Seminar on Symplectic Toric Manifolds

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Contents

Foreword

These notes were conceived to serve as a rough guideline for the student seminar on *symplectic toric manifolds* at ETH Zurich in the Spring of 2019. This seminar comprised twelve lectures and was an introduction to symplectic toric manifolds, i.e., smooth toric varieties from the symplectic viewpoint. It started from basic notions in symplectic geometry, went over the classification of symplectic toric manifolds and closed with some advanced topics.

Geometry of manifolds was the basic prerequisite for this seminar, hence also for these notes. Some familiarity with symplectic geometry is useful to get through faster, though most of the needed definitions and results are stated here.

The study of toric manifolds has many different entrances and has been scoring a wide spectrum of applications. For symplectic geometers, they provide examples of extremely symmetric and completely integrable hamiltonian spaces. In order to distinguish the algebraic from the symplectic approach, we say *symplectic toric manifolds* when focusing on the symplectic and smooth properties.

Native to algebraic geometry, the theory of toric varieties has been around for about thirty years. It was introduced by Demazure in [21] who used toric varieties for classifying some algebraic subgroups. Since 1970 many nice surveys of the theory of toric varieties have appeared (see, for instance, [18, 26, 37, 56]). For the last thirty years, toric geometry became an important tool in physics in connection with mirror symmetry [17] where research has been intensive. As noted by Fulton [26], *toric varieties have provided a remarkably fertile testing ground for general theories*.

In this text, we emphasize the geometry of the *moment map* whose image, the so-called *moment polytope*, determines the symplectic toric manifold by the celebrated classification theorem of Delzant [20]. The notion of a moment map associated to a group action generalizes that of a hamiltonian function associated to a vector field. Either of these notions formalizes the Noether principle, which states that to every symmetry (such as a group action) in a mechanical system, there corresponds a conserved quantity. The concept of a moment map was introduced by Souriau [60] under the French name *application moment* (besides the more widespread English translation to *moment map*, the alternative *momentum map* is also used). Moment maps have been asserting themselves as a main tool to study problems in geometry and topology when there is a suitable symmetry, as illustrated in the book by Gelfand, Kapranov and Zelevinsky [27].

Toric geometry enjoys close connections to a number of different fields, such as algebraic geometry, symplectic geometry, combinatorics, string theory, commutative algebra, integrable systems, algebraic topology, complex geometry, etc.

In Chapter 1, we introduce the basic objects in symplectic and hamiltonian geometry which lead to symplectic toric manifolds.

In Chapter 2, we state and prove Delzant's classification of equivalence classes of symplectic toric manifolds by their *moment polytopes* up to translation, moreover we discuss first examples.

In Chapter 3, we explore *how to understand a toric manifold from its polytope* and look at developments in symplectic geometry following Delzant's theorem.

For their contributions, comments, corrections and interesting questions – some of which have already been incorporated in these notes – I am thankful to the participants of the 2019 seminar, namely: Giovanni Ambrosioni, Yannis Bähni, Joël Beimler, Valentin Bosshard, Gilles Englebert, Alessandro Fasse, Simon Grüning, Amanda Jenny, Shengxuan Liu, Yefei Ma, Angela Maennel, Benjamin Pollitt, Marcella Storino, and Johannes Weidenfeller.

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Chapter 1

Symplectic Preliminaries

1.1 Symplectic Manifolds

Definition 1.1.1. *A* **symplectic form**[1] *on a manifold* M *is a closed 2-form on* M *which is nondegenerate at every point of* M*. A* **symplectic manifold** *is a pair* (M, ω) *where* M *is a manifold and* ω *is a symplectic form on* M*.*

A 2-form ω gives at each point $p \in M$ a skew-symmetric bilinear pairing of tangent vectors at that point,

$$\omega_p : T_pM \times T_pM \to \mathbb{R} \ .$$

Nondegeneracy means that, for any nonzero tangent vector $u \in T_pM$, there is $v \in T_pM$ such that $\omega_p(u, v) \neq 0$. By a skew-symmetric version of the Gram-Schmidt process (see Theorem 1.3.2), we can then conclude that T_pM must be even-dimensional. It follows that a symplectic manifold is necessarily *even-dimensional*. When the manifold M has dimension $2n$, the nondegeneracy of a 2-form ω amounts to the top wedge power, ω^n, being nonzero, i.e., a volume form. It follows that a symplectic manifold is *oriented* by its symplectic form, a volume form being ω^n. For more details on these assertions, see for instance [15, Chapter 1].

[1]If you consult a major English dictionary, you are likely to find that *symplectic* is the name for a bone in a fish's head. However, as clarified in [62], the word *symplectic* in mathematics was coined by Weyl [64, p.165] who substituted the Latin root in *complex* by the corresponding Greek root, in order to label the symplectic group. (In linguistics, a word created this way is called a *calque*.) Weyl thus avoided that this group connote the complex numbers, and also spared us from much confusion that would have arisen, had the name remained the former one in honor of Abel: *abelian linear group*.

Examples

1. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The **standard symplectic form on** $\mathbb{R}^{2n}$ is

$$\omega_0 := \sum_{k=1}^{n} dx_k \wedge dy_k \ .$$

2. Let $M = \mathbb{C}^n$ with linear coordinates $z_1, \ldots, z_n$. The form

$$\omega_0 := \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$$

is a symplectic form on $\mathbb{C}^n$. In fact, this form equals that of the previous example under the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $z_k = x_k + iy_k$.

3. Let Q be any n-dimensional manifold and $M = T^*Q$ its cotangent bundle. If the manifold structure on Q is described by coordinate charts $(\mathcal{U}, q_1, \ldots, q_n)$ with $q_k : \mathcal{U} \to \mathbb{R}$, then, at any $q \in \mathcal{U}$, the differentials $(dq_1)_q, \ldots (dq_n)_q$ form a basis of T_q^*Q. Namely, if $p \in T_q^*Q$, then $p = \sum_{k=1}^{n} p_k (dq_k)_q$ for some real coefficients $p_1, \ldots, p_n$. This induces a map

$$\begin{array}{ccc} T^*\mathcal{U} & \longrightarrow & \mathbb{R}^{2n} \\ (q,p) & \longmapsto & (q_1, \ldots, q_n, p_1, \ldots, p_n) \end{array}$$

and $(T^*\mathcal{U}, q_1, \ldots, q_n, p_1, \ldots, p_n)$ is a coordinate chart for T^*Q; the coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ are the **cotangent coordinates** associated to the coordinates $q_1, \ldots, q_n$ on $\mathcal{U}$. The **canonical symplectic form** on T^*Q is the 2-form given on the coordinate chart $T^*\mathcal{U}$ by

$$\omega := \sum_{k=1}^{n} dq_k \wedge dp_k \ .$$

This form is well-defined globally, because it satisfies $\omega = -d\alpha$ in terms of the so-called **tautological (or Liouville) 1-form** α, intrinsically defined at the point $(q,p) \in T^*Q$ by

$$\alpha_{(q,p)} := p \circ d\pi_{(q,p)} \ ,$$

where $\pi : T^*Q \to Q$ denotes the cotangent bundle projection. Indeed, on the coordinate chart $T^*\mathcal{U}$, the tautological form satisfies $\alpha = \sum_{k=1}^{n} p_k dq_k$.

4. Let $M = S^2$ regarded as the set of unit vectors in $\mathbb{R}^3$. Tangent vectors to S^2 at p may then be identified with vectors orthogonal to p. The **euclidean symplectic form on** S^2 is the form induced by the inner and exterior products:

$$\omega_p(u,v) := \langle p, u \times v \rangle \ , \qquad \text{for } u, v \in T_pS^2 = \{p\}^\perp \ .$$

This form is closed because it is of top degree; it is nondegenerate because $\langle p, u \times v \rangle \neq 0$ when $u \neq 0$ and we take, for instance, $v = u \times p$.

$\diamondsuit$

Exercise 1.1.2. *Check that, in cylindrical coordinates away from the poles ($0 \leqslant \theta < 2\pi$ and $-1 < h < 1$), the euclidean symplectic form on S^2 is the area form given by*

$$\omega_{\text{eucl}} = d\theta \wedge dh \ .$$

This confirms that the total area is 4π.

The natural notion of equivalence in the symplectic category is expressed by a *symplectomorphism*:

Definition 1.1.3. *Let (M_1,ω_1) and (M_2,ω_2) be $2n$-dimensional symplectic manifolds, and let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then φ is a* **symplectomorphism** *if $\varphi^*\omega_2 = \omega_1$. The set of all symplectomorphisms from a symplectic manifold (M,ω) to itself equipped with composition is called the* **group of symplectomorphisms** *of (M,ω) and denoted $\mathrm{Sympl}(M,\omega)$.*

We would like to classify symplectic manifolds up to symplectomorphism. The Darboux theorem (Section 1.3) takes care of this classification locally: the dimension is the only local invariant of symplectic manifolds up to symplectomorphisms. Just as any n-dimensional manifold looks locally like $\mathbb{R}^n$, any $2n$-dimensional *symplectic* manifold looks locally like $(\mathbb{R}^{2n},\omega_0)$. More precisely, any symplectic manifold (M^{2n},ω) is locally symplectomorphic to $(\mathbb{R}^{2n},\omega_0)$. In other words, the prototype of a local piece of a $2n$-dimensional symplectic manifold is $(\mathbb{R}^{2n},\omega_0)$.

1.2 Hamiltonian Flows

A key feature of symplectic forms is that they provide the mechanism to associate to any smooth real function on the underlying manifold $H : M \to \mathbb{R}$ a nontrivial (eventually local) flow that preserves both the symplectic form and the given function. This is the *hamiltonian flow* associated to a *(hamiltonian) function*.

Let (M,ω) be a symplectic manifold.

Definition 1.2.1. *A vector field X on M is a* **symplectic vector field** *if the contraction $\imath_X\omega$ is closed. A vector field X on M is a* **hamiltonian vector field** *if the contraction $\imath_X\omega$ is exact.*

By Poincaré's Lemma, locally on every contractible open set, every symplectic vector field is hamiltonian. If the first de Rham cohomology group is trivial, then globally every symplectic vector field is hamiltonian; in general, $H^1_{\mathrm{deRham}}(M)$ measures the obstruction for symplectic vector fields to be hamiltonian.

The flow of a symplectic vector field X preserves the symplectic form:

$$\mathcal{L}_X\omega = d\underbrace{\imath_X\omega}_{\text{closed}} + \imath_X\underbrace{d\omega}_{0} = 0 \ .$$

If a vector field X is hamiltonian with[2] $\imath_X\omega = -dH$ for some smooth function $H : M \to \mathbb{R}$, then the flow of X also preserves the function H:

$$\mathcal{L}_X H = \imath_X dH = -\imath_X\,\imath_X\omega = 0 \ .$$

[2]The sign here is included to be consistent with Definition 1.2.2.

Therefore, each integral curve $\{\rho_t(x) \mid t \in \mathbb{R}\}$ of X must be contained in a level set of H:

$$H(x) = (\rho_t^* H)(x) = H(\rho_t(x)) \ , \quad \forall t \ .$$

Definition 1.2.2. *A* **hamiltonian function** *for a hamiltonian vector field X on M is a smooth function $H : M \to \mathbb{R}$ such that $\imath_X \omega = -dH$.*

Note the above sign convention, chosen to produce more *positive* pictures later on in this text.

By nondegeneracy of ω, any function $H \in C^\infty(M)$ is a hamiltonian function for some hamiltonian vector field because the equation $\imath_X \omega = -dH$ can be always solved for a smooth vector field X. A hamiltonian vector field X defines a hamiltonian function *up to* a locally constant function.

Examples

1. On the symplectic manifold $(\mathbb{C}^n, \omega_0)$, we translate from linear coordinates $z_1, \ldots, z_n$ to polar coordinates r_k, θ_k on each factor-plane, so that

$$\omega_0 = \frac{i}{2} \sum_{k=1}^n dz_k \wedge d\bar{z}_k = \sum_{k=1}^n r_k dr_k \wedge d\theta_k \ .$$

   Then it is easier to see that the vector field $X = \sum_{k=1}^n \frac{\partial}{\partial \theta_k}$ corresponding to a diagonal rotation is hamiltonian with hamiltonian function given by half of the square of the radius, $H := \frac{1}{2}(|z_1|^2 + \ldots + |z_n|^2)$:

$$\imath_X \omega_0 = -d \underbrace{\left(\tfrac{1}{2}(|z_1|^2 + \ldots + |z_n|^2)\right)}_{H} \ .$$

   Indeed the diagonal rotation preserves the spheres centered at the origin, as well as the area on each factor-plane.

2. On the euclidean symplectic 2-sphere $(S^2, d\theta \wedge dh)$, the vector field $X = \frac{\partial}{\partial \theta}$ is hamiltonian with hamiltonian function $H = -h$ given by the negative of the height function:

$$\imath_X (d\theta \wedge dh) = dh = -d(-h) \ .$$

   The motion generated by this vector field is rotation about the vertical axis, which of course preserves both area and height.

3. On the symplectic 2-torus $(\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$, the vector fields $X_1 = \frac{\partial}{\partial \theta_1}$ and $X_2 = \frac{\partial}{\partial \theta_2}$ are symplectic but not hamiltonian.

♢

Case of exact symplectic manifolds

Definition 1.2.3. *An* **exact symplectic manifold** *is a symplectic manifold* (M, ω) *where the symplectic form* ω *is exact.*

Exercise 1.2.4. *Check that, if* (M, ω) *is an exact symplectic manifold, then the manifold* M *has no compact connected component.*

> **Hint:** *Stokes' Theorem.*

Any cotangent bundle with its canonical symplectic form is an exact symplectic manifold. The standard symplectic forms on $\mathbb{R}^{2n}$ or on $\mathbb{C}^n$ are also exact. Of course, any symplectic manifold is locally exact.

Claim. Let (M, ω) be an exact symplectic manifold with $\omega = -d\alpha$. If the flow of the vector field X preserves α, then X is a hamiltonian vector field.

Proof. Since the flow of X preserves α, we have for the Lie derivative that

$$\mathcal{L}_X \alpha = d\imath_X \alpha + \imath_X d\alpha = 0 \ .$$

Therefore, we have

$$\imath_X \omega = -\imath_X d\alpha = d\imath_X \alpha \ ,$$

hence X is hamiltonian with hamiltonian function $H := -\imath_X \alpha$. □

1.3 Darboux's Theorem and Moser's Argument

Let (M, ω) be a symplectic manifold of dimension $2n$.

Definition 1.3.1. *A* **Darboux chart** *for* M *is a chart* $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ *such that*

$$\omega|_{\mathcal{U}} = \sum_{k=1}^{n} dx_k \wedge dy_k \ .$$

By the Darboux theorem (Theorem 1.3.4), there exists a Darboux chart centered at each point of a symplectic manifold. The modern proof of the Darboux theorem was first noted by Moser [55] and can be broken into the following two key facts, one from linear algebra and the other based on *Moser's argument*.

Theorem 1.3.2. (Standard Form for Skew-symmetric Bilinear Maps)
Let V *be an* m*-dimensional vector space over* $\mathbb{R}$*, and let* $\Omega : V \times V \to \mathbb{R}$ *be a bilinear map. Assume that the map* Ω *is* skew-symmetric*, i.e.,* $\Omega(u, v) = -\Omega(v, u)$*, for all* $u, v \in V$*.*

Then there is a basis $u_1, \ldots, u_\ell, e_1, \ldots, e_n, f_1, \ldots, f_n$ *of* V *such that*

$$\begin{array}{ll} \Omega(u_j, v) = 0 \ , & \text{for all } i \text{ and all } v \in V, \\ \Omega(e_j, e_k) = 0 = \Omega(f_j, f_k) \ , & \text{for all } i, j, \text{ and} \\ \Omega(e_j, f_k) = \delta_{ij} \ , & \text{for all } i, j. \end{array}$$

Proof. This induction proof is a skew-symmetric version of the Gram-Schmidt process.

Let $U := \{u \in V \mid \Omega(u,v) = 0 \text{ for all } v \in V\}$. Choose a basis $u_1, \dots, u_k$ of U, and choose a complementary space W to U in V,

$$V = U \oplus W \ .$$

Take any nonzero $e_1 \in W$. Then there is $f_1 \in W$ such that $\Omega(e_1, f_1) \neq 0$. Assume that $\Omega(e_1, f_1) = 1$. Let

$$\begin{array}{rcl} W_1 & := & \text{span of } e_1, f_1 \\ W_1^\Omega & := & \{w \in W \mid \Omega(w,v) = 0 \text{ for all } v \in W_1\} \ . \end{array}$$

Claim. $W_1 \cap W_1^\Omega = \{0\}$.

Suppose that $v = ae_1 + bf_1 \in W_1 \cap W_1^\Omega$.

$$\left.\begin{array}{l} 0 = \Omega(v, e_1) = -b \\ 0 = \Omega(v, f_1) = \quad a \end{array}\right\} \quad \Longrightarrow \quad v = 0 \ .$$

Claim. $W = W_1 \oplus W_1^\Omega$.

Suppose that $v \in W$ has $\Omega(v, e_1) = c$ and $\Omega(v, f_1) = d$. Then

$$v = \underbrace{(-cf_1 + de_1)}_{\in W_1} + \underbrace{(v + cf_1 - de_1)}_{\in W_1^\Omega} \ .$$

Go on: let $e_2 \in W_1^\Omega$, $e_2 \neq 0$. There is $f_2 \in W_1^\Omega$ such that $\Omega(e_2, f_2) \neq 0$. Assume that $\Omega(e_2, f_2) = 1$. Let $W_2 =$ span of e_2, f_2. Etc.

This process eventually stops because $\dim V < \infty$. We hence obtain

$$V = U \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

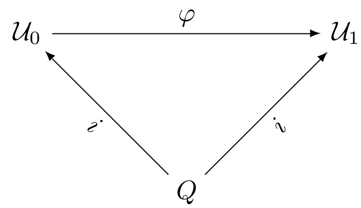
where all summands are orthogonal with respect to Ω, and where W_j has basis e_j, f_j with $\Omega(e_j, f_j) = 1$. □

The dimension of the subspace $U = \{u \in V \mid \Omega(u,v) = 0, \text{ for all } v \in V\}$ does not depend on the choice of basis. That is thus an invariant of (V, Ω), $k := \dim U$. Since $k + 2n = m = \dim V$, the number n is also an invariant of (V, Ω) and this is called the **rank** of Ω.

Normal forms

Theorem 1.3.3. (Darboux-Weinstein Theorem [63]) *Let M be a manifold, Q a submanifold of M, $i : Q \hookrightarrow M$ the inclusion map, ω_0 and ω_1 symplectic forms in M. Suppose that $\omega_0|_p = \omega_1|_p$, $\forall p \in Q$.*

Then there exist neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of Q in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that



commutes and $\varphi^*\omega_1 = \omega_0$.

Proof.

1. Pick a tubular neighborhood $\mathcal{U}_0$ of Q. The 2-form $\omega_1 - \omega_0$ is closed on $\mathcal{U}_0$, and $(\omega_1 - \omega_0)_p = 0$ at all $p \in Q$. By the homotopy formula on the tubular neighborhood, there exists a 1-form μ on $\mathcal{U}_0$ such that $\omega_1 - \omega_0 = d\mu$ and $\mu_p = 0$ at all $p \in Q$.

