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The Fiber Connected Sum

Bachelor Thesis

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Abstract

In 1994, Robert E. Gompf achieved a breakthrough in the construction of symplectic manifolds by presenting the construction of the fiber connected sum. We will consider it in the context of the already then well-known connected sum before applying the construction to show that the fiber connected sum of the symplectic blow-up of a symplectic four-manifold with $(\mathbb{C}P^2, \mathbb{C}P^1)$ is symplectomorphic to the original manifold. In the last part, we will take a closer look at the proof in the case of trivial normal bundles.

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Chapter 1

Introduction

Symplectic geometry has it roots in classical mechanics, more accurately Hamiltonian mechanics, but it has evolved into a major branch of geometry and is now studied for its own sake. This does not mean ideas from physics have completely vanished in the topic; rather, symplectic geometry intertwines concepts from several different fields such as differential and algebraic geometry, physics, and topology. The most obvious difference to Riemannian geometry is the symplectic form, which is skew-symmetric in contrast to the symmetry of a Riemannian inner product. This leads to profoundly different properties, such as Darboux's theorem, asserting that a symplectic structure has no local invariants, or the size of the group of symplectomorphism versus the group of isometries. As the title already suggest, the topic of this thesis falls more into the topological part of symplectic geometry. It is even newer, having emerged after 1970. When Gompf published his influential paper in 1994, many fundamental questions where still only partially answered, such as

Which manifolds can carry symplectic structures?

Do symplectic manifolds exist which are not Kähler?

For the first question some obstructions at that time were known; the most obvious being that the manifold must have even dimension. On noncompact, connected manifolds the existence of an almost complex structure was shown by Gromov to be sufficient[8]. In the closed case one also requires a nontrivial de Rham cohomology class of dimension 2, whose top power is a volume form inducing the same orientation as the almost complex structure.
The second question was first answered in the affirmative by Thurston[21]. However, with the help of a new surgery technique, Gompf constructed various examples of a much greater variety as Thurston did, such as symplectic manifolds not even homotopy equivalent to a complex manifold. In this paper he also proved his celebrated theorem[5].

For any finitely presentable group G there is a closed symplectic 4-manifold M with fundamental group $\pi_1(M) \cong G$.

While surgery on contact manifolds was rather well understood at that time, symplectic manifolds proved to be more resistant. In particular, the connected sum could not be carried over to the category of symplectic manifolds because 2n-spheres S^{2n} do not admit symplectic structures for $n > 1$. Gompf used the normal bundles and tubular neighbourhoods of symplectic submanifolds to circumvent that issue. His main theorem and two preliminary lemmas can be found in the appendix.
However, the main focus of this thesis will not be on the construction but on working out an explicit example. We will consider the symplectic blow up $(\tilde{M}, \tilde{\omega})$ of a symplectic 4-manifold (M, ω) and show that its fiber connected sum along the exceptional divisor with $(\mathbb{C}P^2, \mathbb{C}P^1)$ is symplectomorphic to the original manifold (M, ω). Along the way we will explain the symplectic blow up, touching upon its connection to the complex blow up, yet another interplay between symplectic and algebraic geometry.
Chapter 2 will introduce important concepts needed later on, namely the connected sum and its failure in the symplectic category, as well as some basics from symplectic geometry. In the third chapter we will dive into the blow up and the application of the fiber connected sum. The last chapter will be dedicated to a short survey of an easier instance of Gompf's construction - the case of trivial normal bundles.

Chapter 2

Some Foundations

2.1 Basics of Symplectic Topology

In this subsection we state a few fundamental theorems in symplectic geometry. They can be found in most standard textbooks on the subject so their appearance is more for the sake of completeness. Throughout this thesis, we will assume all manifolds to be smooth and to have empty boundary.

Definition 2.1 *Let M be a 2n-dimensional manifold. A symplectic form ω on M is a nondegenerate closed differential 2-form.*

The symplectic form ω induces a nondegenerate skew-symmetric bilinear form ω_p on each tangent space T_pM of M. This makes obvious the first obstruction mentioned in the motivation: A vector space supporting such a form must have even dimension.
A manifold may admit more than one symplectic form, so the question arises how different symplectic forms on a manifold M may be related. Readers who wish to have more information on this topic may be happier with chapter 3 of *Introduction to Symplectic Topology*[15]. Here we will only present Moser's isotopy, which gives sufficient conditions for two symplectic forms to be isotopic, and explore some of its consequences.

Theorem 2.2 (Moser's Theorem) *[2, p.43] Let M be a compact manifold and assume we have two symplectic forms ω and ω' on M such that there is a smooth family of symplectic forms $\{\omega_t\}_{0\leqslant t\leqslant 1}$ with $\omega_0 = \omega$ and $\omega_1 = \omega'$ and with cohomology class independent of t. Then there exists an isotopy $\{\rho_t : M \to M\}_t$ such that $\rho_t^*\omega_t = \omega$.*

Proof The idea is to construct a time-dependent vector field X on M such that the isotopy is given by the flow of X. More precisely, we want a smooth family $X_t \in \mathrm{Vect}(M)$ such that

$$\frac{d}{dt}\rho_t = X_t(\rho_t)$$

As M is compact, we know that the flow is defined everywhere and above equation is well-defined. We consider what properties the vector field must have such that its flow is the desired isotopy. First of all, we know that $[\omega_t]$ is independent of t, hence, for each $0 \leqslant t \leqslant 1$, there exists $\alpha_t \in H^1_{dR}(M)$ such that

$$\frac{d}{dt}\omega_t = d\alpha_t$$

In the simple case where $\omega_t = \omega + t(\omega' - \omega)$, it is evident that α_t depends smoothly on t; in the general case, the argument is more involved using induction over the cardinality of a finite good cover of M and the Mayer-Vietoris sequence, or Hodge Theory.
Moreover, as we want that $\rho_t^*\omega_t = \omega_0$ and assuming ρ_t is the flow of a time-dependent vector field X_t, the vector field must satisfy

$$0 = \frac{d}{dt}\rho_t^*\omega_t = \rho_t^*(\mathcal{L}_{X_t}(\omega_t) + \frac{d}{dt}\omega_t) = \rho_t^*(d(i_{X_t}\omega_t) + i_{X_t}d\omega_t + \frac{d}{dt}\omega_t)$$

where $i_{X_t}\omega_t(Y) = \omega_t(X_t, Y)$. As $d\omega_t = 0$, this simplifies to

$$0 = \rho_t^*(d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t) = \rho_t^*(d(i_{X_t}\omega_t) + d\alpha_t) = \rho_t^*(d(i_{X_t}\omega_t + \alpha_t))$$

It suffices to find, for each $t \in [0, 1]$, $X_t \in \mathrm{Vect}(M)$ such that

$$i_{X_t}\omega_t + \alpha_t = 0$$

This is called the Moser equation. Each ω_t is symplectic, so in particular non-degenerate, and hence, for each $t \in [0, 1]$, we find such an $X_t \in \mathrm{Vect}(M)$. Moreover, the family $\{X_t\}$ depends smoothly on t because $\{\omega_t\}$ and $\{\alpha_t\}$ do. Now extend $\{X_t\}_t$ smoothly to all $t \in \mathbb{R}$. □

For the next theorem and the construction in chapter 4, we use Moser's and Weinstein's technique. Here the concept of a tubular neighbourhood is essential; the definition is in the appendix.

Theorem 2.3 *[3] Let $N \subseteq M$ be a smooth submanifold as in the tubular neighbourhood theorem and choose a tubular neighbourhood U of N in M. Let $i : N \hookrightarrow U$ be the obvious inclusion. If a closed k-form α on U restricted to N vanishes, i.e., $i^*\alpha = 0$, then $\alpha = d\mu$ for some $\mu \in \Omega^{k-1}(U)$. Moreover, we can choose μ such that $\mu_q = 0$, for all $q \in N$.*

Proof Choose a Riemannian metric g on M and let exp denote the exponential map with respect to that metric. Let $\varepsilon > 0$ such that $\exp|_{\nu(N)_\varepsilon} : \nu(N)_\varepsilon \to U_\varepsilon := \{p \in M : \inf_{q \in N} d(p,q) < \varepsilon\}$ is a diffeomorphism. Without loss of generality we may assume $U = U_\varepsilon$. Define, for $s \in [0,1]$, $\psi_s : U_\varepsilon \to U_\varepsilon$ by

$$\psi_s(\exp(p,v)) := \exp(p, sv).$$

Then $\psi_0(U_\varepsilon) = N$, $\psi_1 = id_{U_\varepsilon}$, $\psi_s|_N = id_N$ and $\{\psi_s\}$ depends smoothly on s. Moreover, for $s > 0$, ψ_s is a diffeomorphism and so we can define $X_s \in \mathrm{Vect}(U_\varepsilon)$ by

$$X_s := \frac{d}{ds}\psi_s \circ (\psi_s)^{-1}$$

Then $\{X_s\}_{0<s\leqslant 1}$ depends smoothly on s and vanishes on N. Unfortunately, we cannot extend it to $[0,1]$ as ψ_0 is not injective. However, we will sidestep this issue as follows. Set

$$X_0(\exp(p,v)) = \frac{d}{ds}|_{s=0}\psi_0(\exp(p,v)) = 0$$

for $(p,v) \in \nu(N)_\varepsilon$ and define a smooth cutoff function $\beta : [0,1] \to [0,1]$ with $\beta(t) = 1$ for $t > \delta$ and $\beta(t) = 0$ for $t < \frac{\delta}{2}$ where $0 < \delta < \frac{1}{2}$ can be chosen arbitrarily small. Now define a new family $\{Y_s\}$ by

$$Y_s := X_{\beta(s)}$$

This family depends smoothly on s and still vanishes on N. Denote by ρ_s its flow which agrees with ψ_s for $s > \delta$.
Now define a homotopy operator $I : \Omega^l(U) \to \Omega^{l-1}(U)$ as follows. For $\eta \in \Omega^l(U)$ set

$$I(\eta) = \int_0^1 \rho_s^*(i_{X_s}\eta)ds.$$

Claim: I as defined above is a homotopy operator between $(i \circ \pi)^*$ and id_U^* where $\pi : U \to N$ is given by $\pi(\exp(p,v)) = p$.

Proof: The exterior derivative commutes with the integral as [0,1] is compact. Fix $\eta \in \Omega^l(U)$. Then we have

$$dI(\eta) + I(d\eta) = \int_0^1 \rho_s^* d(i_{X_s}\eta)ds + \int_0^1 \rho_s^*(i_{X_s}d\eta)ds =$$
$$\int_0^1 \rho_s^*(d(i_{X_s}\eta) + i_{X_s}d\eta)ds = \int_0^1 \rho_s^*(\mathcal{L}_{X_s}\eta)ds$$

where we used Cartan's magic formula in the last equation. But the formulas connecting Lie derivatives and flows give us

$$\int_0^1 \rho_s^*(\mathcal{L}_{X_s}\eta)ds = \int_0^1 (\frac{d}{ds}\rho_s^*\eta)ds = \rho_1^*\eta - \rho_0^*\eta = \psi_1^*\eta - \psi_0^*\eta$$

as on the boundary of $[0,1]$ ψ_s agrees with ρ_s. But $\psi_0 = i \circ \pi$ and $\psi_1 = id_U$, hence I is the desired homotopy operator. Note that if $d\eta = 0$ and $i * \eta = 0$, then $dI(\eta) = \eta$. But this is exactly the case for the k-form α. Hence we can set $\mu := I(\alpha)$. Furthermore, because the vector fields X_s vanish on N for all s, we have that, for all $q \in N$, $\mu_q = 0$. □

Theorem 2.4 (Moser Isotopy in the Symplectic Setting) *[15, p.109]Let M be a manifold and let ω_0 and ω_1 be two closed 2-forms on M that agree on a compact submanifold $N \subset M$ and are nondegenerate on N. Then there exist two neighbourhoods U_0 and U_1 of N in M and a diffeomorphism $\psi : U_0 \to U_1$ with $\psi^*\omega_1 = \omega_0$ and $\psi|_N = id_N$.*

Proof Choose a tubular neighbourhood U of N and set $\alpha := \omega_1 - \omega_0$. If $i : N \to U$ is the obvious inclusion, then $i^*\alpha = 0$ and $d\alpha = 0$. With the theorem above, we obtain a 1-form η on U such that $d\eta = \alpha$. Now define $\omega_t = \omega_0 + td\eta$ for $0 \leqslant t \leqslant 1$. Then $\{\omega_t\}$ is a smooth family of closed 2-forms connecting ω_0 and ω_1 with cohomology class not depending on t. Moreover, because nondegeneracy is an open condition, ω_t is nondegenerate on some neighbourhood of N for all $t \in [0,1]$. With Moser's trick we obtain an isotopy $\{\rho_t : U \to U\}_{0 \leqslant t \leqslant 1}$ such that $\rho_t^*\omega_t = \omega_0$ for all $0 \leqslant t \leqslant 1$. Set $\psi := \rho_1$. Then $\psi^*\omega_1 = \omega_0$ and as $\omega_t = \omega_0$ on N, the ρ_t are equal to the identity on N, hence $\psi|_N = id_N$. Finally set $U_0 = U_1 = U$. This completes the proof. □

Setting $N = \{p\}$ for a point $p \in M$ we obtain Darboux's theorem.

Theorem 2.5 (Darboux's Theorem) *[2, p.46] Let (M^{2n}, ω) be a symplectic manifold. Then for every $p_0 \in M$ there exists an open neighbourhood $U \subset M$ of p_0 and a chart $(U, x_1, y_1, \ldots, x_n, y_n)$ such that*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \tag{2.1}$$

Darboux's theorem shows that locally all symplectic forms are the same, in contrast to, e.g., Riemannian metrics.
Last but not least, we will require Weinstein's Symplectic Neighbourhood

Theorem, which allows us to extend a symplectomorphism of two symplectic submanifolds to a symplectomorphism of sufficiently small neighbourhoods if it is covered by an isomorphism of their normal bundles.

