# SYMPLECTIC TORIC BESTIARY

Bachelor Thesis at the ETH Zürich
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Abstract. A compact connected symplectic manifold endowed with an effective Hamiltonian action of a torus of maximal dimension is uniquely determined by the image of its moment map, which is a polytope of dimension half that of the manifold sitting inside the dual of the Lie algebra of the torus. This is the content of the celebrated Delzant theorem. We use this one-to-one correspondence to collect various results relating combinatorial properties of Delzant polytopes to geometrical properties of symplectic toric manifolds.

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## 1. Introduction and Basic Definitions

Symplectic toric manifolds are extremely symmetric spaces. An example of integrable systems from Hamiltonian mechanics, they are endowed with an effective action of a torus whose dimension is half that of the manifold. Moreover, this action is Hamiltonian, a property that is characterized by the existence of a *moment map*. The geometry of this moment maps encodes every detail of such an action. In fact, the *Delzant theorem* states that both the manifold and the action are uniquely determined by the image of the moment map, which is a polytope in $(\mathbb{R}^n)^*$. Moreover, for every polytope $\Delta$ meeting certain geometrical criteria, there exists a symplectic toric manifold whose moment polytope is $\Delta$. Therefore, many (in theory, all) properties of the space from its moment polytope. We collect results about this species.

In the introductory section, we will shortly summarize the most important notions and tools from symplectic geometry that are needed. For a proper introduction to symplectic geometry, we refer to [CdS,1]. Essential for symplectic toric manifolds

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are the theorems by Atiyah, Guilemin and Sternberg on convexity properties of the moment map of a Hamiltonian toric action and the theory of symplectic reduction.

Section 2 contains the heart of symplectic toric manifold theory: The Delzant theorem. We show how to construct the manifold from the polytope and give examples of classification.

In section 3, we look at a one-to-one correspondence of "symplectic toric submanifolds" to the faces of the polytope. We will also see that a neighborhood of a fixed point has the structure of an open subset in a vector space, on which the torus acts linearly.

Sections 4 and 5 relate properties of the polytope to that of the manifold, namely volume (using the famous Duistermaat-Heckman theorem) and homology (using Morse theory). Section 6 features the interplay between a geometrical operation, the blow-up, and its effect on the polytope, which consists of simply cutting off a part at a corner.

Section 7, building on examples made earlier, contains a classification of the simplest nontrivial class of symplectic toric manifolds, namely those of dimension four.

In section 8, we look at at monotone symplectic toric manifolds, a condition that is mirrored in a direct geometric condition on the polytope.

1.1. **Symplectic Manifolds.** All maps between smooth manifolds will be assumed smooth unless stated otherwise.

A **symplectic form** $\Omega$ on a real vector space $V$ is a map $\Omega : V \times V \to \mathbb{R}$ that is both alternating and nondegenerate, i.e. $\Omega(x,y) = -\Omega(y,x)$ and the map $\Phi : V \to V^*, x \longmapsto \Omega(\cdot, x)$ has trivial kernel. By basic linear algebra, if $V$ admits a symplectic form, then $dim\ V$ is even. In fact, more holds: there is a basis $(v_1, \dots, v_n, w_1, \dots w_n)$ of $V$ such that $\Omega(v_i, w_j) = \delta_{ij}$, $\Omega(v_i, v_j) = 0 = \Omega(w_i, w_j)$ for all $i, j$.

Globalizing this piece of linear algebra yields the following:

**Definition.** A **symplectic manifold** $(M, \omega)$ is a smooth manifold $M$ together with a (smooth) closed symplectic form $\omega \in \Omega^2(M)$; this means that $\omega$ is a 2-form, $d\omega = 0$ (closedness), and it is nondegenerate at each point.

The canonical example is $\left(\mathbb{R}^{2n}, \omega_0\right)$, where

$$\omega_0 = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n \tag{1.1}$$

in coordinates $(x_1, \dots x_n, y_1, \dots, y_n)$. From the complex point of view, $\mathbb{R}^{2n} = \mathbb{C}^n$, and $\omega_0 = \frac{i}{2}\left(dz_1 \wedge d\bar{z}_1 + \ldots + dz_n \wedge d\bar{z}_n\right)$.

In what, follows, let $(M, w)$ be a symplectic manifold, $2n = dim\ \ M$. The existence of a symplectic form implies many topological properties of $M$.

**Proposition 1.1.** *The 2n-form on M given by* $\frac{\omega^n}{n!}$ *is nowhere zero, thus it is a volume form; the induced measure on M is called the* Liouville measure. *Thus, any symplectic manifold is naturally oriented. If M is compact, then by Stokes' Theorem,* $\frac{\omega^n}{n!}$ *is not exact, so* $\omega$ *itself is not exact, therefore* $H^2(M)$ *is nontrivial.*

The Darboux theorem takes care of classifying the local structure of symplectic manifolds:

**Theorem 1.2.** *(Darboux, see* [Aud, II.1.9]*) Let* $p \in M$*. Then there is a coordinate chart* $\phi : U \to U_0 \subseteq \mathbb{R}^{2n}$ *centered at p such that* $\omega|_U = \phi^* i^* \omega_0$*, where* $\omega_0$ *is given as in 1.1 and* $i : U_0 \to \mathbb{R}^{2n}$ *is a neighborhood of* 0 *in* $\mathbb{R}^{2n}$*.*

This means that the only local invariant of a symplectic manifold is its dimension, since all symplectic manifolds locally look the same; all interesting invariants are therefore global in nature!

**Definition.** Let $(M, \omega)$ and $(M', \omega')$ be symplectic manifolds. A diffeomorphism $\Phi : M \to M'$ is said to be a **symplectomorphism** if $\Phi^*\omega' = \omega$. The group of symplectomorphisms of $M$ to itself is denoted by $Sympl(M, \omega)$.

Another way of stating the Darboux theorem is to assert that any two symplectic manifolds of the same dimension are locally symplectomorphic.

1.2. **Tori.** We denote by $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the n-dimensional torus, the quotient of the Lie group $\mathbb{R}^n$ by its closed discrete subgroup $\mathbb{Z}^n$. It is again a Lie group of dimension n. We will also regard $\mathbb{T}^n = S^1 \times \cdots \times S^1$ (n times); the translation between the two variants is given by $(x_1, \dots, x_n) + \mathbb{Z}^n \longmapsto (e^{2\pi i x_1}, \dots, e^{2\pi i x_n})$. There will be many other (possibly confusing) identifications: We will identify the Lie algebra of the torus $\mathfrak{t}^n \backsimeq \mathbb{R}^n$ such that the exponential map becomes $\exp : \mathfrak{t}^n \to \mathbb{T}^n, (x_1, \dots, x_n) \longmapsto (e^{ix_1}, \dots, e^{ix_n})$. We will also identify $(\mathfrak{t}^n)^* \backsimeq (\mathbb{R}^n)^* \backsimeq \mathbb{R}^n$ via the standard inner product. The torus is both compact and connected (and tori are the only compact, connected, abelian Lie groups).

An isomorphism of the torus $\mathbb{T}^n$ is given by a linear map $A : \mathbb{R}^n \mapsto \mathbb{R}^n$ that is an isomorphism of the lattice $\mathbb{Z}^n$, i.e. $A \in GL(n, \mathbb{Z})$. In this context, we define the group of integral affine transformations $AGL(n, \mathbb{Z}) = \mathbb{R}^n \rtimes GL(n, \mathbb{Z})$, whose elements are compositions of linear maps in $GL(n, \mathbb{Z})$ and translations by arbitrary vectors in $\mathbb{R}^n$.

1.3. **Hamiltonian actions and moment maps.**

**Definition.** An action $\psi : G \to \text{Diff}(M)$ of a Lie group $G$ on $M$ is called **symplectic** if it preserves $\omega$, which means that for each $g \in G$, $\psi(g) : M \to M$ is a symplectomorphism of $M$.

We do, however, require somewhat more:

**Definition.** ([CdS,2]) Let $(M, \omega)$ be a symplectic manifold, $G$ a Lie group with Lie algebra $\mathfrak{g}$. A symplectic action $\psi$ of $G$ on $M$ is said to be **Hamiltonian** if there exists a map

$$\mu : M \to \mathfrak{g}^*$$

satisfying:

(1) For each $\xi \in \mathfrak{g}$, let $\xi^{\#}$ be the vector field generated on $M$ by the one-parameter subgroup $\{\exp(t\xi) | t \in \mathbb{R}\}$. Let $\mu^\xi : M \to \mathbb{R}$, $\mu^\xi(p) = \langle \mu(p), \xi \rangle$. Then

$$d\mu^\xi = \imath_{\xi^{\#}}\omega \tag{1.2}$$

.

(2) The map $\mu$ is equivariant with respect to $\psi$ and the coadjoint action $Ad^*$, i.e. $Ad^*_g \circ \mu = \mu \circ \psi_g$.

The map $\mu$ is then called a **moment map** for $\psi$ and the quadruple $(M, \omega, G, \mu)$ is called a **Hamiltonian G-space**. If it exists, $\mu$ is completely determined by the action up to an additive constant.

*Remark.* Some authors define 1.2 with the opposite sign.

**Proposition 1.3.** *Let $G$ be a Lie group and $H \subseteq G$ a closed Lie subgroup of $G$, and let $i : \mathfrak{h} \to \mathfrak{g}$ be the inclusion of lie algebras. Suppose that $(M, \omega, G, \mu)$ is a Hamiltonian G-space. Then the restriction of the G-action to H is Hamiltonian with moment map*

$$i^* \circ \mu : M \to \mathfrak{h}^*$$

*Proof.* 1. Let $\xi \in \mathfrak{h}$. Then 1.2 holds for $\mu$ and the vector $i(\xi) \in \mathfrak{g}$. Precisely:

$$d(\langle \mu(p), i(\xi)\rangle) = \imath_{(i(\xi))^\#}\omega$$

Dualizing on the left side, and using the fact that $(i(\xi))^\# = \xi^\#$,we get

$$d(\langle i^* \circ \mu(p), \xi\rangle) = \imath_{\xi^\#}\omega$$

2. The coadjoint action of $H$ is just the restriction of the coadjoint action of $G$. □

We will only look at the case $G = \mathbb{T}^n$. In this case, as $\mathbb{T}^n$ is abelian, the coadjoint action is trivial, and condition (2) becomes invariance. The image of the moment map of Hamiltonian $\mathbb{T}^n$-action has special properties:

**Theorem 1.4.** *(Atiyah, Guillemin-Sternberg* [GS,2]*) Let $(M, \omega, \mathbb{T}^n, \mu)$ be a compact connected Hamiltonian $\mathbb{T}^n$-space. Then*

(1) *the levels of $\mu$ are connected;*
(2) *the image of $\mu$ is convex;*
(3) *the image of $\mu$ is the convex hull of the fixed points of the action.*

The image is called the **moment polytope** of the action.

