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Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Classification of Surfaces Equipped with Area Forms

Bachelor Thesis

Joël Beimler

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Advisor: Prof. Dr. A. Cannas da Silva
Department of Mathematics, ETH Zürich

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Chapter 1

Introduction

As the title indicates, we are focusing two-dimensional manifolds, endowed with additional structure, and attempt to give a complete classification. More specifically, we require our surface M to be compact, connected, and orientable. Loosely speaking, this means the main question we will have in mind throughout this paper is the following:

When are two surfaces M_1 and M_2 equivalent?

Where of course, as we endow the surfaces with more structure, the objects we wish to relate will vary depending on the context, along with the notions of equivalence.

The thesis is divided into two chapters. In the first one, we would like to see under which conditions two compact, and connected manifolds equipped with an area form, (A_1, α_1) and (A_2, α_2), are equivalent in the sense that there exists an area preserving diffeomorphism. We will show that it is possible to classify, for A fixed, the possible area forms of A by the positive real number

$$\int_A \alpha.$$

Hence the conclusion will be that (A_1, α_1) admits an area preserving diffeomorphism to (A_2, α_2) if and only if A_1 and A_2 are diffeomorphic, and in addition

$$\int_{A_1} \alpha_1 = \int_{A_2} \alpha_2.$$

Following up this question will quickly lead us to the beginnings of symplectic geometry.

In the second chapter, we will consider (A, α, ψ), for A any (compact, connected, orientable) surface, α an area form, and ψ a special type of action on

A by S^1, a so-called hamiltonian action. One might expect to find a lot of variety in this classification as we are letting A, α and ψ vary, but as we will see, requiring ψ to be hamiltonian turns out to be a major constraint, to the point where the only possible triples will be the spheres S^2 of any positive radius, along with multiples of the standard euclidean area form, and rotation around the vertical axis of the sphere for the action ψ.

Our argumentation follows closely A. C. Da Silva's *Lecture Notes on Syplectic Geometry* [1], and a seminar she gave on *Symplectic Toric Manifolds* [2], held at ETH Zürich during the spring semester 2019, which have both been invaluable resources in writing this thesis.

Chapter 2

Classification by Total Area

This part considers a compact, connected, orientable surface A with a choice of area form α. So first, we have to clarify the notion of equivalence between two tuples (A_1, α_1) and (A_2, α_2). We should certainly require A_1 and A_2 to be diffeomorphic, say by φ, and the notion of area being preserved is captured by requiring that the pullback satisfy

$$\varphi^* \alpha_2 = \alpha_1.$$

The question of when two surfaces are diffeomorphic is answered by the usual classification of surfaces via their genus and will not be treated further in this thesis. So throughout, we assume A_1 and A_2 to be diffeomorphic by $\tilde{\varphi}$; then (A_1, α_1) is equivalent to (A_2, α_2) if and only if (A_1, α_1) is equivalent to $(A_1, \tilde{\varphi}^* \alpha_2)$. Hence we may as well fix a surface A, but endow it with two area forms α_1, α_2, and ask under which circumstances there exists a diffeomorphism $\varphi : A \to A$ such that $\varphi^* \alpha_2 = \alpha_1$.

Of course, if there is such a diffeomorphism, the total area is the same:

$$\int_A \alpha_1 = \int_A \varphi^* \alpha_2 = \int_{\varphi(A)} \alpha_2 = \int_A \alpha_2,$$

The assertion is now that we can in fact classify (A, α) by total area:

Theorem 2.1 *Let A be a compact, connected, orientable surface and α_1 and α_2 area forms on A. There exists an area-preserving diffeomorphism $\varphi : A \to A$, that is, such that $\varphi^* \alpha_2 = \alpha_1$, if and only if*

$$\int_A \alpha_1 = \int_A \alpha_2.$$

The goal of this chapter is to prove the other direction, that is, the existence of an area preserving diffeomorphism, provided both area forms on A give the same total area.

2.1 Symplectic Preliminaries

We will use some results from symplectic geometry in the following discussion, which makes it useful to explain how (A, α) can be considered a symplectic manifold.

