem in this case (see [AB2, (4.9)]), we obtain

$$\sum_{k=0}^{n}(-1)^k \text{trace}\, H^k(\psi_g \circ \widetilde{f}) = \sum_{\{p\in M \mid (\psi_g\circ\widetilde{f})(p)=p\}} \frac{1}{\det\big(\mathrm{I}-d(\psi_g\circ\widetilde{f})_p\big)} \ ,$$

where the *determinant is taken over* $\mathbb{C}$, i.e., as the determinant of an endomorphism of a complex vector space, similar to the trace.

Hence, under the above conditions, we have

$$\mathcal{L}(f) = \frac{1}{|G|}\sum_{g\in G}\ \sum_{\{p\in M \mid (\psi_g\circ\widetilde{f})(p)=p\}} \frac{1}{\det\big(\mathrm{I}-d(\psi_g\circ\widetilde{f})_p\big)} \ . \tag{2.3}$$

We can write equation (2.3) as a sum of contributions from the fixed points of $f : X \to X$. Let $p_1, p_2, \ldots, p_\ell$ be the pre-images in M of a fixed point q of $f : X \to X$. Replacing, if necessary, $\widetilde{f}$ by $\psi_g \circ \widetilde{f}$ for some $g \in G$, we may assume $\widetilde{f}(p_i) = p_i$, $i = 1, \ldots, \ell$. Let G_i be the stabilizer of p_i in G. The contribution of q to the Lefschetz number is

$$\frac{1}{|G|}\sum_{i=1}^{\ell}\sum_{g\in G_i} \frac{1}{\det\big(\mathrm{I}-d(\psi_g\circ\widetilde{f})_{p_i}\big)} \ .$$

As p_i and p_1 are in the same G-orbit, G_i is conjugate to G_1. Then, as $\widetilde{f}$ is G-equivariant and the determinant is invariant by conjugation, the i-summands are all equal to the contribution for $i = 1$,

$$\sum_{g\in G_1} \frac{1}{\det\big(\mathrm{I}-d(\psi_g\circ\widetilde{f})_{p_1}\big)} \ .$$

Moreover, we have $|G| = \ell|G_1|$. Therefore, the contribution of q to $\mathcal{L}(f)$ is

$$\frac{1}{|G_1|}\sum_{g\in G_1} \frac{1}{\det\big(\mathrm{I}-d(\psi_g\circ\widetilde{f})_{p_1}\big)} \ ,$$

and equation (2.3) is equivalent to:

Formula 2.4. *Under the above conditions, we have*

$$\mathcal{L}(f) = \sum_{i=1}^{m}\frac{1}{|G_i|}\sum_{g\in G_i} \frac{1}{\det\big(\mathrm{I}-d(\psi_g\circ\widetilde{f})_{p_i}\big)} \ ,$$

where $X^f = \{x_1, \ldots, x_m\}$ *is the fixed-point set of* $f : X \to X$*,* $p_i \in M$ *is any chosen preimage of* x_i*, and* G_i *is the stabilizer of* p_i *in* G*,* $i = 1, \ldots, m$*.*

Formula 2.4 admits the following generalization extending the manifold result of Atiyah and Bott [AB2] (announced also in [AB3, §3]).

Let $\pi : L \to X$ be a holomorphic orbifold line bundle over X, presented as the quotient of a G-equivariant holomorphic line bundle $\widetilde{\pi} : \widetilde{L} \to M$. We denote by $\Psi_g : \widetilde{L} \to \psi_g^* \widetilde{L}$ the holomorphic bundle map corresponding to $\psi_g : M \to M$ for the group element g. Let $H^k(M; \widetilde{L})$ be the homology groups of the elliptic complex obtained by tensoring the Dolbeault complex of M with $\widetilde{L}$:

$$0 \longrightarrow \Gamma(\widetilde{L}) \xrightarrow{1 \otimes \overline{\partial}} \Gamma(\widetilde{L} \otimes_{\mathbb{C}} \textstyle\bigwedge^{0,1}) \xrightarrow{1 \otimes \overline{\partial}} \ldots \Gamma(\widetilde{L} \otimes_{\mathbb{C}} \textstyle\bigwedge^{0,n}) \longrightarrow 0 \; .$$

Then, for the pair X, L, we define again

$$H^k(X; L) := H^k_G(M; \widetilde{L}), \quad k = 0, 1, \ldots, n,$$

where $g \in G$ acts on the complex by $\Psi_g^{-1} \otimes (d\psi_g)^*$.

Assume that the G-equivariant holomorphic map $\widetilde{f} : M \to M$ admits a G-equivariant holomorphic bundle map

$$\widetilde{F} : \widetilde{f}^* \widetilde{L} \longrightarrow \widetilde{L}$$

inducing $F : f^* L \longrightarrow L$. Then the tensor maps

$$\widetilde{F} \otimes \wedge^k d\widetilde{f}^{0,1} \; : \; \widetilde{f}^* (\widetilde{L} \otimes_{\mathbb{C}} \textstyle\bigwedge^{0,k}) \longrightarrow \widetilde{L} \otimes_{\mathbb{C}} \textstyle\bigwedge^{0,k}$$

form a G-equivariant endomorphism of the elliptic complex above, hence induce endomorphisms in homology preserving the G-invariant subgroups

$$H^k(X; L) := H^k_G(M; \widetilde{L}), \quad k = 0, 1, \ldots, n.$$

We thus obtain endomorphisms

$$H^k(f; F) : H^k(X; L) \longrightarrow H^k(X; L), \quad k = 0, 1, \ldots, n.$$

By definition, the *Lefschetz number* of the pair f, F is

$$\mathcal{L}(f; F) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}\, H^k(f; F) \; .$$

By the averaging result (Lemma 2.2), we have

$$\mathcal{L}(f; F) = \sum_{k=0}^{n} (-1)^k \frac{1}{|G|} \sum_{g \in G} \mathrm{trace}\, H^k(\psi_g \circ \widetilde{f}; \Psi_g^{-1} \circ \widetilde{F}) \; .$$