**Definition.** A **symplectic toric manifold** is a compact connected Hamiltonian $\mathbb{T} = \mathbb{T}^n$-space $(M, \omega, \mathbb{T}, \mu)$ of dimension $2n$ such that the action of $\mathbb{T}$ is effective. Two symplectic toric manifolds $(M_i, \omega_i, \mathbb{T}, \mu_i)$ are said to be *equivalent* if there exists a $\mathbb{T}$-equivariant symplectomorphism $\varphi : M_1 \to M_2$ such that $\mu_1 = \mu_2 \circ \varphi$. Evidently, the moment polytopes of equivalent symplectic toric manifolds are identical. Delzant's theorem will provide a converse to this statement.

1.4. **Symplectic reduction.** Symplectic reduction allows us, in some cases, to give the orbit space of a Hamiltonian action a symplectic structure.

**Theorem 1.5** (Marsden-Weinstein-Meyer [MW]). *Let $(M, \omega, G, \mu)$ be a Hamiltonian $G$-space for a compact Lie group $G$. Let $i : \mu^{-1}(0) \hookrightarrow M$ be the inclusion. Assume that $G$ acts freely on $\mu^{-1}(0)$. Then*

- *$M_{red} = \mu^{-1}(0)/G$ is a manifold*
- *$\pi : \mu^{-1}(0) \to M_{red}$ is a principal $G$-bundle*
- *there is a symplectic form $\omega_{red}$ on $M_{red}$ such that $i^*\omega = \pi^*\omega_{red}$.*

Now, suppose that another Lie group $H$ also acts on $M$ in a Hamiltonian fashion, and let $\lambda : M \to \mathfrak{h}^*$ be a moment map for this action. One can ask if the action induces a Hamiltonian action on $M_{red}$. The answer is fairly straightforward:

**Proposition 1.6.** *Suppose that the action of $G$ commutes with that of $H$. Then $H$ acts on $M_{red}$ in the obvious way such that $\pi$ becomes equivariant, and this action is Hamiltonian with moment map $\tilde{\lambda} : M_{red} \to \mathfrak{h}^*$ such that $\tilde{\lambda} \circ \pi = \lambda \circ i$ .*

*Proof.* We define the action by $h \cdot \pi(x) = \pi(h \cdot x)$. This is well-defined by commutativity and because $\mu$ is $H$-invariant (after all, $H$ acts by $G$-equivariant symplectomorphisms). Now let $\xi \in \mathfrak{h}$. Let $\xi'$ be the vector field on $M$ corresponding to $\xi$, and let $\xi^\#$ be the vector field on $M_{red}$. Let $p = \pi(q)$. We may define $\xi^\#$ by $\xi^\#_p = d\pi_q \xi'_q$; this is our only choice if $\pi$ is to be equivariant, and it does not depend on the choice of $q$. Define $\tilde{\lambda}$ by the above formula. This is possible since $\lambda$ is $G$-invariant. For any $v = d\pi(w) \in T_pM_{red}$, we calculate:

$$d\left\langle \tilde{\lambda}, \xi \right\rangle|_p v = \pi^* d(\left\langle \tilde{\lambda}, \xi \right\rangle)|_q w = d\pi^*(\left\langle \tilde{\lambda}, \xi \right\rangle)|_q w = d(\left\langle \tilde{\lambda} \circ \pi, \xi \right\rangle|_q w =$$

$$= d\left\langle \lambda \circ i, \xi \right\rangle|_q w = i^* \imath_{\xi'}\omega|_q w = (i^*\omega)(\xi', \cdot)|_q w = (\pi^*\omega_{red})(\xi', \cdot)|_q w =$$

$$= \omega_{red}|_{\pi(q)}(d\pi\xi'_q, \cdot)v = \omega_{red}(\xi_p^{\#}, \cdot))|_p v = \imath_{\xi^{\#}}\omega_{red}|_p v.$$

Moreover, the map $\tilde{\lambda}$ is equivariant since $Ad_h^* \circ \tilde{\lambda} \circ \pi(x) = Ad_h^* \circ \lambda \circ i(x) = \lambda \circ (h \cdot x) = \lambda \circ i(h \cdot x) = \tilde{\lambda} \circ \pi(h \cdot x) = \tilde{\lambda}(h \cdot \pi(x))$. This proves that $\tilde{\lambda}$ is a moment map for the $H$-action. □

## 2. Delzant's Theorem

We will see that equivalence classes of symplectic toric manifolds are completely determined by the image of the moment map. After stating the Delzant theorem, which makes this classification precise, we will present the construction which proves the "existence" part of the theorem and will be important in further applications.

**Definition.** A **Delzant polytope** $\Delta \subseteq \mathbb{R}^n$ is a polytope (i.e. the convex hull of a finite set of points in $\mathbb{R}^n$) that is

- **simple:** there are n edges meeting at each vertex,
- **rational:** for each vertex p, the edges meeting at p are of the form $\{p + tv_i,\ t \geq 0\}$ for some $v_i \in \mathbb{Z}^n$,
- **smooth:** for each vertex p, the edge vectors may be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

**Theorem 2.1.** *(**Delzant** [Del]) There is a bijective correspondence, given by the moment map, between symplectic toric manifolds $(M, \omega, \mathbb{T}^n, \mu)$ up to equivalence and Delzant polytopes $\Delta = \mu(M)$.*

*Remark.* This bijective correspondence respects all the data, including the choice of moment map $\mu$. Since all possible moment maps for a particular $\mathbb{T}^n$-action differ by a constant, there is a less rigid variant of the above:

**Corollary 2.2.** *There is a bijective correspondence between symplectic toric manifolds $(M, \omega)$ (with a given effective Hamiltonian $\mathbb{T}^n$-action) up to equivalence and the choice of a moment map and Delzant polytopes $\Delta$ up to translation.*

Taking the above further, we might want to fix the action of $\mathbb{T}^n$ only up to isomorphism. An isomorphism $\phi$ of $\mathbb{T}^n$ is an isomorphism of $\mathbb{R}^n$ that restricts to an isomorphism of the lattice $\mathbb{Z}^n$, given by a linear isomorphism $A \in GL(n, \mathbb{Z})$. What is the moment map for the action $\psi \circ \phi$ ? This can be seen as follows: Let $\xi \in \mathfrak{g}$. We let the one-parameter subgroup generated by $\xi$ act on M. This generates the vector field $(A\xi)^{\#}$ (in terms of the action $\psi$). We know that

$$\imath_{(A\xi)^{\#}}\omega = d\mu^{A\xi} = d(\langle \mu, A\xi \rangle) = d(\langle A^* \mu, \xi \rangle),$$

so $A^* \circ \mu$ is the new moment map. We record this as a theorem, giving a coarser (but sometimes useful) adaption of Delzant's theorem.

**Theorem 2.3.** *There is a bijective correspondence between compact connected symplectic manifolds $(M^{2n}, \omega)$, equipped with an effective Hamiltonian action of the n-torus $\mathbb{T}^n$ up to equivalence, a choice of a moment map and an isomorphism of the torus and Delzant polytopes $\Delta \subseteq \mathbb{R}^n$ up to the action of $AGL(n, \mathbb{Z})$.*

### 2.1. **The Delzant construction.**

Because the construction will be of importance later (and is interesting in its own right!), we will go through one half of the proof and construct, for a given Delzant polytope $\Delta \subseteq \mathbb{R}^n$, a symplectic toric manifold $(X_\Delta, \omega, \mathbb{T}^n, \mu)$ such that $\mu(X_\Delta) = \Delta$. The second half ([Del]) would then assert that any symplectic toric manifold with moment polytope $\Delta$ is equivariantly symplectomorphic to $X_\Delta$ (and that the moment polytopes of symplectic toric manifolds are Delzant). The construction follows [CdS,1] and [Gui].

So, let $\Delta$ be a $n$-dimensional Delzant polytope with $d$ facets. Let $u_i \in \mathbb{Z}^n$ be the primitive outward-pointing normal vector to the i-th facet (a vector $v \in \mathbb{Z}^n$ is called primitive if it cannot be written as $v = kw$ with $k \in \mathbb{Z}\backslash\{-1,0,1\}$ and $w \in \mathbb{Z}^n$). Then,

$$\Delta = \{x \in (\mathbb{R}^n)^* | \langle x, u_i \rangle \leq \lambda_i,\ i \in \{1, ...d\}\}$$

for appropriate $\lambda_i \in \mathbb{R}$. Let $\pi : \mathbb{Z}^d \to \mathbb{Z}^n,\ e_i \mapsto u_i$. By the assumptions on $\Delta$, this map is surjective. By extending the above to $\mathbb{R}^d$, one gets an induced map (also called $\pi$) $\mathbb{T}^d \to \mathbb{T}^n$. Denoting the kernel by $N$, we get an exact sequence of tori:

$$0 \longrightarrow N \xrightarrow{i} \mathbb{T}^d \xrightarrow{\pi} \mathbb{T}^n \longrightarrow 0,$$

of Lie algebras (here we use the identifications mentioned in 1.2):

$$0 \longrightarrow \mathfrak{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \longrightarrow 0$$

and its dual:

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{n}^* \to 0.$$

Let now $\mathbb{T}^d$ act on $(\mathbb{C}^d, \omega_0)$ by diagonal multiplication: $(e^{ix_1}, \ldots, e^{ix_d})\cdot(z_1, \ldots, z_d) = (e^{ix_1}z_1, \ldots, e^{ix_d}z_d)$. This action is Hamiltonian, and we choose the moment map

$$J : \mathbb{C}^d \to (\mathbb{R}^d)^*,\ (z_1, \ldots, z_d) \mapsto -\frac{1}{2}(|z_1|^2, \ldots, |z_d|^2) + (\lambda_1, \ldots, \lambda_d).$$

By Proposition 1.3, the restriction of this action to $N$ is Hamiltonian with moment map $i^* \circ J : \mathbb{C}^d \to \mathfrak{n}^*$. Let $Z = (i^* \circ J)^{-1}(0)$.