Definition 2.2 *Let M be any manifold. A* ***symplectic form*** *on M is any closed, non-degenerate two form on M, that is, $\omega \in \Omega^2(M)$ such that*

- $d\omega = 0$,
- *For any $p \in M$ and $u \in T_pM \setminus \{0\}$, the map*

$$\omega_p(u, \cdot) : T_pM \to \mathbb{R}$$

  is not identically zero.

A ***symplectic manifold*** *is a pair (M, ω) for ω a symplectic form on M.*

Equivalence among symplectic manifolds is characterised by symplectomorphisms: a ***symplectomorphism*** *φ from a symplectic manifold (M, ω) to another, (N, θ), is a diffeomorphism $\varphi : M \to N$ such that $\varphi^*\theta = \omega$.*

Proposition 2.3 *Let α an area form on an orientable two-dimensional manifold M. Then α is a symplectic form, making (M, α) into a symplectic manifold.*

Proof α is clearly closed as it is of top degree. As α is an area form, it is nonvanishing, so for any $x \in M$, there exist $u, v \in T_xM$ such that $\omega_x(u, v) \neq 0$. As ω_x is alternating, u and v must be linearly independent, for else $\omega_x(u, v) = \omega_x(u, au) = a\omega_x(u, u) = 0$. Thus (u, v) is a basis of T_xM and for $u' \in T_xM \setminus \{0\}$ arbitrary, write $u' = \lambda_1 u + \lambda_2 v$ for $\lambda_1 \neq 0$ without loss of generality. Then $\omega_x(u', v) = \lambda_1 \omega_x(u, v) \neq 0$. □

Thus our guiding question can be rephrased:

When are (A, α_1) and (A, α_2) symplectomorphic?

In the next section, we will state and prove a theorem by Moser giving a sufficient condition for the existence of such a symplectomorphism, more in fact- the existence of an isotopy. The remainder of the chapter will be spent proving that the conditions for the Moser theorem are, in fact, met in the case of our surface A and two area forms satisfying $\int_M \alpha_1 = \int_M \alpha_2$.

2.2 The Moser Trick

Our problem is similar to a question Moser asked and answered. More generally, he was concerned with compact symplectic manifolds (M, ω_0) and (M, ω_1), and asked whether one could find a symplectomorphism φ which was, in addition, homotopic to id_M. Our treatment of the Moser trick follows chapters 6 and 7 of [1].

Definition 2.4 (Isotopy) *Let $\rho : M \times \mathbb{R} \to M$ a map. ρ is called an **isotopy** if for all $t \in \mathbb{R}$, the map $\rho_t := \rho(\cdot, t) : M \to M$ is a diffeomorphism, and $\rho_0 = \mathrm{id}$.*

If we are given an isotopy ρ, we may define for each $t \in \mathbb{R}$ the vector field

$$v_t(p) = \left.\frac{d}{ds}\right|_{s=t} \rho_s(\rho_t^{-1}(p)), \qquad p \in M.$$

A family of vector fields $(v_t)_t$ is called a **time-dependent vector field**. Here, each vector field v_t satisfies

$$\left.\frac{d}{ds}\right|_{s=t} \rho_s(p) = \left.\frac{d}{ds}\right|_{s=t} \rho_s\left(\rho_t^{-1}(\rho_t(p))\right) = v_t\left(\rho_t(p)\right),$$

That is,

$$\frac{d\rho_t}{dt} = v_t \circ \rho_t. \tag{2.1}$$

If, conversely, we start with a time dependent vector field v_t, it has a time-dependent flow ψ such that for $t_0 \in \mathbb{R}$, $p \in M$ and $t \in \mathbb{R}$ close to t_0, the curve

$$\gamma : t \longmapsto \psi(t, t_0, p)$$

is the unique maximal integral curve of v_t with initial condition $\gamma(t_0) = p$, that is

$$\left.\frac{d}{ds}\right|_{s=t} \psi(s, t_0, p) = v_t(\psi(t, t_0, p))$$

and $\psi(t_0, t_0, p) = p$ for all $t_0 \in \mathbb{R}$ and $p \in M$. Let $\psi_{(s,t)} = \psi(s, t, \cdot)$ and assume M is compact, or that the v_t are compactly supported. Then the maximal integral curves exist for all time $t \in \mathbb{R}$. To see how we can use the time-dependent flow to obtain an isotopy on M, we state two more properties of the flow as in [3], theorem 9.48:

Proposition 2.5 (Properties of time-dependent flow) *Let M be a compact manifold and $(v_t)_t$ a time-dependent vector field on M. Then its flow ψ satisfies*

(a) $\psi_{(s,t)} : M \to M$ *is a diffeomorphism for all* $s, t \in \mathbb{R}$ *with inverse* $\psi_{(t,s)}$,

(b) $\psi_{(t_1,t_0)} \circ \psi_{(t_0,t_2)} = \psi_{(t_1,t_2)}$.

An isotopy is now given simply by

$$\begin{aligned} \rho : \mathbb{R} \times M &\longrightarrow M \\ (t, p) &\longmapsto \psi(t, 0, p). \end{aligned}$$

$\rho_t := \rho(t, \cdot)$ is a diffeomorphism for each t by part (a) of the claim above, and $\rho_0 = \psi(0, 0, \cdot) = \mathrm{id}$.

Furthermore, ρ_t satisfies equation 2.1 by virtue of $t \mapsto \psi(t, 0, p)$ being an integral curve of v_t. Note that if we attempted to define an isotopy via

$$\rho^{t_0}(t, p) = \psi(t, t_0, p)$$

for some $t_0 \neq 0$, we would not have $\rho_0^{t_0} = \mathrm{id}$, so by uniqueness of integral curves, ρ is the unique isotopy corresponding to $(v_t)_t$. Hence for M compact we have a bijective correspondence between

$$\begin{aligned} \{\text{Isotopies of M}\} &\longleftrightarrow \{\text{Time-dependent vector fields on } M\} \\ (\rho_t)_t &\longleftrightarrow (v_t)_t. \end{aligned}$$

We now expand the notion of the Lie derivative to time-dependent vector fields.

Definition 2.6 *Let M a smooth manifold and v a (time-independent) vector field on M with flow θ_t. The **Lie derivative** by v is defined by*

$$\mathcal{L}_v : \Omega(M) \to \Omega(M), \qquad \mathcal{L}_v \omega := \left.\frac{d}{dt}\right|_{t=0} \theta_t^* \omega.$$

If v_t is time-dependent, define similarly for ψ its time-dependent flow

$$\mathcal{L}_{v_t} : \Omega(M) \to \Omega(M), \qquad \mathcal{L}_{v_t} \omega := \left.\frac{d}{ds}\right|_{s=t} \psi_{(s,t)}^* \omega.$$

This is well defined as the flow ψ exists for s close enough to t. From this definition we obtain the following identity:

Lemma 2.7 *Let v_t be a time-dependent vector field on a compact manifold M inducing the unique isotopy ρ_t. Let $\omega \in \Omega(M)$. Then $\frac{d}{dt} \rho_t^* \omega = \rho_t^* \mathcal{L}_{v_t} \omega$.*

Proof Recall that we defined $\rho_t = \psi_{(t,0)}$, and we have $\psi_{(t_0,t_1)} \circ \psi_{(t_1,t_0)} = \psi_{(t_0,t_0)} = \mathrm{id}$:

$$\begin{aligned}\left.\frac{d}{ds}\right|_{s=t} \rho_s^*\omega &= \left.\frac{d}{ds}\right|_{s=t} \psi_{(s,0)}^*\omega \\ &= \left.\frac{d}{ds}\right|_{s=t} (\psi_{(s,0)} \circ \psi_{(0,t)} \circ \psi_{(t,0)})^*\omega \\ &= \left.\frac{d}{ds}\right|_{s=t} \rho_t^*\psi_{(s,t)}^*\omega \\ &= \rho_t^* \left.\frac{d}{ds}\right|_{s=t} \psi_{(s,t)}^*\omega \\ &= \rho_t^*\mathcal{L}_{v_t}\omega,\end{aligned}$$

where the second to last equality used that the pullback ρ_t^* is linear and independent of s. □