Theorem 2.6 (Symplectic Neighbourhood Theorem) *[15, p.120] For j = 1,2, let (M_j, ω_j) be a symplectic manifold with compact symplectic submanifold N_j. Assume there is a symplectomorphism $\varphi : (N_0, \omega_1) \to (N_1, \omega_2)$ covered by an isomorphism of their symplectic normal bundles $\Phi : \nu(N_1) \to \nu(N_2)$. Then φ extends to a symplectomorphism $\psi : (U_1, \omega_1) \to (U_2, \omega_2)$, where U_j is a neighbourhood of N_j, $j = 1, 2$, such that $d\psi$ induces the map Φ on $\nu(N_1)$.*

Proof Choose a Riemannian metric on M_j that is compatible with ω_j.[1] Let $\varepsilon > 0$ be sufficiently small such that for $j = 0, 1$ we have the embeddings $\exp_j : \nu(N_j)_\varepsilon \to M_j$ and set $V_j := \exp_j(\nu(N_j)_\varepsilon)$, where $\nu(N_j)_\varepsilon = \{(q, v) \in \nu(N_j) : |v| < \varepsilon\}$.
Now define

$$\psi' := \exp_1 \circ \Phi \circ (\exp_0)^{-1} : V_0 \to V_1$$

This is a diffeomorphism and its derivative $d\psi'$ induces Φ on $\nu(N_0)$. Then $\psi'^* \omega_1$ agrees with ω_0 on Q and Moser Isotopy give us a diffeomorphism $\rho : U \to U'$ on neighbourhoods U and U' of N_0 in M_0 such that $\rho^* \psi^* \omega_1 = \omega_0$ on U. Shrinking V_0 and V_1 (or U and U') if necessary, we can conclude by setting $\psi = \psi' \circ \rho$ and denoting by U_j the respective shrunk neighbourhood. □

[1] Here we use the existence of an almost complex structure J_i on the symplectic manifold M_i[15, p.153] and define $g_i(u, v) := w(u, Jv)$.

2.2 Connected Sum

Gluing manifolds together is, in principle, not difficult. Doing so and wanting to preserve additional structure is another matter altogether. First of all, if you have smooth manifolds you usually want the resulting manifold also to be smooth, this can be done by identifying not points but collars and smoothing any occurring corners if necessary. If you have an orientation on both manifolds you want to do the gluing in a way such that the sum has an orientation. For that case the connected sum is a well-known tool.
There are two variants of the connected sum, one is a generalisation of the other: the connected sum at a point and the connected sum along a submanifold.

2.2.1 At a Point

Definition 2.7 (Connected Sum at a Point) *[15, p.293]: Let M_1 and M_2 be oriented n-manifolds. Fix a point p_i in each manifold M_i and let $j_i : B(1) \to B^i$ be a neighbourhood of p_i diffeomorphic to the open unit ball (as oriented manifolds, for $i = 1, 2$). Choose $0 < \delta < \varepsilon < 1$ and an orientation reversing diffeomorphism $\varphi : B^1 \to B^2$, f.e., $\varphi = j_2 \circ \Phi \circ j_1^{-1}$ where $\Phi \in O(n) \backslash SO(n)$. Set $B^i_\varepsilon := j_i(B(\varepsilon))$, where $B(\varepsilon)$ is the open ball of radius ε. Now for $q \in B^1_\varepsilon \backslash B^1_\delta$ define*

$$f(q) = \frac{\delta\varepsilon}{|\varphi(q)|^2}\varphi(q).$$

Then f is an orientation preserving diffeomorphism as it sends a sphere of radius $\delta < R < \varepsilon$ to a sphere of radius $\delta\varepsilon R^{-1}$ and we use it as our gluing map, that is, define the connected sum of M_1 and M_2 to be

$$M_1 \# M_2 := (M_1 \backslash cl(B^1_\delta)) \cup_f (M_2 \backslash cl(B^2_\delta))$$

where $cl(B^i_\delta)$ denotes the closure of B^i_δ.
If θ_1 and θ_2 are the orientations on M_1 and M_2 respectively (for example induced by a volume form), then we can define an orientation θ on $M_1 \# M_2$ by setting it equal to θ_i on M_i, $i = 1, 2$. By construction, the two orientations agree on the overlap and hence θ is well-defined.

Claim *$M_1 \# M_2$ is again an oriented m-manifold. Moreover, if both manifolds are connected, the connected sum $M_1 \# M_2$ is independent of the choices made in the construction up to diffeomorphisms.*

The proof will rely on the following theorem of Palais[16], which we only state here.

Theorem 2.8 *[16, p.131] Let M^m be a smooth manifold, $p \in M$, $U \subset M$ an open neighbourhood of p and $\psi : U \to M$ an orientation preserving smooth injective mapping. Assume $d\psi(p)$ is nonsingular and if M is oriented, assume that $d\psi(p)$ to be orientation preserving. Then there exists an open neighbourhood V of p and a smooth isotopy $\{\varphi_s\}_{0 \leqslant s \leqslant 1}$ such that $\varphi_s|_V = \psi|_V$.*

Corollary 2.9 *Any two embeddings into M with the same domain are isotopic, that is, if $j_1, j_2 : U \hookrightarrow M$ are embeddings, there exists a diffeomorphism $\varphi : M \to M$ such that $\varphi \circ j_1 = j_2$ on some open subset $V \subset U$.*

Proof This is first claim is one of the assertions which are very intuitive but whose proof is rather tedious and involves checking some properties case by case. As the connected sum only tangents the topic of this thesis, we refer to Kosinki[13, p.90] for a full proof of the first statement.
Because any two open balls of radius r_1 and $r_2 > 0$, respectively, are diffeomorphic, the diffeomorphism type of the connected sum is independent of the choice of ε and δ. Using this and the theorem by Palais (that is, the corollary), we obtain that the diffeomorphism type does not depend on the choice of embeddings as well.
Furthermore, because $SO(n)$ is path connected, for any two $\Phi_1, \Phi_2 \in O(n)\backslash SO(n)$ there exists a path $A : t \to SO(n)$ with $A(0) = id$ and $A(1) = \Phi_2\Phi_1^{-1}$. Then $\varphi_t := j_2 \circ A(t)\Phi_1 \circ j_1^{-1}$ is an isotopy. Finally, the connected sum does not depend on the choice of points by the assertion of the homogeneity lemma[2].

Remark 2.10 *Assuming M_1 and M_2 to be closed, one can use a simple Mayer-Vietoris argument to show that if $n = dim(M_i) > 2$, then*

$$\dim H^k_{dR}(M_1 \# M_2) = \begin{cases} \dim H^k_{dR}(M_1) + \dim H^k_{dR}(M_2), & 1 \leqslant k \leqslant n-1 \\ \dim H^k_{dR}(M_1) + \dim H^k_{dR}(M_2) - 1, & k = 0, n \end{cases}$$

Hence $H^*_{dR}(M_1 \# M_2)$ does only depend on the de Rham cohomology of M_1 and M_2 as one would expect from the previous proposition.

Remark 2.11 *In particular, if n is odd, then the Euler characteristic $\chi(M_1 \# M_2)$ is simply the sum $\chi(M_1)+\chi(M_2)$, and if n is even, it is given by $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$.*

Remark 2.12 *As we cut out an n-ball, which has a volume (although arbitrarily small), we lose said volume. This can be avoided by connecting M_1 and*

[2]The result can be found in the appendix

M_2 via a cylinder, thus obtaining $vol(M_1 \# M_2) = vol(M_1) + vol(M_2)$ if one chooses the size of the cylinder appropriately.

Remark 2.13 *As we identify the boundaries of two balls of dimension together such that orientation on each hemisphere is preserved, we have a consistent orientation on the resulting n-sphere.*

The connected sum (at a point) can be considered in a wider context. In the setting of surgery one can view it as a $0-$surgery on a disjoint union[17, p.17], or it can be shown that it induces an additive group structure on the oriented cobordism groups as is explained concisely in Hirsch's Differential Topology[9, chapter 7]. However, here we only prove that the connected sum induces an associative commutative operation on the diffeomorphism classes of manifolds with neutral element S^m, which will be the content of the next few propositions. From now on we will assume all manifolds to have dimension $m \geqslant 1$. They are not necessarily symplectic but as usual smooth.

Definition 2.14 *Let $\mathcal{M}_m := \{M : M$ is a smooth connected m-manifold$\}$ be the set of all connected m-manifolds. Define an equivalence relation $M \sim M' :\Leftrightarrow M$ and M' are diffeomorphic. We denote by $[M]$ the equivalence class of $M \in \mathcal{M}_m$ under this relation.*

We will write $[M] * [M'] := [M \# M']$.

Proposition 2.2.1 *The m-sphere S^m acts as a neutral element for $*$.*

Proof Let $M \in \mathcal{M}_m$. We have to show that $M \# S^m$ is diffeomorphic to M. Here we will define the connected sum as was done by Kervaire and Milnor [10]; it is equivalent to our description above.
Let $B(1) := \{x \in \mathbb{R}^m : |x| < 1\}$ be the open unit ball and $i : B(1) \hookrightarrow M$ an orientation preserving embedding, $j : B(1) \hookrightarrow S^m$ an orientation reversing embedding. Set $p_0 := i(0)$ and $z := j(0)$. Then

$$M \# S^m = (M \backslash \{p_0\}) \cup_\sim (S^m \backslash \{z\})$$

where $i(tu) \sim j((1-t)u)$ for $u \in S^{m-1}$ and $t \in (0,1)$. Without loss of generality we may assume, that $z = (0, \cdots, 1)$ and $j(B(1)) = S^m \backslash \{-z\}$. Let $k : B(1) \hookrightarrow S^m \backslash \{z\}$ be an orientation preserving embedding such that for $y = tu \in B(1/2) \backslash B(1/3)$, $k(tu) = j((1-t)u)$ and $k(0) = -z$. Then on $B(1/2) \backslash B(1/3)$, we have that $k(x) \sim i(x)$. Define $\psi : M \to M \# S^m$ by

$$\psi(p) = \begin{cases} p \ , \ \ p \in M \backslash i(B(1/3)) \\ k(i^{-1}(p)) \ , \ \ p \in i(B(1/2)) \end{cases} \tag{2.2}$$

By definition of k, ψ is well-defined and smooth and $\psi(p_0) = -z$. An inverse is given by $\vartheta : M\#S^m$

$$\vartheta(p) = \begin{cases} p \ , \ \ p \in M \backslash i(B(1/3)) \\ i(k^{-1}(p)) \ , \ \ p \in k(B(1/2)) \end{cases} \tag{2.3}$$

□

Proposition 2.2.2 *The operation $*$ is associative.*

Proof Let $M, M', M'' \in \mathcal{M}_m$. Let $j : B(1) \hookrightarrow M$, $j'_1, j'_2 : B(1) \hookrightarrow M'$, and $j'' : B(1) \hookrightarrow M''$ be the embeddings, where j and j'_2 are orientation preserving and j'_1 and j'' are orientation reversing. Because the diffeomorphism types of $(M\#M')\#M''$ and $(M\#M')\#M''$ do not depend on the embeddings, we may assume that $j'_1(B(1)))$ and $j'_2(B(1))$ are disjoint. Then the connected sum with M'' does not not affect the connected sum of M and M'' and by the associativity of the union, we have

$$((M\backslash\{j(0)\}) \cup_\sim (M\backslash\{j'_1(0), j'_2(0)\})) \cup_{\sim'} (M\backslash\{j(0)\}) = \\ (M\backslash\{j(0)\}) \cup_\sim ((M\backslash\{j'_1(0), j'_2(0)\}) \cup_{\sim'} (M\backslash\{j(0)\}))$$

. □

Proposition 2.2.3 *The operation $*$ is commutative.*

Proof Let M, $M' \in \mathcal{M}_m$. Similar to the above let $B(1) := \{x \in \mathbb{R}^m : |x| < 1\}$ be the open unit ball and $i : B(1) \hookrightarrow M$ an orientation preserving embedding, $j : B(1) \hookrightarrow M'$ an orientation reversing embedding. Then $M\#M' = (M\backslash\{p_0\}) \cup_\sim (M'\backslash\{q_0\})$ with where $i(tu) \sim j((1-t)u)$ for $u \in S^{m-1}$ and $t \in (0,1)$. Let Φ be an orientation reversing orthogonal linear transformation of $\mathbb{R}^m$ and define embeddings $i' : B(1) \hookrightarrow M$ and $j' : B(1) \hookrightarrow M'$ by $i'(x) = i(\Phi x)$ and $j'(x) = j(\Phi x)$. Then i' is orientation reversing and j' is orientation preserving and we can use them for the connected sum $M'\#M$. As we identify exactly the same pairs of points in $M'\#M$ as in $M\#M'$, the claim follows immediately. □

2.2.2 Along a Submanifold

The construction above can be extended to submanifolds. However, the submanifold has to satisfy certain conditions. Let M_1^n and M_2^n be two manifolds of the same dimension and let N^{n-k} be another closed smooth manifold. Let $j_1 : N \hookrightarrow M_1$ and $j_2 : N \hookrightarrow M_2$ be two embeddings and let ν_1 be the normal bundle of $j_1(N)$ in M_1, ν_2 the one of $j_2(N)$ in M_2. If they have opposite Euler classes, i.e., $e(\nu_1) = -e(\nu_2)$, we can form the connected sum of M and M' along N as follows: As $e(\nu_1) = -e(\nu_2)$, there exists an orientation reversing bundle isomorphism $\psi : \nu_1 \to \nu_2$. Choose a metric on each normal bundle. Using the Tubular Neighbourhood Theorem, identify a subbundle of each normal bundle with a neighbourhood V_i of $j_i(N)$ in M_i, i = 1,2. Define $\varphi : V_1 \backslash j_1(N) \to V_2 \backslash j_2(N)$ by

$$\varphi(p) = \exp_2(\frac{\psi(\exp_1^{-1}(p))}{|\exp_1^{-1}(p)|^2}) \tag{2.4}$$

As before we can make the following definition.

Definition 2.15 *The connected sum of M_1 and M_2 along N via ψ is*

$$M_1 \#_\psi M_2 := (M_1 \backslash j_1(N)) \cup_\varphi (M_2 \backslash j_2(N)) \tag{2.5}$$

Remark 2.16 *In the construction above M_1 and M_2 need not be different manifolds, we can also form the connected "self-sum" of a single manifold W along N by choosing two embeddings i, j of N with disjoint images and defining as above*

$$\#_\psi W := (W \backslash i(N)) \cup_\varphi (W \backslash j(N)) \tag{2.6}$$

Then the construction of $M_1 \#_\psi M_2$ becomes the special case of $\#_\psi W$ where $W = M_1 \sqcup M_2$.[5]

2.3 Implications for Symplectic Geometry

We claim that any area form ω on a 2-manifold M is automatically a symplectic form. Indeed, we have that it is obviously a 2-form and closed for dimensional reasons, and non-degeneracy follows immediately from the fact that ω is non-vanishing as a volume form. Hence we obtain with the orientation on the "discarded" 2-sphere automatically a symplectic form on S^2. Now we consider the case where $n > 1$.