**Lemma 2.4.** *$Z$ is compact and $N$ acts freely on $Z$.*

Thus, the reduced space $X_\Delta := Z/N$ is a 2n-dimensional symplectic manifold (by Marsden-Weinstein-Meyer). If $j : Z \to \mathbb{C}^d$ is the inclusion and $\pi_\Delta : Z \to X_\Delta$ is the point-orbit projection , then the symplectic form $\omega_\Delta$ on $X_\Delta$ satisfies $\omega_\Delta \circ \pi_\Delta = \omega_0 \circ j$. We will see that there is a Hamiltonian action of $\mathbb{T}^n$ on $X_\Delta$ whose moment polytope is $\Delta$. First, we need some machinery:

**Proposition 2.5.** *Let $z \in \mathbb{C}^d$. Let $I = \{i \in \{1, \ldots d\} | z_i = 0\}$. Let*

$$\mathbb{R}^I = \{x \in \mathbb{R}^d | x_i = 0, i \notin I_z\}.$$

*Then the stabilizer subgroup of $z$ with regard to the action of $\mathbb{T}^d$ is*

$$\mathbb{T}^I := \mathbb{R}^I/(\mathbb{R}^I \cap \mathbb{Z}^d) \subseteq \mathbb{T}^d.$$

*Proof.* Meditate on the the action of the $j$-th subtorus. □

**Lemma 2.6.** *Let $\Delta' = \pi^*\Delta$. Then $Z = J^{-1}(\Delta')$. Thus, $z \in Z$ if and only if $J(z) = \pi^* y$ for some (unique) $y \in \Delta$.*

*Proof.* Let $x \in J^{-1}(\Delta')$, and let $y$ such that $J(x) = \pi^* y$. Then

$$(i^* \circ J)(x) = (i^* \circ \pi^*)(y) = 0.$$

For the other direction, let $x \in Z$. Since $i^*(J(x)) = 0$, $J(x) \in Ker\ i^* = Im\ \pi^*$, and thus $J(x) = \pi^* y$ for some $y \in \mathbb{R}^n$. We want to show that $y \in \Delta$. In fact, $\langle e_i, J(x) \rangle \leq \lambda_i\ \forall i$ (by the definition of J), therefore

$$\langle e_i, \pi^* y \rangle \leq \lambda_i\ \forall i \iff \langle u_i, y \rangle \leq \lambda_i\ \forall i$$

and thus $y \in \Delta$. □

**Lemma 2.7.** *Let $y \in \Delta$. Let $y$ lie in the interior*[1] *of the intersection of the facets $\langle u_i, y\rangle = \lambda_i,\ i \in I$ for $I \subseteq \{1, \dots d\}$ . Let $x = \pi^* y = J(z)$. Then the stabilizer subgroup of $z$ is $T^I$ (Note: for $I = \emptyset$, we define the intersection to be $\Delta$).*

*Proof.* We have to show that $z_i = 0$ precisely for the $i \in I$, for any $z$ as above. But note that

$$\langle u_i, y\rangle = \langle \pi e_i, y\rangle = \langle e_i, \pi^* y\rangle = \langle e_i, x\rangle = x_i = -\frac{1}{2}|z_i|^2 + \lambda_i.$$

Thus, $\langle u_i, y\rangle = \lambda_i$ if and only if $z_i = 0$ for every such $z$. □

Finally, we can prove the theorem.

*Proof.* $J$ is a proper map. In fact, if $C \subseteq (\mathbb{R}^d)^*$ is bounded below in each component by $A \in \mathbb{R}$, then $-\pi |z_i|^2 + \lambda_i$ is bounded below for each $z \in J^{-1}(C)$. Thus, the absolute value of elements of $J^{-1}(C)$ is bounded. $\Delta'$ is compact; therefore, $Z = J^{-1}(\Delta')$ is compact.

It remains to show that $N$ acts freely on $Z$. So, let $y \in \Delta$ lie in the intersection of facets as in lemma 2.7, and let $\pi^* y = J(z)$. By the lemma, the stabilizer subgroup of $z$ is contained in the stabilizer subgroup of $J^{-1}(\pi^*(p))$, where $p$ is any vertex in the that lies in the same face. Thus, it is sufficient to check that for any vertex $p$ such that $p$ lies in the intersection of the facets $\langle u_i, y\rangle = \lambda_i,\ i \in I$, where $|I| = n$.

However, the vectors $u_i$ form a basis of $\mathbb{Z}^n$ (since the $v_j$ going out from $p$ do). The best way to see this is that up to action of $SL(n, \mathbb{Z})$ the $v_j$ are just the standard basis of $(\mathbb{Z}^n)^*$, and thus the $u_i$ are dual to this basis, up to multiplication by $-1$. That means that $\mathfrak{n} \cap \mathrm{span}\{e_i, i \in I\} = \{0\}$, as the map $\pi$ takes $\mathrm{span}\{e_i, i \in I\}$ isomorphically onto $\mathbb{Z}^n$. Therefore the intersection of $N$ with the stabilizer subgroup, which is generated by the $e_i$, is zero. Thus, $N$ acts freely on $Z$. □

**Proposition 2.8.** *The $n$-torus $\mathbb{T}^n$ acts on the quotient space $X_\Delta$ in a Hamiltonian way, and $\mu(X_\Delta) = \Delta$.*

*Proof.* Of course, the quotient $\mathbb{T}^n = \mathbb{T}^d/N$ acts on $X_\Delta$. We turn this into the action of a subgroup: choose any vertex $p$ of $\Delta$. The stabilizer subgroup $\mathbb{T}^d_p$ is mapped isomorphically onto $\mathbb{T}^n$ by $\pi$, therefore we can identify it with $\mathbb{T}^n$. Let $i_0 : \mathbb{T}^n \hookrightarrow \mathbb{T}^d$ be an inclusion such that $\pi \circ i_0 = id$, with the corresponding map $i_0 : \mathfrak{t}^n \to \mathfrak{t}^d$. The moment map $\mu$ then satisfies $\mu \circ \pi_\Delta = i_o^* \circ J \circ j$. Thus, $\mu(X_\Delta) = \mu \circ \pi_\Delta(Z) = i_o^* \circ J(Z) = i_o^* \circ \pi^*(\Delta) = (\pi \circ i_0)^* \Delta = id^* \Delta = \Delta$. □

2.2. **Examples.** The most basic example is **complex projective space** $\mathbb{CP}^n = (\mathbb{C}^{n+1} \backslash \{0\})/\sim$, where $v \sim w \iff v = \lambda w$ for some $\lambda \in \mathbb{C}$. Another way to look at $\mathbb{CP}^n$ is the following: let $S^1$ act on $(\mathbb{C}^{n+1}, \omega_0)$ by multiplication. This action is Hamiltonian, and a moment map is $\mu(z_0, \dots, z_n) = -\frac{1}{2}(\sum |z_i|^2) + \frac{1}{2}$. The circle $S^1$ acts freely on the compact zero level of the moment map $\mu^{-1}(0) = S^{2n+1}$, thus we can perform symplectic reduction and are given the symplectic manifold $(\mathbb{CP}^n, \omega_{FS})$, where $\omega_{FS}$ is the corresponding symplectic form induced by reduction. It is called the *Fubini-Study form* on $\mathbb{CP}^n$. Let $\mathbb{T}^n$ act on $\mathbb{C}^{n+1}$ such that $(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot (z_0, \dots, z_n) = (z_0, e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$. This action is Hamiltonian, a moment map is $\phi(z_0, \dots, z_n) = -\frac{1}{2}(|z_1|^2, \dots, |z_n|^2)$. The action descends onto an action on $\mathbb{CP}^n$. By proposition 1.6, we know what the corresponding moment map $\tilde{\phi}$ must be: If $[z_0, \dots, z_n] \in \mathbb{CP}^n$ such that $(z_0, \dots, z_n) \in S^{2n+1}$, then $\tilde{\phi}([z_0, \dots, z_n]) = \phi(z_0, \dots, z_n) = -\frac{1}{2}(|z_1|^2, \dots, |z_n|^2)$. Since $0 \le \sum_{i=1}^n |z_i|^2 \le 1$, it follows that the image of $\tilde{\phi}$ is the simplex in $\mathbb{R}^n$ with vertices $(0, \dots, 0)$ and $-\frac{1}{2} e_i,\ i \in \{1, \dots n\}$. This can be seen more directly: the fixed points of the action

[1] In this context, we mean that $y$ is in the *interior* of an intersection of facets defined by $I$ if $y$ is not an element of an intersection of facets for some $J$ such that $I \subsetneq J$.

are $[1,0,\dots,0],[0,1,0,\dots],\dots,[0,\dots,0,1]$ and their images are the vertices of the simplex. The above action is effective: the stabilizer of a generic (i.e. $z_i \neq 0\ \forall i$) point $[z_0,\dots,z_n]$ is trivial .

Thus $\mathbb{CP}^n$ is the first example for a symplectic toric manifold.

**Proposition 2.9.** *$\mathbb{CP}^n$ with this action of $\mathbb{T}^n$ is, up to scaling of $\omega$, equivariant symplectomorphism, and an isomorphism of the torus, the unique symplectic toric manifold of dimension $2n$ that has $n+1$ fixed points.*

**Fact 2.10.** *Scaling a symplectic form by a nonzero scalar $\lambda \in \mathbb{R}$, while fixing both the symplectic toric manifold and torus action, allows for a moment map $\lambda\cdot\mu$. This follows directly from the definition of the moment map.*

*Proof.* Using the fact above, we only need to classify $n$-dimensional Delzant polytopes $\Delta$ with $n+1$ vertices, up to the action of $AGL(n,\mathbb{Z})$ (and up to scaling by a constant). If we pick any vertex $p$, since the edges at $p$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$, we can arrange such that $p$ is the origin and the edges meeting at $p$ are $te_i,\ i \in \{1,\dots,n\}, t \in \mathbb{R}$. Therefore, all vertices lie on the coordinate axes; let the vertices be $v_i = a_i e_i$ where $a_i \in \mathbb{R}^+$. First, we normalize such that $a_1 = 1$. Since $-e_1 + a_k e_k \in \mathbb{Q}^n$, as these are the edge vectors, all the $a_k$ become rational, and we can multiply with the largest common denominator to achieve $a_i \in \mathbb{N}$.