Following the strategy as in the previous case without the line bundle, by applying now Theorem 4.12 of Atiyah and Bott in [AB2], we obtain

Formula 2.5. *Under the above conditions, we have*

$$\mathcal{L}(f;F) = \sum_{i=1}^{m} \frac{1}{|G_i|} \sum_{g \in G_i} \frac{\operatorname{trace}\,(\Psi_g^{-1} \circ \widetilde{F})_{p_i}}{\det\bigl(\mathrm{I} - d(\psi_g \circ \widetilde{f})_{p_i}\bigr)} \ ,$$

where $X^f = \{x_1, \dots, x_m\}$*,* $p_i \in M$ *is any preimage of* x_i*, and* G_i *is the stabilizer of* p_i *in* G*,* $i = 1, \dots, m$*. We recall that both determinant and trace are taken over* $\mathbb{C}$*.*

2.2. **Case of General Orbifolds.** Since the contributions to the Lefschetz number are local [AB1, §5], one expects formulas as in §2.1 applying to general orbifolds, i.e., orbifolds that are not globally quotients of a compact manifold by a finite group. Kawasaki [K] first extended Lefschetz formulas to that orbifold case. Sardo Infirri [SI, §4.2] deduced the two formulas below via more elementary arguments.

Let $f : X \to X$ be a holomorphic map from a compact complex n-dimensional orbifold to itself, having only nondegenerate fixed points, $q_1, \dots, q_m$. For simplicity, we assume that X has no singularities away from these fixed points, which is verified in our concrete application in §3. The Lefschetz number of f is, by definition,

$$\mathcal{L}(f) = \sum_{k=0}^{n} (-1)^k \operatorname{trace} H^k(f) \ ,$$

where $H^k(f) : H^k(X) \to H^k(X)$ is the map induced by pullback f^* on the kth Dolbeault cohomology group of X. Each q_i possesses an orbifold chart [S] comprising the following data:

- a neighborhood, X_i, of q_i in X,
- a connected open subset, $M_i \subseteq \mathbb{C}^n$, and
- a finite group, G_i, acting effectively on M_i by linear transformations, $\psi_{i,g}$, such that M_i/G_i is homeomorphic to X_i.

We assume, that the point q_i has exactly one preimage p_i in M_i, so q_i has isotropy group G_i. Moreover, for the map f, there is

- a G_i-invariant neighborhood, $\mathcal{U}_i$, of p_i in M_i, and
- a holomorphic G_i-equivariant lift, $\widetilde{f_i} : \mathcal{U}_i \to M_i$, of $f_i := f|_{\mathcal{U}_i/G_i}$.

Formula 2.6. *Under the conditions in the previous paragraph, we have*

$$\mathcal{L}(f) = \sum_{i=1}^{m} \frac{1}{|G_i|} \sum_{g \in G_i} \frac{1}{\det\bigl(\mathrm{I} - d(\psi_{i,g} \circ \widetilde{f_i})_{p_i}\bigr)} \ ,$$

where the determinant is taken again over $\mathbb{C}$*.*

In addition to the above assumptions about X and f, let $\pi : L \to X$ be a holomorphic orbifold line bundle over X and $F : f^*L \to L$ a holomorphic orbifold bundle map. The Lefschetz number of the pair f, F is, by definition,

$$\mathcal{L}(f;F) = \sum_{k=0}^{n} (-1)^k \text{trace}\, H^k(f;F) ,$$

where $H^k(f;F) : H^k(X;L) \to H^k(X;L)$ are the maps induced by f^* and F on the homology of the Dolbeault complex of X tensored by L. The previous local data at a fixed point q_i of f gets augmented by:

- the line bundle $L_i := \pi^{-1}(X_i) \to X_i$,
- a holomorphic line bundle, $\widetilde{L}_i$ over M_i,
- an action of G_i on $\widetilde{L}_i$ by holomorphic bundle maps, $\Psi_{i,g} : \widetilde{L}_i \to \psi_{i,g}^* \widetilde{L}_i$, such that $L_i = \widetilde{L}_i / G_i$, and
- a holomorphic G_i-equivariant bundle map, $\widetilde{F}_i : \widetilde{f}_i^* \widetilde{L}_i \to \widetilde{L}_i$, lifting the corresponding restriction of F to $f_i^* L_i$.

Formula 2.7. *Conditions and notation being as in §2.2, we have*

$$\mathcal{L}(f;F) = \sum_{i=1}^{m} \frac{1}{|G_i|} \sum_{g \in G_i} \frac{\text{trace}\,(\Psi_{i,g}^{-1} \circ \widetilde{F}_i)_{p_i}}{\det\big(\mathrm{I} - d(\psi_{i,g} \circ \widetilde{f}_i)_{p_i}\big)} .$$

3. Application to a Weighted Projective Space

3.1. **Base Orbifold.** Fix positive integers $q_0, \ldots, q_n$. Let X be the orbifold obtained by dividing $\mathbb{C}^{n+1} \setminus \{0\}$ by the group $\mathbb{C}^*$ of nonzero complex numbers, where $\mathbb{C}^*$ acts by

$$\begin{array}{rcl} \omega & \longmapsto & \rho(\omega) \\ & & \rho(\omega)(z_0, \ldots, z_n) = (\omega^{q_0} z_0, \ldots, \omega^{q_n} z_n) . \end{array}$$

Assuming that the $q_0, \ldots, q_n$ are pairwise relatively prime, the orbifold X is non-singular except, at most, at the $n+1$ points:

$$[1:0:\ldots:0],\ [0:1:0:\ldots:0],\ \ldots,\ [0:\ldots:0:1] , \tag{3.1}$$

which have stabilizers $\mathbb{Z}/q_0, \ldots, \mathbb{Z}/q_n$, respectively.