Lemma 2.17 *Let (M, ω) be a symplectic manifold of dimension 2n. Then ω^n is a volume form.*

Proof ω^n is obviously of top degree, thus we only have to show that it is non-vanishing for all $p \in M$. Choose a coordinate chart $(U, x_1, y_1, \ldots, x_n, y_n)$ such that we can write $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$ on U. Then we have

$$\omega^n = n!(dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n)$$

This is evidently nondegenerate and we obtain the claim. □

Lemma 2.18 *Let (M^{2n}, ω) be a symplectic manifold. If M is closed, then the de Rham cohomology class $[\omega] \in H^2_{dR}(M)$ is nontrivial.*

Proof From the previous lemma we know that ω^n is a volume form on M. Assume $[\omega^n] = 0$, i.e., $\omega^n = d\xi$ for some $\xi \in \Omega^{2n-1}(M)$. As M is compact, we can integrate ω^n over M:

$$0 \neq \int_M \omega^n = \int_M d\xi = \int_{\partial M} \xi = 0$$

where the third equality is due to Stokes' Theorem and the fourth due to the fact that $\partial M = \varnothing$. This contradiction shows that $[\omega^n] \neq 0$.
Assume now ω is exact, $\omega = d\nu$. Then $\omega^n = (d\nu)^n = d(\nu \wedge (d\nu)^{n-1})$. This contradicts the fact that $[\omega^n] \neq 0$. Hence ω is not exact. □

From the last lemma it follows immediately that we cannot have a symplectic form on spheres S^{2n} with $n > 1$, because these spheres have vanishing cohomology groups in dimension 2. This makes it obvious that we cannot apply the connected sum operation in the symplectic category and expect it to be well-defined. To avoid this problem, the fiber connected sum glues along the fibers of the normal bundles instead of collars. For the same reason as argued above, this restricts the summation to codimension 2 submanifolds.

Chapter 3

An Explicit Example

3.1 Blowing Up and Down

3.1.1 The Complex Blow Up

We are going to investigate the connection between the fiber connected sum and the symplectic blow up. More explicitly, we will show that, in dimension 4, the fiber connected sum with $(\mathbb{C}P^2, \mathbb{C}P^1)$ is the inverse operation of the symplectic blow up at a point.
Before we describe the symplectic blow up, we will consider the complex blow up at a point, following the chronological order these constructions were discovered.
The complex blow up (at a point and along a submanifold) is used in algebraic geometry to create divisors and resolve singularities.
We first describe the complex blow up of $\mathbb{C}^n$ at the origin and then consider a complex manifold.

Definition 3.1 *The complex blow up $\tilde{\mathbb{C}}^n$ of $\mathbb{C}^n$ at the origin is defined by*

$$\tilde{\mathbb{C}}^n := \{([z], w) \in \mathbb{C}P^{n-1} \times \mathbb{C}^n : z_i w_j = w_i z_j, i, j = 0, \cdots, n-1\} \tag{3.1}$$

We have two maps induced by the projections onto the respective factor, namely

$$\pi' : \tilde{\mathbb{C}}^n \to \mathbb{C}P^{n-1} : ([z], w) \mapsto [z] \tag{3.2}$$
$$\pi'' : \tilde{\mathbb{C}}^n \to \mathbb{C}^n : ([z], w) \mapsto w \tag{3.3}$$

Then π' exhibits $\tilde{\mathbb{C}}^n$ as the tautological line bundle L over $\mathbb{C}P^{n-1}$, whose fiber over each point $[z] \in \mathbb{C}P^n$ is the line $[z] \subset \mathbb{C}^{n+1}$ itself. This shows that $\tilde{\mathbb{C}}^n$ is a smooth manifold. We define the exceptional divisor Z to be $Z := \pi''^{-1}(0)$.

Definition 3.2 *The complex blow up of M at p_0 is given by*

$$\tilde{M} := (M \backslash p_0) \cup Z \tag{3.4}$$

where $Z := \{\ell = [v] \subset T_{p_0}M : \ell$ is a 1-dimensional complex subspace, $v \in \ell$ $\}$ and we have a map $\pi : \tilde{M} \to M$ being the identity on $\tilde{M}\backslash Z = M\backslash\{p_0\}$ and with $\pi(Z) = \{p_0\}$.
Then Z is called the exceptional divisor.

Claim *$\tilde{M}$ is a complex manifold of complex dimension n.*

Proof Let $\{U_\alpha, \varphi_\alpha\}$ be an atlas of M and by shrinking some U_α if necessary, assume p lies in a single chart $U := U_{\alpha_0}$. Set $\varphi := \varphi_{\alpha_0}$. Without loss of generality, we may assume $\varphi(p_0) = 0$. We will use the complex smooth structure of $\tilde{\mathbb{C}}^n$. Explicitly, define for $k = 0, \cdots, n-1$, $V'_k := \{([z], w) \in \tilde{\mathbb{C}}^n : z_k \neq 0\}$ and $\psi'_k : V'_k \to \mathbb{C}^n$ by

$$\psi'_k([z], w) = (\frac{z_0}{z_k}, \cdots, w_k, \cdots, \frac{z_{n-1}}{z_k}). \tag{3.5}$$

Define $\vartheta : \pi^{-1}(U) \to \tilde{\mathbb{C}}^n$ by

$$\vartheta(p) := \begin{cases} ([\varphi(\pi(p))], \varphi(\pi(p))),\ p \in \pi^{-1}(U)\backslash Z \\ ([d\varphi(p_0)v], 0),\ p = [v] \in Z \end{cases}$$

One checks easily that ϑ is an embedding. Now set $U'_k := U \cap \varphi^{-1}(\psi'_k(V'_k))$ and define charts on $\tilde{M}$ by $V_\alpha := \pi^{-1}(U_\alpha)$ for $\alpha \neq \alpha_0$ and $V_k = \pi^{-1}(U'_k)$, together with the maps $\psi_\alpha = \varphi_\alpha \circ \pi$ on V_α, $\alpha \neq \alpha_0$, and $\psi_k = \psi'_k \circ \vartheta$ on V_k. All maps are diffeomorphisms onto their images and hence $\mathcal{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha\neq\alpha_0} \cup \{(V_k, \psi_k)\}$ is an atlas for $\tilde{M}$. □

The manifold need not be complex, it suffices to have an almost complex structure. Moreover, one can even show that the construction extends to oriented (even dimensional) manifolds when one defines a complex structure compatible with the orientation near the blow up locus. Another proposition asserts that $\tilde{M}$ is diffeomorphic to the connected sum of M with the complex projective plane given the opposite orientation, $M\#\overline{\mathbb{C}P^n}$. We will prove it in dimension 4, with the help of the next two lemmas.

Lemma 3.3 *The normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^2$ is isomorphic to L^*.*

Proof Let h be the Poincaré dual of $\mathbb{C}P^1$ in $\mathbb{C}P^2$. We claim that h is the Euler class (or first Chern class) of the line bundle $H^1 = \{[z_0 : z_1 : z_2; \zeta] \in \mathbb{C}P^2 \times \mathbb{C} : [z_0 : z_1 : z_2; \zeta] = [\lambda z_0 : \lambda z_1 : \lambda z_2; \lambda\zeta]\}$. Too see that, consider the section $s([z_0 : z_1 : z_2]) := [z_0 : z_1 : z_2; z_2]$. It is transverse to the zero section of H^1 and its zero locus is $\mathbb{C}P^1$ (if we embed $\mathbb{C}P^1$ as usual). Hence the normal bundle $\nu(\mathbb{C}P^1)$ is isomorphic to H^1. Moreover, H^1 is isomorphic as a vector bundle to the canonical line bundle $\mathrm{Hom}_{\mathbb{C}}(L, \mathbb{C}) = L^*$. □

Lemma 3.4 *Define* $\psi'(z_0, z_1) := [z_0 : z_1 : 1] \in \mathbb{C}P^2$ *for* $(z_0, z_1) \in B_1$. *Then* $\mathbb{C}P^2 \backslash \psi'(B_1)$ *is diffeomorphic to a tubular neighbourhood of* $\mathbb{C}P^1$ *in* $\mathbb{C}P^2$.

Proof We have that $\mathbb{C}P^2 \backslash \psi'(B_1) = \mathbb{C}P^1 \cup \{[z_0 : z_1 : 1] \in \mathbb{C}P^2 : |z_0|^2 + |z_1|^2 > 1\}$. Define $\tau : L^* \to \mathbb{C}P^2$ by

$$\tau([z_0 : z_1], \lambda) = [z_0 : z_1 : \lambda(z_0 : z_1)] \tag{3.6}$$

This is well-defined, because if $[z_0; z_1] = [z_0' : z_1']$, then $[z_0 : z_1 : \lambda(z_0 : z_1)] = [z_0' : z_1' : \lambda(z_0' : z_1')]$. Now choose $\varepsilon > 0$ so small that for $|z_0|^2 + |z_1|^2 = 1$ $|\lambda(z_0, z_1)| < \varepsilon$, then

$$|\lambda(z_0, z_1)| = 1 \Rightarrow |z_0|^2 + |z_1|^2 > 1$$

Then $\tau : L^*_\varepsilon \to \mathbb{C}P^2 \backslash \psi'(B_1)$ is a diffeomorphism. Because L^* is isomorphic to the normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^2$ and by the uniqueness of tubular neighbourhoods, the assertion follows. □

Proposition 3.1.1 *Let* M *be a complex manifold of complex dimension 2 and let* $p \in M$. *Then the complex blow up* $\tilde{M}$ *of* M *at* p *is diffeomorphic to* $M \# \overline{\mathbb{C}P^2}$ *and the diffeomorphism is orientation-preserving.*

Proof Above we saw that $\tilde{M}$ is a complex manifold and hence if $\varphi : U \to \mathbb{C}^n$ is a chart on M with $p \in U$, we have a chart $\tilde{\varphi} : \tilde{U} := \pi^{-1}(U) \to L$ with

$$\tilde{\varphi}(q) = \begin{cases} ([\varphi \circ \pi(q)], \varphi \circ \pi(q)) \text{ , if } q \in \tilde{U} \backslash Z \\ ([d\varphi(p)v], 0) \text{ , if } q = [v] \in \tilde{U} \cap Z \end{cases}$$

Thus Z is biholomorphic and in particular diffeomorphic to the zero section in L, that is, we can identify a tubular neighbourhood of Z in M with $L_\varepsilon := \{([z], z) \in L : |z| < \varepsilon\}$ with $\varepsilon > 0$ sufficiently small. Let $\Psi : L_\varepsilon \to V \subset \tilde{M}$ be that diffeomorphism. Then Ψ is orientation preserving.

The obvious evaluation gives us a nondegenerate pairing $L^*_{[z]} \otimes L_{[z]} \to \mathbb{C}$ from which we see that $L^* \otimes L$ is the trivial line bundle and hence $c_1(L) = -c_1(L^*)$. Thus we have an orientation-reversing diffeomorphism $\Phi : L^* \to L$. But now we are working with $\overline{\mathbb{C}P^2}$, hence if we let ν_1 be the normal bundle of $\overline{\mathbb{C}P^1}$ in

$\overline{\mathbb{C}P^2}$, we have an orientation preserving diffeomorphism $\Phi' : \nu_1 \to L$.

Define $\psi_2 : B_1 \to \overline{\mathbb{C}P^2}$ by $\psi_2(z_0, z_1) = [z_0 : z_1 : 1]$. Note that ψ_2 is orientation reversing. By the lemma above, we have that $\overline{\mathbb{C}P^2}\backslash\psi_2(B_1)$ is a tubular neighbourhood of $\overline{\mathbb{C}P^1}$ in $\overline{\mathbb{C}P^2}$. Denote the diffeomorphism by $j : \overline{\mathbb{C}P^2}\backslash\psi_2(B_1) \to (\nu_1)_\delta$ and shrink ε or δ if necessary such that $\varepsilon = \delta$.
We use ψ_2 as our orientation reversing embedding and identify $[z] \in \psi_2(B_1\backslash\{0\})$ with $\Psi \circ \Phi' \circ j([z])$.

Define $\varphi : M\#\overline{\mathbb{C}P^2} \to \tilde{M}$ to be the identity on $M\backslash p$ and for $[z] \in \overline{\mathbb{C}P^2}\backslash\psi_2(B_\varepsilon)$ set

$$\varphi([z]) := \Psi \circ \Phi' \circ j([z])$$

By the construction of the connected sum, the two definitions of φ agrees on the overlap and hence constitutes an orientation preserving diffeomorphism. □

Remark 3.5 *As the notions of connected sum and orientation are also valid on topological manifolds, this proposition allows one to extend the definition of blowing up to the category of (even-dimensional) topological manifolds. Explicitly, we define the blow up $\tilde{M}$ of a topological 2n-manifold M to be*

$$\tilde{M} := M\#\overline{\mathbb{C}P^n}$$

3.1.2 The Symplectic Blow Up

We now describe the symplectic blow up.[1]
Let (M^{2n}, ω) be a closed symplectic manifold and let $p \in M$ be a point. Identify an open neighbourhood U of p symplectically with an open neighbourhood V of $0 \in \mathbb{C}^n$ where $\mathbb{C}^n$ carries the standard symplectic structure ω_0. This is equivalent to choosing a symplectic embedding $i'_r : B_r := B^{2n}(r) \to M$. Using the symplectic ball extension theorem, extend this embedding to a symplectic embedding $i_r : B_{r+\delta} \to M$.
Recall the tautological line bundle $L := \{([z], w) \in \mathbb{C}P^{n-1} \times \mathbb{C}^n : z_i w_j = z_j w_i\}$ with the projections $\pi' : L \to \mathbb{C}P^{n-1}$ and $\pi'' : L \to \mathbb{C}^n$. We obtain a family of symplectic forms $\omega_\lambda := \pi''^*\omega_0 + \lambda^2\pi'^*\omega_{FS}$, $\lambda > 0$. Denote by $\tilde{B}_r := (\pi'')^{-1}(B_r)$ and by $\tilde{B}_\delta := (\pi'')^{-1}(B_\delta)$. Note that $(\pi'')|_{L\backslash 0} : L\backslash 0 \to \mathbb{C}^n\backslash 0$ is a diffeomorphism.

Lemma 3.6 $(\tilde{B}_\delta\backslash Z, \omega_r)$ *is symplectomorphic to* $(B_{\delta+r}\backslash B_r, \omega_0)$.

[1]Much of this description was inspired by Jonny Evan's lecture on the symplectic blow up [4] and by chapter 7 in *Introduction to Symplectic Topology* [15].