Suppose that one of the $a_k \neq a_1$. Without loss of generality $k = 2$. Then the edge vectors meeting at $v_1$ are $-e_1, \lambda_2(-a_1e_1 + a_2e_2), \dots, \lambda_n(-a_1e_1 + a_ne_n)$, and those meeting at $v_2$ are $-e_2, \kappa_2(-a_2e_2 + a_1e_1), \dots, \kappa_n(-a_2e_2 + a_ne_n)$, where the $\lambda_i$ and $\kappa_i$ are chosen such that the resulting sets are $\mathbb{Z}$-bases of $\mathbb{Z}^n$. Therefore, their projections onto $\mathbb{Z}^2$ span $\mathbb{Z}^2$. However, the projections of the last (n-2) vectors are simply multiples of $e_1$ and $e_2$ respectively, so we can discard them. Thus, both matrices

$$\begin{pmatrix} -1 & -a_1\lambda_2 \\ 0 & \lambda_2 a_2 \end{pmatrix}, \begin{pmatrix} \kappa_2 a_1 & 0 \\ -\kappa_2 a_2 & -1 \end{pmatrix}$$

are $\mathbb{Z}$-invertible and $\lambda_2 a_2 = \kappa_2 a_1 = 1$. Thus, plugging this into the above bases, we get that $\frac{a_1}{a_2} \in \mathbb{N} \ni \frac{a_2}{a_1}$, which is only possible if $a_1 = a_2$. This proves that all $a_i$ are the same, and thus the polytope is, up to the equivalence above, the moment polytope for the standard toric action on $\mathbb{CP}^n$. Apply Delzant's theorem. □

The second example I will bring are the **Hirzebruch surfaces**, a family of 2-dimensional complex manifolds, which are actually symplectic toric manifolds. Instead of constructing them explicitly, we will use Delzant's theorem:

**Proposition 2.11.** *Let $\Delta \subseteq \mathbb{R}^2$ be a Delzant polytope with 4 vertices. Then up to the action of $AGL(2,\mathbb{Z})$, $\Delta = H_{a,b,n}$, where $H_{a,b,n}$ is the polytope with vertices $(0,0),(0,b),(a+nb,0),(a,b)$, defined for each $(a,b,n) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{N}$.*

*Proof.* It is immediate to check that each of the polytopes $H_{a,b,n}$ is Delzant. On the other hand, let $\Delta \subseteq \mathbb{R}^2$ be Delzant, and suppose that $\Delta$ has 4 vertices. Choose any vertex, translate it to zero, and use $GL(2,\mathbb{Z})$ to turn the edges at zero into the standard basis. Three vertices are already known: $(0,0)$, $(0,b)$ and $(A,0)$ (by reflecting along the line parallel to $(1,1)$ if necessary, we can achieve that $A \geq b$). Let $v_1 = (c_{11}, c_{12}) \in \mathbb{Z}^n$ be the non axial edge vector starting from $(0,b)$, and $v_2 = (c_{21}, c_{22}) \in \mathbb{Z}^n$ be the non axial edge vector starting from $(A,0)$. Since they must meet, $\det(c_{ij}) > 0$; they must form a $\mathbb{Z}$-basis, so $\det(c_{ij}) = 1$. We know that $c_{11} = 1$ and $c_{22} = 1$, by the Delzant condition at $(0,b)$ and $(A,0)$. Therefore, $c_{21}c_{12} = 0$. Without loss of generality, $c_{12} = 0$. Thus, the fourth vertex is $(a,b)$ for some $a \in \mathbb{R}^+$. Then $v_2 = (-n,1)$ for some $n \in \mathbb{Z}$ and $A = a + nb$. □

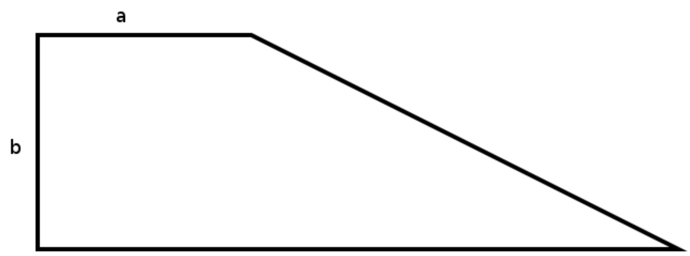


FIGURE 2.1. The Hirzebruch trapezoid $H_{1,1,2}$.

*Note.* For $a = b = 1$, the symplectic toric manifold corresponding to the polytope $H_{1,1,n}$ is called the $n$-th *Hirzebruch surface* $\mathcal{H}_n$. It has the structure of a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$. The polytope $H_{1,1,n}$ is characterized by $u_1 = -e_1, u_2 = e_2, u_3 = e_1 + ne_2, u_4 = -e_2$ and $\lambda = (0, 1, 1+n, 0)$. One can use the Delzant construction above to explicitly exhibit $\mathcal{H}_n$ as a quotient:

$$\mathcal{H}_n = \{z \in \mathbb{C}^4 |\ |z_2|^2 + |z_4|^2 = 2, |z_1|^2 + |z_3|^2 + n|z_4|^2 = 2(1+n)\}/\sim,$$

where $\sim$ is the equivalence relation $(z_1, z_2, z_3, z_4) \sim (e^{i\theta}z_1, e^{-in\theta}e^{i\phi}z_2, e^{i\theta}z_3, e^{i\phi}z_4)$. The projection $p : \mathcal{H}_n \to \mathbb{CP}^1$ is the map induced by the map

$$\tilde{p} : \{z \in \mathbb{C}^4 |\ |z_2|^2 + |z_4|^2 = 2, |z_1|^2 + |z_3|^2 + n|z_4|^2 = 2(1+n)\} \to \mathbb{CP}^1,\ z \longmapsto [z_1, z_3]$$

This is

- well-defined: $(z_1, z_3) \neq 0$, and $\tilde{p}$ is constant on equivalence classes,
- surjective: fix $z_2 = 0$, $|z_4| = \sqrt{2}$, the image of this set is $\{[z_1, z_3]|\ |z_1|^2 + |z_3|^2 = 2\} \simeq \mathbb{CP}^1$,

The fibers are copies of $\mathbb{CP}^1$: Fix $\ell \in \mathbb{CP}^1$. Pick the unique representative $(\hat{z}_1, \hat{z}_3) \in \ell$ with absolute value 1 such that $\widehat{z}_1$ is real and nonzero (if $z_1 = 0$, the proof goes similarly). For each $t \in [2, 2(1+n)]$, this gives us a unique representative of $\ell$ with absolute value $t$, namely $t(\hat{z}_1, \hat{z}_3)$. Define a map $\tilde{q} : \tilde{p}^{-1}(\ell) \to \mathbb{CP}^1, z \longmapsto [(\frac{z_1}{\hat{z}_1})^n z_2, z_4]$. This map is surjective: in fact, note that as $t$ goes from 2 to $2(1+n)$, $|z_4|^2$ goes from 2 to 0 as we stay on the line $t(\hat{z}_1, \hat{z}_3)$. Thus, each point on the sphere $|z_2|^2 + |z_4|^2 = 2$ is contained in the preimage and the map is surjective. It is well-defined after passing to the quotient, as

$$[z_2, z_4] = [(\frac{e^{i\theta}\hat{z}_1}{\hat{z}_1})^n e^{-in\theta}e^{i\phi}z_2, e^{i\phi}z_4]$$

The induced map $q : p^{-1}(\ell) \to \mathbb{CP}^1$ is injective (the main point is that $|z_4|^2$ is directly determined by $|z_1|^2 + |z_3|^2$). So $q$ is at least a homeomorphism. The direct construction of a differentiable inverse shows that it is a diffeomorphism.

- To construct such an inverse, start from $\{z|\ |z_2|^2 + |z_4|^2 = 2\}$ and send $(z_2, z_4)$ to $\pi((2 + 2n - n|z_4|^2)\hat{z}_1, z_2, (2 + 2n - n|z_4|^2)\hat{z}_3, z_4)$. This descends to a map from $\mathbb{CP}^1$ and, as one can check, it is the inverse map to $q$.
- An adaption of the above argument shows that local triviality is satisfied (the map $\tilde{q}$ didn't involve any choices, but it does not work on $p^{-1}([0, 1])$, where an analogous construction with $z_3$ is possible)

The manifold $\mathcal{H}_0$ is simply a product of spheres $\mathbb{CP}^1 \times \mathbb{CP}^1$, while the other $\mathcal{H}_n$ are twisted versions of this product. To be precise: all the manifolds are distinct as complex (or symplectic) manifolds; however, there are only two diffeomorphism classes among them. One way to see this is the following: a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ is simply a choice of gluing the equators of the two half-spheres (over which the bundle is trivial) together. Milnor ([Mil2]) showed that the group of orientation preserving diffeomorphisms of $S^2 = \mathbb{CP}^1$ retracts onto $SO(3)$, and $\pi_1(SO(3)) = \mathbb{Z}_2$.

A way to distinguish the Hirzebruch surface in the symplectic sense is by their Chern class $c_1$. They were originally found in [Hir].

## 3. Invariant Submanifolds

In this section we want to look at $\mathbb{T}^n$-invariant symplectic submanifolds of a given symplectic toric manifold $X_\Delta$ corresponding to a Delzant polytope $\Delta$. It turns out that the answer is very nice and regular. In some special cases, we will also look at *neighborhoods* of these submanifolds in $X_\Delta$. The topology of these neighborhoods depends directly upon combinatorial properties of $\Delta$. The moment map $\mu$ is a proper and closed map (as $X_\Delta$ is compact), and $\mu^{-1}(A)$ is connected for each connected subset $A \subseteq \Delta$; this will make some of the following arguments easier.