2. Consider the family $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + td\mu$ of closed 2-forms on $\mathcal{U}_0$. Shrinking $\mathcal{U}_0$ if necessary, we can assume that ω_t is symplectic for $0 \leqslant t \leqslant 1$.

3. Solve the **Moser equation**: $\imath_{v_t}\omega_t = -\mu$. Notice that $v_t = 0$ on Q.

4. Integrate v_t. Shrinking $\mathcal{U}_0$ again if necessary, there exists an isotopy $\rho : \mathcal{U}_0 \times [0,1] \to M$ with $\rho_t^*\omega_t = \omega_0$, for all $t \in [0,1]$. Since $v_t|_Q = 0$, we have $\rho_t|_Q = \mathrm{id}_Q$.

Set $\varphi = \rho_1$, $\mathcal{U}_1 = \rho_1(\mathcal{U}_0)$. □

Theorem 1.3.4. (Darboux's Theorem) *Let (M, ω) be a $2n$-dimensional symplectic manifold, and let p be any point in M.*
Then there is a coordinate chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that on $\mathcal{U}$

$$\omega = \sum_{k=1}^{n} dx_k \wedge dy_k \ .$$

Proof. Apply the Darboux-Weinstein theorem (Theorem 1.3.3) to $Q = \{p\}$:

Use any symplectic basis for T_pM to construct coordinates $(x_1', \ldots, x_n', y_1', \ldots y_n')$ centered at p and valid on some neighborhood $\mathcal{U}'$, so that

$$\omega_p = \sum dx_j' \wedge dy_j' \Big|_p \ .$$

There are two symplectic forms on $\mathcal{U}'$: the given $\omega_0 = \omega$ and $\omega_1 = \sum dx_j' \wedge dy_j'$. By the Moser theorem, there are neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of p, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that

$$\varphi(p) = p \quad \text{and} \quad \varphi^*(\sum dx_j' \wedge dy_j') = \omega \ .$$

Since $\varphi^*(\sum dx'_j \wedge dy'_j) = \sum d(x'_j \circ \varphi) \wedge d(y'_j \circ \varphi)$, we only need to set new coordinates $x_j = x'_j \circ \varphi$ and $y_j = y'_j \circ \varphi$. □

As a consequence of Theorem 1.3.4, if we prove for $(\mathbb{R}^{2n}, \sum dx_j \wedge dy_j)$ a local assertion which is invariant under symplectomorphisms, then that assertion holds for any symplectic manifold.

1.4 Moment Maps

We start by recalling notions from Lie group actions.

Definition 1.4.1. *An* **action** *of a Lie group G on a manifold M is a group homomorphism*

$$\begin{array}{rccc} \psi : & G & \longrightarrow & \mathrm{Diff}(M) \\ & g & \longmapsto & \psi_g \ , \end{array}$$

where $\mathrm{Diff}(M)$ is the group of diffeomorphisms of M. The **evaluation map** *associated with an action $\psi : G \to \mathrm{Diff}(M)$ is*

$$\begin{array}{rccc} \mathrm{ev}_\psi : & M \times G & \longrightarrow & M \\ & (p, g) & \longmapsto & \psi_g(p) \ . \end{array}$$

The action ψ is **smooth** *if ev_ψ is a smooth map.*

We will always assume that an action is smooth.

Example. Complete vector fields[3] on a manifold M are in one-to-one correspondence with actions of $\mathbb{R}$ on M. In such a case, the diffeomorphism $\psi_t : M \to M$ associated to $t \in \mathbb{R}$ is the time-t map defined by the flow of the vector field X, i.e., for each point $p \in M$, the map $t \mapsto \psi_t(p)$ is the integral curve of X starting at p:

$$\begin{cases} \frac{d}{dt}\psi_t(p)\big|_{t=t_0} = X_{\psi_{t_0}(p)} \\ \psi_0(p) = p \ . \end{cases}$$

◊

Let (M, ω) be a symplectic manifold, and G a Lie group with an action $\psi : G \to \mathrm{Diff}(M)$.

Definition 1.4.2. *The action ψ is a* **symplectic action** *if it is by symplectomorphisms, i.e.,*

$$\psi : G \longrightarrow \mathrm{Sympl}(M, \omega) \subset \mathrm{Diff}(M) \ .$$

[3] A vector field is **complete** if its integral curves through each point exist for *all* time.

Examples

1. On the symplectic 2-sphere $(S^2, d\theta \wedge dh)$ in cylindrical coordinates, the one-parameter group of diffeomorphisms given by rotation around the vertical axis, $\psi_t(\theta, h) = (\theta + t, h)$ $(t \in \mathbb{R})$ is a symplectic action of the group $S^1 \simeq \mathbb{R}/\langle 2\pi \rangle$, as it preserves the area form $d\theta \wedge dh$.

2. On the symplectic 2-torus $(\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$, the one-parameter groups of diffeomorphisms given by rotation around each circle, $\psi_{1,t}(\theta_1, \theta_2) = (\theta_1 + t, \theta_2)$ $(t \in \mathbb{R})$ and $\psi_{2,t}$ similarly defined, are symplectic actions of S^1.

$\diamondsuit$

Let (M, ω) be a symplectic manifold, G a Lie group with an action $\psi : G \to \mathrm{Diff}(M)$, and $\mathfrak{g}$ the Lie algebra of G with dual vector space $\mathfrak{g}^*$.

Definition 1.4.3. *The action ψ is a* **hamiltonian action** *if there exists a map*

$$\mu : M \longrightarrow \mathfrak{g}^*$$

satisfying the following two conditions:

- *For each $X \in \mathfrak{g}$, let $\mu^X : M \to \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X \rangle$, be the component of μ along X, and let $X^\#$ be the vector field on M generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\} \subseteq G$. Then*

  $$d\mu^X = -\imath_{X^\#}\omega$$

  i.e., the function μ^X is a hamiltonian function for the vector field $X^\#$.

- *The map μ is* equivariant *with respect to the given action ψ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$:*

  $$\mu \circ \psi_g = \mathrm{Ad}^*_g \circ \mu\ , \qquad \text{for all } g \in G\ .$$

Then (M, ω, G, μ) is called a **hamiltonian *G*-space** *and μ is called a* **moment map**. *When G is a torus, we will call (M, ω, G, μ) a* **hamiltonian torus space**.

Exercise 1.4.4. *Check that complete symplectic vector fields on M are in one-to-one correspondence with symplectic actions of $\mathbb{R}$ on M, and that, similarly, complete hamiltonian vector fields on M are in one-to-one correspondence with hamiltonian actions of $\mathbb{R}$ on M.*

Examples

1. The previous example of rotation on S^2 illustrates a hamiltonian action of S^1 with moment map given by the negative of the height function, under a suitable identification of the dual of the Lie algebra of S^1 with $\mathbb{R}$.

2. The previous example of rotation on $\mathbb{T}^2$ is not hamiltonian since the one-forms $d\theta_1$ and $d\theta_2$ are not exact.

$\diamondsuit$

Heredity of moment maps

Exercise 1.4.5. *Let G be a Lie group and H a closed subgroup of G, hence what we call a* **Lie subgroup** *of G. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the respective Lie algebras. The projection $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is the map dual to the inclusion $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$. Suppose that (M, ω, G, μ) is a hamiltonian G-space. Show that the restriction of the G-action to H is hamiltonian with moment map*

$$i^* \circ \mu : M \longrightarrow \mathfrak{h}^* \ .$$

Exercise 1.4.6. *Let (M, ω, G, μ) be a hamiltonian G-space and let N be a G-invariant symplectic submanifold of M. Let $\iota : N \to M$ be the inclusion map. Suppose that the restriction of ω to N is symplectic, hence $(N, \iota^*\omega)$ is what we call a* **symplectic submanifold** *of (M, ω). Show that the restriction of μ to N is a moment map making $(N, \iota^*\omega, G, \iota^*\mu)$ into a hamiltonian G-space.*

Moment maps for product actions

Exercise 1.4.7. *Suppose that a Lie group G acts in a hamiltonian way on two symplectic manifolds (M_j, ω_j), $j = 1, 2$, with moment maps $\mu_j : M_j \to \mathfrak{g}^*$. The product manifold $M_1 \times M_2$ has a natural* **product symplectic structure** *given by the sum of the pull-backs of the symplectic forms on each factor, via the two projections. Prove that the diagonal action of G on $M_1 \times M_2$ is hamiltonian with moment map $\mu : M_1 \times M_2 \to \mathfrak{g}^*$ given by*

$$\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2) \ , \quad \text{for } p_j \in M_j \ .$$

Exercise 1.4.8. *Let G_1 and G_2 be Lie groups that act on the same symplectic manifold (M, ω) in a hamiltonian way, with moment maps $\mu_1 : M \to \mathfrak{g}_1^*$ and $\mu_2 : M \to \mathfrak{g}_2^*$, respectively. Assume that these actions, denoted ψ_1 and ψ_2 respectively, commute and that each moment map is invariant with respect to the other action (i.e. $\mu_1 \circ \psi_2 = \mu_1$ and $\mu_2 \circ \psi_1 = \mu_2$). Prove that the action ψ of the* **product group** *$G := G_1 \times G_2$ on M defined by $\psi_{(g_1, g_2)}(p) := (\psi_1)_{g_1}((\psi_2)_{g_2}(p))$ is well-defined and hamiltonian with moment map*

$$\mu : M \to (\mathfrak{g}_1 \oplus \mathfrak{g}_2)^* \simeq (\mathfrak{g}_1)^* \oplus (\mathfrak{g}_2)^* \ , \qquad \mu(p) = (\mu_1(p), \mu_2(p)) \ .$$

The differential of a moment map

We analyse some crucial properties of the differential of a moment map needed in the next section as well as in Section 2.3.

Let (M, ω, G, μ) be a hamiltonian G-space. We denote by $\mathcal{O}$ the G-orbit through a point $p \in M$, by G_p the stabilizer[4] of p, and by $\mathfrak{g}_p$ the Lie algebra of G_p.

[4] The **stabilizer** (group) or **isotropy** of a point p is $G_p := \{g \in G \mid g \cdot p = p\}$.

Lemma 1.4.9. *For the differential of the moment map $\mu : M \to \mathfrak{g}^*$ at p,*

$$d\mu_p : T_pM \longrightarrow \mathfrak{g}^* \ ,$$

where we identify a tangent space to the vector space $\mathfrak{g}^*$ with itself, we have that:

$$\text{(I)}\ \ker d\mu_p = (T_p\mathcal{O})^\omega \qquad \text{and} \qquad \text{(II)}\ \operatorname{im} d\mu_p = (\mathfrak{g}_p)^0 \ ,$$

where $(T_p\mathcal{O})^\omega$ is the symplectic orthocomplement[5] *of $T_p\mathcal{O}$ in the symplectic vector space (T_pM, ω_p), and $(\mathfrak{g}_p)^0$ is the annihilator*[6] *of $\mathfrak{g}_p$.*

The proof of this lemma is contained in the next exercise.

Exercise 1.4.10. *Recall that, by definition of moment map (Definition 1.4.3), we have that*

$$\langle d\mu_p(v), X\rangle = \omega_p(v, X_p^\#) \qquad \text{for all } X \in \mathfrak{g} \text{ and } v \in T_pM.$$

1. *Prove claim (I) in Lemma 1.4.9 by checking that*

$$d\mu_p(v) = 0 \quad \iff \quad \omega_p(v, X_p^\#) = 0 \ ,\ \forall X \in \mathfrak{g} \ .$$

   Note that the tangent space to the G-orbit through p is spanned by all the vectors $X_p^\#$.

2. *By counting dimensions, check that*

$$\begin{aligned} \dim(\ker d\mu_p) &= \dim M - \dim G + \dim G_p \\ \dim(\operatorname{im} d\mu_p) &= \dim G - \dim G_p \ . \end{aligned}$$

3. *Using the dimension count above, for checking claim (II) it is enough to show that*

$$\langle d\mu_p(v), X\rangle = 0 \qquad \forall X \in \mathfrak{g}_p, \forall v \in T_pM \ .$$

4. *Conclude from (II) that the stabilizer group of p is discrete if and only if $d\mu_p$ is surjective.*

5. *Conclude from (I) that the orbit through p is open if and only if $d\mu_p$ is injective.*

---

[5]If W is a subspace of a symplectic vector space (V, Ω), then the **symplectic orthocomplement** of W is the subspace $W^\Omega := \{v \in V \mid \Omega(v, w) = 0 \ \forall w \in W\}$.

[6]The **annihilator** of a linear subspace $W \subset V$ is the subset of V^* defined by $W^0 := \{\xi \in V^* \mid \xi(w) = 0 \ \forall w \in W\}$.

1.5 Hamiltonian Torus Actions

We now concentrate on actions of a **standard torus** or rank $n \geqslant 1$ defined to be the product of n copies of S^1:

$$\mathbb{T}^n := (S^1)^n \ .$$

We write elements of $\mathbb{T}^n$ as n-tuples of complex numbers of absolute value 1,

$$\left(e^{i\theta_1}, \dots, e^{i\theta_n}\right) \ .$$

This now identifies $\mathbb{T}^n$ with the quotient

$$\mathbb{T}^n \simeq \mathbb{R}^n/(2\pi\mathbb{Z})^n \simeq (\mathbb{R}/2\pi\mathbb{Z})^n$$

and we view this as the standard identification of a torus Lie group T with its Lie algebra[7] $\mathfrak{t}$ modulo the *integral lattice* Γ via the *exponential map*:

$$\exp_T : \mathfrak{t} \longrightarrow T \quad \text{has kernel } \Gamma \qquad \Longrightarrow \qquad T \simeq \mathfrak{t}/\Gamma \ ,$$

where here

$$\exp : \mathbb{R}^n \longrightarrow \mathbb{T}^n \ \text{ has kernel } (2\pi\mathbb{Z})^n \ , \quad \exp(\theta_1, \dots, \theta_n) = \left(e^{i\theta_1}, \dots, e^{i\theta_n}\right) \ .$$

Implicitly, we use the standard basis of $\mathbb{R}^n$ as the chosen basis $X_1, \dots, X_n$ of the Lie algebra. This also yields global coordinates (mod 2π) θ_k on $\mathbb{T}^n$. The element

$$[\theta] := [\theta_1, \dots, \theta_n] = \left(e^{i\theta_1}, \dots, e^{i\theta_n}\right) \in \mathbb{T}^n$$

can also be viewed as the element achieved from the *identity* element

$$\mathbb{1} = [0, \dots, 0] = (1, \dots, 1) \in \mathbb{T}^n$$

by flowing along X_1 for time θ_1, along X_2 for time θ_2, ..., and along X_n for time θ_n.

Because the adjoint and coadjoint actions are trivial for a torus $\mathbb{T}^n$ and we are already identifying the Lie algebra with $\mathbb{R}^n$, the dual of the Lie algebra gets also naturally identified with $\mathbb{R}^n$ via the standard pairing (standard inner product). A moment map for an action of $\mathbb{T}^n$ on (M, ω) is simply a map

$$\mu : M \longrightarrow \mathbb{R}^n \ ,$$

whose coordinate functions $\mu_1, \dots \mu_n$ all satisfy:

- μ_k is $\mathbb{T}^n$-invariant, i.e.:

$$\mu_k([\theta] \cdot p) = \mu_k(p) \text{ for all } [\theta] \in \mathbb{T}^n, \ p \in M, \ k = 1, \dots, n, \quad \text{and}$$

[7]Since T is abelian, the Lie algebra $\mathfrak{t}$ of a T is defined as the set of (say left-)invariant vector fields on T (equivalently, as the tangent space at the identity) and the Lie bracket is trivial in this case.

- μ_k is a *hamiltonian function* for the vector field $X_k^\sharp$ on M induced by the k-th standard basis vector of $\mathbb{R}^n$, i.e.:

$$d\mu_k = -\imath_{X_k^\sharp}\omega\ , \quad k = 1, \dots, n\ .$$

If $\mu : M \to \mathbb{R}^n$ is a moment map for a torus action, then clearly any of its translations $\mu + c$ ($c \in \mathbb{R}^n$) is also a moment map for that action. Reciprocally, any two moment maps for a given hamiltonian torus action differ by a constant.

Example. On $(\mathbb{C}, \omega_0 = \frac{i}{2} dz \wedge d\bar{z})$, consider the action of the circle $S^1 = \{t \in \mathbb{C} : |t| = 1\}$ by rotations

$$\psi_t(z) = t^\ell z\ , \qquad t \in S^1\ ,$$

where $\ell \in \mathbb{Z}$ is fixed. The action $\psi : S^1 \to \mathrm{Diff}(\mathbb{C})$ is hamiltonian with moment map (or hamiltonian function) $\mu : \mathbb{C} \to \mathbb{R}$ given by

$$\mu(z) = \tfrac{1}{2}\ell|z|^2\ .$$

This can be easily checked in polar coordinates, since $\omega_0 = r\, dr \wedge d\theta$, $\mu(re^{i\theta}) = \frac{1}{2}\ell r^2$ and the vector field on $\mathbb{C}$ corresponding to the generator 1 of the Lie algebra $\mathbb{R}$ is $X^\# = \ell\frac{\partial}{\partial\theta}$. $\diamondsuit$

Exercise 1.5.1. *Let $\mathbb{T}^n = \{(t_1, \dots, t_n) \in \mathbb{C}^n : |t_j| = 1,$ for all $j\}$ be a torus acting diagonally on $\mathbb{C}^n$ by*

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) = (t_1^{\ell_1} z_1, \dots, t_n^{\ell_n} z_n)\ ,$$

where $\ell_1, \dots, \ell_n \in \mathbb{Z}$ are fixed. Check that this action is hamiltonian with a moment map $\mu : \mathbb{C}^n \to \mathbb{R}^n$ given by

$$\mu(z_1, \dots, z_n) = \tfrac{1}{2}(\ell_1|z_1|^2, \dots, \ell_n|z_n|^2)\ (\ +\ \text{constant}\ )\ .$$

Exercise 1.5.2. *Suppose that $\mathbb{T}^m$ acts linearly on $(\mathbb{C}^n, \omega_0)$ as follows:*

$$(e^{i\theta_1}, \dots, e^{i\theta_m}) \cdot (z_1, \dots, z_n) = \left(e^{i\langle\lambda^{(1)},\theta\rangle} z_1\ , \dots, e^{i\langle\lambda^{(n)},\theta\rangle} z_n\right)\ ,$$

for some weights $\lambda^{(1)}, \dots, \lambda^{(n)} \in \mathbb{Z}^m$.

Show that, this action is hamiltonian with a moment map $\mu : \mathbb{C}^n \to \mathbb{R}^m$ given by

$$\mu(z_1, \dots, z_n) = \tfrac{1}{2}\sum_{j=1}^{n} \lambda^{(j)}|z_j|^2\ (\ +\ \text{constant}\ )\ .$$

Convexity

It is a remarkable feature of compact connected hamiltonian torus spaces that the image of a moment map is a convex polytope. This was discovered and proved independently at about the same time by Atiyah and by Guillemin and Sternberg, following work of Kostant [40] for the case of coadjoint orbits.

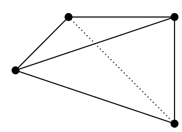
Theorem 1.5.3. (Atiyah [6], Guillemin-Sternberg [30]) *Let (M, ω) be a compact connected symplectic manifold with a hamiltonian action of an m-torus, $\mathbb{T}^m$, and with moment map $\mu : M \to \mathbb{R}^m$. Then:*

(a) the levels of μ are connected;

(b) the image of μ is convex;

(c) the image of μ is the convex hull of a finite number of points, that are images of the fixed points of the action.