The standard diagonal action of S^1 on $\mathbb{C}^{n+1}$,

$$\begin{array}{rcl} e^{2\pi i t} & \longmapsto & \widetilde{f}_t \\ & & \widetilde{f}_t(z_0, \ldots, z_n) := (e^{2\pi i t} z_0, \ldots, e^{2\pi i t} z_n) , \end{array}$$

induces an action, f_t, on X. We assume that at most one of the q_i's is equal to 1. We will look at maps f_t for $t \neq 0$ in a neighborhood of $t = 0$. In this case, the fixed points of each f_t are again the $n+1$ points (3.1) and these are nondegenerate.

On the cross-section $z_n = c$, the diffeomorphism $\psi_q := \rho(e^{2\pi i \frac{q}{q_n}})$ with $q \in \mathbb{Z}$ satisfies

$$\psi_q(z_0, \ldots, z_{n-1}, c) = (e^{2\pi i \frac{q q_0}{q_n}} z_0, \ldots, e^{2\pi i \frac{q q_{n-1}}{q_n}} z_{n-1}, c) \ ,$$

whereas the diffeomorphism $\widetilde{f}_t$ satisfies

$$\begin{aligned} \widetilde{f}_t(z_0, \ldots, z_{n-1}, c) &= (e^{2\pi i t} z_0, \ldots, e^{2\pi i t} z_{n-1}, e^{2\pi i t} c) \\ &\sim (e^{2\pi i t(1-\frac{q_0}{q_n})} z_0, \ldots, e^{2\pi i t(1-\frac{q_{n-1}}{q_n})} z_{n-1}, c) \ . \end{aligned}$$

3.2. **Line Bundle.** Let L be the holomorphic line bundle over X associated with the representation

$$\gamma : \mathbb{C}^* \longrightarrow \mathrm{Aut}(\mathbb{C}) \ , \quad \gamma(\omega)s = \omega^{-d} s \ ,$$

i.e., $L = (\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}) /\!\sim$ for the equivalence relation $\sim$ defined by

$$(z, s) \sim \big(\rho(\omega)z, \gamma(\omega^{-1})s\big) \, , \omega \in \mathbb{C}^* \ ,$$

that is,

$$(z_0, \ldots, z_n, s) \sim \big(\omega^{q_0} z_0, \ldots, \omega^{q_n} z_n, \omega^d s\big) \, , \omega \in \mathbb{C}^* \ .$$

We will assume that $d = \ell \cdot q_0 \cdots q_n$ for some integer ℓ, so that all fibers of L be complex lines.[2]

We define an action, F_t, of S^1 on L induced by letting S^1 act by f_t^{-1} on the first factor of $\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}$ and trivially on the second factor. In particular, we have

$$F_t[(0, \ldots, 0, 1), s] = [(0, \ldots, 0, e^{-2\pi i t}), s] \sim [(0, \ldots, 0, 1), e^{2\pi i t \frac{d}{q_n}} s] \ ,$$

so the action of $e^{2\pi i t} \in S^1$ on the fiber of L above $[0 : \ldots : 0 : 1]$ is given by multiplication by $e^{2\pi i t \frac{d}{q_n}}$.

We define the action Ψ of $\mathbb{Z}/q_n$ on the L-fiber over $[0 : \ldots : 0 : 1]$ to be trivial.

3.3. **General Formula.** The relevant data for the contribution of the fixed point $[0 : \ldots : 0 : 1]$ to the Lefschetz number of the pair f_t, F_t is hence

$$\Psi_q^{-1} \circ F_t = \text{multiplication by } e^{2\pi i t \frac{d}{q_n}} : L_{[0:\ldots:0:1]} \to L_{[0:\ldots:0:1]}$$

and

$$d(\psi_q \circ \widetilde{f}_t)_{[0:\ldots:0:1]} =$$
$$\mathrm{diag}(e^{2\pi i \frac{q q_0}{q_n}}, \ldots, e^{2\pi i \frac{q q_{n-1}}{q_n}}) \cdot \mathrm{diag}(e^{2\pi i t(1-\frac{q_0}{q_n})}, \ldots, e^{2\pi i t(1-\frac{q_{n-1}}{q_n})}) \ .$$

[2]For $\omega \in \mathbb{Z}/q_n$, we have

$$((0, \ldots, 0, 1), s) \sim \big(\rho(\omega)(0, \ldots, 0, 1), \gamma(\omega^{-1})s\big) = \big((0, \ldots, 0, 1), \omega^d s\big) \ .$$

Hence, in order for the fiber of L over $[0 : \ldots : 0 : 1]$ to be a complex line, we need $q_n | d$. Similarly for the other singular points.

Summing over $q = 0, 1, \ldots, q_n - 1$ for the q_n-roots of unity, $\omega = e^{2\pi i \frac{q}{q_n}}$, the contribution from $[0 : \ldots : 0 : 1]$ in Formula 2.7 reads

$$\frac{1}{q_n} \sum_{q=0}^{q_n-1} \frac{e^{2\pi i t \frac{d}{q_n}}}{\prod_{m \neq n} (1 - e^{2\pi i (1 - \frac{q_m}{q_n}) t} \cdot e^{2\pi i \frac{q q_m}{q_n}})} .$$

Similar computations yield similar results for the other fixed points. Adding up all contributions, we obtain the Lefschetz number:

$$\mathcal{L}(f_t; F_t) = \sum_{r=0}^{n} \frac{1}{q_r} \sum_{q=0}^{q_r-1} \frac{e^{2\pi i t \frac{d}{q_r}}}{\prod_{m \neq r} (1 - e^{2\pi i (1 - \frac{q_m}{q_r}) t} \cdot e^{2\pi i \frac{q q_m}{q_r}})} . \tag{3.2}$$

On the other hand, by definition, the Lefschetz number is

$$\mathcal{L}(f_t; F_t) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}\left(H^k(f_t; F_t) : H^k(X; L) \to H^k(X; L)\right) .$$

3.4. **Cohomology.** The orbifold X is a good complex orbifold, since it is the quotient of ordinary complex projective space $\mathbb{CP}^n$ by the coordinatewise action of $G := \mathbb{Z}/q_0 \times \ldots \times \mathbb{Z}/q_n$. The quotient map is simply[3]

$$\begin{array}{ccc} \mathbb{CP}^n & \longrightarrow & X \\ [z_0 : \ldots : z_n] & \longmapsto & [z_0^{q_0} : \ldots : z_n^{q_n}] . \end{array}$$