**Proposition 3.1.** *For each $p \in \Delta$, $\mu^{-1}(p)$ is a single $\mathbb{T}^n$-orbit. Said differently,* $\Delta$ is *the orbit space of the $\mathbb{T}^n$-action. The dimension of $\mu^{-1}(p)$ is equal to the dimension of the (open) face that p is contained in.*

*Remark.* Delzant has proven this in [Del] for any symplectic toric manifold, without recourse to the construction of $X_\Delta$.

*Proof.* The moment map is, by definition, $\mathbb{T}^n$-invariant. So the only thing left to prove is:

If $\mu(x) = \mu(y)$ for $x, y \in X_\Delta$, then $x = gy$ for some $g \in \mathbb{T}^n$.

Consider the standard action of $\mathbb{T}^d = N \times \mathbb{T}^n$ on $\mathbb{C}^d$, with moment map $J : \mathbb{C}^d \to (\mathbb{R}^d)^*$ If $J(z) = J(z')$, then $|z_i| = |z'_i|$ $\forall i$. It follows that $\frac{z_i}{z'_i} = e^{i\theta_i}$ and thus $z = (e^{i\theta_1}, \ldots, e^{i\theta_d})z'$. This proves that the preimage of each point is a single orbit for this action.

Now we return to the general case. Let $x, y$ as above. Let $x', y' \in Z$ such that $\pi_\Delta(x') = x, \pi_\Delta(y') = y$. Then $J(x') = \pi^*(z)$ for some $z \in \Delta$. In particular,

$$\mu(x) = \mu(\pi_\Delta(x')) = (i_0^* \circ J \circ j)(x') = i_0^* \pi^*(z) = z,$$

thus $J(x') = \pi^*(\mu(x))$ and similarly $J(y') = \pi^*(\mu(y))$, thus $x'$ and $y'$ are in the same $\mathbb{T}^d$-orbit and $x' = (h, g)y'$ for $(h, g) \in N \times \mathbb{T}^n = \mathbb{T}^d$. Therefore,

$$x = \pi_\Delta(x') = \pi_\Delta((h, g)y' = \pi_\Delta((0, g)y' = g\pi_\Delta(y') = gy.$$

The result about the dimension follows from lemma 2.7. □

Not only that, all the orbits are isotropic[2] submanifolds. In particular, the preimage of each *interior* point of $\Delta$ is a Lagrangian[3] submanifold. In fact, let $\xi,\eta \in \mathfrak{t}^n$. Then $\omega(\xi^\#, \eta^\#) = (i_{\xi^\#}\omega)(\eta^\#) = d\mu^\xi(\eta^\#) = 0$, since $\mu$ (thus, in particular, $\mu^\xi$) is constant along orbits.

By proposition 3.1, any $\mathbb{T}^n$-invariant submanifold must be the preimage of some subset of $\Delta$.

[2]An isotropic subspace of a symplectic vector space $(V, \omega)$ is a subspace $U \subset V$ such that $\omega|_{U \times U} = 0$. An isotropic submanifold $N \subset M$ is a submanifold such that $T_pN \subset T_pM$ is an isotropic subspace at each $p \in N$.

[3]A Lagrangian submanifold $L \subset M$ of a symplectic manifold $(M, \omega)$ is an isotropic submanifold that satisfies 2 *dim* $L =$ *dim* $M$.

*Remark.* Any invariant submanifold $M = \mu^{-1}(A)$ such that $A$ contains interior points must be of the same dimension as $M$ itself; if it is compact, it is the whole space. More generally, any invariant symplectic submanifold $M$ such that $\mu(M)$ contains interior points of faces of dimension $k$ must have dimension at least $2k$. The preimage of the interior $\mu^{-1}(\text{Int } \Delta)$ is a principal $\mathbb{T}^n$-bundle with contractible base Int $\Delta$. Thus $\mu^{-1}(\text{Int } \Delta) = \text{Int } \Delta \times \mathbb{T}^n$.

If we want to find more interesting examples (in particular, **compact** invariant submanifolds), we have to look at the boundary. We can actually classify all "symplectic toric submanifolds" of a given symplectic toric manifold $X_\Delta$ by faces of $\Delta$.

**Theorem 3.2.** *Let $F \subseteq \Delta$ be a face of dimension $0 \leq k \leq n$. Then $M = \mu^{-1}(F)$ is a compact, connected, invariant, symplectic submanifold of dimension $2k$. Let $S$ be the stabilizer of $M$; then the torus $\mathbb{T}^n/S$ acts effectively on $M$ in a Hamiltonian way, with moment polytope $i^*F$ (where $i^*$ is a rational projection onto a $k$-dimensional subspace that is injective when restricted to $F$), turning $M$ into a symplectic toric manifold. All compact, connected, invariant, symplectic submanifolds are of this type.*

For example, the preimage of an edge is an invariant sphere $\mathbb{CP}^1$. Note that you cannot expect the volume of $F$ to preserved by $i^*$. For example, if $E \subset \Delta$ is an edge, it's preimage will be a sphere of area $\ell$, the *affine length* of $E$ (which is defined as the unique length of $E$ after it has been moved onto a coordinate axis by a transformation $A \in AGL(2, \mathbb{Z})$

*Proof.* Note first that $M$ is compact, connected, and $\mathbb{T}^n$-invariant, by the properties of $\mu$. Without loss of generality, let $F$ be the intersection of facets $\langle u_i, y\rangle = \lambda_i,\ i = 1, \ldots, n-k$. We know that $\pi_\Delta^{-1}(M) = \{0\} \times \mathbb{C}^{d-n+k} \cap Z$. But since $\{0\} \times \mathbb{C}^{d-n+k}$ is a symplectic invariant submanifold of $\mathbb{C}^d$ and $N$ acts freely on $\pi_\Delta^{-1}(M)$, we can perform symplectic reduction on $\pi_\Delta^{-1}(M)$ with regards to the $N$-action (and arrive back at $M$, of course). Thus $M$ is a symplectic manifold of dimension $2(d-n+k) - (d-n) - (d-n) = 2k$.

It is even a symplectic toric manifold. Pick any vertex $p$ lying in $F$, and let $v_1^*, \ldots, v_n^*$ be the $\mathbb{Z}$-basis forming the edges at $p$, such that $v_1^*, \ldots, v_k^*$ span $F$. The inclusion $i : span\{v_1, \ldots, v_k\} \to \mathfrak{t}^n$ is dual to a projection $i^*$, and $i^*F \simeq F$. The subgroup $H$ whose Lie algebra is spanned by $v_1, \ldots, v_k$ therefore acts in a Hamiltonian way with moment polytope $i^*F$. Since $i^*F$ is nondegenerate, the action is effective. Therefore, $H \cap S = \{0\}$ and we can identify $H \approx_\pi \mathbb{T}^n/S$.

It remains to show that every such submanifold is the preimage of a face $F$. In fact, let $N = \mu^{-1}(A)$ be a submanifold as above. Pick an interior point $p$ in a face $F$ of maximal dimension $k$ (and $p \in A$). Then the dimension of $N$ is precisely twice the dimension of $F$: it is as least $2k$, as $\mu^{-1}(p)$ is an isotropic subspace of dimension $k$. It is also at most $2k$, since $A$ contains no points in higher-dimensional (open) faces. Thus $N = \mu^{-1}(F)$, as both are compact and connected. □

3.1. **Neighborhoods of fixed points.** The Darboux theorem says that any two symplectic manifolds of the same dimension are locally the same. In the presence of a symplectic - or, equivalently, for the local picture, Hamiltonian - action of a Lie group, this is no longer true. However:

**Theorem 3.3.** *(Equivariant Darboux theorem,* [GS,1]*) Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold equipped with a symplectic action of a compact Lie group $G$, and let $q$ be a fixed point. Then there exists a $G$-invariant chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$*

*centered at $q$ and $G$-equivariant with respect to a linear action of $G$ on $\mathbb{R}^{2n}$ by symplectomorphisms such that*

$$\omega|_U = \sum_{k=1}^{n} dx_k \wedge dy_k \tag{3.1}$$

*Note.* Symplectomorphisms of $\mathbb{R}^{2n}$ with this symplectic form preserve the complex inner product, so $G$ will act as a subgroup of $U(n)$.

A close inspection of the proof of this reveals that the vector space $\mathbb{R}^{2n}$ is actually the tangent space $T_qM$, on which $G$ acts linearly by its differential. Since $\mathbb{T}^n$ is abelian, any (complex) linear representation of it is a direct sum of one-dimensional representations. We get the following toric adaptation of the above:

**Theorem 3.4.** *(*[CdS,2, Theorem 3.1.2] *or* [Aud, Proof of IV.4.12]*) Let $(M^{2n}, \omega, \mathbb{T}^m, \mu)$ be a Hamiltonian $\mathbb{T}^m$-space, where $q$ is a fixed point. Then there exists a chart $(U, x_1, \dots, x_n, y_1, \dots, y_n)$ centered at $q$ and weights $\lambda^{(1)}, \dots, \lambda^{(n)} \in \mathbb{Z}^m$ such that 3.1 holds and*

$$\mu|_U = \mu(q) - \frac{1}{2}\sum_{k=1}^{n} \lambda^{(k)}(x_k^2 + y_k^2) \tag{3.2}$$

*Precisely, this means that $\mathbb{T}^m$ acts on the space $\{x_k = y_k = 0, k \neq j\}$ by*

$$(e^{i\theta_1}, \dots, e^{i\theta_m}) \cdot v = e^{i\sum_k \lambda_k^{(j)}\theta_k} v, \tag{3.3}$$

*where we identify $z_j = x_j + iy_j$.*

In the case of an *effective* action of $\mathbb{T}^n$,the weights form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$. In fact, the matrix $(\lambda_j^{(i)})$ has trivial kernel, as the action is effective. Assume by contradiction that $e_l$ is not an integral linear combination of the $\lambda^{(k)}$. However, we know that $e_l = \sum_k a_k \lambda^{(k)}$ for $a_k \in \mathbb{Q}$. Then $(e^{2\pi i a_1}, \dots, e^{2\pi i a_n}) \cdot e_j = e^0 e_j$ if $j \neq l$ and $(e^{2\pi i a_1}, \dots, e^{2\pi i a_n}) \cdot e_l = e^{2\pi i} e_l = e_l$, so $(e^{2\pi i a_1}, \dots, e^{2\pi i a_n})$ acts trivially without being the identity (as not all $a_k$ are in $\mathbb{Z}$), contradiction.