Proof Let $pr : \mathbb{C}^n\backslash 0 \to \mathbb{C}P^{n-1}$ be the usual projection. Consider $\rho := \frac{i}{2}\partial\bar{\partial}\log(|z|^2)$. We will show that $pr^*\omega_{FS} = \rho$. The Fubini-Study form on the usual coordinate chart $U_l = \{[z] \in \mathbb{C}P^{n-1} : z_l \neq 0\}$ is given by $\rho_l := \frac{i}{2\pi}\partial\bar{\partial}\log(\frac{\sum_{k=0}^{n-1}|z_k|^2}{|z_l|^2})$. Above equality holds if and only if it holds on every coordinate chart, hence we can assume we are working on U_l for some $0 \leqslant l \leqslant n-1$. Computing this, we obtain

$$\omega_{FS} = \frac{i}{2\pi}(\frac{\sum_{j=0}^{n-1} dz_j \wedge d\bar{z}_j}{\sum_{j=0}^{n-1}|z_j|^2} - \frac{\sum_{j=0}^{n-1}\bar{z}_j dz_j \wedge z_j d\bar{z}_j}{(\sum_{j=0}^{n-1}|z_j|^2)^2})$$

The dependence on z_l vanishes (as it better do for ω_{FS} to be well-defined on all of $\mathbb{C}P^{n-1}$) and this term is exactly equal to ρ apart from the domain which is rectified by the pullback pr^*.
Define $F' : B_\delta\backslash 0 \to B_{\delta+r}\backslash B_r$ by $F(z) = z\sqrt{1+\frac{r^2}{z^2}}$. Then

$$F'^*\omega_0 = \frac{i}{2}\sum_{k=0}^{n-1} F'^*dz_k \wedge F'^*d\bar{z}_k = \frac{i}{2}\sum_{k=0}^{n-1}[(1+\frac{r^2}{|z|^2})dz_k \wedge d\bar{z}_k - \frac{r^2}{|z|^4}\sum_{l=0}^{n-1} dz_k \wedge d\bar{z}_k]$$

$$= \frac{i}{2}\sum_{k=0}^{n-1}[dz_k \wedge d\bar{z}_k - \frac{r^2}{|z|^4}\sum_{l=0}^{n-1} dz_k \wedge d\bar{z}_k + \frac{r^2}{|z|^2}dz_k \wedge d\bar{z}_k] = \frac{i}{2}\partial\bar{\partial}(|z|^2 + r^2\log(|z|^2)) = \omega_0 + r^2 pr^*\omega_{FS}$$

Now we can pull back by $(\pi''|_{\tilde{B}_\delta\backslash Z})^{-1}$ to obtain the claim. For later reference set $F := F' \circ \pi''|_{\tilde{B}_\delta\backslash Z}$. □

Definition 3.7 *Define the symplectic blow up $\tilde{M}$ of weight r of M at p to be*

$$\tilde{M} := (M\backslash i_r(B_r)) \cup_{i_r \circ F} \tilde{B}_\delta \tag{3.7}$$

where F is the symplectomorphism from the lemma above. Then the symplectic forms agree on the overlap and we have a well-defined symplectic form $\tilde{\omega}$ on the blow up.

Define a map $\pi_M : \tilde{M} \to M$ by letting it be the identity on $M\backslash i(B_r)$, for $p' \in \tilde{B}_\delta\backslash Z$, set $\pi_M(p') = i_r(F(p'))$ and, for all $p' \in Z$, set $\pi_M(p') = i_r(0) = p$. Then π_M is smooth and $\pi_M^*\omega$ is a closed 2-form on $\tilde{M}$. This allows us a nicer description of the cohomology class of $\tilde{\omega}$ in terms of $\pi_M^*[\omega]$ and Z.
From now on we assume our manifold to have real dimension 4.

Lemma 3.8 *The cohomology class of the symplectic form $\tilde{\omega}$ is given by $[\tilde{\omega}] = \pi_M^*[\omega] - r^2\pi[\tau_Z]$ where $[\tau_Z]$ denotes the Poincaré dual of $[Z]$ in $\tilde{M}$.*

Proof We use the Mayer-Vietoris sequence to show that $H^2(\tilde{M}) \cong H^2(M) \oplus H^2(Z)$. Write $\tilde{M} = M \backslash i(B_r) \cup \tilde{B}_\delta$ and set $N := M \backslash i(B_r)$. We then have an exact sequence

$$\cdots \to H^{k-1}\left(N \cap \tilde{B}_\delta\right) \to H^k\left(\tilde{M}\right) \to H^k(N) \oplus H^k\left(\tilde{B}_\delta\right) \to H^k\left(N \cap \tilde{B}_\delta\right) \to \cdots$$

Now $N \cap \tilde{B}_\delta = i(B_{r+\delta}) \backslash i(B_r)$ which is homotopy equivalent to S^3, $\tilde{B}_\delta$ deformation retracts onto Z and hence for k = 2, we have

$$0 \to H^2\left(\tilde{M}\right) \to H^2(N) \oplus H^2(Z) \to 0 \tag{3.8}$$

Now another simple Mayer-Vietoris argument shows that $H^k(N) = H^k(M)$ for $k \neq 4$.
Because $\tilde{\omega} = \pi_M^* \omega$ on $M \backslash i(B_r)$ and because $[\tau_Z]$ generates $H^2(Z)$, we have that $[\tilde{\omega}] = \pi_M^*[\omega] + a[\tau_Z]$, where $a = \int_Z \tilde{\omega}$. But the explicit description of $\tilde{\omega}$ on $\tilde{B}_\delta$ gives us

$$a = \int_Z \tilde{\omega} = r^2 \int_Z \pi'^* \omega_{FS} = r^2 \pi Z \cdot Z \int_{\mathbb{C}P^1} \omega_{FS} = -r^2 \pi \tag{3.9}$$

where the last equation will be proved in the next lemma and we assume that the Fubini Study metric ω_{FS} is so scaled that $\int_{\mathbb{C}P^1} \omega_{FS} = \pi$. □

Lemma 3.9 *Z has self-intersection number -1 in $\tilde{M}$.*

Proof We only have to show that Z has self-intersection number -1 in $\tilde{B}_\delta$, respectively L. But Z is the zero section of L, hence $Z \cdot Z$ is simply the Euler number of L. Consider the smooth section $s : \mathbb{C}P^1 \to L$ given by $s([z_0 : z_1]) := ([z_0 : z_1], (\frac{|z_0|^2}{|z|^2}, \frac{\bar{z}_0 z_1}{|z|^2}))$. Then $s^{-1}(0) = \{[0:1]\}$ and

$$Ds([0:1]) = (d\bar{z}_0)_{[0:1]}$$

which is orientation reversing. Thus the index of s at $[0 : 1]$ is -1 and the assertion follows. □

We will sum along the exceptional divisor Z with $\mathbb{C}P^2$. In the next section we will check the requirements explicitly.
Before we start, we have to consider the issue of volume. As M and $\mathbb{C}P^2$ are closed manifolds, the second power of their respective symplectic forms constitutes a volume form, as does the second power of the symplectic form on the blow up. For the manifold after the fiber connected sum to be symplectomorphic to M, we need them to have the same volume (with respect to their respective symplectic structure). This can be achieved by scaling the symplectic form on $\mathbb{C}P^2$ accordingly, to determine by what constant we need to determine the volume lost during the blow up.

Lemma 3.10 *The volume of the blow-up $\tilde{M}$ is given by the formula[12, p.4]*

$$\int_{\tilde{M}} \tilde{\omega}^2 = \int_M \omega^2 - r^4\pi^2 \tag{3.10}$$

Proof Inded, we have that

$$\int_{\tilde{M}} \pi_M^*\omega \wedge \tau_Z = \int_Z \pi_M^*\omega = 0$$

and hence

$$\begin{aligned} \int_{\tilde{M}} \tilde{\omega}^2 &= \int_{\tilde{M}} (\pi_M^*\omega - r^2\pi\tau_Z)^2 = \int_{\tilde{M}} \pi_M^*(\omega \wedge \omega) + r^4\pi^2 \int_Z \tau_Z \\ &= \int_M \omega^2 + r^4\pi^2 Z \cdot Z = \int_M \omega^2 - r^4\pi^2 \end{aligned}$$

□

3.2 Fiber Connected Sum of the Blow Up with $(\mathbb{C}P^2, \mathbb{C}P^1)$

The aim of this subsection is to establish that the blow down of $\tilde{M}$ along the exceptional divisor is symplectomorphic to the symplectic sum of $\tilde{M}$ with $\mathbb{C}P^2$ along $\mathbb{C}P^1$.

3.2.1 Outline of the General Construction

We will briefly outline the construction. A more detailed description can be found in the appendix.

Let (M^n, ω_M) and (N^{n-2}, ω_N) be two closed symplectic manifolds. Assume we have two symplectic embeddings $j_1 : N \hookrightarrow M$ and $j_1 : N \hookrightarrow M$. We require them to have disjoint image and that the normal bundles ν_1 of $j_1(N)$ and ν_2 of $j_2(N)$ have opposite Euler classes, i.e., $e(\nu_1) = -e(\nu_2)$. This is of course satisfied if they have trivial normal bundles. This condition guarantees the existence of an orientation-reversing bundle isomorphism $\psi : \nu_1 \to \nu_2$. With the help of Čech cohomology one can find an SO(2)-vector bundle $E \to N$ such that $e(E) = e(\nu_1)$. We build a 2-sphere S from a subbundle of E together with the same subbundle but opposite orientation by clutching and then identify tubular neighbourhoods of $j_1(N)$ and $j_2(N)$ symplectically with the north and south pole of S respectively (after we perturb ω_M slightly near $j_2(N)$). This sphere is then used to symplectically identify a small neighbourhood of $j_1(N)$ with one of $j_2(N)$. One can show that the perturbed symplectic form induces a family of isotopic symplectic forms on the manifold $\#_\psi M$ obtained

from M by gluing the neighbourhoods via S.
The perturbation of ω_M can be kept as localised as one wishes and the embedding $j_1(N)$ is not affected by it. It is necessary as the identification of $j_2(N)$ with the north pole of S is in general not symplectic.

3.2.2 The Summation

We now check explicitly the requirements stated by the main theorem. First of all, the normal bundles of the embeddings of $\mathbb{C}P^1$ need have inverse Euler classes. We have seen in lemma 3.3 that $\nu(\mathbb{C}P^1)$ is isomorphic to L^* as an oriented line bundle and by construction $\nu(Z) = L$, hence this condition is satisfied.

Let $(\mathbb{C}P^2, r^2\omega_{FS})$ be the complex projective space with the scaled symplectic form and $(N, \omega_N) := (\mathbb{C}P^1, r^2\omega_{FS})^2$ and define $W := \tilde{M} \sqcup_\vartheta \mathbb{C}P^2$. We have the two symplectic embeddings $j_1 : \mathbb{C}P^1 \to \mathbb{C}P^2$ and $j_2 : \mathbb{C}P^1 \to \tilde{B}_\delta \subset \tilde{M}$, where

$$j_1([z_0 : z_1]) = [z_0 : z_1 : 0]$$

and

$$j_2([z_0 : z_1]) = ([z_0 : z_1], 0).$$

Then $\nu_1 = H^1$ and $\nu_2 = L$ (with the opposite orientation), so we can set $\vartheta([z_0 : z_1; \xi]) := ([z_0 : z_1], (\xi z_0, \xi z_1))$.
Scaling the fiber metrics on H^1 and L if necessary, we may assume $\nu_2^0 = (\nu_2)_{\pi^{-1/2}}$ is diffeomorphic to $\tilde{B}_\delta$, so we can set $V_2 = \tilde{B}_\delta$. Let V_1 be a tubular neighbourhood of $\mathbb{C}P^1$ in $\mathbb{C}P^2$ diffeomorphic to $(\nu_1)_{\pi^{-1/2}}$ (with the induced metric). It is obvious that one can extend the embedding of ν_1^0 to an embedding of $\nu_1 = H^1$. In the case of ν_2, one uses a chart on $\tilde{M}$ containing the blow up locus.
On $i_r(B_\delta \backslash B_r) \cong \tilde{B}_\delta \backslash Z$, ω_M agrees with $\omega_r = \pi'^*\omega_0 + r^2\pi''^*\omega_{FS}$, which is SO(2)-invariant on each fiber of ν_1^0, and if we denote by $\varphi_i : U_i\mathbb{C}P^1 \to \mathbb{C}$ the standard chart, $i = 0, 1$, we have

$$r^2 \int_{(\nu_2)_{[z_0:1]}} \omega_{FS} = r^2 \int_{(\nu_2)_{[z_0:1]}} \varphi_1^*\omega_0 = r^2 \int_{|z|<(r+\delta)} \frac{i}{2} dz \wedge d\bar{z} = \pi(r+\delta)^2 r^2$$

and

$$r^2 \int_{(\nu_2)_{[1:z_1]}} \omega_{FS} = r^2 \int_{(\nu_2)_{[1:z_1]}} \varphi_0^*\omega_0 = r^2 \int_{|z|<(r+\delta)} \frac{i}{2} dz \wedge d\bar{z} = \pi(r+\delta)^2 r^2$$

[2] By a slight abuse of notation we will denote the Fubini Study form on the complex projective line and plane both by ω_{FS}.

Hence the area t_0 of each fiber with respect to ω_M is $t_0 = \pi(r+\delta)^2 r^2$ and thus independent of the fiber. Therefore we have the second and third part of the theorem. The 2-form ζ is the difference between $\pi_M^* \omega_M$ and $\tilde{\omega}$, namely $r^2(\pi')^* \tau_Z$. It is closed and is supported in an arbitrarily small neighbourhood of $\tilde{B}_\delta$ (as we can choose our gluing in $\tilde{M}$ to be on an arbitrarily small collar). For the O(2)-bundle isomorphism $\bar{\varphi}$ we can use ψ, and we let K be the empty set. Then $\varphi([z_0 : z_1; \xi]) = \iota(\bar{\varphi}([z_0 : z_1; \xi]) = \iota([z_0 : z_1], (\xi z_0, \xi z_1)) = ([z_0 : z_1], \sqrt{\frac{1}{\pi|\xi|^2} - 1}(\xi z_0, \xi z_1))$ and if $\varphi_i : U_i \subset \mathbb{C}P^1 \to \mathbb{C}$ denotes the standard chart, we have

$$\#_\vartheta W = \tilde{M} \backslash Z \cup_\varphi \mathbb{C}P^2 =$$
$$((M \backslash i_r(B_r)) \cup_{i_r \circ F} (\tilde{B}_\delta \backslash Z)) \cup_\varphi \ \mathbb{C}P^2 = (M \backslash i_r(B_r)) \cup_{i_r \circ F \circ \varphi} \mathbb{C}P^2$$

Let ω be the induced symplectic form of the symplectic sum. We will not check part one of the main theorem. But we will show that M is diffeomorphic to $\#_\psi W$.