Similarly, the line bundle $L \to X$ is the quotient of $\mathcal{O}(\ell) \to \mathbb{CP}^n$ with quotient map

$$\begin{array}{ccc} \mathcal{O}(\ell) & \longrightarrow & L \\ [(z_0, \ldots, z_n), s] & \longmapsto & [(z_0^{q_0}, \ldots, z_n^{q_n}), s^{q_0 \cdots q_n}] . \end{array}$$

For any $\ell \geq -n$, we have $H^k(\mathbb{CP}^n; \mathcal{O}(\ell)) = 0$ for $k > 0$ (see, for instance, [Ha, Theorem 5.1]), therefore also $H^k(X; L) = 0$ for $k > 0$. As for $H^0(X; L)$ this is the global G-invariant holomorphic sections of $\mathcal{O}(\ell)$, and these are linearly spanned by the monomial sections $[(z_0, \ldots, z_n), z_0^{m_0} \cdots z_n^{m_n}]$ satisfying the law

$$(\omega^{q_0} z_0)^{m_0} \cdots (\omega^{q_n} z_n)^{m_n} = \omega^d z_0^{m_0} \cdots z_n^{m_n} , \quad \text{for all } \omega \in \mathbb{C}^* ,$$

which implies

$$q_0 m_0 + \ldots + q_n m_n = d .$$

The dimension of $H^0(X; L)$ is thus the number of integer lattice points $(m_0, \ldots, m_n)$ satisfying $q_0 m_0 + \ldots + q_n m_n = d$, $m_0, \ldots, m_n \geq 0$. We denote this number by

$$N_n(q_0, \ldots, q_n, \ell)$$

recalling that we have $d = \ell q_0 \ldots q_n$.

[3]However, the S^1-action on X is not the quotient of an action on $\mathbb{CP}^n$.

3.5. **Limit as $t \to 0$.** Although our formula (3.2) does not hold for $t = 0$ since f_0 leaves all points fixed, we can take its limit and thus compute the above dimension $N_n(q_0, \ldots, q_n, \ell)$.

When $t \to 0$, the left-hand side of (3.2) becomes

$$\begin{aligned} \lim_{t\to 0} \mathcal{L}(f_t; F_t) &= \lim_{t\to 0} \operatorname{trace} H^0(f_t; F_t) \\ &= \dim H^0(X; L) = N_n(q_0, \ldots, q_n, \ell) \ . \end{aligned}$$

When $t \to 0$, the right-hand side of (3.2) becomes a sum of two types of terms, denoted $A_n(q_0, \ldots, q_n)$ and $B_n(q_0, \ldots, q_n, \ell)$, respectively:

$$\lim_{t\to 0} \sum_{r=0}^{n} \frac{1}{q_r} \sum_{q=0}^{q_r-1} \frac{e^{2\pi i t \frac{d}{q_r}}}{\prod_{m\neq r}(1 - e^{2\pi i (1-\frac{q_m}{q_r})t} \cdot e^{2\pi i \frac{q q_m}{q_r}})}$$

(3.3)

$$= \underbrace{\sum_{r=0}^{n} \frac{1}{q_r} \sum_{q=1}^{q_r-1} \frac{1}{\prod_{m\neq r}(1 - e^{2\pi i \frac{q q_m}{q_r}})}}_{A_n(q_0,\ldots,q_n)} + \underbrace{\lim_{t\to 0} \sum_{r=0}^{n} \frac{1}{q_r} \frac{e^{2\pi i t \frac{d}{q_r}}}{\prod_{m\neq r}(1 - e^{2\pi i (1-\frac{q_m}{q_r})t})}}_{B_n(q_0,\ldots,q_n,\ell)} .$$

The last limit can be computed via a Laurent series for each summand:

$$\frac{b_{n,r}}{t^n} + \ldots + \frac{b_{1,r}}{t} + b_{0,r} + \ldots \ .$$

Since the total sum is finite, the terms with negative powers of t in these series must cancel out as $t \to 0$, and we end up with

$$B_n(q_0, \ldots, q_n, \ell) = \sum_{r=0}^{n} b_{0,r} \ .$$

Moreover, the A_n sum may be rewritten in terms of sums over nontrivial q_r-roots of unity as

$$A_n(q_0, \ldots, q_n) = \sum_{r=0}^{n} \frac{1}{q_r} \sum_{\eta^{q_r}=1, \eta\neq 1} \frac{1}{\prod_{m\neq r}(1 - \eta^{q_m})} \ ,$$

which is related to generalized Dedekind sums, as we will see in the next section.

4. Number-Theoretic Consequences

4.1. **The Case $n = 2$.** For the first interesting case, formula (3.3) reads:

$$\#\{(m_0, m_1, m_2) \in (\mathbb{Z}_0^+)^3 \mid q_0 m_0 + q_1 m_1 + q_2 m_2 = d\} \ =$$

$$\underbrace{\sum_{r=0}^{2}\frac{1}{q_r}\sum_{q=1}^{q_r-1}\frac{1}{\prod_{m\neq r}(1-e^{2\pi i\frac{qq_m}{q_r}})}}_{A}\;+\;\underbrace{\lim_{t\to 0}\sum_{r=0}^{2}\frac{1}{q_r}\frac{e^{2\pi it\frac{d}{q_r}}}{\prod_{m\neq r}(1-e^{2\pi i(1-\frac{q_m}{q_r})t})}}_{B}.$$

We will deal with each of the terms A and B in turn.