Under the moment map $\mu$, the (complex) eigenspaces are sent to the edges, and more generally, the direct sum of $k$ eigenspaces is sent to a $k$-dimensional face. This gives a complete local picture of the $\mathbb{T}^n$-action and is one of the ingredients in the reverse direction of the Delzant theorem.

## 4. Volume

There is a direct connection between the (symplectic) volume of the manifold $X_\Delta$ and the ordinary Euclidean volume of the polytope $\Delta$. In the Delzant case, this becomes especially simple.

**Definition.** Let $\mathfrak{m}$ be the Liouville measure on Borel subsets of $(M, \omega)$, i.e. $\mathfrak{m}(U) = \int_U \frac{\omega^n}{n!}$. Define the **Duistermaat-Heckman measure** $\mathfrak{m}_{DH}$ on Borel subsets of $(\mathbb{R}^n)^*$ by $\mathfrak{m}_{DH}(U) = \mathfrak{m}(\mu^{-1}(U))$ (if the moment map is proper, for example if $M$ is compact).

The main ingredient is the celebrated Duistermaat-Heckman theorem. One corollary of it is the following proposition:

**Theorem 4.1.** *(*[DH]*) Let $M^{2n}$ be connected and $T = \mathbb{T}^\ell$ act effectively (in a Hamiltonian way) on $M$. For each $\xi \in \mathfrak{t}^n$, denote by $M_\xi = \mu^{-1}(\xi)/T$ the reduced space, and by $\omega_\xi$ the symplectic form on $M_\xi$. Then $d\mathfrak{m}_{DH} = f(\xi)d\xi$, where $d\xi$ denotes the Lebesgue measure. For each regular value $\xi$ of $\mu$, the density $f$ satisfies*

$$f(\xi) = \int_{M_\xi} \frac{\omega_\xi^{n-\ell}}{(n-\ell)!}. \tag{4.1}$$

*Remark.* This theorem only takes the nice form given here when the volume of the torus is normalized to be 1. Depending on the source, it might also be $(2\pi)^\ell$, which would have to be multiplied in as a constant. This normalization is really the decision: is $exp(x) = e^{ix}$ or $exp(x) = e^{2\pi ix}$? There are good arguments for both sides. We have chosen the former variant to ease the presentation of this theorem.

*Note.* This is *not* the Duistermaat-Heckman theorem itself, which can be found in [DH].

**Corollary 4.2.** *Let $\Delta$ be a Delzant polytope. Then $\mathfrak{m}(X_\Delta) = \mathfrak{m}_{DH}(\Delta) = vol(\Delta)$: The symplectic volume of $X_\Delta$ is the Euclidean volume of $\Delta$.*

*Proof.* All interior points of the polytope are regular values of $\mu$ (on the other hand, boundary points aren't regular values). This follows, for example, from the basic lemma below. The preimage of $\Delta \backslash \text{Int } \Delta$ is a finite union of codimension-2 submanifolds and thus has measure 0. The preimage of $\Delta^c$ is empty. All we need, therefore, is $f(\xi)$, where $\xi \in \text{Int } \Delta$. However, by Proposition 3.1, $\mu^{-1}(\xi)$ is a single orbit, so the reduced space is a point and has measure $1 = f(\xi)$. Thus

$$\text{vol}(\Delta) = \int_\Delta 1 d\xi = \int_\Delta f(\xi) d\xi = \mathfrak{m}_{DH}(\Delta) = \mathfrak{m}(X_\Delta).$$

□

**Lemma 4.3.** *Let $(M, \omega, G, \mu)$ be a Hamiltonian G-space. Let $p \in M$. Then the kernel of $d\mu_p$ is the symplectic orthocomponent of the tangent space to the G-orbit through p, and the image of $d\mu_p$ is the annihilator of $\mathfrak{g}_\mathfrak{p}$ (the Lie algebra of the stabilizer group $G_p$). In particular, the action is free at p if and only if $d\mu_p$ is surjective.*

*Proof.* Note that $\omega_p(\xi^\#, v) = \langle d\mu_p v, \xi \rangle$ and count dimensions. □

## 5. Morse Functions and Homology

It has been very fruitful to study symplectic toric manifolds via Morse Theory[4]. There is already a $(\mathbb{R}^n)^*$-valued function on a symplectic toric manifold $\Delta$. By pairing it with a suitable $\xi \in \mathbb{R}^n$, we can turn it into a Morse function. We choose $\xi$ such that its components are independent over $\mathbb{Q}$ (following [AB]). This has the following effects:

- the subgroup $\{\exp(t\xi), t \in \mathbb{R}\}$ is dense in $\mathbb{T}^n$; thus, the fixed points of the $\mathbb{T}^n$-action are precisely the fixed points of the $\exp(t\xi)$-action,
- $\xi$ is neither parallel nor orthogonal to any of the edges of $\Delta$, and
- the vertices of $\Delta$ have different projections onto $\xi$.

**Lemma 5.1.** *The function $\mu^\xi$ is a Morse function, whose critical points are the fixed points of the $\mathbb{T}^n$-action. Its index at a critical point p is twice the number of edge vectors $v_k$ at p such that $\langle v_k, \xi \rangle < 0$.*

*Proof.* Since the function $\mu^\xi$ is a Hamiltonian function for the vector field $\xi^\#$, the critical points of $\mu^\xi$ are precisely the fixed points of the flow of $\xi^\#$. Using Theorem 3.4, the local picture of $\mu^\xi$ near a critical point $p$ is

$$\mu^\xi(x, y) = \langle \mu(p), \xi \rangle - \frac{1}{2} \sum_{k=1}^n \left\langle \lambda^{(k)}, \xi \right\rangle (x_k^2 + y_k^2) = \langle \mu(p), \xi \rangle + \frac{1}{2} \sum_{k=1}^n \langle v_k, \xi \rangle (x_k^2 + y_k^2) \tag{5.1}$$

By the choice of $\xi$, $\langle v_k, \xi \rangle \neq 0, \forall k$. The Hessian of $\mu^\xi$ in this coordinate system is a diagonal matrix, in which each of the $\langle v_k, \xi \rangle$ appears twice. This proves the lemma. □

[4]For an introduction to Morse Theory, see the classic [Mil].

**Theorem 5.2** (Morse Inequalities, [Mil]). *Let $f$ be a Morse function on a compact manifold $M$. For $n \in \mathbb{N}$, let $b_n(M) = dim\ H_n(M, \mathbb{Q})$ be the $n$-th Betti number of $M$ and let $C_n$ be the number of critical points of $f$ with index $n$. Then the following inequalities hold:*

$$(a)\ b_n(M) \leq C_n \tag{5.2}$$

$$(b)\ \sum_n (-1)^n b_n(M) = \sum_n (-1)^n C_n \tag{5.3}$$

*Remark.* These are Morse's original inequalities. The above can be strengthened to the statement:

*$M$ has the homotopy type of a CW complex with one cell of dimension $n$ for each critical point of index $n$.*

**Definition.** A Morse function such that the inequality(5.2) becomes equality is called a **perfect Morse function**.

If all its indices are even, a Morse function is perfect, since all the signs in (5.3) are positive. Thus:

**Theorem 5.3.** *Let $X_\Delta$ be a symplectic toric manifold. Then $H_k(M;\mathbb{Z}) = 0$ if $k$ is odd, and $H_{2k}(M;\mathbb{Z}) \approx \mathbb{Z}^{A_k}$, where $A_k$ is the number of vertices which have exactly $k$ inward-pointing edge vectors $v_i$ such that $\langle v_i, \xi\rangle < 0$.*

*Proof.* The cellular chain complex of the CW complex associated with the Morse function $\mu^\xi$ is

$$0 \to \mathbb{Z}^{A_n} \to 0 \to \mathbb{Z}^{A_{n-1}} \to \ldots \to 0 \to \mathbb{Z}^{A_0} \to 0.$$

This CW complex is homotopy equivalent to $M$, so their homology is the same. □

Note that the homology groups of a symplectic toric manifold are all torsion-free; therefore, $H_k(M;\mathbb{Z}) = H^{n-k}(M;\mathbb{Z}) = H_{n-k}(M;\mathbb{Z}) = H^k(M;\mathbb{Z})$ by Poincaré duality. Theorem 5.3 says that $H_1(M,\mathbb{Z}) = 0$. In fact:

**Proposition 5.4.** *Symplectic toric manifolds are simply connected.*

*Proof.* The CW complex has a single 0-cell and no 1-cells. Moreover, the map $\pi_1(X^1, x_0) \xrightarrow{i_*} \pi_1(X, x_0)$ induced by inclusion is surjective for a CW complex $X$. □

The above is only the tip of the iceberg. Using the language of equivariant cohomology and localization, it is possible to compute the entire cohomology *ring* of a symplectic toric manifold by the fixed point data given by the vertices ([KW]). There are some quite explicit formulas for the cohomology ring of a reduced space ([TW]).

## 6. Symplectic Blow-ups

Symplectic **blow-up** is a special case of symplectic cutting. It is a local operation that has a very explicit effect on the Delzant polytope.

First, we will perform the construction in $\mathbb{C}^n$ (walking in the steps of [Gui]; [McD-Sa] has a rigorous treatment). The $2n$-manifold $X = \{(p,v) | v \in p\} \subseteq \mathbb{CP}^{n-1} \times \mathbb{C}^n$ is the blow-up of $\mathbb{C}^n$ at the origin. There is an embedding $\imath : \mathbb{CP}^{n-1} \to X$, $p \mapsto (p, 0)$ and a projection $\beta : X \to \mathbb{C}^n$, $(p,v) \mapsto v$, which is bijective off the image of $\imath$ and collapses the image of $\imath$ to 0. Furthermore, the maps $\imath$ and $\beta$ are $U(n)$-equivariant with respect to the natural action of $U(n)$ on $\mathbb{CP}^{n-1}$, $\mathbb{C}^n$ and $X$. Now consider the standard symplectic form $\omega_0$ on $\mathbb{C}^n$.