If now in addition $\omega = \omega_t$ is also time-dependent, we can prove the following:

Proposition 2.8 *Let $(\omega_t)_{t\in\mathbb{R}}$ be a smooth family of d-forms, v_t a time-dependent vector field and ρ the isotopy generated by v_t. Then we have*

$$\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right).$$

Proof Recall that for a smooth real function $f : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\frac{d}{dt}f(t,t) = \left.\frac{d}{ds}\right|_{s=t} f(s,t) + \left.\frac{d}{ds}\right|_{s=t} f(t,s)$$

by the chain rule. Then for $x \in M$ and $u, v \in T_xM$ fixed, we may consider

$$(s,t) \longmapsto (\rho_s^*\omega_t)_x(u,v),$$

which is precisely a smooth function $\mathbb{R}^2 \to \mathbb{R}$, and thus

$$\frac{d}{dt}\rho_t^*\omega_t = \left.\frac{d}{ds}\right|_{s=t} \rho_s^*\omega_t + \left.\frac{d}{ds}\right|_{s=t} \rho_t^*\omega_s.$$

By lemma 2.7, we have $\left.\frac{d}{ds}\right|_{s=t} \rho_s^*\omega_t = \rho_t^*\mathcal{L}_{v_t}\omega_t$, and by linearity of the pullback, it follows that $\left.\frac{d}{ds}\right|_{s=t} \rho_t^*\omega_s = \rho_t^* \left.\frac{d\omega_s}{ds}\right|_{s=t}$. Together, this is

$$\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*\mathcal{L}_{v_t}\omega_t + \rho_t^*\frac{d\omega_t}{dt} = \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right).$$ □

We are now able to state and prove a version of Moser's theorem:

Theorem 2.9 (Moser) *Suppose M is a compact manifold and ω_0, ω_1 two symplectic forms on M such that*

- $[\omega_0] = [\omega_1] \in H^2_{dR}(M)$;
- *For all $t \in [0,1]$, the form $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic.*

Then there exists an isotopy $\rho : \mathbb{R} \times M \to M$ such that $\rho_t^* \omega_t = \omega_0$ for all $t \in [0,1]$.
Note that in particular, $\varphi := \rho_1 : (M, \omega_0) \to (M, \omega_1)$ is a symplectomorphism.

The proof of this theorem uses the so-called **Moser trick**:
Suppose we have an isotopy as in the theorem. We can then define the unique time-dependent vector field associated to ρ as discussed above:

$$v_t := \frac{d\rho_t}{dt} \circ \rho_t^{-1}$$

Recall that $(v_t)_t$ satisfies

$$\left.\frac{d}{ds}\right|_{s=t} \rho_s = v_t \circ \rho_t.$$

Hence by proposition 2.8, we have $\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right)$, and thus

$$\frac{d}{dt}\rho_t^*\omega_t = 0 \iff \mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt} = 0 \tag{2.2}$$

as ρ_t is a diffeomorphism.
If, conversely, we start with a time-dependent vector field v_t such that 2.2 holds, we consider the isotopy ρ generated by v_t; If M is compact, then ρ_t exists for all time $t \in \mathbb{R}$, and satisfies $\frac{d}{dt}\rho_t^*\omega_t = 0$, which says nothing but that $\rho_t^*\omega_t$ is independent of t. That is

$$\rho_t^*\omega_t = \rho_0^*\omega_0 = \omega_0.$$

Hence to prove the theorem, one need only solve 2.2 for v_t.