A:

We can write

$$A=\sum_{r=0}^{2}\frac{1}{q_r}\sum_{\eta^{q_r}=1,\eta\neq 1}\frac{1}{\prod_{m\neq r}(1-\eta^{q_m})}\;.$$

Setting

(4.1) $\quad q_2\equiv k_0q_1 \mod q_0,\quad q_0\equiv k_1q_2 \mod q_1,\quad q_1\equiv k_2q_0 \mod q_2,$

we find

$$A=\sum_{r=0}^{2}\frac{1}{q_r}\sum_{\eta^{q_r}=1,\eta\neq 1}\frac{1}{(1-\eta)(1-\eta^{k_r})}\;.$$

Now, using the Eisenstein formula (see [E] and also [Z, §1])

$$\left(\left(\frac{p}{q}\right)\right)=\frac{1}{2q}\sum_{\eta^{q}=1,\eta\neq 1}\frac{1+\eta}{1-\eta}\eta^{p}$$

we obtain an alternative expression for the Dedekind sum as in [RG, p.15]:

$$s(k_r,q_r)=-\frac{1}{q_r}\sum_{\eta^{q_r}=1,\eta\neq 1}\frac{1}{(1-\eta)(1-\eta^{k_r})}+\frac{q_r-1}{4q_r}\;.$$

Therefore, we find

$$A=\sum_{r=0}^{2}\left(\frac{q_r-1}{4q_r}-s(k_r,q_r)\right)\;. \tag{4.2}$$

B:

Each summand in B is of the form

$$\frac{1}{q_r}\cdot\frac{e^{\omega t}}{(1-e^{\omega_1 t})(1-e^{\omega_2 t})}$$

for which the constant term in the Laurent expansion[4] is

$$b_{0,r} = \frac{1}{q_r}\left(\frac{1}{4} - \frac{1}{2}\frac{\omega}{\omega_1} - \frac{1}{2}\frac{\omega}{\omega_2} + \frac{1}{2}\frac{\omega^2}{\omega_1\omega_2} + \frac{1}{12}\frac{\omega_1}{\omega_2} + \frac{1}{12}\frac{\omega_2}{\omega_1}\right) .$$

Therefore, we have

$$B = \sum_{r=0}^{2} b_{0,r} = \frac{1}{4}\left(\frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{q_2}\right) + \frac{\ell}{2}(q_0+q_1+q_2) + \frac{\ell^2}{2} q_0 q_1 q_2 + \frac{1}{12}\cdot\frac{q_0^2+q_1^2+q_2^2}{q_0q_1q_2} . \tag{4.3}$$

Finally, we compute the left-hand side. Since $d = \ell q_0 q_1 q_2$, we have

$$\#\{(m_0,m_1,m_2) \in (\mathbb{Z}_0^+)^3 \mid q_0m_0 + q_1m_1 + q_2m_2 = \ell q_0q_1q_2\}$$
$$= \sum_{m_2=0}^{\ell q_0 q_1} \#\{(m_0,m_1) \in (\mathbb{Z}_0^+)^2 \mid q_0m_0 + q_1m_1 = (\ell q_0 q_1 - m_2)q_2\} .$$

For given $0 \le m_2 \le \ell q_0 q_1$, we denote simply by $\#$ the number of solutions $(m_0,m_1) \in (\mathbb{Z}_0^+)^2$ of the equation

$$q_0m_0 + q_1m_1 = (\ell q_0q_1 - m_2)q_2 . \tag{4.4}$$

Claim: *The number $\#$ is equal to $\left\lfloor \frac{(\ell q_0 q_1 - m_2)q_2}{q_0q_1} \right\rfloor + 1 - \varepsilon(m_2)$, where the integer-valued function $\varepsilon(m_2)$ satisfies*

$$\varepsilon(m_2) = 0 \quad \text{if } m_2 \text{ is a multiple of } q_0 \text{ or of } q_1, \text{ and}$$
$$\varepsilon(m_2) + \varepsilon(\ell q_0q_1 - m_2) = 1 \quad \text{if } m_2 \text{ is neither a multiple of } q_0 \text{ nor of } q_1.$$

Proof. Whenever m_2 is a multiple of q_1, say $m_2 = sq_1$ with $s \in \mathbb{Z}_0^+$, equation (4.4) is equivalent to

$$q_0m_0 + q_1m_1 = (\ell q_0 - s)q_1q_2 ,$$

and has a *first* solution $(m_0,m_1) = (0, (\ell q_0 - s)q_2)$. Since q_0 and q_1 are relatively prime, all further solutions (m_0,m_1) are of the form

$$(kq_1, (\ell q_0 - s)q_2 - kq_0) \text{ with } k = 1,2,\ldots,\left\lfloor \frac{(\ell q_0 - s)q_2}{q_0} \right\rfloor .$$

[4]This limit term can be written in terms of Bernoulli numbers B_n defined by

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1} , \qquad \text{so } B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6}, \ldots$$

In particular, the Laurent expansion of $\frac{1}{1-e^{\omega t}}$ is

$$\frac{1}{1-e^{\omega t}} = -\sum_{n=0}^{\infty} \frac{B_n}{n!} \omega^{n-1} t^{n-1} .$$

Therefore, the number of solutions is

$$\# = \left\lfloor \frac{(\ell q_0 - s)q_2}{q_0} \right\rfloor + 1 = \left\lfloor \frac{(\ell q_0 q_1 - m_2)q_2}{q_0 q_1} \right\rfloor + 1 \ .$$

A similar count holds when m_2 is a multiple of q_0, so in these cases we have $\varepsilon(m_2) = 0$.