Claim *M is diffeomorphic to $\#_\psi W$*

Proof We have that

$$\#_\psi W = M \backslash i_r(B_r) \cup_{i_r \circ F \circ \varphi} \mathbb{C}P^2 \backslash V \cong M \backslash i_r(B_r) \cup_{f'} B_1$$

Indeed, by lemma 3.4, we know that $\mathbb{C}P^2 \backslash V \cong B_1 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 \leqslant 1\}$. If we denote by τ this diffeomorphism, then we can set $f' := i_r \circ F \circ \varphi \circ \tau$ to obtain the equation. Thus we replace $i_r(B_r)$ with B_1 (by appropriate gluing) which does not change the diffeomorphism type of M. ▫

Now Gompf's result asserts that ω is isotopic to ω_M and hence (M, ω_M) is symplectomorphic to $(\#_\psi W, \omega)$. This shows that the fiber connected sum of $(\tilde{M}, Z)$ with $(\mathbb{C}P^2, \mathbb{C}P^1)$ is the so called blow down of $\tilde{M}$ along Z. As the name already suggests, the blow down is the inverse operation to the blow up and roughly speaking consists of cutting out the exceptional divisor and sewing a ball back in. Hence any closed symplectic manifold M of dimension 2n can be blown down as long as it contains a submanifold symplectomorphic to $\mathbb{C}P^{n-1}$ whose normal bundle in M is the tautological line bundle. Because this reduces the second Betti number of M, one can blow down M only finitely many times.

Chapter 4

Fiber Connected Summation with Trivial Normal Bundles

We consider here the easier version of Gompf's construction, namely when the normal bundles are trivial. It has already considerable consequences and is moreover a nice application of Weinstein's Symplectic Neighbourhood Theorem.

Let (M^m, ω_M) and (N^{m-2}, ω_N) be two closed symplectic manifolds and assume we are given a symplectic embedding $j : N \hookrightarrow M$. Furthermore, assuming that the normal bundle is trivial, we have a framing $X = (X_1, X_2)$ of N in M, where we identify N with $j(N)$. N and M are both oriented by their symplectic forms and we choose the normal vector fields X_1, X_2 such that for each $p \in N$,

$$T_pN \oplus < X_1(p) > \oplus < X_2(p) >= T_pM$$

as oriented vector spaces.
Denote by $B_\varepsilon := \{x \in \mathbb{R}^2 : |x| < \varepsilon\}$ the open ball of radius ε. We have the standard symplectic structure $\omega_0 = dx_1 \wedge dx_2$ on B_ε. Let $\pi' : N \times B_\varepsilon \to N$ and $\pi'' : N \times B_\varepsilon \to B_\varepsilon$ denote the projections. Then the symplectic structures on N and B_ε pull back to give the product symplectic structure on $N \times B_\varepsilon$, namely $\omega' = \pi'^*\omega_N + \pi''^*\omega_0$.

Lemma 4.1 *[6] If we let (N, ω_N) and $(B_\varepsilon, \omega_0)$ be as above, then, for $\varepsilon > 0$ sufficiently small, there exists a symplectic embedding $\varphi : N \times B_\varepsilon \to M$, such that $\varphi(p, 0) = j(p)$, for all $p \in N$, and φ maps the product normal framing of $N \times B_\varepsilon$ onto X (up to isotopy).*

Proof We will first construct an embedding that maps the product framing onto X but is not symplectic, and afterwards invoke Weinstein's Symplectic

Neighbourhood theorem. Choose a Riemannian metric g on M compatible with ω_M[1] and let exp denote the exponential map with respect to g. Let ν denote the normal bundle of $j(N)$ in M and let $\delta > 0$ be sufficiently small so that $\exp|_{\nu_\delta}$ is an embedding into M. Define $\varphi' : N \times B_\delta \to M$ by

$$\varphi'(p, x) = \exp(j(p), x_1 X_1(p) + x_2 X_2(p))$$

for $(p, x) \in N \times B_\delta$. Then it is an embedding by the hypotheses and equal to j on the zero section of $\nu(N)_\delta$. Moreover, if $p \in N$, then $\partial_{x_i} \varphi'(p, 0) = X_i(p)$, $i = 1, 2$, hence we have our desired (not yet symplectic) embedding. Now consider $\varphi'|^*_{N \times \{0\}} \omega_M$. Because φ' agrees with j on $N \times \{0\}$, this equals $\pi'^* \omega_N + \alpha \pi''^* \omega_0$ on $N \times \{0\}$. Now $\alpha > 0$ because ω_0 is the area form of $T_x B_\delta$ and the product orientation is mapped onto the orientation given by the normal framing. Rescaling the fibers B_δ, we may assume $\alpha = 1$. Because N is a symplectic submanifold of $N \times B_\delta$, there is $0 < \varepsilon < \delta$ and a diffeomorphism $\psi : N \times B_\varepsilon \to N \times B_\delta$ (shrinking δ if necessary) isotopic to the identity such that $\pi'^* \omega_N + \pi''^* \omega_0 = \psi^* \varphi'^* \omega_M$ and ψ is equal to the identity on $N \times \{0\}$. Then $\varphi := \varphi' \circ \psi$ is our desired symplectic embedding. □

As in the case of the ordinary connected sum at a point we have to interchange boundaries before we glue in order for the orientations to match up. Here dimension 2 is special because in this case it can be done symplectically. For the next lemma it will be convenient to use polar coordinates (r, θ) on B_ε. In these coordinates, $\omega_0 = r dr \wedge d\theta$.

Lemma 4.2 *The smooth map* $\psi : B_\varepsilon \backslash \{0\} \to B_\varepsilon \backslash \{0\}$ *given by* $\psi(r, \theta) = (\sqrt{\varepsilon^2 - r^2}, -\theta)$ *is a symplectomorphism.*

Proof

$$\psi^* \omega_0 = (r \circ \psi) d(r \circ \psi) \wedge d(\theta \circ \psi) = (-1)^2 \sqrt{\varepsilon^2 - r^2} \left(\frac{r}{\sqrt{\varepsilon^2 - r^2}} dr \right) \wedge d\theta = r dr \wedge d\theta$$

□

Definition 4.3 *We define the fiber connected sum of* (M, ω_1) *and* (M_2, ω_2) *along* N *to be*

$$M_1 \#_\psi M_2 := (M_1 \backslash j_1(N)) \cup_\varphi (M_2 \backslash j_2(N)) \tag{4.1}$$

where we glue via $\varphi := \varphi_2 \circ (id_N \times \psi) \circ (\varphi_1|_{\varphi_1(N \times B_\varepsilon \backslash 0)})^{-1} : \varphi_1(N \times (B_\varepsilon \backslash 0)) \to \varphi_2(N \times (B_\varepsilon \backslash 0))$. *Then the symplectic form* ω *on* $M_1 \#_\psi M_2$ *is simply* ω_1 *on* $M_1 \backslash j_1(N)$ *and* ω_2 *on* $M_2 \backslash j_2(N)$.

[1] It exists because there exists an almost complex structure J on M compatible with ω_M, see [15, p.153]

Again, the volume of the fiber connected sum depends on the choice of ε, where the volume is taken with respect to the respective symplectic form. However we can require the symplectic sum to be additive regarding volume (in a more natural way than for the connected sum) by choosing the tubular neighbourhoods (that is, φ_i) appropriately. Gompf showed that, if we require the volumes to add, any two symplectic forms on $M_1 \#_\psi M_2$ are isotopic.[5, p.26]

Appendix A

Appendix

A.1 Differential Topology

We present some fundamental theorems from differential topology which are still an important part of the background needed for the topic. Some of the shorter proofs are included as they illustrate some methods of proof in basic differential topology but longer winded ones are omitted for reasons of space.

Theorem A.1 (Homogeneity Lemma) *[14, p.22] Let q_0, q_1 be two points of a connected manifold M. Then there exists an isotopy $h : M \to M$ such that $h(q_0) = q_1$.*

The notion of the normal bundle surfaces again and again in differential and symplectic topology. It gives us an explicit geometric meaning of the Poincaré dual of a submanifold and in some cases a neat description of a neighbourhood of the submanifold.

Definition A.2 *Let M^m be a smooth manifold, N^n a smooth submanifold of M. We define the normal bundle $\nu(N) \xrightarrow{p} N$, as follows. For $p \in N$ let the fiber, the so called normal space at p, be the quotient space $\nu(N)_p := T_pM/T_pN$. We can embed N in $\nu(N)$ via the zero section $i_0(p) = (p, 0)$. A tubular neighbourhood of N is a neighbourhood of N in M which is diffeomorphic to a neighbourhood of N in $\nu(N)$ (where we identify N with $i_0(N)$) via a diffeomorphism ψ which restricts to the identity on N, i.e., such that the following diagram commutes:*

$$\begin{array}{ccc} & N & \\ \swarrow & & \searrow \\ U_0 & \xrightarrow{\psi} & U \end{array}$$

where the unlabeled arrows denote the inclusions $i : N \hookrightarrow U$ and $i_0 : N \hookrightarrow U_0$.

Remark A.3 *If $N \subset M$ is a symplectic submanifold, then we can, for $p \in N$, identify $\nu(N)_p$ with $T_pN^{\omega_p} = \{v \in T_pM : \forall u \in T_pN \ \omega_p(u, v) = 0\}$.*

Now we have the versatile tubular neighbourhood theorem, which is probably the most mentioned statement in this thesis and on which Gompf's construction fundamentally relies.

Theorem A.4 (Tubular Neighbourhood Theorem) *[19] Let M be a smooth manifold, $N \subset M$ a compact smooth submanifold. Then there exists a tubular neighbourhood of N in M. Explicitly, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the map $\psi_\varepsilon : \nu(N)_\varepsilon := \{(p, v) \in \nu(N) : |v| < \varepsilon\} \to U_\varepsilon := \{q \in M : \inf_{p \in N} d(p, q) < \varepsilon\}$ given by*

$$\psi_\varepsilon(p, v) := \exp_p(v)$$

is a diffeomorphism.[19]

Proof Let g be a Riemannian metric on M and let $\psi : \nu(N) \to M$ be the exponential map with respect to g. Denote by ψ_ε its restriction to $\{(p, v) \in \nu(N) : |v| < \varepsilon\}$.

1. Claim: ψ_ε is a local diffeomorphism for $\varepsilon > 0$ sufficiently small.
   We have that $\psi(p, 0) = p$ and $d\psi(p, 0)(w, u) = w + u$ which is injective because $w \in T_pN$ and $u \in \nu(N)_p$ and therefore bijective. Hence there exists by the Inverse Function Theorem an open neighbourhood U_p of $(p, 0)$ such that $\psi|_{U_p}$ is a local diffeomorphism. Let $\varepsilon_p > 0$ be such that $\forall (q, v) \in U_p : |v| < \varepsilon_p$ and choose a finite subcover $U_{p_1}, \ldots, U_{p_k}$ of the cover $\{U_p\}_{p \in N}$ of the zero section. Now set $\varepsilon := \min\{\varepsilon_{p_1}, \ldots, \varepsilon_{p_k}\}$ to obtain the claim.

2. Claim: ψ_ε is injective for $\varepsilon > 0$ sufficiently small.
   Assume not. Then there exist $(p_i)_{i \in \mathbb{N}} \subset N$, $(p'_i)_{i \in \mathbb{N}} \subset N$, and $(v_i)_{i \in \mathbb{N}}$, $(v_i)_{i \in \mathbb{N}}$ such that $v_i \in \nu(N)_{p_i}$, $v'_i \in \nu(N)_{p'_i}$ and $\lim_{i \to \infty} v_i = \lim_{k \to \infty} v'_i = 0$ such that $(p_i, v_i) \neq (p'_i, v'_i)$ but $\psi(p_i, v_i) = \psi(p'_i, v'_i)$. N is compact, hence (p_i) has a convergent subsequence (p_{i_k}), as does (p'_i). Write $p^* := \lim_{i \to \infty} p_i$ and $p'^* := \lim_{k \to \infty} p_{i'_k}$. As $\exp_{p_{i_k}}(v_{i_k}) = \exp_{p'_{i_k}}(v'_{i_k})$, we have that
   $$d(p_{i_k}, p'_{i_k}) \leqslant |v_{i_k}| + |v'_{i_k}|$$
   hence $d(p^*, p'^*) = 0$ which contradicts the fact that ψ_ε is local diffeomorphism for $\varepsilon > 0$ small enough by the first claim.

3. Claim: ψ is surjective for $\varepsilon > 0$ sufficiently small.
   Let $p \in U_\varepsilon$ where $\varepsilon > 0$ is so small that ψ_ε is injective and a local

diffeomorphism. Because N is compact, there is $q \in N$ such that $d(p,q) = \inf_{q' \in N} d(p,q')$. Then there is a unique $v \in T_qM$ such that $p = \exp_q(v)$. Assume first that $|v| < inj(p;M)$. Let $w \in T_qN$ with $|w| < inj(p;M)$ and let $\gamma : \mathbb{R} \to N$ be a smooth map such that $\gamma(0) = p$ and $\dot{\gamma}(0) = w$. Assume $d(\gamma(t),p) < inj(p,M)$ for all $t \in \mathbb{R}$. Then exists a unique path $u : \mathbb{R} \to T_pM$ such that $\gamma(t) = \exp_p(u(t))$ and $|u(t)| = d(\gamma(t),p)$ for all $t \in \mathbb{R}$. Because $d(\gamma(t),p) > d(\gamma(0),p)$ for all $t > 0$, there exists a unique function $\lambda : \mathbb{R} \to (0,1]$ such that for all $t \in \mathbb{R}$ $d(q,p) = d(p, \exp_p(\lambda(t)u(t))$. Define $\alpha : \mathbb{R} \to M$ by $\alpha(s) = \exp_p(su(0)) = \exp_q((1-s)w)$ and $\beta : \mathbb{R} \to M$ by $\beta(t) = \exp_p(\lambda(t)u(t))$. Then $\alpha(1) = \beta(0)$ and by the Gauss Lemma[20], $\dot{\alpha}(1)$ is orthogonal to $\dot{\beta}(0)$. But $\dot{\alpha}(1) = -w$ and $\dot{\beta}(0) = v$ because λ attains its maximum at 0. Hence $w \in \nu(N)_q$. If $|w| \geqslant inj(p;M)$ repeat the argument with $p_\varepsilon = \exp_q(\varepsilon w)$. □

Definition A.5 *Let M be an m-dimensional manifold. A cover $\{U_i\}_{i \in I}$ is called a good cover if each intersection bigcap$^n_{k=0} U_{i_0}$, $n \in \mathbb{N}$, $i_k \in I$, is either empty or diffeomorphic to $\mathbb{R}^m$.*

Theorem A.6 *[19] Let M^m be a smooth manifold. Then M admits a good cover.*

The last three theorems concern de Rham cohomology.