The image $\mu(M)$ of the moment map is called the **moment polytope**. A proof of Theorem 1.5.3 following Atiyah can be found in [49].



Remark. Although for the standard torus $\mathbb{T}^n$ both the Lie algebra and its dual are naturally identified with $\mathbb{R}^n$, we will distinguish $\mathbb{R}^n$ from $(\mathbb{R}^n)^*$ and write for the natural pairing $\langle \cdot, \cdot \rangle : (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$. In particular, a moment map will be denoted $\mu : M \to (\mathbb{R}^n)^*$. ♢

1.6 Symplectic Toric Manifolds

Effective hamiltonian tori actions

An action of a group G on a manifold M is called **effective** (or *faithful*) if it is injective as a map $G \to \mathrm{Diff}(M)$, i.e., each group element $g \neq \mathbb{1}$ moves at least one point, that is, $\cap_{p \in M} G_p = \{\mathbb{1}\}$.

The following two results use the crucial fact that any effective action $\mathbb{T}^m \to \mathrm{Diff}(M)$ has at least one orbit of dimension m; a proof may be found in [12, Ch.IV,§5]. In fact, more is true for an effective torus action on a connected M: the set of points where the action has trivial stabilier is open and dense; a proof may be found in [28, Corollary B.48].

Corollary 1.6.1. *Under the conditions of the convexity theorem (Theorem 1.5.3), if the $\mathbb{T}^m$-action is effective, then there must be at least $m + 1$ fixed points.*

Proof. By Exercise 1.4.10, at any point p of an m-dimensional orbit, the stabilizer is discrete, so $d\mu_p$ is surjective. This means that a the moment map is a submersion, i.e., $(d\mu_1)_p, \ldots, (d\mu_m)_p$ are linearly independent. Hence, $\mu(p)$ is an interior point of $\mu(M)$, and $\mu(M)$ is a nondegenerate convex polytope. Any nondegenerate convex polytope in $\mathbb{R}^m$ must have at least $m + 1$ vertices. The vertices of $\mu(M)$ are images of fixed points. □

Theorem 1.6.2. *Let $(M, \omega, \mathbb{T}^m, \mu)$ be a hamiltonian $\mathbb{T}^m$-space. If the $\mathbb{T}^m$-action is effective, then $\dim M \geqslant 2m$.*

Proof. Since the moment map is constant on an orbit $\mathcal{O}$, for $p \in \mathcal{O}$ the exterior derivative

$$d\mu_p : T_pM \longrightarrow \mathfrak{g}^*$$

maps $T_p\mathcal{O}$ to 0. Thus

$$T_p\mathcal{O} \subseteq \ker d\mu_p = (T_p\mathcal{O})^\omega \ ,$$

where $(T_p\mathcal{O})^\omega$ is the symplectic orthocomplement of $T_p\mathcal{O}$ (see Lemma 1.4.9). This shows that orbits $\mathcal{O}$ of a hamiltonian torus action are always *isotropic* submanifolds[8] of M. In particular, by symplectic linear algebra we have that $\dim \mathcal{O} \leqslant \frac{1}{2}\dim M$. Now consider an m-dimensional orbit. □

Exercise 1.6.3. *Suppose that $\mathbb{T}^m$ acts linearly on $(\mathbb{C}^n, \omega_0)$ as follows:*

$$(e^{i\theta_1}, \dots, e^{i\theta_m}) \cdot (z_1, \dots, z_n) = \left(e^{i\langle\lambda^{(1)},\theta\rangle} z_1 \ , \dots , e^{i\langle\lambda^{(n)},\theta\rangle} z_n\right) \ ,$$

for some weights $\lambda^{(1)}, \dots, \lambda^{(n)} \in \mathbb{Z}^m$. *In Exercise 1.5.2, we have seen that this action is hamiltonian with a moment map given by*

$$\mu(z_1, \dots, z_n) = \tfrac{1}{2} \sum_{j=1}^{n} \lambda^{(j)} |z_j|^2 \ (\ + \text{ constant }) \ .$$

(a) Show that, if this action is effective, then the weights $\lambda^{(1)}, \dots, \lambda^{(n)}$ $\mathbb{Z}$-span $\mathbb{Z}^m$.

(b) Conclude that, if such an action is effective, then any moment map μ is a submersion, i.e., each differential $d\mu_z : \mathbb{C}^n \to \mathbb{R}^n$ ($z \in \mathbb{C}^n$) is surjective.

Definition of symplectic toric manifold

The so-called *symplectic toric manifolds* fit in the optimal case of effective hamiltonian tori actions:

Definition 1.6.4. *A* **symplectic toric manifold** *is a compact connected symplectic manifold (M, ω) equipped with an effective hamiltonian action of a standard torus $\mathbb{T}^n$ of dimension equal to half the dimension of the manifold,*

$$\dim \mathbb{T} = \frac{1}{2} \dim M \ ,$$

and with a choice of a corresponding moment map $\mu : M \to (\mathbb{R}^n)^*$.

---

[8]A submanifold Q of a symplectic manifold (M, ω) is **isotropic**, if the restriction of ω to Q is trivial. This means that at each $p \in Q$ the pairing of two vectors tangent to Q by ω_p gives 0. We say that T_pQ is an **isotropic subspace** of the symplectic vector space (T_pM, ω_p). An isotropic submanifold Q necessarily has $\dim Q \leqslant \frac{1}{2}\dim M$, and when $\dim Q = \frac{1}{2}\dim M$ we say that Q is **lagrangian**.

In the examples below, we choose a scaling factor giving the *Fubini-Study form* on $\mathbb{CP}^n$ as

$$\omega_{\text{FS}} = \frac{i}{2}\partial\bar{\partial}\ln(1+|z|^2)$$

with respect to standard charts with n coordinates z_j, $0 \leqslant j \leqslant n$, $j \neq k$, on each open set

$$\mathcal{U}_k = \{[z_0 : \ldots : z_{k-1} : 1 : z_{k+1} : \ldots : z_n] \in \mathbb{CP}^n\} \quad \longrightarrow \quad \mathbb{C}^n \ .$$

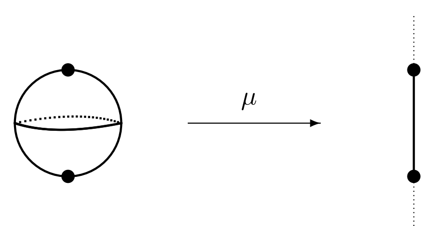
In particular for $n = 1$, we have that the sphere $\mathbb{CP}^1$ has $\omega_{\text{FS}} = \frac{1}{4}\omega_{\text{eucl}}$ and total area π with respect to ω_{FS}, whereas the euclidean area of a unit sphere in $\mathbb{R}^3$ is 4π.

Examples of symplectic toric manifolds

1. The circle S^1 acts on the 2-sphere $(S^2, \omega_{\text{eucl}} = d\theta \wedge dh)$ by rotations

$$e^{i\alpha} \cdot (\theta, h) = (\theta + \alpha, h)$$

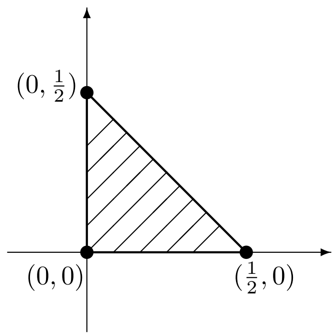
with moment map $\mu = -h$ equal to minus the height function and moment polytope $[-1, 1]$.



Equivalently, the circle S^1 acts on $\mathbb{CP}^1 = \mathbb{C}^2 \backslash \{0\} / \sim$ with the Fubini-Study form $\omega_{\text{FS}} = \frac{1}{4}\omega_{\text{eucl}}$, by $e^{i\alpha} \cdot [z_0 : z_1] = [z_0 : e^{i\alpha} z_1]$. This is hamiltonian with moment map $\mu[z_0 : z_1] = \frac{1}{2} \cdot \frac{|z_1|^2}{|z_0|^2+|z_1|^2}$, and moment polytope $\left[0, \frac{1}{2}\right]$.

2. Let $(\mathbb{CP}^2, \omega_{\text{FS}})$ be 2-(complex-)dimensional complex projective space equipped with the Fubini-Study form defined in Section 1.7. The $\mathbb{T}^2$-action on $\mathbb{CP}^2$ by $(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$ has moment map

$$\mu[z_0 : z_1 : z_2] = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, \frac{|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2}\right) \ .$$



The fixed points get mapped as

$$\begin{array}{ccc} [1:0:0] & \longmapsto & (0,0) \\ [0:1:0] & \longmapsto & (\frac{1}{2},0) \\ [0:0:1] & \longmapsto & (0,\frac{1}{2}) \end{array}$$

Notice that the stabilizer of a preimage of the edges is S^1, while the action has trivial stabilizers at preimages of interior points of the moment polytope.

3. More generally, on $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ with diagonal action of $\mathbb{T}^n$ as $(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot [z_0 : z_1 : \ldots : z_n] = [z_0 : e^{i\theta_1} z_1 : \ldots : e^{i\theta_n} z_n]$ we have as moment map

$$\mu[z_0 : z_1 : \ldots : z_n] = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2+|z_1|^2+\ldots+|z_n|^2}, \ldots, \frac{|z_n|^2}{|z_0|^2+|z_1|^2+\ldots+|z_n|^2}\right) ,$$

whose image is an n-dimensional simplex. ♢

Products of symplectic toric manifolds are naturally symplectic toric manifolds. For instance, on the product manifold $(\mathbb{CP}^1)^n$ with product symplectic structure given by the Fubini-Study form on each factor and with diagonal action of $\mathbb{T}^n$, we have as moment map

$$\mu(z_1, \ldots, z_n) = \frac{1}{2}\left(|z_1|^2, \ldots, |z_n|^2\right) ,$$

whose image is an n-dimensional cube. In particular, the moment polytope for the $\mathbb{T}^2$-action on $\mathbb{CP}^1 \times \mathbb{CP}^1$ as

$$(e^{i\theta}, e^{i\eta}) \cdot ([z_0 : z_1], [w_0 : w_1]) = ([z_0 : e^{i\theta} z_1], [w_0 : e^{i\eta} w_1])$$

is a square.

Equivalence between symplectic toric manifolds

The equivalence between symplectic toric manifolds is given by equivariant symplectomorphisms.

Definition 1.6.5. *Two symplectic toric manifolds, $(M_k, \omega_k, \mathbb{T}^n, \mu_k)$, $k = 1, 2$, are* **isomorphic** *if there exists an equivariant*[9] *symplectomorphism $\varphi : M_1 \to M_2$.*

Isomorphic symplectic toric manifolds are often undistinguished. Note that the torus is fixed and that the moment maps necessarily differ by a constant, in the sense that

$$\mu_1 = \mu_2 \circ \varphi + c \quad \text{for some } c \in (\mathbb{R}^n)^*.$$

(For general hamiltonian torus actions, moment maps are unique up to a constant).

In the next chapter, we give the classification of equivalence classes of symplectic toric manifolds by their moment polytopes up to translation.

Normal form near a fixed point

First note that an equivariant version of the Darboux-Weinstein theorem holds (see [63] and [32, §II.22][10]): *If a compact group G acts on the manifold M, the submanifold Q is G-invariant, and ω_0 and ω_1 are symplectic forms in M invariant under this action and equal at all points of Q, then the diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ claimed in Theorem 1.3.3 may be chosen to be G-equivariant.*

Indeed, by picking a G-invariant riemannian metric (for instance, by averaging over G any riemannian metric), we can obtain a G-invariant tubular neighborhood $\mathcal{U}_0$ of Q. Since the homotopy operator will commute with the G-action, we produce a G-invariant 1-form μ as in the proof of Theorem 1.3.3. As all input is G-invariant, the solution v_t of the Moser equation will be also G-invariant, and its flow ρ will be G-equivariant (i.e., commute with the G-action), thus proving what we call the **equivariant Darboux-Weinstein theorem**.

Now, we will concentrate on the case of a fixed point p of a hamiltonian torus space $(M, \omega, \mathbb{T}^k, \mu)$. At p, there is an induced representation (i.e., a linear action) of the group on the tangent space T_pM given by differentiating the action. This is called the **isotropy representation** at the point p. Using a $\mathbb{T}^k$-invariant *compatible almost complex structure* J[11] making (T_pM, J_p) into a complex vector space (by defining $iv := J_p v$), we view the isotropy representation as a complex representation. Since the group is a torus $\mathbb{T}^k$, we thus get a *weight space decomposition* (see, for instance, [13, §II.8]):

$$T_pM = \bigoplus_{j=1}^{n} W_j \ ,$$

where the torus acts on the complex 1-dimensional subspace $W_j \subseteq T_pM$ by

$$(e^{i\theta_1}, \ldots, e^{i\theta_k}) \cdot w_j = e^{i\langle \lambda^{(j)}, \theta \rangle} w_j \ ,$$

[9] Equivariance here means $\varphi([\theta] \cdot p) = [\theta] \cdot \varphi(p)$.

[10] Theorem 22.2 in [32] should also require that ω_0 and ω_1 agree in x; see [19].

[11] An **almost complex structure** on a manifold M is a smooth collection of linear maps $J_p : T_pM \to T_pM$ with $J_p^2 = -I$ at each $p \in M$. On any symplectic manifold (M, ω), there exists an almost complex structure J, such that the bilinear form defined by $\langle \cdot, \cdot \rangle := \omega(\cdot, J\cdot)$ is symmetric and positive definite, hence a riemannian metric; see, for example, [15]. Then J is called a **compatible almost complex structure** and such a triple $(\omega, J, \langle , \rangle)$ is called a **compatible triple**. When a compact Lie group acts symplectically on (M, ω), we can start from a G-invariant riemannian metric, to build an equivariant compatible J (note that $\omega(\cdot, J\cdot)$ tends to be a different G-invariant riemannian metric from the starting one).

for some weight $\lambda^{(j)} \in (\mathbb{Z}^k)^*$, with $\theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$. Notice that, in our notation, the weights $\lambda^{(j)}$ are not necessarily distinct; a **weight space** is the direct sum of all summands W_j with the same weight. Then we can adapt Darboux's theorem to model a $\mathbb{T}^k$-invariant neighborhood of p by euclidean space with standard symplectic form and linear action by those weights:

Theorem 1.6.6. (Toric Darboux) *Let $(M, \omega, \mathbb{T}^k, \mu)$ be a $2n$-dimensional hamiltonian torus space, and let p be a fixed point. Let $\lambda^{(1)}, \dots, \lambda^{(n)} \in (\mathbb{Z}^k)^*$ be the weights (given with multiplicity) of the isotropy representation of $\mathbb{T}^k$ on T_pM.*

Then there is a $\mathbb{T}^k$-invariant neighborhood $\mathcal{U}$ of p in M and coordinate functions $(x_1, \dots, x_n, y_1, \dots, y_n)$ centered at p with respect to which we have:

(a)

$$\omega|_{\mathcal{U}} = \sum_{j=1}^{n} dx_j \wedge dy_j = \tfrac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j \ ,$$

where we use the symbols $z_j := x_j + iy_j$, $\overline{z}_j := x_j - iy_j$,

(b) the action becomes the linear action of $\mathbb{T}^k$ with the given weights:

$$(e^{i\theta_1}, \dots, e^{i\theta_k}) \cdot (x_1, \dots, x_n, y_1, \dots, y_n) = (\widetilde{x}_1, \dots, \widetilde{x}_n, \widetilde{y}_1, \dots, \widetilde{y}_n) \ ,$$

where

$$\begin{pmatrix} \widetilde{x}_j \\ \widetilde{y}_j \end{pmatrix} = \begin{pmatrix} \cos\langle\lambda^{(j)}, \theta\rangle & -\sin\langle\lambda^{(j)}, \theta\rangle \\ \sin\langle\lambda^{(j)}, \theta\rangle & \cos\langle\lambda^{(j)}, \theta\rangle \end{pmatrix} \begin{pmatrix} x_j \\ y_j \end{pmatrix} \ ,$$

or, in terms of the symbolic notation with z_j, $\overline{z}_j$,

$$(e^{i\theta_1}, \dots, e^{i\theta_k}) \cdot (z_1, \dots, z_n) = \left(e^{i\langle\lambda^{(1)}, \theta\rangle} z_1, \dots, e^{i\langle\lambda^{(n)}, \theta\rangle} z_n\right)$$

$$(e^{i\theta_1}, \dots, e^{i\theta_k}) \cdot (\overline{z_1}, \dots, \overline{z_n}) = \left(e^{-i\langle\lambda^{(1)}, \theta\rangle} \overline{z_1}, \dots, e^{-i\langle\lambda^{(n)}, \theta\rangle} \overline{z_n}\right) \quad \text{and}$$

(c) the moment map becomes

$$\mu|_{\mathcal{U}} = \mu(p) + \frac{1}{2} \sum_{j=1}^{n} \lambda^{(j)} (x_j^2 + y_j^2) = \mu(p) + \frac{1}{2} \sum_{j=1}^{n} \lambda^{(j)} |z_j|^2 \ .$$

Proof. We take $Q = \{p\}$ and apply the equivariant Darboux-Weinstein theorem with ω_0 being the given symplectic form and ω_1 a constant symplectic form on a Darboux chart around p constructed as follows:

1. Let J be the $\mathbb{T}^k$-equivariant compatible almost complex structure and $\langle \cdot, \cdot \rangle := \omega(\cdot, J\cdot)$ the corresponding $\mathbb{T}^k$-invariant riemannian metric we are working with. Note that different weight spaces are mutually orthogonal with respect to $\langle , \rangle$ and ω. By the Gram-Schmidt argument, we can choose summands W_j as above within the same weight space to be mutually orthogonal as well. Now we choose a unit vector $u_j \in W_j$ and set $v_j := Ju_j$. Then

   $$(u_1, \dots, u_n, v_1, \dots v_n)$$

   is a symplectic basis for T_pM.

2. By using the exponential map with respect to the metric, we construct coordinates $(x_1', \dots, x_n', y_1', \dots y_n')$ centered at p and valid on some neighborhood $\mathcal{U}'$, with $\left.\frac{\partial}{\partial x_j'}\right|_p = u_j$ and $\left.\frac{\partial}{\partial y_j'}\right|_p = v_j$. Then we have
$$\omega_p = \left.\sum_{j=1}^n dx_j' \wedge dy_j'\right|_p .$$
Note that the exponential map is $\mathbb{T}^k$-equivariant.

3. There are two $\mathbb{T}^k$-invariant symplectic forms on $\mathcal{U}'$, namely the given $\omega_0 := \omega$ and $\omega_1 := \sum dx_j' \wedge dy_j'$, that agree at p. By the equivariant Darboux-Weinstein theorem, there are $\mathbb{T}^k$-invariant neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of p, and a $\mathbb{T}^k$-equivariant diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that
$$\varphi(p) = p \quad \text{and} \quad \varphi^* \left(\sum dx_j' \wedge dy_j'\right) = \omega .$$
Since $\varphi^*(\sum dx_j' \wedge dy_j') = \sum d(x_j' \circ \varphi) \wedge d(y_j' \circ \varphi)$, we set new coordinates $x_j := x_j' \circ \varphi$ and $y_j := y_j' \circ \varphi$.