**Definition.** An $\varepsilon$-blow-up of $\mathbb{C}^n$ at the origin is a pair $(X, \omega)$, where $\omega$ is a $U(n)$-invariant symplectic form on X such that $\omega - \beta^*\omega_0$ has compact support and $\imath^*\omega = \varepsilon\omega_{FS}$.

**Lemma 6.1.** *([Gui] and [GS,3]) Given $\delta > 0$, there exists an $\varepsilon > 0$ and an $\varepsilon$-blow-up of $\mathbb{C}^n$ such that $\omega = \beta^*\omega_0$ on the set $|z| \geq \delta$.*

Thus, blow-ups are actually possible! In a general symplectic manifold $(M, \omega)$, we can perform the blow-up at any point via a Darboux chart (we can choose $\epsilon$ small enough such that the support of $\omega - \beta^*\omega_0$ is contained inside the chart). In the case of a Hamiltonian $G$-space, where $G$ is a compact Lie group, we can apply the equivariant Darboux theorem. Remember that linear symplectomorphisms of $\mathbb{C}^n$ preserve the complex inner product; therefore, $G$ acts as a subgroup of $U(n)$ and there is a Hamiltonian action of $G$ on $X$ (namely, the one induced by the Hamiltonian actions of $G$ on $\mathbb{C}^n$ and on $\mathbb{CP}^{n-1}$).

If $M$ is a symplectic toric manifold, the $\epsilon$ blow-up of $M$ is another symplectic toric manifold:

**Proposition 6.2.** *Let $\Delta$ be a Delzant polytope, let $X_\Delta$ be the corresponding symplectic toric manifold, and let $p = \mu(q)$ be a vertex of $\Delta$. Denote by $v_1, \ldots, v_n$ the primitive inward-pointing edge vectors at $p$. Then the moment polytope of the $\varepsilon$-blow-up of $X_\Delta$ at $q$ (for $\varepsilon$ sufficiently small) is the polytope $\Delta_\varepsilon$, whose vertices are the same as those of $\Delta$, except that $p$ has been replaced by the vertices $p + \frac{\varepsilon}{2}v_i,\ i = 1, \ldots, n$.*

*Proof.* It is sufficient to calculate the new vertices, i.e. the images of the fixed points of the the action on the blow-up space $X_\varepsilon$ under the new moment map $\mu_\varepsilon$. Using the equivariant Darboux theorem, we are in the situation of $\mathbb{T}^n$ acting on $(\mathbb{C}^n, \omega_0)$ with moment map

$$\mu|_U = p + \frac{1}{2}\sum_{k=1}^{n} v_k(x_k^2 + y_k^2) \tag{6.1}$$

The only fixed point of this action is 0. It induces an action on $X_\varepsilon$, whose fixed points are the images of the eigenspaces in $\mathbb{CP}^{n-1}$, i.e. the coordinate planes $\{x_k = y_k = 0, k \neq j\}$ for all $j$. The symplectic form $\varepsilon\omega_{FS}$ on $\mathbb{CP}^{n-1}$ is induced by symplectic reduction with regard to the $S^1$-action on $\mathbb{C}^n$ at the zero level of the moment map

$$\phi(z) = -\frac{1}{2}(\sum_k |z_k|^2) + \frac{\varepsilon}{2}$$

Fix $j$, and denote by $\pi : \phi^{-1}(0) \to \mathbb{CP}^{n-1}$ the projection onto $\mathbb{CP}^{n-1}$. The vector $\sqrt{\varepsilon}e_j \in \phi^{-1}(0)$ lies in the plane $\{x_k = y_k = 0, k \neq j\}$, and thus $\pi(\sqrt{\varepsilon}e_j)$ is the $j$-th fixed point. Moreover, $\mu_\epsilon(\pi(\sqrt{\varepsilon}e_j)) = \mu(\sqrt{\varepsilon}e_j) = p + \frac{\varepsilon}{2}v_j$. □

A generalization of the above, **symplectic cutting** cuts the polytope along a (rational) hyperplane $(i^*)^{-1}(\varepsilon)$ (the details of the construction are in [Ler]). If the hyperplane is chosen such that the halves resulting from the cut - namely $\Delta_{\geq\varepsilon} := \Delta \cap \{x | i^*(x) \geq \varepsilon\}$ and $\Delta_{\leq\varepsilon} := \Delta \cap \{x | i^*(x) \leq \varepsilon\}$ are still Delzant, the preimages of the cut polytopes are symplectic toric manifolds, by Delzant's theorem. On the other hand, there is an intrinsic operation on the manifold - the symplectic cutting - that closes off the preimage of either half and turns it into a compact connected symplectic manifold with a torus action and moment polytopes $\Delta_{\geq\varepsilon}$ and $\Delta_{\leq\varepsilon}$, respectively. Moreover, there is an equivariant symplectomorphism from $\mu^{-1}(\Delta_{>\varepsilon})$ into the preimage of the same set in the cut space, and the same for the other half.

Among other things, one relevant application of symplectic cutting is the following: consider a face $F \subset \Delta$. By Theorem 3.2, its preimage $M = \mu^{-1}(F)$ is a symplectic toric submanifold. How is it embedded into $X_\Delta$, what is its normal bundle? A priori, this answer may depend on all of $\Delta$, as the construction in Chapter 2 is global. But by symplectic cutting parallel to $F$ and close to it, one can see that the normal bundle only depends on the edge vectors at the edges of $F$.

## 7. Symplectic Toric Four-Manifolds

Two-dimensional symplectic toric manifolds are trivial: all 1-dimensional Delzant polytopes are intervals, and thus all symplectic toric manifolds of dimension 2 are differently-sized copies of $\mathbb{CP}^1 = S^2$. We will present results which, in a certain sense, classify four-dimensional symplectic toric manifolds by classifying two-dimensional Delzant polytopes.

**Definition.** A **corner chopping** on a Delzant polytope at a corner $p$ is the polytope resulting from a symplectic blow-up at $\mu^{-1}(p)$ . That is, $p$ is replaced with $p + \varepsilon v_i$, where the $v_i$ are the primitive inward-pointing edge vectors at $p$.

**Theorem 7.1.** *Let $\Delta$ be a Delzant polytope in $\mathbb{R}^2$. Then:*

- *if $\Delta$ has three edges, then there is a unique $\lambda > 0$ such that $\Delta$ is $AGL(2, \mathbb{Z})$-congruent to the simplex with vertices 0 and $\lambda e_i$, $i = 1, \ldots, n$,*
- *if $\Delta$ has four edges, then there exist numbers $a, b > 0$ and an integer $n \geq 0$ such that $\Delta$ is $AGL(2, \mathbb{Z})$-congruent to the polytope $H_{a,b,n}$, and*
- *if $\Delta$ has $k \geq 5$ edges, then there exist numbers $a, b > 0$ such that $\Delta$ is $AGL(2, \mathbb{Z})$-congruent to a Delzant polytope obtained from a Hirzebruch trapezoid $H_{a,b,n}$ with $n \in \mathbb{N}$ odd by a sequence of $(k - 4)$ corner choppings.*

We have already proven the first to statements. For the more involved proof of the last one, see [KKP].

**Corollary 7.2.** *Let $(M^4, \omega)$ be a symplectic manifold endowed with an effective Hamiltonian action of $\mathbb{T}^2$. Then $M$ is equivariantly symplectomorphic to*

- *$(\mathbb{CP}^2, \lambda\omega_{FS})$ for a unique $\lambda \in \mathbb{R}^*$ if the action has three fixed points,*
- *the Hirzebruch surface $X_{H_{a,b,k}}$ for certain numbers $a, b > 0$ and an integer $n \geq 0$ if the action has four fixed points, or*
- *a $(k-4)$-fold blow-up of a Hirzebruch surface $X_{H_{a,b,n}}$ for $a, b > 0$ and $n \in \mathbb{N}$ odd if the action has $k \geq 5$ fixed points.*

By Theorem 3.2, the inverse image of an edge $e \subset \Delta$ is a sphere $\mathbb{CP}^1 \subset X_\Delta$, whose area is equal to the affine length of $e$. How does it lie in $X_\Delta$?

This property only depends on the edge vectors at the two vertices of $E$, by the remark on symplectic cutting in the previous section. Via the action of $AGL(2, \mathbb{Z})$, we can achieve that that the edge $e$ has outward normal $-e_1$, and its clockwise and counterclockwise neighbors have outward normals $-e_1 + ke_2$ and $-e_2$. Let $(0, a)$ be the second vertex of $\Delta$ that lies on the y-axis. Then:

**Proposition 7.3.** *$\mu^{-1}((-\infty, \epsilon) \times \mathbb{R} \cap \Delta)$ embeds equivariantly and symplectically into the subspace $\phi^{-1}((-\infty, \epsilon) \times \mathbb{R} \cap \Box)$, where $\Box$ is the Hirzebruch trapezoid with vertices $(0, 0), (0, a), (\epsilon, 0), (e, a + k\epsilon)$ and $\phi$ is the moment map of the corresponding Hirzebruch surface.*

*Proof.* Symplectic cutting. $\Box$

**Corollary 7.4.** *The normal bundle of each invariant sphere $S = \mu^{-1}(e)$ in a four-dimensional symplectic toric manifold is uniquely determined by $k$, where $k$ is the number assigned to the edge $e$ as above.*

We can give this number a more explicit meaning, as it classifies the normal bundle of $S$.