Theorem A.7 (Poincaré Lemma) *[19] Let M be a contractible manifold. Then there exists a collection of linear maps $P : \Omega^k(M) \to \Omega^{k-1}(M)$, one for each $k \in \mathbb{N}$, such that*

$$id^*_M = d \circ P + P \circ d \tag{A.1}$$

In particular, M has the deRham cohomology of a point, that is, $H^0(M) = \mathbb{R}$ and $H^k(M) = 0$ for $k \neq 0$.

Proof Evidently, a contractible manifold is nonempty and connected. Let $F : [0,1] \times M \to M$ be the homotopy between the constant map $c : M \to \{p\}$ where $p \in M$ and id_M. Define $f_t(p) := F(t,p)$ for $p \in M$ and $0 \leqslant t \leqslant 1$. Let $P_0 : \Omega^0(M) \to \Omega^{-1}(M)$ simply be the zero map. As $\Omega^{-1}(M) = \{0\}$, this map is clearly the only choice. Let $\alpha \in \Omega^1(M)$ and define P_1 via

$$(P_1\alpha)(p) = \int_0^1 \alpha_p(\partial_t f_t(p))dt \tag{A.2}$$

In general, define P_k via

$$(P_k\eta)_p(v_1, \ldots, v_{k-1}) = \int_0^1 \eta_p(\partial_t f_t(p), df_t(p)v_1, \ldots, df_t(p)v_{k-1})dt \tag{A.3}$$

for $\eta \in \Omega^k(M)$, $p \in M$, $v_1, \ldots, v_{k-1} \in T_pM$. Define $h_t\eta := i_{\partial_t f_t(p)} f_t^* \eta$. Then $P_k\eta = \int_0^1 h_t \eta dt$. We claim that P_k satisfies the requirements. Indeed, by Cartan (see below)

$$f_1^*\alpha - f_0^*\alpha = \int_0^1 \frac{d}{dt} f_t^* \alpha dt = \int_0^1 (dh_t\alpha + h_t d\alpha)dt = dP\alpha + Pd\alpha \tag{A.4}$$

□

But $f_1 = id_M$ and $f_0 = c$. Moreover, as c is constant, obviously $dc = 0$, hence $\forall \alpha \in \Omega^*(M) : c^*\alpha = 0$.
The second statement follows directly from the definition of the deRham cohomology.

Theorem A.8 (Cartan) *[19]: Let $\{f_t\} \subset C^\infty(M)$ depend smoothly on t. If we define $h_t : \Omega^k(M) \to \Omega^{k-1}(M)$ by $h_t\omega = i_{\partial_t f_t} f_t^* \omega$, then we have that $\frac{d}{dt} f_t^* = d \circ h_t + h_t \circ d$.*

Theorem A.9 *[19] Let M^m be a connected oriented manifold. Assume $\omega \in \Omega_c^m(M)$ is a compactly supported closed m-form. Then the following equivalence holds.*

1. $\int_M \omega = 0$
2. *There exists $\tau \in \Omega_c^{m-1}(M)$ such that $\omega = d\tau$.*

A.2 Gompf's Construction

While we already considered the construction in the case of trivial bundles, our example requires the theorem in full generality (although we could have used that dimension 4 is slightly easier, see p.16 in [5]). Hence we state the main theorem as well as two preliminary lemmas here. The lemmas are proved but the proof of the main theorem is only sketched. Furthermore, we prove a well known result from algebraic geometry, the isomorphism between isomorphism classes of line bundles over a manifold and its integral cohomology classes of degree two.

Lemma A.10 *[5, p.18] Let (V^n, ω_V) and (M^n, ω_M) be symplectic manifolds (not necessarily compact) and let $N \subset V$ be a compact, codimension-2 symplectic submanifold. Suppose $f : V \to M$ is a smooth orientation preserving embedding such that $f|_N$ is symplectic. Then there exists an isotopy rel N with compact support[1] from f to an embedding $f' : V \to M$ that is symplectic in a neighbourhood of N. If f is already symplectic in a neighbourhood of a compact subset $K \subset N$, then we may assume that the isotopy has support in a preassigned neighbourhood of $\overline{N \backslash K}$ in V.*
We can assume that f' varies smoothly with ω_V and ω_M and given a smooth family $f_r : (V, \omega_{V,r}) \to (M, \omega_{M,r})$, $0 \leqslant t \leqslant 1$, satisfying the the hypotheses of the lemma for each r (assuming N, K, and a neighbourhood of K to be fixed), suppose that f_0' and f_1' are constructed as above, allowing different choices in the construction. Then we can makes choices for each $0 < r < 1$ so that the family f_r' depends smoothly on r, and, on some (fixed) neighbourhood of N, each f_r' is symplectic.[2]

Proof Let $f : (V, \omega_V) \to (M, \omega_M)$ be as in the claim. Let $j : N \hookrightarrow V$ be the obvious inclusion and define $\eta := f^*\omega_M - \omega_V$. Then η is a closed 2-form on V and by the assumption on f, $j^*\eta = 0$. For $0 \leqslant t \leqslant 1$ define $\omega_t = \omega_V - t\eta$. We can make three observations

1. $j^*\omega_t = j^*\omega_0 = j^*\omega_V$
2. By the above observation, the forms ω_t agree on TN.
3. As f is orientation preserving and a symplectic form is also an area form, each ω_t is an area form on the normal spaces of N.

These facts imply that the ω_t are nondegenerate on $TV|_N$ and because nondegeneracy is an open condition, they are also nondegenerate on an open neighbourhood U of N in V (respectively on $TV|_U$). Now we can use Moser's isotopy to obtain an isotopy $\{\tilde{\rho}_t : U \to U\}_{0 \leqslant t \leqslant 1}$ such that $\tilde{\rho}_t{}^*\omega_t = \omega_0$ for all

[1] In the case of an isotopy $\{\rho_t\}$ we define $supp(\{\rho_t\}) := \overline{\bigcup_{0 \leqslant t \leqslant 1} \{p \in V : \rho_t(p) \neq p\}}$
[2] Robert E.Gompf, A New Construction of Symplectic Manifolds

$0 \leqslant t \leqslant 1$. But we want an isotopy $\rho_t : V \to V$. To obtain it, we reconsider the construction of $\{\tilde{\rho}_t\}$ in the homotopy formula: We define a smooth family of vector fields X_t. Now instead of simply defining them on U, we choose a smaller open neighbourhood U' of N such that $\overline{U'} \subset U$ and a smooth bump function $\beta : V \to [0, 1]$ such that $\beta|_{\overline{U'}} \equiv 1$ and β vanishes outside of U. Then for $0 \leqslant t \leqslant 1$ and $p \in U$ define

$$Y_t(p) := \beta(p)X_t(p) \tag{A.5}$$

and $Y_t \equiv 0$ on $M \backslash U$. As Y_t has compact support, we can integrate it on V, even though V is not compact. Let $\rho_t : V \to V$ be its flow, $0 \leqslant t \leqslant 1$. Then $\{\rho_t\}$ has the same properties as $\{\rho_t\}$, simply on a slightly smaller neighbourhood of U' of Q. Now set $f' := f \circ \rho_1 : V \to M$ to get the desired symplectic embedding of a neighbourhood (U') of Q.
Now assume we have a compact subset $K \subset N$ such that f is already symplectic on a neighbourhood $\tilde{K}$ of K. Then $\omega_t|_{\tilde{K}} = \omega_V|_{\tilde{K}}$. Then Y_t (defined as above) vanishes on $\tilde{K}$ and hence $\rho_t|_{\tilde{K}} = id_{\tilde{K}}$. This gives us the second statement.
Now assume we have a smooth family of symplectic forms on V and M, indexed by $s \in \mathbb{R}$, $\{\omega_{V,s}\}$ and $\{\omega_{M,s}\}$. Fix N, K, and $\tilde{K}$. For each s we have a closed 2-form η_s and thus a family $\{Y_t^s\}_{s,t} \in$ Vect(M) and the corresponding flow ρ_t^s. As the symplectic forms depend smoothly on s, so does η_s and hence also Y_t^s and ρ_t^s. In particular, ρ_1^s depends smoothly on s and hence so does $f_s' := f \circ \rho_1^s$.
Given a smoothly family $\{f_r\}_{0 \leqslant r \leqslant 1}$ where each f_r satisfies the hypotheses of the lemma, fix N, K, and $\tilde{K}$, suppose we have constructed f_0' and f_1' as above, possibly with the isotopies having different support and f_0' and f_1' being symplectic in different neighbourhoods of Q. If we define, for each r, $\eta_r := f_r^*\omega_M - \omega_V$, then η_r depends smoothly on r. This smoothness pulls through, that is, the $\omega_{r,t}$ depend smoothly on r as do the $Y_{r,t}$ and $\rho_{r,t}$. For the neighbourhood simply denote by U_r the neighbourhood on which f_r' is symplectic and set $r_0 := \{r \in [0, 1] : U_r \subseteq U_r'$ for all $r' \in [0, 1]\}$. Then U_{r_0} satisfies the requirement. □

Assume we have a closed, symplectic manifold (N, ω_N). Before we state the second lemma, we construct via the clutching construction a sphere bundle over N, which will later serve as a model for the tubular neighbourhoods of the images of the embeddings of N in M. Let α be an arbitrary class in $H^2(N; \mathbb{Z})$. We will show that there is an SO(2)-vector bundle $E \to M$ such that its Euler class is α. While there is a more elegant proofs using sheaf theory (f.e., in Griffiths' and Harris' Principles of Algebraic Geometry[7]), we will prove it directly here and only use Čech cohomology. We will identify an SO(2)-vector bundle with a complex line bundle, using the identifications $\mathbb{R}^2 \cong \mathbb{C}$ and $SO(2) \cong U(1)$, where $U(1)$ is the unitary group. Then the Euler class $e(E)$ of the SO(2)-vector bundle is by definition the first Chern class

$c_1(L)$ of the complex line bundle corresponding to E.

Claim *[11, p.91] There exists a $SO(2)$-vector bundle $p : E \to N$ over N with Euler class $e(E) = \alpha$.*

Proof [3] Define $\mathscr{L}$ to be the set of all equivalence classes of complex line bundles $L \to N$ where $L_1 \sim L_2$ if there exists a bundle isomorphism between them. We can identify $\mathscr{L}$ with $H^2(N;\mathbb{Z})$ via Čech cohomology. Thus we first digress slightly to build up the necessary machinery.
We use here a slightly weaker notion of good cover, namely we require arbitrary intersections of sets in the cover to be either empty or contractible. Let $\mathscr{U} = \{U_i\}_{i\in I}$ be a good cover of N; we may assume without loss of generality that $\mathscr{U}$ is countable. For $k \in \mathbb{N}$ define $\mathscr{I}^k := \{(i_0,\dots,i_k) \in I^{k+1} : U_{i_0} \cap \dots \cap U_{i_k} \neq \varnothing\}$. We now construct a cohomology theory on this intersection data. Choose an abelian group A and define a k-chain to be a function $c : \mathscr{I}^k \to A : (i_0,\dots,i_k) \mapsto c_{i_0,\dots,i_k}$. These k-chains form an abelian group under addition, denoted by $C^k(\mathscr{U};A)$. Define a boundary operator $d : C^k(\mathscr{U};A) \to C^{k+1}(\mathscr{U};A)$ by

$$dc_{i_0,\dots,i_{k+1}} = \sum_{j=0}^{k+1} (-1)^j c_{i_0,\dots,\hat{i}_j,\dots,i_{k+1}} \tag{A.6}$$

Then $d^2 = 0$ and we obtain a cochain complex

$$\cdots \to C^{k-1}(\mathscr{U};A) \xrightarrow{d} C^k(\mathscr{U};A) \xrightarrow{d} C^{k+1}(\mathscr{U};A) \to \cdots$$

We define the cohomology groups $H^k(\mathscr{U};A)$, $k \geqslant 0$, as usual. Using that for any refinement $\mathscr{V}$ of $\mathscr{U}$ we have restrictions $H^k(\mathscr{U};A) \to H^k(\mathscr{V};A)$[4], we can take the inductive limit over all covers of N and define this to be $\check{H}^k(N;A)$.[5] Moreover, because $\mathscr{U}$ is a good cover, $H^k(\mathscr{U};A) \cong \check{H}^k(N;A)$ by the Leray Theorem[7, p.40] and it is isomorphic to the de Rham cohomology group of N.[19, p.165]
We are interested in cohomology classes in $H^2(\mathscr{U};A)$, so we consider briefly what elements lie in that group. First, for $a \in C^2(\mathscr{U};A)$ to be a cocycle, it has to satisfy

$$a_{i_1i_2i_3} - a_{i_0i_2i_3} + a_{i_0i_1i_3} - a_{i_0i_1i_2} = 0 \tag{A.7}$$

for all elements in $\mathscr{I}^2$ and if it is a boundary, the following equation holds

$$a_{i_0i_1i_2} = b_{i_1i_2} - b_{i_0i_2} + b_{i_0i_1} \tag{A.8}$$

[3] We proceed by a method presented by Kostand and use background material from Salamon [19].

[4] Denote by $\mathscr{I}^k(\mathscr{V})$ the same set as $\mathscr{I}^k$ for $\mathscr{U}$ above. Then $\mathscr{I}^k(\mathscr{V}) \subset \mathscr{I}^k$, hence we can restrict our c simply to $\mathscr{I}^k(\mathscr{V})$.