□

Exercise 1.6.7. *Show that for a symplectic toric manifold the weights of the isotropy representation at a fixed point, $\lambda^{(1)}, \dots, \lambda^{(n)}$, form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*

Hint: *Toric Darboux and Exercise 1.6.3.*

1.7 Symplectic Reduction

Symplectic reduction is a fundamental construction of (new) symplectic manifolds starting from (old) symplectic manifolds with a hamiltonian group action by taking quotients in the symplectic sense.

Symplectic reduction is also the key for Delzant's proof of existence in his classification theorem, by providing the construction of a symplectic toric manifold out of the data encoded in an appropriate polytope.

First we recall *orbit spaces*. Let $\psi : G \to \mathrm{Diff}(M)$ be any action. The **orbit** of G through $p \in M$ is $\{\psi_g(p) \mid g \in G\}$.

Exercise 1.7.1. *If q is in the orbit of p, then their stabilizers G_q and G_p are conjugate subgroups. In particular, when G is abelian, all points in the same orbit have the same stabilizer.*

Definition 1.7.2. *We say that the action of G on M is:*

- **transitive** *if there is just one orbit,*
- **free** *if all stabilizers are trivial $\{\mathbb{1}\}$,*

- **locally free** *if all stabilizers are discrete.*

Let $\sim$ be the orbit equivalence relation; for $p, q \in M$,

$$p \sim q \quad \iff \quad p \text{ and } q \text{ are on the same orbit.}$$

The space of orbits $M/G := M/\sim$ is called the **orbit space**. Let

$$\begin{array}{rccl} \Pi : & M & \longrightarrow & M/G \\ & p & \longmapsto & \text{orbit through } p \end{array}$$

be the **point-orbit projection**.

We equip M/G with the weakest topology for which Π is continuous, i.e., $\mathcal{U} \subseteq M/G$ is open if and only if $\Pi^{-1}(\mathcal{U})$ is open in M. This is called the **quotient topology**. This topology can be *bad*. For instance:

Example. Let $G = \mathbb{C}\backslash\{0\}$ act on $M = \mathbb{C}^n$ by

$$\lambda \longmapsto \psi_\lambda = \text{multiplication by } \lambda \ .$$

The orbits are the punctured complex lines (through non-zero vectors $z \in \mathbb{C}^n$), plus one so-called *unstable* orbit through 0, which has a single point. The orbit space is

$$M/G = \mathbb{CP}^{n-1} \sqcup \{\text{point}\} \ .$$

The quotient topology restricts to the usual topology on $\mathbb{CP}^{n-1}$. The only open set containing $\{\text{point}\}$ in the quotient topology is the full space, hence the topology in M/G is *not* Hausdorff.

However, it suffices to remove 0 from $\mathbb{C}^n$ to obtain a Hausdorff orbit space:

$$\Big(\mathbb{C}^n\backslash\{0\}\Big)\Big/\Big(\mathbb{C}\backslash\{0\}\Big) = \mathbb{CP}^{n-1} \ .$$

$\diamondsuit$

We next address the previous example once again but from a compact and symplectic (yet not complex) viewpoint:

Example. Let $\omega = \frac{i}{2}\sum dz_k \wedge d\bar{z}_k = \sum dx_k \wedge dy_k = \sum r_k dr_k \wedge d\theta_k$ be the standard symplectic form on $\mathbb{C}^n$. Consider the following S^1-action on $(\mathbb{C}^n, \omega)$:

$$\theta \in S^1 \longmapsto \psi_\theta = \text{multiplication by } \theta \ .$$

The vector field generated by this action is

$$X^\# = \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2} + \cdots + \frac{\partial}{\partial\theta_n} \ .$$

This vector field is hamiltonian, i.e., the action ψ is hamiltonian with moment map

$$\begin{array}{rccl} \mu : & \mathbb{C}^n & \longrightarrow & \mathbb{R} \\ & z & \longmapsto & \frac{||z||^2}{2} + \text{constant} \end{array}$$

since

$$\imath_{X^\#}\omega = -\sum r_k dr_k = -\tfrac{1}{2}\sum d(r_k^2) = -d\mu \ .$$

If we conveniently choose the constant to be $-\frac{1}{2}$, then $\mu^{-1}(0) = S^{2n-1}$ is the unit sphere. The orbit space of the zero level of the moment map is

$$\mu^{-1}(0)/S^1 = S^{2n-1}/S^1 = \mathbb{CP}^{n-1} \ .$$

This description induces a symplectic form on $\mathbb{CP}^{n-1}$ as a particular instance of the following major theorem; see below. ♢

Meyer on one side and Marsden and Weinstein on the other proved independently the following mathematical formulation of the reduction process from physics. Later in this text, we will only be concerned with the case where the Lie group is a torus.

Theorem 1.7.3. (Marsden-Weinstein [47], Meyer [51]) *Let (M, ω, G, μ) be a hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*

(a) the orbit space $M_{\mathrm{red}} = \mu^{-1}(0)/G$ is a manifold,

(b) $\Pi : \mu^{-1}(0) \to M_{\mathrm{red}}$ is a principal G-bundle, and

(c) there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \Pi^*\omega_{\mathrm{red}}$.

Note that this theorem does not assume that $\mu^{-1}(0)$ is a regular level, but this is a consequence of G acting freely on this level; see Exercise 1.4.10.

For a proof of Theorem 1.7.3, see for instance [15]. Here is just a sketch of the idea for the case $G = S^1$ and $\dim M = 4$ going back to Bott.

In this case the moment map is $\mu : M \to \mathbb{R}$. Let $p \in \mu^{-1}(0)$. Choose local coordinates:

- θ along the orbit through p,
- μ given by the moment map, and
- η_1, η_2 pullback of coordinates on $\mu^{-1}(0)/S^1$.

Then the symplectic form can be written

$$\omega = A\ d\theta \wedge d\mu + B_j\ d\theta \wedge d\eta_j + C_j\ d\mu \wedge d\eta_j + D\ d\eta_1 \wedge d\eta_2 \ .$$

Since $d\mu = \imath\left(\frac{\partial}{\partial\theta}\right)\omega$, we must have $A = 1$, $B_j = 0$. Hence,

$$\omega = d\theta \wedge d\mu + C_j\ d\mu \wedge d\eta_j + D\ d\eta_1 \wedge d\eta_2 \ .$$

Since ω is symplectic, we must have $D \neq 0$. Therefore, $i^*\omega = D\ d\eta_1 \wedge d\eta_2$ is the pullback of a symplectic form on M_{red}.

Definition 1.7.4. *The pair $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ is called the* **symplectic reduction** *of (M, ω) with respect to G and μ (or the* reduced space*, or the* symplectic quotient*, or the* Marsden-Weinstein-Meyer quotient*, etc.).*

Example. Consider the S^1-action on $(\mathbb{R}^{2n+2}, \omega_0)$ which, under the usual identification of $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$, corresponds to multiplication by $e^{i\theta}$. This action is hamiltonian with a moment map $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$ given by

$$\mu(z) = \tfrac{1}{2}||z||^2 - \tfrac{1}{2} \ .$$

Symplectic reduction yields complex projective space $\mu^{-1}(0)/S^1 = \mathbb{CP}^n$ equipped with the so-called **Fubini-Study symplectic form** $\omega_{\text{red}} = \omega_{\text{FS}}$. ♢

Exercise 1.7.5. *Recall that $\mathbb{CP}^1 \simeq S^2$ as real 2-dimensional manifolds. Check that*

$$\omega_{\text{FS}} = \frac{1}{4}\omega_{\text{eucl}} \ ,$$

where $\omega_{\text{eucl}} = d\theta \wedge dh$ is the euclidean area form on the unit sphere S^2.

We consider here two basic extensions of the procedure of symplectic reduction. There is a further major extension to the case of *symplectic toric orbifolds*, which we briefly address in Chapter 3. Reduction for product groups (a.k.a. reduction in stages) will be needed in Chapter 2.

Reduction for product groups

Let G_1 and G_2 be compact connected Lie groups whose actions on a manifold M commute, and let $G = G_1 \times G_2$. Then $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{g}^* \simeq \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$. Suppose that (M, ω, G, μ) is a hamiltonian G-space with moment map

$$\mu : M \longrightarrow \mathfrak{g}_1^* \oplus \mathfrak{g}_2^* \ .$$

Write $\mu = (\mu_1, \mu_2)$ where $\mu_k : M \to \mathfrak{g}_k^*$ for $k = 1, 2$; cf. Exercise 1.4.8. The fact that μ is equivariant implies that μ_1 is invariant under G_2 and μ_2 is invariant under G_1. Now reduce (M, ω) with respect to the G_1-action. Let

$$Z_1 = \mu_1^{-1}(0) \ .$$

Assume that G_1 acts freely on Z_1. Let $M_1 = Z_1/G_1$ be the reduced space and let ω_1 be the corresponding reduced symplectic form. The action of G_2 on Z_1 commutes with the G_1-action. Since G_2 preserves ω, it follows that G_2 acts symplectically on (M_1, ω_1). Since G_1 preserves μ_2, G_1 also preserves $\mu_2 \circ \iota_1 : Z_1 \to \mathfrak{g}_2^*$, where $\iota_1 : Z_1 \hookrightarrow M$ is inclusion. Thus $\mu_2 \circ \iota_1$ is constant on fibers of $Z_1 \stackrel{p_1}{\to} M_1$. We conclude that there exists a smooth map $\nu : M_1 \to \mathfrak{g}_2^*$ such that $\nu \circ p_1 = \mu_2 \circ \iota_1$.

Exercise 1.7.6. *Show that:*

(a) the map ν is a moment map for the action of G_2 on (M_1, ω_1), and

(b) if G acts freely on $\mu^{-1}(0,0)$, then G_2 acts freely on $\nu^{-1}(0)$, and there is a natural symplectomorphism

$$\mu^{-1}(0,0)/G \ \simeq \ \nu^{-1}(0)/G_2 \ .$$

The map ν is called the **reduced moment map**.

Example. Consider the hamiltonian S^1-action on $(\mathbb{C}^{n+1}, \omega_0)$ by multiplication by $e^{i\theta}$, for which symplectic reduction yields complex projective space $\mu^{-1}(0)/S^1 = \mathbb{CP}^n$ (see example above). Now $\mathbb{T}^{n+1}$ acts also on $(\mathbb{C}^{n+1}, \omega_0)$ by diagonal multiplication and this hamiltonian action commutes with the S^1-action. Hence, it descends to the reduced space $\mathbb{CP}^n$. The reduced moment map is given by

$$\begin{array}{ccc} \mathbb{CP}^n & \longrightarrow & \mathbb{R}^{n+1} \\ [z_0 : z_1 : \ldots : z_n] & \longmapsto & \frac{1}{2}\left(|z_0|^2, |z_1|^2, \ldots, |z_n|^2\right) \end{array}$$

where we choose $(z_0, z_1, \ldots, z_n) \in \mu^{-1}(0)$. ♢

This technique of performing reduction with respect to one factor of a product group at a time is called **reduction in stages**. It may be extended to reduction by a normal subgroup $H \subset G$ and by the corresponding quotient group G/H.

Reduction at other levels

Suppose that a compact Lie group G acts on a symplectic manifold (M, ω) in a hamiltonian way with moment map $\mu : M \to \mathfrak{g}^*$. Let $\xi \in \mathfrak{g}^*$. To reduce at the level ξ of μ, we need $\mu^{-1}(\xi)$ to be preserved by G, or else take the G-orbit of $\mu^{-1}(\xi)$, or equivalently take the inverse image $\mu^{-1}(\mathcal{O}_\xi)$ of the coadjoint orbit through ξ, or else take the quotient by the maximal subgroup of G which preserves $\mu^{-1}(\xi)$. Of course the level 0 is always preserved. Also, when G is a torus, any level is preserved and reduction at ξ for the moment map μ, is equivalent to reduction at 0 for a shifted moment map $\phi : M \to \mathfrak{g}^*$, $\phi(p) := \mu(p) - \xi$.

For the case of *torus* actions, are all levels equally easy, since the coadjoint action is trivial.

Example. Consider again the hamiltonian S^1-action on $(\mathbb{C}^{n+1}, \omega_0)$ by multiplication by $e^{i\theta}$ with moment map

$$\mu(z) = \frac{1}{2}||z||^2 - \frac{1}{2} \ ,$$

for which symplectic reduction at level 0 yields complex projective space

$$\mu^{-1}(0)/S^1 = \mathbb{CP}^n$$

equipped with the Fubini-Study symplectic form (see example above).

If we now reduce at another level $\xi > -\frac{1}{2}$, we obtain as reduced space the same smooth manifold

$$\mu^{-1}(\xi)/S^1 \simeq \mathbb{CP}^n \ ,$$

but the symplectic form will be scaled. ♢

Chapter 2

Delzant's Classification

2.1 Unimodular Polytopes

Foreword on level of generality

Our discussion below has *moment polytopes* in sight. Those are polytopes in the target space of a moment map for the action of a torus, i.e., of a compact connected abelian Lie group, T. If $\mathfrak{t}$ is the Lie algebra of the torus, then the polytopes are in $\mathfrak{t}^*$.

However, for the sake of concreteness, we will often address the case of polytopes in $\mathbb{R}^n$, after using two identifications:

- A **splitting** of the n-dimensional torus T, i.e., a choice of an isomorphism $\mathfrak{t} \to \mathbb{R}^n$, under which the integral lattice $\Gamma := \ker(\exp_T : \mathfrak{t} \to T)$ gets identified with the lattice $(2\pi\mathbb{Z})^n$; cf. [13, Section I.3]. This amounts to choosing a $\mathbb{Z}$-basis $X_1, \ldots, X_n$ for the integral lattice. A splitting induces a Lie group isomorphism $T \simeq \mathbb{T}^n$ given by the exponential maps,

$$\exp_T(\theta_1 X_1 + \ldots + \theta_n X_n) \longmapsto \left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)$$

  where $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$. The dual vector space of the Lie algebra, $\mathfrak{t}^* := \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ gets then identified with $(\mathbb{R}^n)^*$.

- The **euclidean identification** $(\mathbb{R}^n)^* \simeq \mathbb{R}^n$ via the standard (euclidean) inner product, under which the natural pairing $\mathfrak{t}^* \times \mathfrak{t} \to \mathbb{R}$, $(\xi, X) \mapsto \xi(X)$ translates into the dot product $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(u, v) \mapsto \sum u_k v_k$.

The **weight lattice** of T is (up to 2π scaling) the dual of the integral lattice, that is, $\Gamma^* := \mathrm{Hom}_{\mathbb{Z}}(\Gamma, 2\pi\mathbb{Z})$. Under the above identifications, $\Gamma^* \subset \mathfrak{t}^*$ gets identified with $(\mathbb{Z})^n \subset \mathbb{R}^n$.

Whenever needed in the discussion below, we will come back to the more abstract level, for instance to define *annihilator* or to distinguish tangent vectors to a face of the polytope (elements of $\mathfrak{t}^*$) from vectors normal to facet (elements of $\mathfrak{t}$).

Convex polytopes

A **convex polytope** $\Delta \subset \mathbb{R}^n$ is the convex hull[1] of a finite set of points in $\mathbb{R}^n$. A **convex polyhedron** is a subset of $\mathbb{R}^n$ that is the intersection of a finite number of affine half-spaces. It is a theorem, usually attributed to Weyl and Minkowski, that convex polytopes coincide with compact convex polyhedra. The proof of the Weyl-Minkowski theorem in $\mathbb{R}^2$ may be left as an exercise. However, its proof for higher $\mathbb{R}^n$ is involved, even though the claim is intuitive.

A **face** of a convex polytope Δ is a nonempty intersection of Δ with a closed halfspace whose boundary is disjoint from the interior of Δ. In particular, the whole polytope is a face of itself. The dimension of a face is the dimension of its affine hull. A **vertex** is a 0-dimensional face, an **edge** is a 1-dimensional face and a **facet** is a face of codimension 1 with respect to the dimension of the polytope.

The subspace of $\mathfrak{t}^*$ modelling the affine hull of a face is called the **tangent space** to that face. The **annihilator** of a face is the annihilator of its tangent space, hence is a subspace of $\mathfrak{t}$. So the dimension of a face is the dimension of its tangent space and is the codimension of its annihiliator.

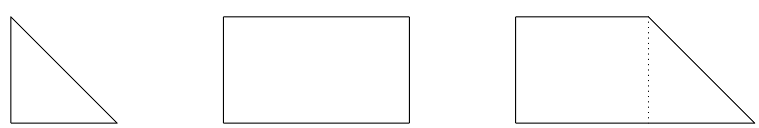
We now define the class of polytopes which arise in the classification of symplectic toric manifolds.

Definition 2.1.1. *A* **unimodular polytope** Δ *in* $\mathbb{R}^n$ *is a convex polytope satisfying:*

- **simplicity**, *i.e., there are n edges meeting at each vertex;*
- **rationality**, *i.e., the edges meeting at the vertex τ are rational in the sense that each edge is of the form $\tau + tu_k$, $t \geqslant 0$, where $u_k \in \mathbb{Z}^n$;*
- **smoothness**, *i.e., for each vertex, the corresponding $u_1, \dots, u_n$ can be chosen to form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.*[2]

Unimodular polytopes are, in the context of symplectic toric manifolds, sometimes also referred to as *Delzant polytopes*.

Examples of unimodular polytopes in $\mathbb{R}^2$:

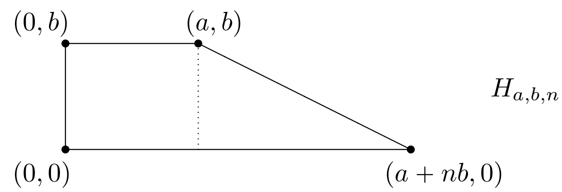


The pictures above represent polytopes in $\mathbb{R}^2$ with standard lattice $\mathbb{Z}^2$, i.e., standard horizontal and vertical cartesian axes with same scale. The dotted vertical line in the trapezoidal example is there just to stress that it is a picture of a rectangle plus an *isosceles* triangle. For "taller" triangles, smoothness would be violated. "Wider"

[1] The **convex hull** of a given subset X of a vector space is the intersection of all convex sets containing X or, equivalently, the set of all convex combinations of points in X.

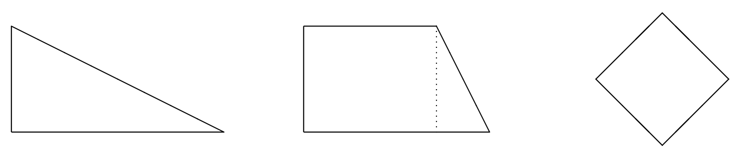
[2] This condition amounts to having a basis of $\mathbb{R}^n$ by integral edge vectors $u_1, \dots, u_n$ forming a parallelepiped of volume 1, and is usually referred to as **unimodularity**.

triangles may still be unimodular as in the examples below, denoted $H_{a,b,n}$, as long as the slope of the hypothenuse satisfies an integrality condition given by $n = 0, 1, 2, \ldots$. The positive real parameters a and b are the width and height of the left rectangle. We call these examples **Hirzebruch trapezoids**. In particular, $H_{a,b,0}$ is just a rectangle.

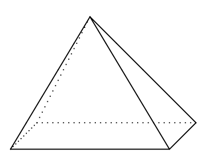


♢

Examples of polytopes that are not unimodular:



Once again, the pictures above represent polytopes in $\mathbb{R}^2$ with standard lattice $\mathbb{Z}^2$. The picture on the left fails the smoothness condition on the upper vertex, whereas the one in the middle fails the smoothness condition on the two right vertices, and the one on the right fails the smoothness condition on all vertices. Moreover, the following pyramid in $\mathbb{R}^3$ fails the simplicity condition.