Suppose now that m_2 is neither a multiple of q_0, nor of q_1, and consider it together with the integer $m_2' := \ell q_0 q_1 - m_2$, satisfying $0 < m_2, m_2' < \ell q_0 q_1$. We denote by $\#'$ the number of solutions $(m_0', m_1') \in (\mathbb{Z}_0^+)^2$ of the equation

$$q_0 m_0' + q_1 m_1' = (\ell q_0 q_1 - m_2')q_2 = m_2 q_2 \ .$$

We list the solutions to Equation (4.4) in increasing order of the first term in the pair, starting with a solution called (m_0, m_1):

$$(m_0, m_1), (m_0 + q_1, m_1 - q_0), \ldots (m_0 + (\# - 1)q_1, m_1 - (\# - 1)q_0) \ ,$$

and we do the same for the second equation, starting with a solution called (m_0', m_1'):

$$(m_0', m_1'), (m_0' + q_1, m_1' - q_0), \ldots (m_0' + (\#' - 1)q_1, m_1' - (\#' - 1)q_0) \ .$$

Note that we must start with $0 < m_0, m_0' < q_1$, as well as end with $0 < m_1 - (\# - 1)q_0, m_1' - (\#' - 1)q_0 < q_0$. But adding the two equations for the first pairs of solutions (m_0, m_1), (m_0', m_1'), we get

$$q_0(m_0 + m_0') + q_1(m_1 + m_1') = \ell q_0 q_1 q_2 \ ,$$

hence $m_0 + m_0'$ must be a multiple of q_1, so it must be

$$m_0 + m_0' = q_1 \quad \text{and} \quad m_1 + m_1' = (\ell q_2 - 1)q_0 \ .$$

Similarly, adding the two equations for the last pairs of solutions, we get that it must be

$$m_1 - (\# - 1)q_0 + m_1' - (\#' - 1)q_0 = q_0 \ ,$$

from what follows, with the knowledge of $m_1 + m_1' = (\ell q_2 - 1)q_0$, that

$$\# + \#' = \ell q_2 \ .$$

But this is what we needed to prove,[5] since

$$\# = \ell q_2 - \left\lfloor \frac{m_2 q_2}{q_0 q_1} \right\rfloor - \varepsilon(m_2) \quad \text{and} \quad \#' = \left\lfloor \frac{m_2 q_2}{q_0 q_1} \right\rfloor + 1 - \varepsilon(m_2') \ .$$

□

It follows that

$$\begin{aligned} &\#\{(m_0, m_1, m_2) \in (\mathbb{Z}_0^+)^3 \mid q_0 m_0 + q_1 m_1 + q_2 m_2 = \ell q_0 q_1 q_2\} \\ = &\sum_{m_2=0}^{\ell q_0 q_1} \left(\left\lfloor \frac{(\ell q_0 q_1 - m_2) q_2}{q_0 q_1} \right\rfloor + 1 - \varepsilon(m_2) \right) \ . \end{aligned}$$

From the properties of the integer-valued function $\varepsilon(m_2)$, we have

$$\begin{aligned} \sum_{m_2=0}^{\ell q_0 q_1} \varepsilon(m_2) &= \frac{1}{2} \sum_{m_2=0}^{\ell q_0 q_1} (\varepsilon(m_2) + \varepsilon(\ell q_0 q_1 - m_2)) \\ &= \frac{1}{2} \#\{m_2 \in [0, \ell q_0 q_1] \cap \mathbb{Z} \text{ s.t. } q_0 \not| m_2 \text{ and } q_1 \not| m_2\} \\ &= \frac{\ell(q_0 - 1)(q_1 - 1)}{2} \ . \end{aligned}$$

Also, since (see, for instance, [RG, p.32])

$$\sum_{k=1}^{p-1} \left\lfloor \frac{kq}{p} \right\rfloor = \frac{(p-1)(q-1)}{2} \qquad \text{for } p, q \text{ relatively prime,}$$

or, equivalently,

$$\sum_{k=1}^{p-1} \left\lfloor -\frac{kq}{p} \right\rfloor = -\frac{(p-1)(q+1)}{2} \qquad \text{for } p, q \text{ relatively prime,}$$

[5]We further show that $\varepsilon(m_2)$ only takes the values 0 or 1. Again let (m_0, m_1) be the solution of Equation (4.4) with the smallest first entry. Since q_0 and q_1 are relatively prime, any other solution is of the form $(m_0 + kq_1, m_1 - kq_0)$ with $k = 1, 2, \ldots, \# - 1$. Whereas the first term from the (m_0, m_1) solution contributes $q_0 m_0$ to the sum in the equation, the following solutions contribute $q_0 m_0 + k q_0 q_1$. Geometrically, $\# - 1$ will hence be the number of times that a segment of length $q_0 q_1$ fits inside the interval $[0, (\ell q_0 q_1 - m_2) q_2]$ to the right of the point $q_0 m_0$. Since $q_0 m_0 < q_0 q_1$, we conclude that either

$$\# - 1 = \left\lfloor \frac{(\ell q_0 q_1 - m_2) q_2}{q_0 q_1} \right\rfloor \quad \text{or} \quad \# - 1 = \left\lfloor \frac{(\ell q_0 q_1 - m_2) q_2}{q_0 q_1} \right\rfloor - 1 \ ,$$

that is, $\varepsilon(m_2)$ is 0 or 1. In particular, when m_2 is a multiple of q_1 (so the first solution has $m_0 = 0$), the segments of length $q_0 q_1$ start at the origin, and when m_2 is a multiple of q_0 (so the last solution has $m_1 - kq_0 = 0$), the segments of length $q_0 q_1$ finish at $(\ell q_0 q_1 - m_2) q_2$, hence in both of these cases we have $\varepsilon(m_2) = 0$.

we get

$$\sum_{m_2=0}^{\ell q_0 q_1} \left\lfloor \frac{(\ell q_0 q_1 - m_2) q_2}{q_0 q_1} \right\rfloor = \frac{\ell^2}{2} q_0 q_1 q_2 + \frac{\ell}{2}(q_2 - q_0 q_1 + 1) \ .$$

We conclude that

$$\begin{aligned} &\#\{(m_0, m_1, m_2) \in (\mathbb{Z}_0^+)^3 \mid q_0 m_0 + q_1 m_1 + q_2 m_2 = \ell q_0 q_1 q_2\} \\ &\qquad = \frac{\ell^2}{2} q_0 q_1 q_2 + \frac{\ell}{2}(q_0 + q_1 + q_2) + 1 \ . \end{aligned} \tag{4.5}$$

4.2. **Classic Reciprocity Laws and Pick's Theorem.** According to the previous section, for $n = 2$ formula (3.3) says

$$\frac{\ell^2}{2} q_0 q_1 q_2 + \frac{\ell}{2}(q_0 + q_1 + q_2) + 1 = A + B \tag{4.6}$$

where A is given by (4.2) and B is given by (4.3). This can hence be rewritten as

$$\sum_{r=0}^{2} s(k_r, q_r) = \frac{1}{12} \cdot \frac{q_0^2 + q_1^2 + q_2^2}{q_0 q_1 q_2} - \frac{1}{4} \ , \tag{4.7}$$

which is the Rademacher reciprocity law [HZ, p.96].