[5] This is well-defined as we can define a partial ordering $\prec$ on covers of N by defining $\mathscr{V} \prec \mathscr{U} :\Leftrightarrow \mathscr{U}$ is a refinement of $\mathscr{V}$.

for a $b \in C^1(\mathscr{U};A)$.
From now on, we assume that $A = \mathbb{Z}$. Take a vector bundle $L \to N$ with local trivialisations $\psi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}$ and transition functions $\psi_{ji} : (U_i \cap U_j) \times \mathbb{C} \to (U_i \cap U_j) \times \mathbb{C}$. By definition we have $\psi_{ji}(x)\psi_{ik}(x) = \psi_{jk}(x)$ if $U_i \cap U_j \cap U_k \neq \emptyset$ and $x \in U_i \cap U_j \cap U_k$. $(*)$
Because we have a good cover, such a nonempty intersection is contractible and hence in particular contractible. Thus the logarithm is defined[18, p.127]. Hence define $f_{ji} := \frac{1}{2\pi i}\log(\psi_{ji})$. Now $(*)$ results in the following equation for the f_{ij}:

$$\exp(2\pi i[f_{i_0i_1} - f_{i_0i_2} + f_{i_1i_2}]) = 1 \tag{A.9}$$
$$\Rightarrow f_{i_0i_1} - f_{i_0i_2} + f_{i_1i_2} \in \mathbb{Z} \tag{A.10}$$

As the f_{ij} are smooth and in particular continuous and $\mathbb{Z}$ is discrete, this implies that $f_{i_0i_1} - f_{i_0i_2} + f_{i_1i_2}$ is constant on $U_{i_0} \cap U_{i_1} \cap U_{i_2}$. Hence this expression defines an element $h \in C^2(\mathscr{U};\mathbb{Z})$, i.e., $h_{i_0i_1i_2} = f_{i_0i_1} - f_{i_0i_2} + f_{i_1i_2}$. We check that h is a cocycle and determines an element $[h] \in H^2(\mathscr{U};\mathbb{Z}) \cong \check{H}^2(N;\mathbb{Z})$.

$$\begin{gathered}(dh)_{i_0i_1i_2i_3} = h_{i_1i_2i_3} - h_{i_0i_2i_3} + h_{i_0i_1i_3} - h_{i_0i_1i_2} = \\ [f_{i_1i_2} - f_{i_1i_3} + f_{i_2i_3}] - [f_{i_0i_2} - f_{i_0i_3} + f_{i_2i_3}] + \\ [f_{i_0i_1} - f_{i_0i_3} + f_{i_1i_3}] - [f_{i_0i_1} - f_{i_0i_2} + f_{i_1i_2}] = 0\end{gathered}$$

Thus we define our bijection $\kappa : \mathscr{L} \to H^2(\mathscr{U};\mathbb{Z})$ by sending $[L]$ to the class $[h]$ constructed as above from the system of local trivialisations.

Claim *$\kappa([L])$ does not depend on the choice of representative L.*

Proof Assume $L \sim L'$ with local trivialisations $\{(U_i, \psi_i)\}$ and $\{(V_j, \varphi_j)\}$ respectively where $\{U_i\}$ and $\{V_j\}$ are good covers. Taking a refinement if necessary, we may assume $U_i = V_i$. Then, by the definition of being equivalent, there exists a diffeomorphism $\Phi : L \to L'$ which restricts to a linear isomorphism on each fiber. Hence there exists linear functions $\lambda_i : U_i \to \mathbb{C}^*$ such that

$$\psi_{ij} = \lambda_i\varphi_{ij}(\lambda_j)^{-1}$$

if $U_i \cap U_j \neq \emptyset$.[6]
From this it follows that if $U_i \cap U_j \neq \emptyset$, then

$$f_{ij} = \frac{1}{2\pi i}\log(\psi_{ij}) = \frac{1}{2\pi i}[\log(\lambda_i) + \log(\varphi_{ij}) - \log(\lambda_j)] \tag{A.11}$$
$$\Rightarrow f_{ji} - f_{jk} + f_{ik} = \frac{1}{2\pi i}[\log(\varphi_{ji}) - \log(\varphi_{jk}) + \log(\varphi_{ik})] = f'_{ji} - f'_{jk} + f'_{ik} \tag{A.12}$$

[6] Id est, $\varphi_{ij}(x) = \lambda_i(x)\psi_{ij}(x)(\lambda_j(x))^{-1}$ for $x \in U_i \cap U_j$.

Thus h is independent of the choice of representative (and thus also of the trivialisations) and κ is well-defined. □

Claim *κ is injective.*

Proof Assume $\kappa([L]) = \kappa([L'])$. Then $h = h' + db$ for some $b \in C^1(\mathscr{U};\mathbb{Z})$. Explicitly,

$$h_{i_0i_1i_2} = h'_{i_0i_1i_2} + (b_{i_1i_2} - b_{i_0i_2} + b_{i_0i_1}) = f'_{i_0i_1} - f'_{i_0i_2} + f'_{i_1i_2} + b_{i_1i_2} - b_{i_0i_2} + b_{i_0i_1}$$

Hence $f_{i_0i_1} = f'_{i_0i_1} + b_{i_0i_1}$ for $(i_0, i_1) \in \mathscr{I}^1$. This is equivalent to

$$\psi_{ij} = \exp(2\pi i b_{ij})\varphi_{ij} \tag{A.13}$$

But this shows that the transition functions differ by a nonzero holomorphic function and hence the corresponding line bundles are isomorphic. □

Claim *κ is surjective.*

Proof Let $[h] \in H^2(\mathscr{U};\mathbb{Z})$. Choose a partition of unity $\{\rho_i : N \to [0,1]\}$ subordinate to $\mathscr{U}$ and define $f_{ij} := \sum_{k\in I} h_{ijk}\rho_k$ on $U_i \cap U_j \neq \emptyset$. Then f_{ij} is smooth and we can define $\psi_{ij} := \exp(2\pi i f_{ij})$. Because $dh = 0$, we have that $\psi_{ij}\psi_{jk} = \psi_{ik}$ and $\psi_{ij} = (\psi_{ji})^{-1}$. Hence the $\{\psi_{ij}\}$ satisfy the conditions for the local transition maps of a complex line bundle. But from this data we can build a complex line bundle L.[7] such that $\kappa([L]) = [h]$. □

Claim $\kappa([L]) = c_1(L)$.

Proof This follows directly from the definition of the Euler class and the fact that $c_1(L) = e(L_\mathbb{R})$ where $L_\mathbb{R}$ is the underlying real vector bundle of L.[1, p.72] □

So fix a closed symplectic manifold (N, ω_N), a class $e \in H^2(N;\mathbb{Z})$ and choose a SO(2)-vector bundle $p_E : E \to N$ such that $e(E) = e$. Define a norm on E via local trivialisations, that is, if $\varphi : p^{-1}(U) \to U \times \mathbb{R}^2$ is a local trivialisation and $\pi' : U \times \mathbb{R}^2 \to \mathbb{R}^2$ the projection on the second factor, then for $v \in p^{-1}(U)$ define

$$|v| := \sqrt{< \pi'(\varphi(v)), \pi'(\varphi(v)) >} \tag{A.14}$$

where $< \cdot, \cdot >$ denotes the standard inner product on $\mathbb{R}^2$. This is well-defined because the transition functions take values in SO(2). With respect to this metric define the disk bundle $E^0 := \{v \in E : |v| < \pi^{-\frac{1}{2}}\}$. Note that E and hence E^0 are obviously orientable as vector bundles and as manifolds as N is oriented by its symplectic form. Let $\overline{E^0}$ be E^0 with the opposite orientation. Let $i_0 : N \hookrightarrow E^0$ and $i_\infty : N \hookrightarrow \overline{E^0}$ be the zero sections with images N_0 and

[7] Set $L := \bigcup U_i \times \mathbb{C}/\sim$ where we identify $(p, x) \in U_i$ with $(p, \psi_{ij}(x)) \in U_j$, if $p \in U_i \cap U_j$.

N_∞ respectively and glue the two disk subbundles together as follows. Let ι be the map defined at the beginning of the section and note that ι maps $E^0 \backslash N_0$ to $\overline{E^0} \backslash N_\infty$. Then define

$$S := E^0 \cup_\iota \overline{E^0} \tag{A.15}$$

to obtain a 2-sphere bundle $p : S \to N$. Then $E^0 = S \backslash N_\infty$ and the SO(2)-action on E induces an SO(2)-action on S.

We are now ready to state and prove the second lemma.

Lemma A.11 *[5, p.20] There is a closed, SO(2)-invariant 2-form η on S with $i_0^* \eta = 0$ and η restricts to a symplectic form of area 1 on each fiber. For any such η the following holds.*

1. *There is a constant $t_1 > 0$ such that the family $\{\omega_t := p^*\omega_N + t\eta : 0 < t \leqslant t_1\}$ consists of SO(2)-invariant symplectic forms on S.*
2. *For any neighbourhood W of N_0 in (E^0, ω_{t_1}), there exists $t_0 \in (0, t_1]$, such that, for all $t \in (0, t_0]$, (E^0, ω_t) embeds symplectically in W rel N_0, and the embedding is isotopic rel N_0 to id_{E^0}. The embedding is determined by the construction, depends smoothly on t, η, and ω_N, and equals id_{E^0} for $t = t_0 = t_1$ if $W = E^0$.*

Moreover, one can choose η such that $\eta|_{E^0}$ extends smoothly to a closed form on E that is symplectic on each fiber of E. For any such η and t_1 small enough, the resulting extensions of ω_t for $t \in (0, t_1]$ will be symplectic on the closure $cl(E^0)$ of E^0 in E. The resulting embeddings in 2. will extend to embeddings of $cl(E^0)$.

Proof To construct the symplectic form ω_t we use a method introduced by Thurston.

1. First, let $[\beta] := \tau_{N_0} \in H^2_{dR}(S)$, the class of the Poincaré dual of N_0, be represented by a closed 2-form β, such that $\int_F \beta = 1$ for each fiber F. (S is oriented and compact and hence we have a volume form integrating to 1 on S. Now locally $S \cong N \times F \cong N_0 \times F$.) Choose local trivialisations $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{S}^2$ where $\{U_i\}_{i \in I}$ is a good cover of N and a partition of unity $\{\rho_i\}$ subordinate to that cover. Let $j : F \hookrightarrow S$ be the canonical inclusion.[8] Then $\pi_{\mathbb{S}^2} \circ \varphi_i \circ j : F \to \mathbb{S}^2$ is a diffeomorphism and

$$\int_F \varphi_i^* \pi_{\mathbb{S}^2}^* \omega_{\mathbb{S}^2} = 1$$

[8] As is conventional we will leave out the j^* in the integral.

Due to the contractibility of U_i this implies that there is $\alpha_i \in \Omega^1_{dR}(p^{-1}(U_i))$ such that $d\alpha_i = \varphi_i^* \pi_{\mathbb{S}^2}^* \omega_{\mathbb{S}^2} - \beta$ on $p^{-1}(U_i)$. Define now $\eta' := \beta + d(\sum_{i \in I}(\rho_i \circ p)\alpha_i)$ and $\eta'' := \eta' - (i_0 \circ p)^* \eta'$. Then $d\eta'' = 0$ and $i_0^* \eta'' = 0$. To obtain SO(2)-invariance we simply integrate

$$\eta := \int_{SO(2)} r(\theta)\eta'' d\theta$$

This η satisfies the requirements of the lemma.

2. We claim that the closed 2-forms $\omega_t = p^*\omega_N + t\eta$ are nondegenerate for t small enough.
Let F be a fiber and denote by F^t the ω_t-normal space of F. First observe that, because $p^*\omega_t - p^*\omega_s = (t-s)\eta$, F^t does not depend on t. Moreover, $p^*\omega_N$ is nondegenerate on F^t, as on F^t $p^*\omega_N = -t\eta$ which restricts to a symplectic form on each fiber. Because nondegeneracy is an open condition, for each $p \in S$ there exists an open neighbourhood $U_p \subset S$ of p and $t_p > 0$ such that, for all $0 < t \leqslant t_p$, ω_t is nondegenerate on U_p. We can cover S by these open neighbourhoods and due to compactness find a finite subcover $\{U_{p_1}, \ldots, U_{p_k}\}$. Now set $t_1 := \min\{t_{p_1}, \ldots, t_{p_k}\}$.

3. We claim that $\eta|_{E^0}$ can be extended to all of E.
First we construct a 1-form on $\mathbb{R}^2 \backslash 0$ that extends to a 1-form on the 2-sphere without zero and whose differential is defined on all of $\mathbb{R}^2$.
Define $\alpha := \frac{1}{2}(r^2 - \frac{1}{\pi})d\theta \in \Omega^1(\mathbb{R}^2 \backslash 0)$. Then $d\alpha = rdr \wedge d\theta$ is the standard area (and symplectic) form on $\mathbb{R}^2 \backslash 0$. If D is the disc of radius $\pi^{-\frac{1}{2}}$, then α vanishes on the boundary ∂D. Although we can extend $d\alpha$ to all of $\mathbb{R}^2$, this is not possible with α because of the factor $\frac{1}{2\pi}d\theta$. Now consider $\alpha|_{D\backslash 0}$. We construct a 2-sphere $\mathbb{S}^2$ with ι as in the discussion before this lemma.[9] Then we can define on $D\backslash 0$ $\alpha' := \alpha|_{D\backslash 0}$ and because $\alpha|_{\partial D} = 0$, we can extend α' to 0 on $\mathbb{S}^2 \backslash D$. Because $d\alpha$ is the standard symplectic form on $\mathbb{R}^2$, $d\alpha' = \omega_{\mathbb{S}^2}$ (as we can set $d\alpha' = d\alpha$ at 0).
Consider again β as in the first step of the proof. As β is Poincaré-dual to N_0 we can assume that it vanishes outside of an arbitrarily small neighbourhood of N_0, hence has support disjoint from N_∞. Because we chose our U_i to be contractible, their first Betti number is zero. Consider the forms α_i. On $p^{-1}(U_i)$ we have that $\varphi_i^* \pi_{\mathbb{S}^2}^* \omega_{\mathbb{S}^2} - \beta$ is exact and for any $V \subset S \backslash N_\infty$ this implies that on $V \cap p^{-1}(U_i)$

$$\begin{aligned} &\varphi_i^* \pi_{\mathbb{S}^2}^* d\alpha' - d\alpha_i = \varphi_i^* \pi_{\mathbb{S}^2}^* \omega_{\mathbb{S}^2} - d\alpha_i = 0 \\ &\Rightarrow \varphi_i^* \pi_{\mathbb{S}^2}^* \alpha' - \alpha_i = df \end{aligned}$$

for some $f \in C^\infty(U_i \cap V, \mathbb{R})$. If we consider $\alpha_i - df$ instead of α_i in the construction above, we obtain the same solution; hence we can assume

[9] The 2-sphere $\mathbb{S}^2$ here is assumed to have radius $\pi^{-\frac{1}{2}}$, so that it has volume 1.

without loss of generality that $\varphi_i^* \pi_{\mathbb{S}^2}^* \alpha' = \alpha_i$. Now set $\alpha_i := \varphi_i^* \pi_{\mathbb{S}^2}^* \alpha'$ on N_∞. This is well-defined and thus α_i extends smoothly to $E|_{U_i}$. Moreover, on $U_i \backslash supp(\beta)$ $d\alpha_i = \varphi_i^* \pi_{\mathbb{S}^2}^* \omega_{\mathbb{S}^2}$ and hence $d\alpha_i$ is symplectic on that set.
Now construct η just as in the first step, simply using the modified β and the α_i. Then $\eta|_{E^0}$ extends to all of E and is still symplectic on each fiber. Step 2 also goes through with this new η and so the first part of the lemma still holds.