♢

Exercise 2.1.2. *Show that, up to translations and linear transformations in* $\mathrm{GL}(2;\mathbb{Z})$, *the unimodular polytopes in* $\mathbb{R}^2$ *with three vertices are the isosceles right triangles.*

> **Hint:** *By translation, we bring one of the vertices to the origin. A transformation in* $\mathrm{GL}(2;\mathbb{Z})$ *will align the edges at the origin with the coordinate axes, in particular we get a right-angled triangle. How are the lengths of the two legs related?*

Conclude that the equivalence classes of unimodular polytopes in $\mathbb{R}^2$ *up to* $\mathrm{GL}(2;\mathbb{Z})$ *and translation by arbitrary vectors in* $\mathbb{R}^2$ *is the one-parameter family of triangles with vertices* $(0,0)$, $(a,0)$ *and* $(0,a)$, *where* $a > 0$.

Exercise 2.1.3. *Describe the class of unimodular polytopes in* $\mathbb{R}^2$ *with four vertices, up to translations and linear transformations in* $\mathrm{GL}(2;\mathbb{Z})$.

Hint: *Choose any vertex, translate it to the origin and use $\mathrm{GL}(2;\mathbb{Z})$ to turn the corresponding edge vectors into the standard basis. Then the vertices will be of the form $(0,0)$, $(c,0)$, $(0,b)$, (a,d) with $a,b,c,d>0$. By a reflection if necessary, assume that $c>b$. By convexity and the smoothness condition at $(c,0)$, $(0,b)$, the non-axial primitive edge vectors at those vertices must be of the form $u_1=(C,1)$ and $u_2=(1,B)$ with $C,B\in\mathbb{Z}$. Because those non-axial edges meet at (a,d) in a convex fashion, we must have that the determinant of the matrix with columns u_1 and u_2 must be negative, i.e., $BC<1$. Then the Delzant condition at (a,d) gives that that determinant must be -1, i.e., $BC=0$. Follow on, checking the cases $B=0$ and $C=0$.*

Coming back to the abstract case, we have that a convex polytope $\Delta\subset\mathfrak{t}^*$ is **unimodular** if and only if, for each vertex v, there is a $\mathbb{Z}$-basis $\eta_1,\ldots,\eta_n$ of the *weight lattice* Γ^* and a neighborhood $\mathcal{U}$ of v, such that

$$\Delta\cap\mathcal{U}=\{v+t_1\eta_1+\ldots+t_n\eta_n \mid t_i\in[0,\epsilon)\}\ .$$

Notice that then this basis is unique for each vertex (up to vector order). Notice also that the above condition encompasses simplicity, rationality and smoothness.

2.2 Delzant's Theorem

Recall that a $2n$-dimensional symplectic toric manifold is a compact connected symplectic manifold (M^{2n},ω) equipped with an effective hamiltonian action of a standard n-torus $\mathbb{T}^n$ and with a corresponding moment map. Two symplectic toric manifolds are called *isomorphic* – and thus considered *equivalent* – if they are equivariantly symplectomorphic; cf. Section 1.6. Moreover, it is a consequence of Theorem 1.6.6, that the moment polytope of a symplectic toric manifold is a unimodular polytope in $\mathbb{R}^n$; cf. Exercise **??**.

Delzant's theorem classifies (equivalence classes of) symplectic toric manifolds in terms of the combinatorial data encoded by a unimodular polytope (up to translation). This correspondence is given by the moment polytopes.

Theorem 2.2.1. (Delzant [20]) *Symplectic toric manifolds are classified up to equivalence by unimodular polytopes up to translation. More specifically, the bijective correspondence between these two sets is given by the moment map:*

$$\begin{array}{rcl}\{\textit{symplectic toric manifolds}\} & \xrightarrow{1-1} & \{\textit{unimodular polytopes}\}\\ \textit{(mod equivalence)} & & \textit{(mod translation)}\\ (M^{2n},\omega,\mathbb{T}^n,\mu) & \longmapsto & \mu(M)\ .\end{array}$$

For the proof of the existence part we follow Delzant [20] and for the uniqueness we will follow Lerman [43].

Preparation of Delzant's construction

In order to prepare the construction of a symplectic toric manifold from a unimodular polytope, we view convex polytopes as compact convex polyhedra, according to the Weyl-Minkowski theorem. For this passage, we use the following exercise, translating the unimodularity condition into a similar condition in terms of normal vectors to the facets.

Exercise 2.2.2. *Consider one vertex of a unimodular polytope in $\mathbb{R}^n$ and a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ built up of the edge vectors meeting at that vertex,*

$$u_1, u_2, \ldots, u_n .$$

Show that then there are n corresponding facets meeting at that vertex (each one containing all but one of the u_k vectors) and that the primitive inward-pointing normal vectors to these facets also form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

> **Hint:** *By a change of basis $A \in \mathrm{GL}(n;\mathbb{Z})$, you may assume that $u_1, u_2, \ldots, u_n$ is the standard basis. Then the corresponding primitive inward-pointing normal vectors to the facets meeting at that vertex are also the vectors from the standard basis. Observe that those normal vectors have changed by $(A^T)^{-1}$.*

Strictly speaking, a moment polytope is in $\mathfrak{t}^*$, edge vectors are elements of $\mathfrak{t}^*$, whereas normal vectors to facets are elements of $\mathfrak{t}$. The identifications from Section 2.1 still hold, yet we will here distinguish $\mathfrak{t} \simeq \mathbb{R}^n$ and $\mathfrak{t}^* \simeq (\mathbb{R}^n)^*$.

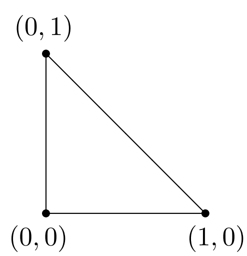
Let Δ be a unimodular polytope in $(\mathbb{R}^n)^*$ and with d facets. Let $v_k \in \mathbb{Z}^n$, $k = 1, \ldots, d$, be the primitive[3] inward-pointing normal vectors to the facets of Δ. Then, we can describe Δ as an intersection of halfspaces

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_k\rangle \geqslant c_k,\ k = 1, \ldots, d\} \quad \text{for some } c_k \in \mathbb{R} ,$$

where $\langle \cdot, \cdot \rangle : (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$ is the natural pairing.

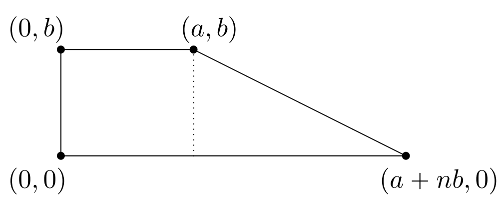
Example. For the picture below, we have

$$\begin{aligned} \Delta &= \{x \in (\mathbb{R}^2)^* \mid x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_1 + x_2 \leqslant 1\} \\ &= \{x \in (\mathbb{R}^2)^* \mid \langle x, \underbrace{(1,0)}_{v_1}\rangle \geqslant 0 \ , \ \langle x, \underbrace{(0,1)}_{v_2}\rangle \geqslant 0 \ , \ \langle x, \underbrace{(-1,-1)}_{v_3}\rangle \geqslant -1\} \ . \end{aligned}$$



$\diamondsuit$

Exercise 2.2.3. *Describe the polytope $H_{a,b,n}$ as an intersection of four hyperplanes.*



[3] A lattice vector $v \in \mathbb{Z}^n$ is **primitive** if it cannot be written as $v = \ell u$ with $u \in \mathbb{Z}^n$, $\ell \in \mathbb{Z}$ and $|\ell| > 1$; for instance, $(1,1)$, $(4,3)$, $(1,0)$ are primitive, but $(2,2)$, $(4,6)$ are not.

Exercise 2.2.4. *This is a generalization of Exercise 2.1.2. Show that, up to translations and linear transformations in* $\mathrm{GL}(n;\mathbb{Z})$*, the unimodular polytopes in* $\mathbb{R}^n$ *with* $n+1$ *vertices are the simplices with vertices at the origin and at the points with all coordinates* 0 *except one equal to* a $(a > 0)$*. In particular, in* $\mathbb{R}^3$ *that is the set of simplices with vertices* $(0,0,0)$, $(a,0,0)$, $(0,a,0)$, $(0,0,a)$.

> **Hint:** *Choose any vertex, translate it to the origin and use* $\mathrm{GL}(n;\mathbb{Z})$ *to turn the corresponding edge vectors into the standard basis. Now all vertices lie on the coordinate axes and let* $(a,0,0,\ldots,0)$ *be one of the vertices. By a permutation, you may assume that this is a vertex closest to the origin. By rationality, the other vertices will be of the form* $(0,k_2a,0,\ldots,0)$, $(0,0,k_3a,0,\ldots,0)$, *etc with* $k_2,k_3,\ldots,k_n \in \mathbb{N}$*. The smoothness condition at the vertex* $(a,0,0,\ldots,0)$ *forces that* $k_2 = k_3 = \ldots = k_n = 1$.

2.3 Proof of Existence

We prove the existence part (or surjectivity) in Delzant's theorem, by using symplectic reduction to associate to an n-dimensional unimodular polytope Δ a symplectic toric manifold $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ with moment polytope Δ, following [20, 29].

Let Δ be a unimodular polytope with d facets. Let $v_k \in \mathbb{Z}^n$, $k = 1,\ldots,d$ (where $d > n$), be the primitive inward-pointing normal vectors to the facets. For some $c_k \in \mathbb{R}$, we can write

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_k\rangle \geqslant c_k,\ k = 1,\ldots,d\} .$$

Let $e_1 = (1,0,\ldots,0),\ldots,e_d = (0,\ldots,0,1)$ be the standard basis of $\mathbb{R}^d$. Consider

$$\begin{array}{rccc} \Pi: & \mathbb{R}^d & \longrightarrow & \mathbb{R}^n \\ & e_k & \longmapsto & v_k \ . \end{array}$$

Then the map Π is onto and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$ since, for each vertex, the v_k's corresponding to the facets meeting at that vertex form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$; see Exercise 2.2.2.

Therefore, Π induces a surjective map, still called Π, between tori:

$$\begin{array}{ccccc} \mathbb{R}^d/(2\pi\mathbb{Z}^d) & \xrightarrow{\ \Pi\ } & \mathbb{R}^n/(2\pi\mathbb{Z}^n) & & \\ \| & & \| & & \\ \mathbb{T}^d & \longrightarrow & \mathbb{T}^n & \longrightarrow & \mathbb{1} \ . \end{array}$$

Let N be the kernel of Π, a $(d-n)$-dimensional Lie subgroup of $\mathbb{T}^d$, with inclusion $i : N \hookrightarrow \mathbb{T}^d$. Let $\mathfrak{n}$ be the Lie algebra of N. The exact sequence of Lie groups

$$\mathbb{1} \longrightarrow N \xrightarrow{\ i\ } \mathbb{T}^d \xrightarrow{\ \Pi\ } \mathbb{T}^n \longrightarrow \mathbb{1}$$

induces an exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\ i\ } \mathbb{R}^d \xrightarrow{\ \Pi\ } \mathbb{R}^n \longrightarrow 0$$

with dual exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\Pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{n}^* \longrightarrow 0 \ .$$

Now consider $\mathbb{C}^d$ with symplectic form $\omega_0 = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k$, and standard hamiltonian action of $\mathbb{T}^d$ given by

$$(e^{i\theta_1}, \ldots, e^{i\theta_d}) \cdot (z_1, \ldots, z_d) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_d} z_d) \ .$$

The moment map is $\phi : \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^*$ defined by

$$\phi(z_1, \ldots, z_d) = \tfrac{1}{2}(|z_1|^2, \ldots, |z_d|^2) + \text{ constant } ,$$

where we will choose the constant to be $(c_1, \ldots, c_d)$, where the c_k are the constants above. By Exercise 1.4.5, the subgroup N acts on $\mathbb{C}^d$ in a hamiltonian way with moment map

$$i^* \circ \phi : \mathbb{C}^d \longrightarrow \mathfrak{n}^* \ .$$

Let $Z = (i^* \circ \phi)^{-1}(0)$ be the zero-level set. Note that Z is connected, because $(i^*)^{-1}(0)$ is a linear subspace of $\mathbb{R}^d$ and the fibers $\phi^{-1}(x)$ are path-connected.

Claim 1. The submanifold Z is compact and N acts freely on Z.

We postpone the proof of this claim until further down.

Now Z is the zero-level of a moment map for the action of the torus N on $\mathbb{C}^d$. Knowing that N acts freely on Z ensures that this is a *regular level*; this is a consequence of Exercise 1.4.10. Hence, Z is a *submanifold* of $\mathbb{C}^d$ of (real) dimension $2d - (d - n) = d + n$. We now use the following theorem from Lie theory; see for instance [41, Chapter 21 and Exercise 21-6]:

Theorem 2.3.1. *If a compact Lie group N acts freely on a manifold Z, then the orbit space Z/N is a manifold and the point-orbit map $p : Z \to Z/N$ is a principal N-bundle.*

In our case, Z is a *compact* $(d + n)$-dimensional manifold, so the orbit space $M_\Delta := Z/N$ is a compact manifold of (real) dimension

$$\dim M_\Delta = \dim Z - \dim N = (d + n) - (d - n) = 2n.$$

The point-orbit map $p : Z \to M_\Delta$ is a principal N-bundle over M_Δ. Consider the diagram

$$\begin{array}{ccc} Z & \stackrel{j}{\hookrightarrow} & \mathbb{C}^d \\ {\scriptstyle p}\downarrow & & \\ M_\Delta & & \end{array}$$

where $j : Z \hookrightarrow \mathbb{C}^d$ is inclusion. The Marsden-Weinstein-Meyer theorem (Theorem 1.7.3) guarantees the existence of a symplectic form ω_Δ on M_Δ satisfying

$$p^* \omega_\Delta = j^* \omega_0 \ .$$

Since Z is connected, the compact symplectic $2n$-dimensional manifold $(M_\Delta, \omega_\Delta)$ is also connected.

Proof of Claim 1. The set Z is clearly closed, hence in order to show that it is compact it suffices (by the Heine-Borel theorem) to show that Z is bounded. Let Δ' be the image of Δ by Π^*. We will show that $\phi(Z) = \Delta'$.

Lemma 2.3.2. *Let $y \in (\mathbb{R}^d)^*$. Then:*

$$y \in \Delta' \iff y \in \phi(Z) \ .$$

Proof. The value y is in the image of Z by ϕ if and only if both of the following conditions hold:

1. y is in the image of ϕ;
2. $i^* y = 0$.

Using the expression for ϕ and the dual exact sequence, we see that these conditions are equivalent to:

1. $\langle y, e_k \rangle \geqslant c_k$ for $k = 1, \ldots, d$;
2. $y = \Pi^*(x)$ for some $x \in (\mathbb{R}^n)^*$.

Suppose that the second condition holds, so that $y = \Pi^*(x)$. Then

$$\begin{aligned}
\langle y, e_k \rangle \geqslant c_k, \ \forall k &\iff \langle \Pi^*(x), e_k \rangle \geqslant c_k, \ \forall k \\
&\iff \langle x, \Pi(e_k) \rangle \geqslant c_k, \ \forall k \\
&\iff \langle x, v_k \rangle \geqslant c_k, \ \forall k \\
&\iff x \in \Delta \ .
\end{aligned}$$

Thus,

$$y \in \phi(Z) \iff y \in \Pi^*(\Delta) = \Delta' \ .$$

This concludes the proof of Lemma 2.3.2. □

Since we have that Δ' is compact, that ϕ is a proper map[4] and that $\phi(Z) = \Delta'$, we conclude that Z must be bounded, and hence compact.

It remains to show that N acts freely on Z.

Pick a vertex τ of Δ, and let $I = \{k_1, \ldots, k_n\}$ be the set of indices for the n facets meeting at τ. Pick $z \in Z$ such that $\phi(z) = \Pi^*(\tau)$. Then τ is characterized by n equations $\langle \tau, v_k \rangle = c_k$ where k ranges in I:

$$\begin{aligned}
\langle \tau, v_k \rangle = c_k &\iff \langle \tau, \Pi(e_k) \rangle = c_k \\
&\iff \langle \Pi^*(\tau), e_k \rangle = c_k \\
&\iff \langle \phi(z), e_k \rangle = c_k \\
&\iff i\text{-th coordinate of } \phi(z) \text{ is equal to } c_k \\
&\iff \tfrac{1}{2}|z_k|^2 + c_k = c_k \\
&\iff z_k = 0 \ .
\end{aligned}$$

[4] A map between topological spaces is called **proper** if inverse images of compact subsets are compact.

Hence, those z's are points whose coordinates in the set I are zero, and whose other coordinates are nonzero. Without loss of generality, we may assume that $I = \{1, \dots, n\}$. The stabilizer of z is

$$(\mathbb{T}^d)_z = \{(e^{i\theta_1}, \dots, e^{i\theta_n}, 1, \dots, 1) \in \mathbb{T}^d\} .$$

As the restriction $\Pi : (\mathbb{R}^d)_z \to \mathbb{R}^n$ maps the vectors $e_1, \dots, e_n$ to a $\mathbb{Z}$-basis $v_1, \dots, v_n$ of $\mathbb{Z}^n$ respectively, at the level of groups the map $\Pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$ must be bijective. Since $N = \ker(\Pi : \mathbb{T}^d \to \mathbb{T}^n)$, we conclude that $N \cap (\mathbb{T}^d)_z = \{\mathbb{1}\}$, i.e., $N_z = \{\mathbb{1}\}$. Hence, all N-stabilizers at points mapping to vertices are trivial. But this was the worst case, since other stabilizers $N_{z'}$ $(z' \in Z)$ are contained in stabilizers for points z which map to vertices. This concludes the proof of Claim 1. □

Given a unimodular polytope Δ, we have constructed a symplectic manifold $(M_\Delta, \omega_\Delta)$ where $M_\Delta := Z/N$ is a compact $2n$-dimensional manifold and ω_Δ is the reduced symplectic form.

Claim 2. The manifold $(M_\Delta, \omega_\Delta)$ inherits a hamiltonian $\mathbb{T}^n$-action with a moment map μ_Δ having image $\mu_\Delta(M_\Delta) = \Delta$.

Proof of Claim 2. Let z be such that $\phi(z) = \Pi^*(\tau)$ where τ is a vertex of Δ, as in the proof of Claim 1. Let $\sigma : \mathbb{T}^n \to (\mathbb{T}^d)_z$ be the inverse for the earlier bijection $\Pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$. Since we have found a *section*, i.e., a right inverse for Π, in the exact sequence

$$\mathbb{1} \longrightarrow N \xrightarrow{i} \mathbb{T}^d \underset{\xleftarrow{\sigma}}{\xrightarrow{\Pi}} \mathbb{T}^n \longrightarrow \mathbb{1} ,$$

the exact sequence *splits*, i.e., becomes like a sequence for a product, as we obtain an isomorphism

$$(i, \sigma) : N \times \mathbb{T}^n \xrightarrow{\simeq} \mathbb{T}^d .$$

The action of the $\mathbb{T}^n$ factor (or, more rigorously, $\sigma(\mathbb{T}^n) \subset \mathbb{T}^d$) descends to the quotient $M_\Delta = Z/N$.