**Lemma 7.5.** *(*[KKP, Lemma 2.10 (2)]*) The self-intersection number of $S$ in $X_\Delta$, i.e. the Euler number of the normal bundle of $S$ in $X_\Delta$, equals $-k$.*

*Proof.* Consider the action of the circle $\{0\} \times S^1 \subset \mathbb{T}^2$; its moment is $\pi_2 \circ \mu$, and let $E = TX_\Delta|_s/TS$ be the normal bundle of $S$. We want to know $c_1(E)$ or equivalently $\langle c_1(E), S\rangle$ (the bundle has complex rank 1). On the tangent space to $S$, the circle acts by rotation at the height levels. To calculate $\langle c_1(E), S\rangle$, we only need the weights of the action on the normal space at the fixed points. We can (using the equivariant Darboux theorem) identify these normal spaces with the preimage of the non-vertical edges starting at the two vertices. Since

$$\pi_2(e_1) = 0 \text{ and } \pi_2(e_1 + ke_2) = k,$$

we have that $\int_S c_1(E) = 0 - k = -k$. □

Finally, we can apply the Morse theory studied in chapter 5:

**Proposition 7.6.** *Let $\Delta$ have $d$ vertices. Then dim $H_0(X_\Delta) =$ dim $H_4(X_\Delta) = 1$ and dim $H_2(X_\Delta) = d - 2$. $H_2(X_\Delta)$ is generated by the preimages of the edges of $\Delta$.*

*Proof.* After choosing a suitable $\xi \in \mathbb{R}^n$, the Morse function $\mu^\xi$ has a unique global maximum and minimum. All other vertices must be critical points of index two. For the last statement, use Morse theory as in [KKP]. □

## 8. Monotone Polytopes

Finally, we will look at a special type of symplectic toric manifolds and polytopes. This idea is presented thoroughly in [McD].

**Definition.** The **first chern class** $c_1(M) \in H^2(M)$ of a symplectic manifold $(M, \omega)$ is the first chern class of the the complex line bundle $\bigwedge^n TM \overset{p}{\to} M$ , where $TM$ becomes a complex vector bundle by choice of an almost complex structure $J$ compatible[5] with $\omega$ (this does not depend on the choice of $J$). By definition, the first chern class of a complex line bundle, or equivalently, a principal $S^1$-bundle $E \to M$, is $\frac{1}{2\pi}[\beta]$, where $\beta \in \Omega^2(M)$ satisfies $p^*\beta = d\alpha$ and $\alpha$ is a connection form[6] on $E$.

A symplectic manifold $(M, \omega)$ is called **monotone** if $[\omega] = \lambda c_1(M)$, where $c_1(M)$ is the chern class of the manifold $M$ (we can restrict ourselves to $\lambda = 1$). A Delzant polytope $\Delta$ is called **monotone** if it is the moment polytope of a monotone symplectic toric manifold. There is a direct geometric condition for monotonicity:

**Proposition 8.1.** *(*[McD, Lemma 3.3]*, original proof in* [EP, Proposition 1.8]*) A Delzant polytope $\Delta$ is monotone if and only if the following conditions are satisfied:*

- *$\Delta$ is an integral polytope in $\mathbb{R}^n$ with a* ***unique*** *interior integral point $u_0$,*
- *$\Delta$ satisfies the* **vertex-Fano** *condition: for each vertex $p_j$,*

$$u_0 = p_j + \sum_i e_{ij}, \tag{8.1}$$

*where the $e_{ij}$ are the primitive integral vectors at $p_j$ pointing along the edges.*

---

[5]An almost complex structure $J$ is a smooth map $TM \to TM$ respecting the fibers and linear on each fiber such that $J^2 = -1$. It is said to be compatible if $\omega(u, Jv)$ is a Riemannian metric on $M$. Compatible almost structures exist ([CdS,1]) and form a contractible set.

[6]See the appendix of [Che] for details on this and the rest of this construction.

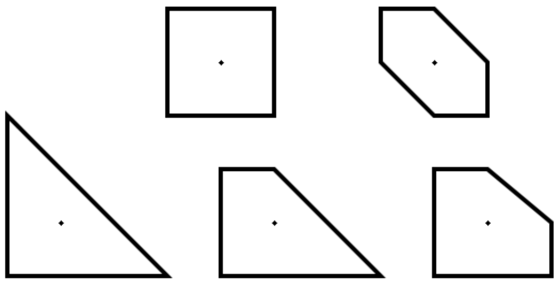

FIGURE 8.1. All the monotone 2-polytopes, up to integral affine transformations. The manifolds are: $\mathbb{CP}^1 \times \mathbb{CP}^1$(the square) and the $i$-fold monotone blow-up of $\mathbb{CP}^2$, $i = 0, 1, 2, 3$. The dot marks the location of the point $u_0$.

*Proof.* We prove the converse, which was left out in [McD]. So suppose that the manifold $(X_\Delta, \omega)$ is monotone, i.e. $[\omega] = c_1(X_\Delta)$. The affine length of an edge $E$ is

$$\ell_{\text{aff}}(E) = \int_{\mu^{-1}(E)} \omega = \int_{\mu^{-1}(E)} [\omega] = \int_{\mu^{-1}(E)} c_1(X_\Delta),$$

which is well-defined since $\mu^{-1}(E)$ is a closed manifold. In particular, as $c_1$ is integral, $\ell_{\text{aff}}(E)$ is integral and thus $\Delta$ is an integral polytope, as $E$ was arbitrary. We show that there is an interior point $u_0$ that satisfies 8.1. As remarked in [McD], it follows automatically that $u_0$ is unique.

Now let $p = p_0$ be any vertex. Choose an edge $E$, whose vertices are $p_0$ and $p_1$, and suppose (possibly re-numbering the vertices) that the the other edges at $p_0$ with edge vectors $e_{0i}$ end at the vertices $p_i$, $2 \leq i \leq n$. Move $\Delta$ by the action of $AGL(n, \mathbb{Z})$ such that $p_0$ is the origin and $e_{0i} = e_i$, $1 \leq i \leq n$, where $e_i$ denotes the $i$-th standard basis vector. Then $p_1 = (a, 0, \ldots, 0)$ for some $a \in \mathbb{N}$. In this coordinate system and for the vertex $p_0$, 8.1 becomes $u_0 = (1, \ldots, 1)$. If we prove that evaluating 8.1 at $p_1$ gives the same result, we are done (iterating this recipe for every vertex). Note that in this case, $u_0$ must also lie in the interior of $\Delta$, which is the interior of the intersection of half-spaces associated to the facets of $\Delta$ (and $u_0$ will lie in each of those, by 8.1).

Consider the vertices $p_0 = w_1, w_2, \ldots, w_n$ adjacent to $p_1$. As in [McD], the primitive integral edge vectors at $p_1$ in the direction of those vertices are $-e_1$ and

$$e'_{1j} = (b_j, 0, \ldots, 0, 1, 0, \ldots, 0),$$

where the 1 appears in the $j$-th entry. Let $S^1$ act on $X_\Delta$ by the inclusion into the first factor of $\mathbb{T}^n$, whose moment map is $\pi_1 \circ \phi$. By the Bott residue formula,

$$a = \int_{\mu^{-1}(E)} c_1(X_\Delta) = 1 - \sum b_j + 1.$$

This implies that

$$p_1 + \sum e'_{1j} = (a, 0, \ldots, 0) - e_1 + \sum (b_j, 0, \ldots, 0, 1, 0, \ldots, 0) =$$

$$= (a - 1 + \sum b_j, 1, \ldots, 1) = (1, \ldots, 1).$$

□

## References


[AB] M.F. Atiyah and R. Bott, The moment map and equivariant cohomology, Topology **23**, No. 1, 1-28 (1984)

[Aud] Michele Audin, Torus Actions on Symplectic Manifolds, Second revised edition, Progress in Mathematics **93**, Birkhäuser, Basel (2004)

[CdS,1] Ana Cannas da Silva, Lectures on Symplectic Geometry, Lecture Notes in Mathematics **1764**, Springer-Verlag, Berlin (2001)

[CdS,2] Ana Cannas da Silva, Symplectic Toric Manifolds, *in* Symplectic Geometry of Integrable Hamiltonian Systems, Advanced Courses in Mathematics - CRM Barcelona, Birkhäuser, Basel (2003)

[Che] S. Chern, Complex manifolds without potential theory, Second edition, Springer-Verlag, New York (1995)

[Del] Thomas Delzant, Hamiltoniens périodiques et images convexes de l'application moment, Bull. Soc. Math. France **116**, 315-339 (1988)

[DH] J.J Duistermaat and G.J Heckman, On the variation in the cohomology of the symplectic form of the reduced phase space, Invent. math. **69**, 259-268 (1982)

[EP] Michael Entov and Leonid Polterovich, Rigid subsets of symplectic manifolds, arXiv:0704.0105v2 [math.SG] (2008)

[Ler] Eugene Lerman, Symplectic cuts, Math. Res. Lett. **2,** 247-258 (1995)

[Gui] Victor Guillemin, Moment Maps and Combinatorial Invariants of Hamiltonian $T^n$-spaces, Progress in Mathematics **122**, Birkhäuser, Boston (1994)

[GS,1] Victor Guillemin and S. Sternberg, Symplectic Techniques in Physics, second edition, Cambridge University Press, Cambridge (1990)

[GS,2] Victor Guillemin and S. Sternberg, Convexity properties of the moment mapping, Invent. Math. **67**, 491-513 (1982)

[GS,3] Victor Guillemin and S. Sternberg, Birational equivalence in the symplectic category, Invent. Math. **97**, 485-522 (1989)

[Hir] F. Hirzebruch, Über eine Klasse von einfachzusammenhängenden komplexen Mannigfaltigkeiten, Mathematische Annalen, Volume **124**, Number 1, Pages (1951)

[KW] L.C. Jeffrey and F.C. Kirwan, Localization for nonabelian group actions. Topology **34**, 291-327 (1995)

[KKP] Yael Karshon, Liat Kessler and Martin Pinsonnault, A compact symplectic four-manifold admits only finitely many inequivalent toric actions, J. Symplectic Geom **5**, Number 2, 139-166 (2007)

[MW] J. Marsden and A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Mathematical Phys. **5**, 121-130 (1974)

[McD] Dusa McDuff, Displacing lagrangian toric fibers via probes, arXiv:0904.1686v4 [math.SG] (2011)

[McD-Sa] Dusa McDuff and Dietmar Salamon, Introduction to Symplectic Topology, Oxford University Press, New York (1995)

[Mil] John Milnor, Morse Theory, Annals of Mathematical Studies **51**, Princeton University Press, Princeton (1963)

[Mil2] John Milnor, On simply connected 4-manifolds, International Symposium on Algebraic Topology **122-128**, Universidad Nacional Autónoma de México and UNESCO, Mexico City (1958)

[TW] Susan Tolman and Jonathan Weitsman, The cohomology rings of abelian symplectic quotients, arXiv:math/9807173v1 [math.DG] (1998)