4. We construct the vector field whose flow will be the desired symplectic embeddings in the second part of the lemma.
Define $I : \Omega^k(E^0) \to \Omega^{k-1}(E^0)$ as in theorem 2.3 and set $\mu := I(\eta)$. Let Y_t be the solution of $i_{Y_t}\omega_t = -\mu$, for $0 < t \leqslant t_1$. By construction, Y_t, $0 < t \leqslant t_1$ is a time-dependent smooth vector field on E^0. Moreover, Y_t is SO(2)-invariant because η is and the integral and pullback commute with the SO(2)-action. Because μ vanishes on N_0, so does Y_t for each $t \in (0, t_1]$. Due to the solution of differential equations we can integrate, for any fixed $t_0 \in (0, 1]$ and any compact SO(2)-invariant subset $K \subset E^0$, Y_t to a flow $\psi : J \times K \to E^0$ such that $\psi_{t_0} = id_K$, where J is a neighbourhood of t_0. Because K is invariant under the SO(2)-action, ψ is SO(2)-equivariant: Let $g \in \mathrm{SO}(2)$ and $(t', p) \in K \times J$. Then $g \cdot \psi(t, p)$ also satisfies
$$\begin{cases} \frac{d}{dt}(g \cdot \psi(t,p)) = Y_t(\psi(t,p)). \\ g \cdot \psi(t',p) = g \cdot p \end{cases}$$
just as $\psi(t, g \cdot p)$ does because Y_t is invariant under the group action. By the uniqueness of the flow, it follows that $g \cdot \psi(t, p) = \psi(t, g \cdot p)$.
Note also that
$$\frac{d}{dt}\psi_t^*\omega_t = \psi_t^*(\mathcal{L}_{Y_t}\omega_t) + \psi_t^*\eta = \psi_t^*(\eta - d\mu) = 0 \tag{A.16}$$
Hence $\psi_t^*\omega_t$ is independent of t.

5. We show that Y_t integrates for each $t_0 \in (0, t_1]$, to a global flow $\psi : [t_0, t_1] \times E^0 \to E^0$.
To do so, we need to prove that no flow lines escape out of E^0 to N_∞. We will use the SO(2)-structure and symmetry of the constructed bundle. Let for $x \in E^0$ $S(x) := \{g \cdot x : g \in SO(2)\}$ denote the circle traced by the SO(2)-orbit through x. Define $D(x)$ to be the closure of the disk in E^0 circumscribed by that circle (if $x \in N_0$ set $D(x) := \{x\}$) and define $A : E^0 \to \mathbb{R}$ by
$$A(x) := \int_{D(x)} \eta \tag{A.17}$$
Note that for each $x \in E^0 : 0 \leqslant A(x) < \int_{E_x} \eta = \int_F \eta = 1$, hence $A : E^0 \to [0, 1)$. Moreover, A is invariant under the SO(2)-action because

both $D(x)$ and η are and A increases proportionally to the radius of $D(x)$. Thus it is surjective. We will show that the η-area of the flow lines of Y_t is bounded and hence our global flow is well-defined.
Fix $x_0 \in E^0$ and $t_0 \in (0, t_1]$ and let $\psi : K \times J \to E^0$ be the flow with $\psi_{t_0} = id_K$. As $D(x_0)$ is compact and SO(2)-invariant, we can take $K = D(x_0)$. Because ψ_t is SO(2)-equivariant, we have that $S(\psi_t(x)) = \psi_t(S(x))$ for each $x \in K$ and each $t \in J$. Thus we have that

$$A(\psi_t(x)) = \int_{D(\psi_t(x))} \eta = \int_{\psi_t(D(x))} \eta = \int_{D(x)} \psi_t^* \eta \tag{A.18}$$

Note that

$$\int_{D(x)} \omega_t = \int_{D(x)} p^*\omega_N + t\int_{D(x)} \eta = \int_{p(x)} \omega_N + t\int_{D(x)} \eta = tA(x)$$

Using that $\psi_t^*\omega_t = \omega_{t_0}$ we obtain the following equation

$$tA(\psi_t(x)) = \int_{\psi_t(D(x))} \omega_t = \int_{D(x)} \psi_t^*\omega_t = \int_{D(x)} \omega_{t_0} = t_0A(x)$$

$$\Rightarrow A(\psi_t(x)) = \frac{t_0}{t}A(x) \tag{A.19}$$

for any $x \in D(x_0)$ and $t \in J$. Hence $A(\psi_t(x))$ decreases with time.
We claim that A is proper. Indeed let $B := [a, b] \subset [0, 1)$ be compact. Then $b < 1$ and hence $A^{-1}(B)$ is closed and $\partial E^0 \cap A^{-1}(B) = \varnothing$. Thus $A^{-1}(B)$ is compact and A is proper.
Hence, as the sequence $A(\psi_t(x))$ converges in $[0, 1)$ (as it decreases monotonically and is bounded from below), so does $\psi_t(x)$ in E^0. Hence all flow lines stay in E^0 and we have a global flow $\psi : [t_0, t_1] \times E^0 \to E^0$ for any fixed $t_0 \in (0, t_1]$.

6. We prove the second part of the lemma.
First note that $A(\psi_t(x)) = \frac{t_0}{t}A(x)$, hence choosing t_0 small enough, we can force $\psi_{t_1-t}(E^0)$ to lie in any chosen neighbourhood W of N_0 for any $t < t_0$. Now $\psi_{t_1-t} : (E^0, \omega_t) \to (W, \omega_{t_1})$ is an embedding as it is a flow and it is symplectic because

$$\psi_{t_1}^*\omega_{t_1} = \psi_t^*\omega_t \Leftrightarrow \psi_{t_1-t}^*\omega_{t_1} = \omega_t$$

Furthermore, ψ_{t_1-t} is obviously isotopic rel N_0 to id_{E^0} because $\psi_{t_1-t}(N_0) = N_0$ $(A(\psi_{t_1-t}(x)) = 0$ for any $x \in N_0)$.
Because we can extend $\eta|_{E^0}$ to all of E we can extend ω_t and Y_t to all of of E for $t \in (0, t_1]$ and thus also ψ. □

Remark A.12 *If we define ω_t as above for $t \in (0, t_1]$, then $i_0^*\omega_t = \omega_N$ and hence i_0 is symplectic. This does not necessarily hold for i_∞; indeed $i_\infty^*[\eta] = -e$.*

Proof If we look at the construction of η we see that $[\eta'] = [\beta] = \tau_{N_0}$ and thus $[\eta''] = \tau_{N_0} - p^* i_0^* \tau_{N_0}$. Hence

$$\begin{aligned} i_\infty^*[\eta''] = i_\infty^* \tau_{N_0} - i_\infty^* p^* i_0^* \tau_{N_0} = \\ i_\infty^* \tau_{N_0} - i_0^* \tau_{N_0} = -i_0^* \tau_{N_0} = -e. \end{aligned}$$

□

Because $g \cdot i_\infty(p) = i_\infty(p)$ for any $g \in SO(2)$ and any $p \in N$, we obtain the claim.

Definition A.13 *We will call $(M_1 \# M_2, \omega)$ the symplectic sum of M_1 and M_2 along N via ψ. If M is connected, $(\#_\psi M, \omega)$ will be referred to as the symplectic self-sum.*

Theorem A.14 *[Symplectic Sum][5, p.13] Let (M^m, ω_M) and (N^{n-2}, ω_N) be two closed symplectic manifolds and let $j_1, j_2 : N \hookrightarrow M$ be two symplectic embeddings such that $j_1(N) \cap j_2(N) = \varnothing$ and such that $e(\nu_2) = -e(\nu_1)$, where ν_i is the normal bundle of $j_i(N)$ in M, i = 1,2. For any orientation-reversing bundle isomorphism $\psi : \nu_1 \to \nu_2$, $\#_\psi M$ admits a canonical symplectic form ω which is induced by ω_M after we perturbed it slightly near $j_2(N)$.*
Explicitly, we have a unique isotopy class of symplectic forms on $\#_\psi M$ that contains forms ω with the following properties

1. *If Q is the cobordism between M and $\#_\psi M$ and $i' : \#_\psi M \to Q$ is the obvious inclusion and Ω is the class on Q induced by ω_M, then*

$$[\omega] = i'^* \Omega \in H^2(\#_\psi M) \tag{A.20}$$

2. *We can choose fiber metrics on the two normal bundles and identify ν_1^0 and ν_2^0 with disjoint tubular neighbourhoods V_1 and V_2 of $j_1(N)$ and $j_2(N)$ respectively, such that the embedding $\nu_1^0 \hookrightarrow M$ extends to an embedding $\nu_1 \hookrightarrow M$.*
3. *The 2-form ω_M is SO(2)-invariant on ν_1^0 and symplectic on the closure of each fiber of ν_1^0 with area t_0 independent of the fiber. The 2-forms $(1-s)\omega_M + s(p_1^* \omega_N)$, $s \in [0,1)$, are symplectic on $cl(\nu_1^0)$ where $p_1 : \nu_1 \to N$.*
4. *There exists a closed 2-form ζ on M with $supp(\zeta) \subset V_2$ such that, for all $t \in [0, t_0]$, the 2-form $\omega_M + t\zeta$ is symplectic on both M and $j_2(N)$.*
5. *There is an O(2)-bundle isomorphism $\varphi : \nu_1 \to \nu_2$ that is fiber-isotopic to ψ, such that outside of a compact subset K of V_1, the map $\varphi : \iota \circ \varphi : V_1 \backslash j_1(N) \to V_2 \backslash j_2(N)$ is symplectic with respect to the symplectic form $\tilde{\omega}_M := \omega_M + t_0 \zeta$ on M. Then $\#_\psi M$ is obtained from $(M \backslash (K \cup j_2(N)), \tilde{\omega}_M)$ by gluing via φ.*

The form ω depends smoothly on ω_M and ω_N (and hence on j_1 and j_2), and it can be constructed with each V_i lying inside any preassigned neighbourhood of $j_i(N)$. It can be assumed that $[\zeta] \in H^2(M)$ is Poincaré dual to $(j_2)_*[N] \in H_{n-2}(M;\mathbb{R})$. In fact, this is necessarily true unless $e(\nu_1)$ vanishes over $\mathbb{R}$ on more than one component of N.

Proof Instead of gluing directly one uses the normal bundles of the embeddings to glue fiberwise.
Let $e = e(\nu_1)$ and construct the disk bundle E, the forms η and $\{\omega_t\}$ and the sphere bundle S over N as in the second lemma. Because $e(E) = e(\nu_1)$, there is an orientation preserving bundle isomorphism $\Phi : E \to \nu_1$. Let it induce a metric on ν_1 such that $\nu_1^0 \cong E^0$ and we thus have an embedding $f : E^0 \to M$. Moreover, $f \circ i_0 = j_1$ where $i_0 : N \to S$ is the zero section N_0 of E^0 seen a subset of S (the "south pole"). With the help of the two lemmas we obtain a smooth family of symplectic embeddings $f_t : (E^0, \omega_t) \to (M, \omega_M)$, all satsifying $f_t \circ i_0 = j_1$. Then we can choose one of these for our identification.
The symplectic idenitification of a neighbourhood of $j_2(N)$ with $\overline{E^0}$ is trickier. If we denote by $i_\infty : N \to \overline{E^0} \hookrightarrow S$ the other zero section, then i_∞ is no longer symplectic. Ignoring that at first, we note that we have an embedding $g : S\backslash N_0 \to M$ with $g \circ i_\infty = j_2$ because $e(\nu_2) = e(\overline{E})$. We then have the following diagram

$$\begin{array}{ccccccccc} N & \xrightarrow{i_0} & E^0 & \xrightarrow{\subset} & S & \xleftarrow{\supset} & \overline{E^0} & \xleftarrow{i_\infty} & N \\ & \searrow^{j_1} & \downarrow f_t & & & & g \downarrow & \swarrow^{j_2} & \\ & & M & & & & M & & \end{array}$$

To ameliorate the situation at the north pole, we perturb the symplectic form ω_M slightly near $j_2(N)$, keeping it so localised that our other symplectic embeddings are still symplectic with respect to this new form $\tilde{\omega_M}$. Denote by N_∞ the image of i_∞, then $g|_{N_\infty} : (N_\infty, p^*\omega_N) \to (M, \tilde{\omega_M})$ is symplectic. Again we apply both lemmas to get a family of symplectic embeddings $g_t : (\overline{E^0}, \omega_t) \to (M, \tilde{\omega_M})$. Choose t_0 so small that $\tilde{g} := g_{t_0}$ and $\tilde{f} := f_{t_0}$ are defined and note that both are defined on $S\backslash(N_0 \cup N_\infty)$. Let U be a neighbourhood of N_∞ in S and set $W := \tilde{g}(U\backslash N_\infty)$. Note that $\tilde{f}$ is defined on $U\backslash N_\infty = \tilde{g}^{-1}(W)$ and thus we can set $K' := (\tilde{f}(\tilde{g}^{-1}(W))$ and as $W \subset \tilde{g}(\overline{E^0})$, $\tilde{g}^{-1}|_W$ is defined and smooth. Hence we have a diffeomorphism $\varphi := \tilde{f} \circ \tilde{g} : W \to K'$. We then define $K := f(E^0\backslash U)$ and

$$\#_\psi M := (M\backslash(K \cup j_2(N)))/\sim \qquad \text{(A.21)}$$

□

where $x \sim \varphi(x)$ for $x \in K'$.
This is a rough sketch of the construction. Now one has to show that this

is indeed diffeomorphic to the connected sum and that all the claims of the theorem hold. However, there are many subtleties involved and an outline would not do it justice while a full explanation would go beyond the scope of a bachelor thesis.

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