It remains to show that the $\mathbb{T}^n$-action on M_Δ is hamiltonian with appropriate moment map.

Consider the diagram

$$\begin{array}{ccccccc} Z & \overset{j}{\hookrightarrow} & \mathbb{C}^d & \xrightarrow{\phi} & (\mathbb{R}^d)^* \simeq \mathfrak{n}^* \oplus (\mathbb{R}^n)^* & \xrightarrow{\sigma^*} & (\mathbb{R}^n)^* \\ {\scriptstyle p}\downarrow & & & & & & \\ M_\Delta & & & & & & \end{array}$$

where the last horizontal map is simply projection onto the second factor. Since the composition of the horizontal maps is constant along N-orbits, it descends to a map

$$\mu_\Delta : M_\Delta \longrightarrow (\mathbb{R}^n)^*$$

which satisfies

$$\mu_\Delta \circ p = \sigma^* \circ \phi \circ j \ .$$

By Exercise 1.7.6 on reduction for product groups, this is a moment map for the action of $\mathbb{T}^n$ on $(M_\Delta, \omega_\Delta)$. Finally, the image of μ_Δ is:

$$\mu_\Delta(M_\Delta) = (\mu_\Delta \circ p)(Z) = (\sigma^* \circ \phi \circ j)(Z) = (\sigma^* \circ \Pi^*)(\Delta) = \Delta \ ,$$

because $\phi(Z) = \Pi^*(\Delta)$ and $\sigma^* \circ \Pi^* = (\Pi \circ \sigma)^* = \mathrm{id}$. □

The above $\mathbb{T}^n$-action is effective because $\mathbb{T}^d$, and hence $\mathbb{T}^n$, acts freely on the open dense subset

$$\phi^{-1}\left(\Pi^*(\Delta^o)\right) \subset Z \ ,$$

where Δ^o denotes the interior of Δ.

We conclude that $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ is the required symplectic toric manifold corresponding to Δ.

2.4 Discussion of Delzant's Correspondence

Delzant's theorem asserts that the map

$$\begin{array}{rcl} \{\text{symplectic toric manifolds}\} & \longrightarrow & \{\text{unimodular polytopes}\} \\ \text{(mod equivalence)} & & \text{(mod translation)} \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) & \longmapsto & \mu(M) \ . \end{array}$$

is well-defined and bijective.

- In the previous section, we saw that it is indeed surjective.
- We will see now that it is well-defined.
- In the section after next, we will see that it is indeed injective.

Moreover, in this section, we later review the main idea behind Delzant's construction, check that the moment polytope is the orbit space of a symplectic toric manifold, and discuss concrete examples.

In Section 1.6, we had already observed that equivalent (i.e. isomorphic) symplectic toric manifolds have the same moment map up to a constant, hence have the same moment polytope up to translation. It remains to show that the moment polytope is *unimodular*.

Proposition 2.4.1. *Let $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ be a symplectic toric manifold. Then the image Δ of μ is a unimodular polytope.*

Proof. By the Atiyah-Guillemin-Sternberg convexity theorem (Theorem 1.5.3) the image Δ is the convex hull of the images of the fixed points of the action.

Let τ be a vertex of Δ. Then there is $p \in M$ fixed by $\mathbb{T}^n$ and with $\mu(p) = \tau$.

By the toric Darboux theorem (Theorem 1.6.6), we can find a Darboux chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that:

- the neighborhood $\mathcal{U}$ is $\mathbb{T}^n$-invariant,
- the symplectic form becomes $\omega_{\mathcal{U}} = \sum_k dx_k \wedge dy_k = \frac{i}{2} \sum_k dz_k \wedge d\bar{z}_k$ where $z_k = x_k + iy_k$, $\bar{z}_k = x_k - iy_k$,
- in these coordinates the action of $\mathbb{T}^n$ is linear:

$$(e^{i\theta_1}, \ldots, e^{i\theta_k}) \cdot (z_1, \ldots, z_n) = \left(e^{i\sum_j \lambda_j^{(1)}\theta_j} z_1, \ldots, e^{i\sum_j \lambda_j^{(n)}\theta_j} z_n\right) ,$$

  where $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^n$ are the corresponding weights,
- thus the moment map has the form:

$$\mu_{\mathcal{U}}(z_1, \ldots, z_n) = \tau + \tfrac{1}{2} \sum_{k=1}^{n} \lambda^{(k)} |z_k|^2 .$$

Moreover, the weights $\lambda^{(1)}, \ldots, \lambda^{(n)}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ because the action is effective (cf. Exercise 1.6.3 (a)). This shows that the image of this neighborhood $\mathcal{U}$ by μ is of the form

$$\tau + \sum_{k=1}^{n} t_k \lambda^{(k)} \quad \text{with} \quad t_k \geqslant 0 ,$$

which by itself satisfies simplicity, rationality and smoothness.

Moreover, by the Atiyah-Guillemin-Sternberg theorem the levels of μ are connected, in particular, the level $\mu^{-1}(\tau)$ is connected. The above form shows that $\mu^{-1}(\tau) = \{p\}$, therefore the preimage of a neighborhood of τ is completely described by the model above and Delzant's conditions are globally satisfied. □

Exercise 2.4.2. *Use the previous proof to show that the fixed points of a symplectic toric manifold are isolated and the moment map of a symplectic toric manifold maps the fixed points of the action bijectively onto the vertices of the moment polytope. (This last fact will be generalized in Theorem 2.4.5.)*

Idea behind Delzant's construction:

The main idea of Delzant's construction is that the space $\mathbb{R}^d$ is *universal* in the sense that any n-dimensional (nondegenerate) polytope Δ with d facets can be obtained by intersecting the positive orthant $\mathbb{R}^d_+$ with an affine plane A. (We now identify $\mathbb{R}^n$ with its dual.) Given Δ, to construct A first write Δ as:

$$\Delta = \{x \in \mathbb{R}^n \mid \langle x, v_k \rangle \geqslant c_k,\ k = 1, \ldots, d\} .$$

Define

$$\begin{array}{rccc} \Pi : & \mathbb{R}^d & \longrightarrow & \mathbb{R}^n \\ & e_k & \longmapsto & v_k \end{array} \qquad \text{with dual map} \qquad \Pi^* : \ \mathbb{R}^n \longrightarrow \mathbb{R}^d .$$

Then $\Pi^* - \lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^d$ is an injective affine map, where $\lambda = (c_1, \ldots, c_d)$. Let A be the image of $\Pi^* - \lambda$. Then A is an n-dimensional affine space.

Lemma 2.4.3. *We have the equality* $(\Pi^* - \lambda)(\Delta) = \mathbb{R}^d_+ \cap A$.

Proof. Let $x \in \mathbb{R}^n$. Then

$$\begin{aligned}
(\Pi^* - \lambda)(x) \in \mathbb{R}^d_+ &\iff \langle \Pi^*(x) - \lambda, e_k \rangle \geqslant 0, \forall i \\
&\iff \langle x, \Pi(e_k) \rangle - c_k \geqslant 0, \forall i \\
&\iff \langle x, v_k \rangle \geqslant c_k, \forall i \\
&\iff x \in \Delta \ .
\end{aligned}$$

□

We conclude that $\Delta \simeq \mathbb{R}^d_+ \cap A$.

Now $\mathbb{R}^d_+$ is the image of the moment map for the standard hamiltonian action of $\mathbb{T}^d$ on $\mathbb{C}^d$

$$\begin{aligned}
\phi : \mathbb{C}^d &\longrightarrow \mathbb{R}^d \\
(z_1, \ldots, z_d) &\longmapsto \tfrac{1}{2}(|z_1|^2, \ldots, |z_d|^2)
\end{aligned}$$

and we assume that Δ is *Delzant*. Then the following facts hold:

- The set $\phi^{-1}(A) \subset \mathbb{C}^d$ is a compact submanifold. Let $i : \phi^{-1}(A) \hookrightarrow \mathbb{C}^d$ denote inclusion. Then $i^*\omega_0$ is a closed 2-form whose kernel is an integrable distribution. The corresponding foliation is called the **null foliation**.

- The null foliation of $i^*\omega_0$ is a principal fibration, so we take the quotient:

$$\begin{array}{ccc}
N \circlearrowright & \phi^{-1}(A) & \\
& \downarrow p & \\
& M_\Delta & := \phi^{-1}(A)/N
\end{array}$$

  with induced (reduced) symplectic form ω_Δ satisfying $p^*\omega_\Delta = i^*\omega_0$.

- The (non-effective) action of $\mathbb{T}^d = N \times \mathbb{T}^n$ on $\phi^{-1}(A)$ has a "moment map" with image $\phi(\phi^{-1}(A)) = \Delta$. By "moment map" we mean a map satisfying the usual definition even though the closed 2-form is not symplectic.

There is a remaining action of $\mathbb{T}^n$ on M_Δ which is hamiltonian with a moment map $\mu_\Delta : M_\Delta \to \mathbb{R}^n$ defined by the commutative diagram

$$\begin{array}{ccccc}
\phi^{-1}(A) & \overset{j}{\hookrightarrow} & \mathbb{C}^d & \overset{\phi}{\longrightarrow} & \mathbb{R}^d \\
p \downarrow & & & & \downarrow \mathrm{pr}_2 \\
M_\Delta & & \overset{\mu_\Delta}{\dashrightarrow} & & \mathbb{R}^n
\end{array}$$

where $\mathrm{pr}_2 : \mathbb{T}^d = N \times \mathbb{T}^n \to \mathbb{T}^n$, resp. $\mathrm{pr}_2 : \mathbb{R}^d = \mathfrak{n} \times \mathbb{R}^n \to \mathbb{R}^n$ is projection onto the second factor.

The moment polytope of a symplectic toric manifold is its orbit space:

Exercise 2.4.4. *As an experiment, consider the standard $\mathbb{T}^3$-action on $(\mathbb{CP}^3, \omega_{FS})$,*

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2 : e^{i\theta_3} z_3] ,$$

with moment map

$$\mu[z_0 : z_1 : z_2 : z_3] = \tfrac{1}{2(|z_0|^2+|z_1|^2+|z_2|^2+|z_3|^2)} \left(|z_1|^2, |z_2|^2, |z_3|^2\right) .$$

Exhibit explicitly the subsets of $\mathbb{CP}^3$ for which the stabilizer under this action is $\{\mathbb{1}\}$, a circle, a 2-torus and $\mathbb{T}^3$. Show that the images of these subsets under the moment map are the interior, the facets, the edges and the vertices of $\Delta = \mu(\mathbb{CP}^3)$, respectively. Given $x \in \Delta$, how many $\mathbb{T}^3$-orbits is $\mu^{-1}(x)$?

Theorem 2.4.5. *[20, Lemma 2.2] For any $x \in \Delta$, we have that:*

- *$\mu_\Delta^{-1}(x)$ is a single $\mathbb{T}^n$-orbit;*
- *the dimension of that orbit is equal to the dimension of the smallest face to which x belongs; and*
- *the stabilizer of that orbit is the torus whose Lie algebra is the annihilator in $\mathfrak{t}$ of that face.*

In particular, for a symplectic toric manifold, the moment polytope Δ is the orbit space and the moment map is the point-orbit map.

Proof. First consider the standard $\mathbb{T}^d$-action on $\mathbb{C}^d$ with moment map $\phi : \mathbb{C}^d \to (\mathbb{R}^d)^*$,

$$\phi(z_1, \ldots, z_d) = \tfrac{1}{2}(|z_1|^2, \ldots, |z_d|^2) + (c_1, \ldots, c_d) .$$

Then $\phi^{-1}(y)$ is a single $\mathbb{T}^d$-orbit for any $y \in \phi(\mathbb{C}^d)$, its stabilizer (i.e., the stabilizer of any point on that orbit) is

$$\{(t_1, \ldots, t_d) \in \mathbb{T}^d \mid t_k = 1 \text{ whenever } \langle y, e_k \rangle > c_k\}$$

and its dimension is equal to the number of indices k with $\langle y, e_k \rangle > c_k$. The only fixed point is the origin mapping to the only vertex of the image.

Now let $x_0 \in \Delta \subset (\mathbb{R}^n)^*$, take $y_0 = \Pi^*(x_0) \in (\mathbb{R}^d)^*$ and recall from Lemma 2.3.2 and the definition of Z that

$$y_0 \in \Delta' := \Pi^*(\Delta) \quad \Longleftrightarrow \quad y_0 \in \phi(Z) \quad \Longleftrightarrow \quad \phi^{-1}(y_0) \subseteq Z .$$

Then $\mu_\Delta^{-1}(x_0) = \phi^{-1}(y_0)/N$. But $\phi^{-1}(y_0)$ is a single $\mathbb{T}^d$-orbit where $\mathbb{T}^d = N \times \mathbb{T}^n$, hence $\mu_\Delta^{-1}(x_0)$ is a single $\mathbb{T}^n$-orbit.

Let F be the smallest face to which x_0 belongs and let m be the codimension of F. The face F is given as

$$F = \Delta \cap \{x \in (\mathbb{R}^n)^* \mid \langle x, v_k \rangle = c_k,\ k \in I_F\}$$

for some index subset $I_F \subset \{1, \ldots, d\}$ with cardinality $|I_F| = m$. Then y_0 belongs to the face of Δ' given by

$$\Delta' \cap \{y \in (\mathbb{R}^d)^* \mid \langle y, e_k \rangle = c_k,\ k \in I_F\}$$

and thus has as stabilizer the m-dimensional subtorus

$$T_F := \{(t_1, \ldots, t_d) \in \mathbb{T}^d \mid t_k = 1 \text{ whenever } k \notin I_F\}$$

and the $\mathbb{T}^d$-orbit of y_0, namely $\phi^{-1}(y_0)$, is $(d-m)$-dimensional. It follows that the $\mathbb{T}^n$-orbit of x_0, namely $\mu_\Delta^{-1}(x_0)$, has stabilizer $\Pi(T_F)$. Since N acts freely on Z, we see that $\Pi(T_F)$ is also an m-dimensional torus and the orbit $\mu_\Delta^{-1}(x_0)$ has dimension

$$\dim\left(\mu_\Delta^{-1}(x_0)\right) = \underbrace{\dim\left(\phi^{-1}(y_0)\right)}_{d-m} - \underbrace{\dim N}_{d-n} = n - m\ .$$

Since the Lie algebra of T_F is $\mathfrak{t}_F :=$ span$\{e_k \mid k \in I_F\}$ and $\Pi(e_k) = v_k$, the Lie algebra of the stabilizer $\Pi(T_F)$ is

$$\Pi(\mathfrak{t}_F) = \text{span}\{v_k \mid k \in I_F\}\ ,$$

which is the annihilator of the tangent space to F,

$$\{x \in (\mathbb{R}^n)^* \mid \langle x, v_k \rangle = 0,\ k \in I_F\}\ .$$

□

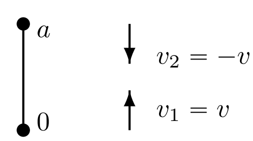
Concrete instances of Delzant's construction:

We will follow through the details of Delzant's construction for specific cases.

Example. We consider the case of $\Delta = [0, a] \subset \mathbb{R}^*$ $(n = 1, d = 2)$. Let $v = 1$ be the standard basis vector in $\mathbb{R}$. Then Δ is described by

$$\langle x, v \rangle \geqslant 0 \qquad \text{and} \qquad \langle x, -v \rangle \geqslant -a\ ,$$

so we have $v_1 = v$, $v_2 = -v$, $c_1 = 0$ and $c_2 = -a$.



The projection

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\Pi} & \mathbb{R} \\ e_1 & \longmapsto & v \\ e_2 & \longmapsto & -v \end{array}$$

has kernel equal to the span of $(e_1 + e_2)$, so that N is the diagonal subgroup of $\mathbb{T}^2 = S^1 \times S^1$. The exact sequences become

$$\begin{array}{ccccccccc} \mathbb{1} & \longrightarrow & N & \xrightarrow{i} & \mathbb{T}^2 & \xrightarrow{\Pi} & S^1 & \longrightarrow & \mathbb{1} \\ & & t & \longmapsto & (t,t) & & & & \\ & & & & (t_1,t_2) & \longmapsto & t_1 t_2^{-1} & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n} & \xrightarrow{i} & \mathbb{R}^2 & \xrightarrow{\Pi} & \mathbb{R} & \longrightarrow & 0 \\ & & x & \longmapsto & (x,x) & & & & \\ & & & & (x_1,x_2) & \longmapsto & x_1 - x_2 & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}^* & \xrightarrow{\Pi^*} & (\mathbb{R}^2)^* & \xrightarrow{i^*} & \mathfrak{n}^* & \longrightarrow & 0 \\ & & x & \longmapsto & (x,-x) & & & & \\ & & & & (x_1,x_2) & \longmapsto & x_1 + x_2\ . & & \end{array}$$

We choose the moment map for the standard $\mathbb{T}^2$-action on $\mathbb{C}^2$:

$$\phi(z_1,z_2) = \tfrac{1}{2}(|z_1|^2, |z_2|^2) + \underbrace{(0,-a)}_{(c_1,c_2)}\ .$$

The action of the diagonal subgroup $N = \{(e^{i\theta}, e^{i\theta}) \in S^1 \times S^1\}$ on $\mathbb{C}^2$,

$$(e^{i\theta}, e^{i\theta}) \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)\ ,$$

has moment map

$$(i^* \circ \phi)(z_1,z_2) = \tfrac{1}{2}(|z_1|^2 + |z_2|^2) - a\ ,$$

with zero-level set

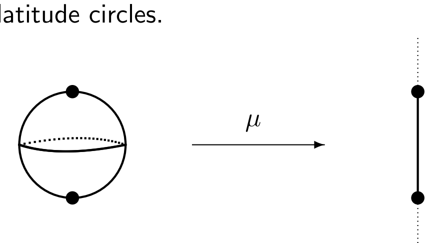
$$Z = (i^* \circ \phi)^{-1}(0) = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 2a\}\ .$$

Hence, the reduced space is a projective space:

$$(i^* \circ \phi)^{-1}(0)/N = \mathbb{CP}^1\ .$$

One can further check that the induced symplectic form is a multiple of the Fubini-Study form: $\omega_\Delta = 2a\omega_{\text{FS}}$; cf. Sections 1.6 and 1.7.

Here we see clearly the point-orbit correspondence given by the moment map. The boundary points of the moment polytope $\Delta = [0,a]$ correspond to the fixed points – North pole and South pole – whereas interior points correspond to free orbits – the other latitude circles.

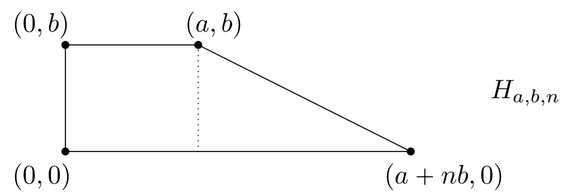


◊

Exercise 2.4.6. *Let Δ be the n-simplex in $\mathbb{R}^n$ spanned by the origin and the standard basis vectors $(1,0,\ldots,0),\ldots,(0,\ldots,0,1)$. Show that the corresponding symplectic toric manifold is n-dimensional complex projective space, $M_\Delta = \mathbb{CP}^n$.*

Exercise 2.4.7. *Which $2n$-dimensional toric manifolds have exactly $n+1$ fixed points?*

Example. We consider Delzant's construction for the case of $\Delta = H_{a,b,n} \subset (\mathbb{R}^2)^*$. The manifolds we will obtain are known as **Hirzebruch surfaces**, $M_\Delta = \mathcal{H}_{a,b,n}$ [35].