We now set $q_2 = 1$ and take $k_0 = q_1$, $k_1 = q_0$ and $k_2 = 1$. The fact that $q_i \equiv 1 \mod q_j$ for $i \neq j$ guarantees that condition (4.1) is satisfied by these choices. In this case, formula (4.7) reduces to

$$\sum_{r=0}^{2} s(k_r, q_r) = s(q_1, q_0) + s(q_0, q_1) = \frac{1}{12}\left(\frac{q_0}{q_1} + \frac{1}{q_0 q_1} + \frac{q_1}{q_0}\right) - \frac{1}{4} \tag{4.8}$$

which is the Dedekind reciprocity law [RG, p.4].

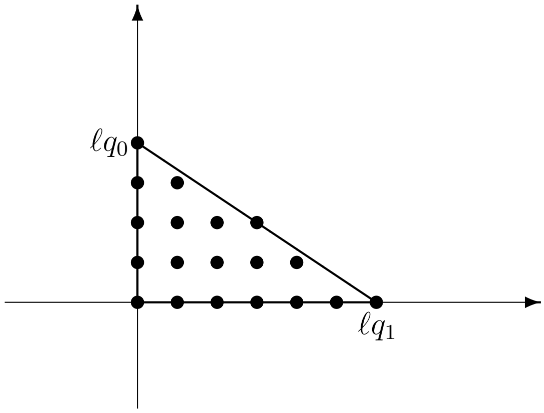
Consider the triangle $\Delta = \{(x_0, x_1) \in (\mathbb{R}_0^+)^2 \mid q_0 x_0 + q_1 x_1 \leq \ell q_0 q_1\}$ with vertices $(\ell q_1, 0)$, $(0, \ell q_0)$ and $(0, 0)$. When $q_2 = 1$, formula (4.5) becomes

$$\#(\Delta \cap \mathbb{Z}^2) = \text{Area } \Delta + \frac{\ell}{2}(q_0 + q_1 + 1) + 1 \ ,$$

which is an instance of Pick's theorem,

$$\text{Area } \Delta = I + \frac{1}{2} B - 1 \ ,$$

where I is the number of lattice points in the interior of Δ and $B = \ell(q_0 + q_1 + 1)$ is the number of lattice points on the boundary of Δ.



4.3. **Generalized Dedekind Sums.** When $\ell = 0$, i.e., $d = 0$ and the line bundle L is trivial, formula (3.3) reduces to

$$1 = \sum_{r=0}^{n} \frac{1}{q_r} \sum_{q=1}^{q_r - 1} \frac{1}{\prod_{m \neq r}(1 - e^{2\pi i \frac{q q_m}{q_r}})} + \lim_{t \to 0} \sum_{r=0}^{n} \frac{1}{q_r} \frac{1}{\prod_{m \neq r}(1 - e^{2\pi i (1 - \frac{q_m}{q_r}) t})}$$

which is equivalent to
(4.9)

$$\sum_{r=0}^{n} \frac{1}{q_r} \sum_{\eta^{q_r}=1, \eta \neq 1} \frac{1}{\prod_{m \neq r}(1 - \eta^{q_m})} = 1 - \lim_{t \to 0} \sum_{r=0}^{n} \frac{1}{q_r} \frac{1}{\prod_{m \neq r}(1 - e^{2\pi i (1 - \frac{q_m}{q_r}) t})} .$$

The last limit can be evaluated by the Laurent series argument. We define a generalized Dedekind sum

$$\delta_n(q_r; q_i, i \neq r) = \sum_{\eta^{q_r}=1, \eta \neq 1} \frac{1}{\prod_{m \neq r}(1 - \eta^{q_m})} , \quad \text{and define}$$

$$\alpha_n(q_0, \ldots, q_n) = \sum_{r=0}^{n} \frac{1}{q_r} \delta_n(q_r; q_i, i \neq r) .$$

When $n = 1, 2, 3, 4$ we find the following explicit generalized reciprocity laws.

$$\begin{aligned}
\alpha_1(q_0, q_1) &= 1 - \frac{1}{2}\left(\frac{1}{q_0} + \frac{1}{q_1}\right) \\
\alpha_2(q_0, q_1, q_2) &= 1 - \frac{1}{4}\left(\frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{q_2}\right) - \frac{1}{12} \cdot \frac{q_0^2 + q_1^2 + q_2^2}{q_0 q_1 q_2} \\
\alpha_3(q_0, q_1, q_2, q_3) &= 1 - \frac{1}{8}\left(\frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}\right) \\
&\quad - \frac{1}{24}\left(\frac{q_0 + q_1}{q_2 q_3} + \frac{q_0 + q_2}{q_1 q_3} + \frac{q_0 + q_3}{q_1 q_2}\right. \\
&\quad \left. + \frac{q_1 + q_2}{q_0 q_3} + \frac{q_1 + q_3}{q_0 q_2} + \frac{q_2 + q_3}{q_0 q_1}\right) \\
\alpha_4(q_0, q_1, q_2, q_3, q_4) &= 1 - \frac{1}{16}\sum \frac{1}{q_i} - \frac{1}{48} \cdot \frac{1}{q_0 q_1 q_2 q_3 q_4} \sum_{i \neq j < k \neq i} q_i^2 q_j q_k \\
&\quad - \frac{1}{144} \cdot \frac{1}{q_0 q_1 q_2 q_3 q_4} \sum_{i<j} q_i^2 q_j^2 \\
&\quad + \frac{1}{720} \cdot \frac{1}{q_0 q_1 q_2 q_3 q_4} \sum q_i^4
\end{aligned}$$

Remark. Using other methods, Hirzebruch and Zagier [HZ, p.100-101] gave results for generalized Dedekind sums of type δ_n for n even, namely

$$\sum_{r=0}^{n} \frac{(-1)^{\frac{n}{2}}}{q_r} \sum_{k=1}^{q_r - 1} \prod_{m \neq r} \cot \frac{\pi k q_m}{q_r} = 1 - \frac{\ell_n(q_0, \ldots, q_n)}{q_0 \cdots q_n} \tag{4.10}$$

where ℓ_n is a certain polynomial in $n+1$ variables which is symmetric in its variables, even in each variable, and homogeneous of degree n.