The polytope $\Delta = H_{a,b,n}$ is described by

$$\begin{aligned} \Delta &= \{x \in (\mathbb{R}^2)^* \mid x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_1 + nx_2 \leqslant a + nb,\ x_2 \leqslant b, \} \\ &= \{x \in (\mathbb{R}^2)^* \mid \langle x, \underbrace{(1,0)}_{v_1}\rangle \geqslant \underbrace{0}_{c_1},\ \langle x, \underbrace{(0,1)}_{v_2}\rangle \geqslant \underbrace{0}_{c_2}, \\ &\qquad \langle x, \underbrace{(-1,-n)}_{v_3}\rangle \geqslant \underbrace{-a-nb}_{c_3},\ \langle x, \underbrace{(0,-1)}_{v_4}\rangle \geqslant \underbrace{-b}_{c_4} \} . \end{aligned}$$

The projection

$$\begin{array}{rcl} \mathbb{R}^4 & \stackrel{\Pi}{\longrightarrow} & \mathbb{R}^2 \\ e_1 & \longmapsto & v_1 = (1,0) \\ e_2 & \longmapsto & v_2 = (0,1) \\ e_3 & \longmapsto & v_3 = (-1,-n) \\ e_4 & \longmapsto & v_4 = (0,-1) \end{array}$$

has kernel equal to the span of $\{e_2 + e_4, e_1 + ne_2 + e_3\}$, so that

$$N := \{\left(e^{i\beta}, e^{i(\alpha+n\beta)}, e^{i\beta}, e^{i\alpha}\right)\} \subset \mathbb{T}^4,$$

the exact sequences are

$$\begin{array}{cccccccc} \mathbb{1} & \longrightarrow & N & \stackrel{i}{\longrightarrow} & \mathbb{T}^4 & \stackrel{\Pi}{\longrightarrow} & \mathbb{T}^2 & \longrightarrow \mathbb{1} \\ & & (a,b) & \longmapsto & (b, ab^n, b, a) & & & \\ & & & & (t_1,t_2,t_3,t_4) & \longmapsto & (t_1 t_3^{-1}, t_2 t_3^{-n} t_4^{-1}) & \\[2ex] 0 & \longrightarrow & \mathfrak{n} & \stackrel{i}{\longrightarrow} & \mathbb{R}^4 & \stackrel{\Pi}{\longrightarrow} & \mathbb{R}^2 & \longrightarrow 0 \\ & & (x,y) & \longmapsto & (y, x+ny, y, x) & & & \\ & & & & (x_1,x_2,x_3,x_4) & \longmapsto & (x_1\text{-}x_3, x_2\text{-}nx_3\text{-}x_4) & \\[2ex] 0 & \longrightarrow & (\mathbb{R}^2)^* & \stackrel{\Pi^*}{\longrightarrow} & (\mathbb{R}^4)^* & \stackrel{i^*}{\longrightarrow} & \mathfrak{n}^* & \longrightarrow 0 \\ & & (x,y) & \longmapsto & (x, y, \text{-}x\text{-}ny, \text{-}y) & & & \\ & & & & (x_1,x_2,x_3,x_4) & \longmapsto & (x_2 + x_4, x_1 + nx_2 + x_3)\ . & \end{array}$$

and the action of N has moment map

$$(i^* \circ \phi)(z_1, z_2, z_3, z_4) = \tfrac{1}{2}(|z_2|^2 + |z_4|^2, |z_1|^2 + n|z_2|^2 + |z_3|^2) + (-b, -a - nb) ,$$

with zero-level set

$$\begin{aligned} Z = (i^* \circ \phi)^{-1}(0) = &\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \\ &|z_2|^2 + |z_4|^2 = 2b, |z_1|^2 + n|z_2|^2 + |z_3|^2 = 2(a + nb)\} . \end{aligned}$$

Hence, the reduced space is a so-called *Hirzebruch (complex) surface*

$$\mathcal{H}_{a,b,n} := Z/\sim ,$$

where the equivalence relation given by N is

$$(z_1, z_2, z_3, z_4) \sim \left(e^{i\beta} z_1, e^{i(\alpha+n\beta)} z_2, e^{i\beta} z_3, e^{i\alpha} z_4\right) .$$

♢

Remark. One can see that each 4-dimensional manifold $\mathcal{H}_{a,b,n}$ is a sphere bundle over a sphere, by considering the projection $p : \mathcal{H}_{a,b,n} \to \mathbb{CP}^1$ induced by the map

$$\begin{array}{rccc} \widetilde{p} : & Z & \longrightarrow & \mathbb{CP}^1 \\ & (z_1, z_2, z_3, z_4) & \longmapsto & [z_1 : z_3] . \end{array}$$

The map p is well-defined because $\widetilde{p}$ is invariant by N. Moreover, one can check that the fibers of p are copies of $\mathbb{CP}^1$ and that it is locally trivial, hence actually a fibration. In particular, for $n = 0$ we get a product of spheres,

$$\mathcal{H}_{a,b,0} \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$$

and all $\mathcal{H}_{a,b,n}$ for n even are diffeomorphic to this,[5] whereas all $\mathcal{H}_{a,b,n}$ for n odd are diffeomorphic to $\mathcal{H}_{a,b,1}$, the nontrivial S^2-bundle over S^2. Note that all these $\mathcal{H}_{a,b,n}$'s are distinct as complex manifolds, as well as symplectic manifolds (as well as symplectic toric manifolds). ♢

Exercise 2.4.8. *What are all the 4-dimensional symplectic toric manifolds that have exactly four fixed points?*

> **Hint:** *We simply need to classify unimodular polygons with four vertices. Consider Exercise 2.1.3, the observation after Theorem 2.4.5 and the previous example.*

[5]Orientable sphere bundles over the sphere S^2 are trivializable over each half-sphere and hence obtained by gluing two trival bundles over a disk along the boundary by a map from the equator S^1 to the group of orientation preserving diffeomorphisms of S^2. Milnor showed that this group of diffeomorphisms retracts onto $\mathrm{SO}(3)$, and we have that $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2\mathbb{Z}$. So there are only two diffeomorphism classes of such bundles: the class of the trivial bundle $S^2 \times S^2$ and the class of *the* nontrivial bundle.

2.5 Lerman's Construction

In the 1990's Eugene Lerman gave an alternative version of Delzant's construction using his *symplectic cutting* trick. Whereas we will deal with the original symplectic cutting technique in Chapter 3, we will now follow the exposition in [50, Ch.7, §5] to do cutting w.r.t. a unimodular polytope by working with the cotangent bundle of the torus, $T^*(\mathbb{T}^n)$.

Symplectic cutting w.r.t. a unimodular polytope:

Let Δ be a unimodular polytope given as an intersection of halfspaces as

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_k \rangle \geqslant c_k,\ k = 1, \ldots, d\} ,$$

where the $v_k \in \mathbb{Z}^n$, $k = 1, \ldots, d$, are the primitive inward-pointing normal vectors to the facets of Δ and where $c_k \in \mathbb{R}$. We set $\lambda := (c_1, \ldots, c_d)$.

We trivialize the tangent bundle $T(\mathbb{T}^n)$ as $\mathbb{T}^n \times \mathfrak{t}^n$ by invariant vector fields and, correspondingly, the cotangent bundle $T^*(\mathbb{T}^n)$ as $\mathbb{T}^n \times (\mathfrak{t}^n)^*$. We equip $T^*(\mathbb{T}^n) \simeq \mathbb{T}^n \times (\mathfrak{t}^n)^*$ with the symplectic form given by

$$\sum_{k=1}^{n} d\xi_k \wedge d\theta_k$$

with respect to cotangent coordinates $(\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n)$, to which we refer as **action coordinates** ξ_k and **angle coordinates** θ_k. Note the sign convention for this symplectic form.

We call **standard action** to the action of $\mathbb{T}^n$ on its cotangent bundle $T^*\mathbb{T}^n$ by the lift of its multiplication action on itself.[6] W.r.t. the action-angle coordinates above, the element $(e^{i\theta'_1}, \ldots, e^{i\theta'_n}) \in \mathbb{T}^n$ acts by

$$(\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n) \longmapsto (\theta_1 + \theta'_1, \ldots, \theta_n + \theta'_n, \xi_1, \ldots, \xi_n) .$$

This action is hamiltonian with moment map given by projection onto the second factor $\mathrm{pr}_2 : \mathbb{T}^n \times (\mathfrak{t}^n)^* \to (\mathfrak{t}^n)^*$, $(\theta, \xi) \mapsto \xi$.

Now consider the Lie group homomorphism

$$\begin{array}{rcl} \rho_\Delta : \mathbb{T}^d & \longrightarrow & \mathbb{T}^n \\ (e^{i\alpha_1}, \ldots, e^{i\alpha_d}) & \longmapsto & \exp\left(\sum_{k=1}^{d} \alpha_k v_k\right) . \end{array}$$

Notice that the differential of this homomorphism is the linear map from Section 2.3

$$\begin{array}{rrcl} D\rho_\Delta \simeq \Pi : & \mathfrak{t}^d \simeq \mathbb{R}^d & \longrightarrow & \mathfrak{t}^n \simeq \mathbb{R}^n \\ & e_k & \longmapsto & v_k . \end{array}$$

[6] If $\varphi : X \to X$ is a diffeomorphism, then its (cotangent) *lift* is $\varphi^\sharp : T^*X \to T^*X$, $(x, \xi) \mapsto \left(\varphi(x), ((d\varphi_x)^{-1})^* \xi\right)$. Then $\varphi^\sharp$ is a symplectomorphism for any (nonzero) multiple of the canonical symplectic form, $\sum dx_k \wedge d\xi_k$.

The action of $\mathbb{T}^d$ on $T^*(\mathbb{T}^n)$ via the composition of $\rho_\vartriangle$ with the standard $\mathbb{T}^n$-action is hamiltonian with moment map $\nu_\vartriangle$ given by the composition of the projection pr_2 with the adjoint of $D\rho_\vartriangle$ up to a constant. We fix the following moment map:

$$\begin{array}{rcccl} \nu_\vartriangle = (D\rho_\vartriangle)^* \circ \mathrm{pr}_2 & : & \mathbb{T}^n \times (\mathfrak{t}^n)^* & \to & (\mathfrak{t}^d)^* \\ & & (\theta,\xi) & \longmapsto & \sum\limits_{k=1}^{d} \langle \xi, v_k \rangle e_k - \lambda \ , \end{array}$$

where we added the constant $-\lambda$ for later convenience.

Let $\mathbb{T}^d$ act on $(\mathbb{C}^d, \frac{i}{2}\sum dz_k \wedge d\bar{z}_k)$ by

$$(e^{i\alpha_1}, \ldots, e^{i\alpha_d}) \cdot (z_1, \ldots, z_d) = (e^{-i\alpha_1} z_1, \ldots, e^{-i\alpha_d} z_d)$$

with moment map

$$(z_1, \ldots, z_d) \longmapsto -\tfrac{1}{2}(|z_1|^2, \ldots, |z_d|^2) \ .$$

Proposition 2.5.1. *Given a unimodular polytope Δ, consider the product manifold*

$$(T^*\mathbb{T}^n) \times \mathbb{C}^d$$

with:

- *product symplectic form*

$$\sum_{k=1}^{n} d\xi_k \wedge d\theta_k + \tfrac{i}{2} \sum_{k=1}^{d} dz_k \wedge d\bar{z}_k \ ,$$

- *product action of $\mathbb{T}^d$, where $\mathbb{T}^d$ acts on each factor as above, and*
- *moment map*

$$((\theta,\xi), z) \quad \longmapsto \quad \sum_{k=1}^{d} \langle \xi, v_k \rangle e_k - \lambda - \tfrac{1}{2}(|z_1|^2, \ldots, |z_d|^2) \ .$$

Then the $\mathbb{T}^d$-action is free on the zero level set of the moment map, so the reduced space is a symplectic manifold. Moreover, this reduced space is naturally a $2n$-dimensional symplectic toric manifold with moment map image Δ.

We denote this reduced space by $(E^\Delta, \omega^\Delta, \mathbb{T}^n, \mu^\Delta)$ and call it **Lerman's symplectic toric manifold associated with Δ**.

Proof. Let $((\theta,\xi), z) \in T^*\mathbb{T}^n \times \mathbb{C}^d$ be a point in the zero level, i.e.,

$$\sum_{k=1}^{d} \langle \xi, v_k \rangle e_k = \lambda + \tfrac{1}{2}(|z_1|^2, \ldots, |z_d|^2) \ .$$

If $z_k \neq 0$, then the kth factor of $\mathbb{T}^d$ has trivial stabilizer on $((\theta, \xi), z)$. Thus we need only worry about the set I of indices k with $z_k = 0$. For such an index $k \in I$, we have that $\langle \xi, v_k \rangle = c_k$. Let

$$T_I := \{(t_1, \dots, t_d) \in \mathbb{T}^d \mid t_k = 1 \text{ whenever } k \notin I\} \ .$$

By the Delzant condition (see the end of the proof of Claim 1 in Section 2.3), the restriction to T_I of the homomorphism ρ_Δ,

$$\rho_\Delta|_{T_I} : T_I \longrightarrow \mathbb{T}^n \ ,$$

is injective. Therefore, since $\mathbb{T}^n$ acts freely on $T^*(\mathbb{T}^n)$, so does T_I. This shows that the $\mathbb{T}^d$-action on the zero level set is free, so, by the Marsden-Weinstein-Meyer theorem (Theorem 1.7.3), the reduced space is a symplectic manifold, $(E^\Delta, \omega^\Delta)$.

This reduced space inherits a hamiltonian $\mathbb{T}^n$-action induced by the standard $\mathbb{T}^n$-action on $T^*(\mathbb{T}^n) = \mathbb{T}^n \times (\mathfrak{t}^n)^*$ with moment map $[(\theta, \xi), z] \mapsto \xi$ (this is well-defined since the $\mathbb{T}^d$-action preserves ξ). A point $\xi \in (\mathfrak{t}^n)^*$ is in the image by the moment map of some $((\theta, \xi), z) \in T^*\mathbb{T}^n \times \mathbb{C}^d$ from the zero level exactly when we can find z such that

$$\sum_{k=1}^{d} \langle \xi, v_k \rangle e_k = \lambda + \tfrac{1}{2}(|z_1|^2, \dots, |z_d|^2) \ ,$$

that is, when

$$\langle \xi, v_k \rangle \geqslant c_k \ , \quad k = 1, \dots, d \ ,$$

that is, when $\xi \in \Delta$. □

Interpretation of Lerman's construction:

We may view Δ as a *manifold with corners* in $(\mathfrak{t}^n)^*$. For any $x \in \Delta$, let F be the smallest face to which x belongs. Then the tangent space $T_x\Delta$ is the subspace of $(\mathfrak{t}^n)^* \simeq (\mathbb{R}^n)^*$ tangent to F.

The **interior** of Δ is the set of points given by strict inequalities:

$$\Delta^o := \{x \in (\mathbb{R}^n)^* \mid \langle x, v_k \rangle > c_k, \ k = 1, \dots, d\}$$

and this is a manifold (simply an open subset of euclidean space).

Essentially, what we do is take the product $\mathbb{T}^n \times \Delta$. Let x lie in the interior $\mathbb{T}^n \times \Delta^o$. The tangent space at x is $\mathfrak{t}^n \times (\mathfrak{t}^n)^* \simeq \mathbb{R}^n \times (\mathbb{R}^n)^*$. Define ω^o_x by:

$$\omega^o_x(v, \xi) = -\xi(v) = -\omega^o_x(\xi, v) \quad \text{and} \quad \omega^o_x(v, v') = \omega^o_x(\xi, \xi') = 0 \ ,$$

for all $v, v' \in \mathfrak{t}^n$ and $\xi, \xi' \in (\mathfrak{t}^n)^*$. Then ω^o is a closed nondegenerate 2-form on the interior of $\mathbb{T}^n \times \Delta$.

We will see that we can *close* the open subset $\mathbb{T}^n \times \Delta^o$ in a smooth and symplectic way.

At corners, there are tangent directions missing in $(\mathfrak{t}^n)^*$, so the extension of ω^o above would be a degenerate pairing. The missing directions at each corner point

x are the normal directions to the facets of Δ meeting at that point. For all ξ in the tangent space to the kth facet, we have $\omega(v_k, \xi) := -\xi(v_k) = 0$, where v_k is the vector defining that facet, and v_k spans the annihilator of that tangent space. We fix the degeneracy by eliminating in the $\mathfrak{t}^n$ component of the tangent space the directions of the vectors v_k defining the facets that meet at the point x. To do this, we collapse the orbit of the subgroup of $\mathbb{T}^n$ generated by those v_k's. This is a blow-down process and the result is a smooth compact manifold. We thus simultaneously eliminate corners and singularities of ω .

Finally, $\mathbb{T}^n$ acts on $\mathbb{T}^n \times \Delta$ by multiplication on the $\mathbb{T}^n$ factor. The moment map for this action is projection onto the Δ factor. We thus obtain $(E^\Delta, \omega^\Delta, \mathbb{T}^n, \mu^\Delta)$.

Note that the interior $\mathbb{T}^n \times \Delta^o$ with symplectic form ω^o embeds symplectically into $(E^\Delta, \omega^\Delta)$ and here we have **action-angle coordinates**, namely the ξ_k's and the θ_k's, with respect to which the symplectic form is

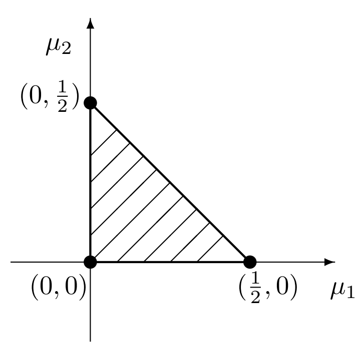
$$\omega^\Delta|_{\mathbb{T}^n \times \Delta^o} = \omega^o = \sum_{k=1}^n d\xi_k \wedge d\theta_k \ .$$

Example. Consider

$$(S^2, \omega = dh \wedge d\theta, S^1, \mu = h) \ ,$$

where S^1 acts on S^2 by rotation (with vector field $\frac{\partial}{\partial\theta}$). The image of μ is the line segment $I = [-1, 1]$. The product $S^1 \times I$ is an open-ended cylinder. By collapsing each circle end of the cylinder to a point, we recover the 2-sphere. Note that the notation for the symplectic form is only valid in the interior $(-1, 1) \times S^1$, so that is actually $\omega^o = dh \wedge d\theta$ presuming the extension. $\diamondsuit$

Example. We want to build $\mathbb{CP}^2$ from $\mathbb{T}^2 \times \Delta$ where Δ is the right-angled isosceles triangle below, following the above construction.



Consider, for instance, the edge of the triangle lying on the x-axis, whose tangent vectors ξ satisfy $\langle \xi, v_1 \rangle = 0$, where $v_1 = (0, 1) \in \mathfrak{t}^2$. For points of that edge, we collapse the subgroup of $\mathbb{T}^2$ generated by v_1, namely, the second circle factor. Similarly, for the edge of the triangle lying on the y-axis we collapse the first circle factor in $\mathbb{T}^2$, and for the hypothenuse we collapse the diagonal circle $\{(e^{i\theta}, e^{i\theta}) \in \mathbb{T}^2\}$. At the vertices (points lying in two facets), we collapse the whole $\mathbb{T}^2$.

All together, after the above collapses, the map

$$\begin{array}{ccc} \mathbb{T}^2 \times \Delta & \longrightarrow & \mathbb{CP}^2 \\ (e^{i\theta_1}, e^{i\theta_2}), (\mu_1, \mu_2) & \longmapsto & \left[\sqrt{1-2(\mu_1+\mu_2)} : \sqrt{2\mu_1}e^{i\theta_1} : \sqrt{2\mu_2}e^{i\theta_2}\right] \ . \end{array}$$

provides an equivariant symplectomorphism. ♢

Exercise 2.5.2. *Build $\mathbb{CP}^n$ from $\mathbb{T}^n \times \Delta^n$ where Δ^n is simplex with vertices at the origin and at the points*

$$(\tfrac{1}{2}, 0, \ldots, 0), (0, \tfrac{1}{2}, 0, \ldots, 0), \ldots (0, \ldots, 0, \tfrac{1}{2}) \ .$$

Exercise 2.5.3. *Build a Hirzebruch surface from $\mathbb{T}^2 \times \Delta$ where $\Delta = H_{a,b,n}$ is the polytope in Section 2.4.*

2.6 Proof of Uniqueness

For Delzant's theorem, it remains to prove:

Theorem 2.6.1. *Let $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ be a symplectic toric manifold with moment polytope $\Delta := \mu(M)$. Then $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ is equivariantly symplectomorphic to Lerman's symplectic toric manifold associated with Δ, $(E^\Delta, \omega^\Delta, \mathbb{T}^n, \mu^\Delta)$, defined in Section 2.5*

The original proof due to Delzant uses a sheaf-theoretic argument. We will sketch here an alternative proof going back to ideas of Lerman [43] and Meinrenken [50].

Warning: The proof-outline below uses concepts such as *compatible almost complex structure*, *principal bundles*, *symplectic neighborhood* and *cut space* which we have not yet addressed in these notes. Symplectic cutting is a construction based on symplectic reduction, of which we saw an instance in Section 2.5, and which will be discussed in Sections 3.2 and 3.3.

Sketch of proof of Theorem 2.6.1. The idea is to present $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ as a cut space w.r.t. its moment polytope Δ of a hamiltonian torus space $(\widetilde{M}^{2n}, \widetilde{\omega}, \mathbb{T}^n, \widetilde{\mu})$ with *free* $\mathbb{T}^n$-action and moment map image containing a neighborhood of Δ. Then the moment map $\widetilde{\mu} : \widetilde{M}^{2n} \to \mathbb{R}^n$ may be viewed as a lagrangian (torus) fibration over its image and we can introduce action-angle coordinates $(\xi_k,\ \theta_k)$, thus identifying $(\widetilde{M}^{2n}, \widetilde{\omega}, \mathbb{T}^n, \widetilde{\mu})$ up to equivariant symplectomorphism with an open subset of

$$(T^*\mathbb{T}^n, \sum d\xi_k \wedge d\theta_k, \mathbb{T}^n, \mathrm{pr}_2) \ .$$

It will follow that $(M^{2n}, \omega, \mathbb{T}^n, \mu)$ is equivariantly symplectomorphic to Lerman's symplectic toric manifold $(E^\Delta, \omega^\Delta, \mathbb{T}^n, \mu^\Delta)$.

Here is a short description of how to construct such a space $(\widetilde{M}^{2n}, \widetilde{\omega}, \mathbb{T}^n, \widetilde{\mu})$.

Let

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_k \rangle \geqslant c_k,\ k = 1, \ldots, d\} \ ,$$

let $k_1 \in \{1, \ldots, d\}$, consider the corresponding facet

$$\Delta_1 := \Delta \cap \{x \in (\mathbb{R}^n)^* \mid \langle x, v_{k_1} \rangle = c_{k_1}\}$$

and let $S = \mu^{-1}(\Delta_1) \subset M$ be the preimage of that facet. Then S is a connected component of the fixed point set of the 1-dimensional subgroup

$$\{\exp(tv_{k_1}) \mid t \in \mathbb{R}\}$$

and is a symplectic submanifold of codimension 2. We denote by ω_S the symplectic form on S. We consider its *symplectic normal bundle*[7], TS^ω. We can equip TS^ω with the structure of an hermitian line bundle (this involves choosing a *compatible almost complex structure* on M) and thus extract its unit circle bundle, $\Pi_Q : Q \to S$, which is a $\mathbb{T}^n$-equivariant principal S^1-bundle and satisfies $TS^\omega = Q \times_{S^1} \mathbb{C}$. We choose a corresponding $\mathbb{T}^n$-invariant connection form α, i.e., a $\mathbb{T}^n$-invariant 1-form on Q satisfying, for the vertical vector field v generated by the circle action:

$$\imath_v \alpha = 1 \quad \text{and} \quad \imath_v d\alpha = 0 \ .$$

Consider the closed 2-form on $Q \times \mathbb{C}$:

$$\omega_{Q\times\mathbb{C}} := \Pi_Q^* \omega_S + \omega_{\mathbb{C}} + \tfrac{1}{2} d\left(|z|^2 \alpha\right) \ ,$$

where $\omega_{\mathbb{C}}$ denotes the standard symplectic structure in $\mathbb{C}$. This 2-form is invariant for the circle action and vanishes on the vertical vector field,

$$\imath_{v-\frac{\partial}{\partial\theta}} \omega_{Q\times\mathbb{C}} = \imath_{-\frac{\partial}{\partial\theta}} \omega_{\mathbb{C}} + \imath_v r dr \wedge \alpha = r dr - r dr = 0 \ ,$$

so it descends to a closed 2-form, ω_{TS^ω}. Moreover, ω_{TS^ω} is nondegenerate near its zero section, S_0. It follows that there exists an equivariant symplectomorphism between tubular neighborhoods of S in M and of S_0 in TS^ω.

Now the symplectic normal bundle TS^ω may be viewed as a *cut space* w.r.t. the interval $[0, +\infty)$ of the hamiltonian S^1-space $Q \times \mathbb{R}$ equipped with the symplectic form

$$\omega_{Q\times\mathbb{R}} := \Pi_Q^* \omega_S + d(t\alpha) \ ,$$

where t is the coordinate function on $\mathbb{R}$ and defines the moment map. There is a natural $\mathbb{T}^n$-equivariant diffeomorphism between $Q \times \mathbb{R}^+$ and $TS^\omega \backslash S_0$ preserving the 2-forms. We can thus glue $M \backslash S$ with a small neighborhood of Q in $Q \times \mathbb{R}$, to obtain a new hamiltonian $\mathbb{T}^n$-space $(M_1, \omega_1, \mathbb{T}^n, \mu_1)$ with one orbit type stratum less. The original space is obtained from this M_1 by cutting with respect to the affine half-space

$$\mathcal{H}_1 := \{x \in (\mathbb{R}^n)^* \mid \langle x, v_{k_1} \rangle \geqslant c_{k_1}\} \ .$$

Continuing in this fashion for each facet of Δ, we obtain a sequence of spaces $M_1, M_2, \ldots, M_d$ with the property that the final space M_d has a *free* action and

[7]The **symplectic normal bundle** of a symplectic submanifold $S \subset M$ is the vector bundle over S whose fiber at each point p is given by the symplectic orthogonal of T_pS in (T_pM, ω_p).

each M_k is equivariantly symplectomorphic to the cut space of M_{k-1} w.r.t. $\mathcal{H}_k$, setting $M_0 := M$. Hence, we have

$$M \simeq (M_1)_{\mathcal{H}_1} \simeq (M_2)_{\mathcal{H}_1 \cap \mathcal{H}_2} \simeq \ldots \simeq (M_d)_\Delta$$

and we set $M_d =: \widetilde{M}$. □

Exercise 2.6.2. *What would be the classification of symplectic toric manifolds if, instead of the equivalence relation defined in Section 1.6, one considered to be equivalent those* $(M_j, \omega_j, \mathbb{T}^n, \mu_j)$, $j = 1, 2$, *related by:*

(a) a $\mathbb{T}^n$*-equivariant symplectomorphism* φ *sucht that* $\mu_1 = \mu_2 \circ \varphi$*?*

(b) an isomorphism $\lambda : \mathbb{T}^n \to \mathbb{T}^n$ *and a* λ*-equivariant*[8] *symplectomorphism* $\varphi : M_1 \to M_2$*?*

Hint: *The* **general affine group**, $\mathrm{AGL}(n;\mathbb{Z}) := \mathbb{R}^n \rtimes \mathrm{GL}(n;\mathbb{Z})$, *is the group of all invertible affine integral transformations, whose elements are compositions of linear maps in* $\mathrm{GL}(n;\mathbb{Z})$ *and translations by arbitrary vectors in* $\mathbb{R}^n$.

[8] λ-equivariance means that $\varphi(t \cdot p) = \lambda(t) \cdot \varphi(p)$ for all $p \in M_1$ and $t \in \mathbb{T}^n$. An isomorphism of $\mathbb{T}^n$ is given by an element of $\mathrm{GL}(n;\mathbb{Z})$ (those are the linear maps $\mathbb{R}^n \to \mathbb{R}^n$ that are isomorphisms of the lattice $(2\pi\mathbb{Z})^n$).

Bibliography

[1] Abreu, M., Kähler geometry of toric manifolds in symplectic coordinates. Symplectic and contact topology: interactions and perspectives, 1–24, Fields Inst. Commun., 35, Amer. Math. Soc. (2003).

[2] Abreu, M., Macarini, L., Contact homology of good toric contact manifolds. Compos. Math. 148 (2012), 304–334.

[3] Abreu, A., Macarini, M., Remarks on Lagrangian intersections in toric manifolds. Trans. Amer. Math. Soc. 365 (2013), 3851–3875.

[4] Abraham, R., Marsden, J. E., *Foundations of Mechanics*, second edition, Addison-Wesley, Reading, 1978.

[5] Arnold, V., *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics **60**, Springer-Verlag, New York, 1978.

[6] Atiyah, M., Convexity and commuting Hamiltonians, *Bull. London Math. Soc.* **14** (1982), 1-15.

[7] Atiyah, M., Bott, R., The moment map and equivariant cohomology, *Topology* **23** (1984), 1-28.

[8] Atiyah, M., Macdonald, I., *Introduction to Commutative Algebra*, Addison-Wesley, Reading, 1969.

[9] Audin, M. Torus actions on symplectic manifolds. Second revised edition. Progress in Mathematics 93, Birkhäuser (2004).

[10] Audin, M., *Spinning Tops. A Course on Integrable Systems*, Cambridge Studies in Advanced Mathematics **51**, Cambridge University Press, Cambridge, 1996.

[11] Bott, R., Tu, L., *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics **82**, Springer-Verlag, New York-Berlin, 1982.

[12] Bredon, G., *Introduction to Compact Transformation Groups*, Pure and Applied Mathematics **46**, Academic Press, New York-London, 1972.

[13] Bröcker, T., tom Dieck, T., *Representations of Compact Lie Groups*, Graduate Texts in Mathematics **98**, Springer-Verlag, New York, 1985.

[14] Cannas da Silva, A., A Chiang-type lagrangian in $\mathbb{CP}^2$. Lett. Math. Phys. 108 (2018), 699–710.

[15] Cannas da Silva, A., *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics **1764**, Springer-Verlag, Berlin, 2001.

[16] Cannas da Silva, A., Symplectic toric manifolds. Symplectic geometry of integrable Hamiltonian systems (Barcelona, 2001), 85–173, Adv. Courses Math. CRM Barcelona, Birkhäuser (2003).

[17] Cox, D., Recent developments in toric geometry, *Algebraic Geometry – Santa Cruz 1995*, 389-436, Proc. Sympos. Pure Math. **62**, part 2, Amer. Math. Soc., Providence, 1997.

[18] Danilov, V., The geometry of toric varieties, *Uspekhi Mat. Nauk* **33** (1978), no. 2 (200), 85-134, 247, English translation: *Russian Math. Surveys* **33** (1978), no. 2, 97-154.

[19] Dellnitz, M., Melbourne, I., The equivariant Darboux theorem, *Exploiting symmetry in applied and numerical analysis*, 163–169, Lectures in Appl. Math., **29**, Amer. Math. Soc., Providence, 1993.

[20] Delzant, T., Hamiltoniens périodiques et images convexes de l'application moment, *Bull. Soc. Math. France* **116** (1988), 315-339.

[21] Demazure, M., Sous-groupes algébriques de rang maximum du groupe de Cremona, *Ann. Sci. École Norm. Sup.* (4) **3** (1970), 507-588.

[22] Duistermaat, J.J., On global action-angle coordinates, *Comm. Pure Appl. Math.* **33** (1980), 687-706.

[23] Duistermaat, J.J., Equivariant cohomology and stationary phase, *Symplectic Geometry and Quantization* (Sanda and Yokohama, 1993), edited by Maeda, Y., Omori, H. and Weinstein, A., 45-62, *Contemp. Math.* **179**, Amer. Math. Soc., Providence, 1994.

[24] Duistermaat, J.J., Heckman, G., On the variation in the cohomology of the symplectic form of the reduced phase space, *Invent. Math.* **69** (1982), 259-268; Addendum, *Invent. Math.* **72** (1983), 153-158.

[25] Fukaya, K., Oh, Y.-G., Ohta, H., Ono, K., Lagrangian Floer theory on compact toric manifolds: survey. Surveys in differential geometry, 17, 229–298, Int. Press (2012).

[26] Fulton, W., *Introduction to Toric Varieties*, Annals of Mathematics Studies **131**, Princeton University Press, Princeton, 1993.

[27] Gelfand, I., Kapranov, M., Zelevinsky, A., *Discriminants, Resultants, and Multidimensional Determinants*, Mathematics: Theory & Applications, Birkhäuser Boston, 1994.

[28] Guillemin, V., Ginzburg, V., Karshon, Y., *Moment Maps, Cobordisms, and Hamiltonian Group Actions*, American Mathematical Society, Providence, 2002.

[29] Guillemin, V., *Moment Maps and Combinatorial Invariants of Hamiltonian T^n-spaces*, Progress in Mathematics **122**, Birkhäuser, Boston, 1994.

[30] Guillemin, V., Sternberg, S., Convexity properties of the moment mapping, *Invent. Math.* **67** (1982), 491-513.

[31] Guillemin, V., Sternberg, S., Birational equivalence in the symplectic category, *Invent. Math.* **97** (1989), 485-522.

[32] Guillemin, V., Sternberg, S., *Symplectic Techniques in Physics*, second edition, Cambridge University Press, Cambridge, 1990.

[33] Haug, L., On the quantum homology of real Lagrangians in Fano toric manifolds. Int. Math. Res. Not. IMRN 2013, no. 14, 3171–3220.

[34] Hausmann, J.-C., Holm, T., Puppe, V., Conjugation spaces. Algebr. Geom. Topol. 5 (2005), 923–964.

[35] Hirzebruch, F., Über eine Klasse von einfachzusammenhängenden komplexen Mannigfaltigkeiten. Mathematische Annalen **124** (1951), 77-86.

[36] Karshon, Y., Lerman, E., Non-compact symplectic toric manifolds, *SIGMA Symmetry Integrability Geom. Methods Appl.* **11** (2015), Paper 055, 37 pp.

[37] Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B., *Toroidal Embeddings, I*, Lecture Notes in Mathematics **339**, Springer-Verlag, Berlin-New York, 1973.

[38] Kirwan, F., *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Mathematical Notes **31**, Princeton University Press, Princeton, 1984.

[39] Karshon, Y., Lerman, E., Non-compact symplectic toric manifolds. SIGMA Symmetry Integrability Geom. Methods Appl. 11 (2015), Paper 055.

[40] Kostant, B., On convexity, the Weyl group and the Iwasawa decomposition. Ann. Sci. École Norm. Sup. **6** (1973), 413–455.

[41] Lee, J. M., *Introduction to Smooth Manifolds*, second edition, Springer, 2013.

[42] Lerman, E., Contact toric manifolds. J. Symplectic Geom. 1 (2003), 785–828.

[43] Lerman, E., Symplectic cuts, *Math. Res. Lett.* **2** (1995), 247-258.

[44] Lerman, E., Geodesic flows and contact toric manifolds. Symplectic geometry of integrable Hamiltonian systems (Barcelona, 2001), 175–225, Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel, 2003.

[45] Lerman, E., Meinrenken, E., Tolman, S., Woodward, C., Nonabelian convexity by symplectic cuts, *Topology* **37** (1998), 245-259.

[46] Lerman, E., Tolman, S., Hamiltonian torus actions on symplectic orbifolds and toric varieties, *Trans. Amer. Math. Soc.* **349** (1997), 4201-4230.

[47] Marsden, J., Weinstein, A., Reduction of symplectic manifolds with symmetry, *Rep. Mathematical Phys.* **5** (1974), 121-130.

[48] McDuff, D., Examples of simply-connected symplectic non-Kählerian manifolds, *J. Differential Geom.* **20** (1984), 267-277.

[49] McDuff, D., Salamon, D., *Introduction to Symplectic Topology*, third edition. Oxford University Press, Oxford, 2017.

[50] Meinrenken, E., Symplectic geometry. Lecture notes in the author's website `www.math.toronto.edu/mein/teaching/LectureNotes/sympl.pdf`.

[51] Meyer, K., Symmetries and integrals in mechanics, *Dynamical Systems* (Proc. Sympos., Univ. Bahia, Salvador, 1971), 259-272. Academic Press, New York, 1973.

[52] Mikhalkin, G., Examples of tropical-to-lagrangian correspondence, Eur. J. Math. **5** (2019), 1033–1066.

[53] Milnor, J., *Morse Theory*, based on lecture notes by M. Spivak and R. Wells, Annals of Mathematics Studies **51**, Princeton University Press, Princeton, 1963.

[54] Morse, M., The foundations of a theory in the calculus of variations in the large, *Trans. Amer. Math. Soc.* **30** (1928), 213-274.

[55] Moser, J., On the volume elements on a manifold, *Trans. Amer. Math. Soc.* **120** (1965), 286–294.

[56] Oda, T., *Convex Bodies and Algebraic Geometry – An Introduction to the Theory of Toric Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **15**, Springer-Verlag, Berlin, 1988.

[57] Oesinghaus, J., *Symplectic Toric Bestiary*, Bachelor Thesis at ETH Zurich, 2012.

[58] Satake, I., On a generalization of the notion of manifold, *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), 359-363.

[59] Scott, P., The geometries of 3-manifolds, *Bull. London Math. Soc.* **15** (1983), 401-487.

[60] Souriau, J.-M. , *Structure des Systèmes Dynamiques*, Maîtrises de Mathématiques, Dunod, Paris 1970.

[61] Spivak, M., *A Comprehensive Introduction to Differential Geometry*, Vol. I, second edition, Publish or Perish, Inc., Wilmington, 1979.

[62] Weinstein, A., *Lectures on Symplectic Manifolds*, Regional Conference Series in Mathematics **29**, Amer. Math. Soc., Providence, 1977.

[63] Weinstein, A., Symplectic manifolds and their Lagrangian submanifolds. *Advances in Math.* **6** (1971), 329–346.

[64] Weyl, H., *The Classical Groups. Their Invariants and Representations*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, 1997.

[65] Wells, R.O., *Differential Analysis on Complex Manifolds*, second edition, Graduate Texts in Mathematics **65**, Springer-Verlag, New York-Berlin, 1980.