Formula (4.10) is related to the δ_n's and α_n's by

$$\begin{aligned}
& \sum_{r=0}^{n} \frac{(-1)^{\frac{n}{2}}}{q_r} \sum_{k=1}^{q_r-1} \prod_{m\neq r} \cot \frac{\pi k q_m}{q_r} \qquad\qquad \text{using } \cot\theta = i\frac{e^{2i\theta}+1}{e^{2i\theta}-1} \\
= & \sum_{r=0}^{n} \frac{1}{q_r} \sum_{\eta^{q_r}=1,\eta\neq 1} \prod_{m\neq r} \frac{\eta^{q_m}+1}{\eta^{q_m}-1} \\
= & \sum_{r=0}^{n} \frac{1}{q_r} \sum_{\eta^{q_r}=1,\eta\neq 1} \sum_{s=0}^{n} \sum_{I\subseteq\{0,\ldots,n\}\setminus r,\#I=s} \frac{(-2)^s}{\prod_{i\in I}(1-\eta^{q_i})} \\
= & \sum_{s=0}^{n} (-2)^s \sum_{r=0}^{n} \frac{1}{q_r} \sum_{I\subseteq\{0,\ldots,n\}\setminus r,\#I=s} \delta_s(q_r; q_i, i\in I) \\
= & \sum_{s=0}^{n} (-2)^s \sum_{I\subseteq\{1,\ldots,n\},\#I=s} \alpha_s(q_i, i\in I) \ .
\end{aligned}$$

For instance, when $n = 2$,

$$\ell_2(q_0, q_1, q_2) = \frac{1}{3}\sum_{i=0}^{2} q_i^2$$

and when $n = 4$,

$$\ell_4(q_0, q_1, q_2, q_3, q_4) = \frac{1}{18}\left(\sum_{i=0}^{4} q_i^2\right)^2 - \frac{7}{90}\sum_{i=0}^{4} q_i^4 \ .$$

4.4. **Counting Lattice Points.** Considering again a line bundle for arbitrary positive ℓ, formula (3.3) provides an expression for the number $N_n := N_n(q_0, \ldots, q_n, \ell)$ of non-negative integral solutions $(m_0, \ldots, m_n)$ of the equation $q_0 m_0 + \ldots + q_n m_n = \ell q_0 \cdots q_n$, namely

(4.11)

$$N_n = \underbrace{\sum_{r=0}^{n} \frac{1}{q_r} \sum_{q=1}^{q_r-1} \frac{1}{\prod_{m\neq r}(1-e^{2\pi i \frac{q q_m}{q_r}})}}_{A_n} \ + \ \underbrace{\lim_{t\to 0} \sum_{r=0}^{n} \frac{1}{q_r} \frac{e^{2\pi i t \frac{d}{q_r}}}{\prod_{m\neq r}(1-e^{2\pi i (1-\frac{q_m}{q_r})t})}}_{B_n} \ .$$

From the case $d = \ell = 0$ (see formula (4.9)), we have

$$A_n = 1 \ - \ \lim_{t\to 0} \sum_{r=0}^{n} \frac{1}{q_r} \frac{1}{\prod_{m\neq r}(1-e^{2\pi i(1-\frac{q_m}{q_r})t})} \ ,$$

and thus both A_n and B_n can be computed from the Laurent series argument. For $n \leq 4$ we have the following explicit results.

$$\begin{aligned}
N_0 &= 1 \\
N_1 &= \ell + 1 \\
N_2 &= \frac{\ell^2}{2} q_0 q_1 q_2 + \frac{\ell}{2}(q_0 + q_1 + q_2) + 1 \\
N_3 &= \frac{\ell^3}{6}(q_0 q_1 q_2 q_3)^2 + \frac{\ell^2}{4} q_0 q_1 q_2 q_3 (q_0 + q_1 + q_2 + q_3) \\
&\quad + \frac{\ell}{12}(q_0^2 + q_1^2 + q_2^2 + q_3^2) \\
&\quad + \frac{\ell}{4}(q_0 q_1 + q_0 q_2 + q_0 q_3 + q_1 q_2 + q_1 q_3 + q_2 q_3) + 1 \\
N_4 &= \frac{\ell^4}{24}\left(\prod q_i\right)^3 + \frac{\ell^3}{12}\left(\prod q_i\right)^2 \left(\sum q_i\right) \\
&\quad + \frac{\ell^2}{24}\left(\prod q_i\right)\left(\sum q_i^2 + 3\sum_{i<j} q_i q_j\right) \\
&\quad + \frac{\ell}{24}\left(\sum_{i \neq j} q_i^2 q_j + 3 \sum_{i<j<k} q_i q_j q_k\right) + 1
\end{aligned}$$

Working out N_n directly for each n, by decomposing into sums generalizing the procedure in §4.1, e.g.

$$N_3 = \sum_{x=0}^{\ell q_0 q_1 q_2 q_3} \#\{q_0 m_0 + q_1 m_1 = x\} \cdot \#\{q_2 m_2 + q_3 m_3 = \ell q_0 q_1 q_2 q_3 - x\},$$

and equating similar powers of ℓ in (4.11), we can iteratively obtain higher-dimensional formulas.

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Department of Mathematics, ETH Zurich, Raemistrasse 101 8092 Zurich, Switzerland

Email address: acannas@math.ethz.ch