Proof In the case where $\omega_t = (1-t)\omega_0 + t\omega_1$, we have $\frac{d\omega_t}{dt} = \omega_1 - \omega_0$. The cohomology assumption $[\omega_0] = [\omega_1]$ then tells us that $\omega_1 - \omega_0 = d\mu$ for some $\mu \in \Omega^1(M)$.
Using Cartan's magic formula, we have

$$\mathcal{L}_{v_t}\omega_t = d\imath_{v_t}\omega_t + \imath_{v_t}\underbrace{d\omega_t}_{=0},$$

where $\imath_{v_t}$ denotes the interior product and $d\omega_t = 0$ as ω_t is symplectic and thus, in particular, closed.
With this, equation 2.2 becomes

$$\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt} = 0 \iff d\imath_{v_t}\omega_t + d\mu = 0,$$

so it is sufficient to solve $\imath_{v_t}\omega_t = -\mu$. This we can do pointwise to obtain a unique, smooth v_t. □

2.3 Conditions on the Area Forms

Remember we wish to prove that if for two area forms on our surface A, we have $\int_A \alpha_1 = \int_A \alpha_2$, then (A, α_1) and (A, α_2) are symplectomorphic. A is a compact manifold and α_1 and α_2 are symplectic forms, so if we can verify the two conditions of Moser's theorem, we will obtain the desired symplectomorphism. It is a matter of some computations to reduce the condition that $(1-t)\alpha_1 + t\alpha_2$ is symplectic for all $t \in [0,1]$ to the first condition; however, proving

$$\int_A \alpha_1 = \int_A \alpha_2 \implies [\alpha_1] = [\alpha_2]$$

will take some more refined arguments. Note that the other direction is easy: If M is a compact manifold of dimension n and $[\omega_0] = [\omega_1] \in H^n(M)$, then $\omega_1 - \omega_0 = d\mu$ for some $\mu \in \Omega^{n-1}(M)$, so

$$\int_M \omega_1 - \int_M \omega_0 = \int_M d\mu = \int_{\partial M} \mu = 0$$

by Stokes' theorem and because M is assumed to have no boundary. Hence $\int_M \omega_1 = \int_M \omega_0$.

2.3.1 The Sympleticity Condition

Proposition 2.10 *Suppose ω_0 and ω_1 are two area forms on a surface M which induce the same orientation. Then all convex combinations*

$$\omega_t = (1-t)\omega_0 + t\omega_1$$

are symplectic.

Proof We prove that ω_t is an area form for all $t \in [0,1]$, whence it is symplectic by proposition 2.3. So assume by contradiction that ω_t is not an area form for some $t \in (0,1)$, so that there exists $x \in M$ such that $(\omega_t)_x \equiv 0$. Then $0 = (1-t)(\omega_0)_x + t(\omega_1)_x$, so

$$(\omega_0)_x = \underbrace{\frac{-t}{1-t}}_{<0}(\omega_1)_x.$$

If (v_1, v_2) is any positively oriented basis of T_xM with respect to ω_0, we must have $(\omega_0)_x(v_1, v_2) > 0$ as well as $(\omega_1)_x(v_1, v_2) > 0$ as they both induce the same orientation. This contradicts the equation above. □

Lemma 2.11 *Let ω_0 and ω_1 be symplectic forms on a surface M such that $[\omega_0] = [\omega_1]$. Then they induce the same orientation on M.*

Proof The cohomology assumption gives $\omega_0 - \omega_1 = d\mu$ for some $\mu \in \Omega^1(M)$. Then $d\mu$ is not an area form since it is exact: indeed, Stokes' theorem gives

$$\int_M d\mu = \int_{\partial M} \mu = 0$$

as M is assumed to have no boundary. Hence there exists $x \in M$ with $d\mu_x \equiv 0$. Choose a positively oriented basis (v_1, v_2) of T_xM with respect to ω_0. Then

$$0 < (\omega_0)_x(v_1, v_2) = (\omega_1)_x(v_1, v_2) + \underbrace{d\mu_x(v_1, v_2)}_{=0},$$

which implies that also $(\omega_1)_x(v_1, v_2) > 0$. Thus (v_1, v_2) is also positively oriented with regard to ω_1, so the two forms induce the same orientation. □

Hence if we can prove that $\int_A \alpha_1 = \int_A \alpha_2 \implies [\alpha_1] = [\alpha_2]$, this discussion implies that all convex combinations of α_1 and α_2 are symplectic.

2.3.2 The Cohomology Condition

Here, we finally show that $\int_A \alpha_1 = \int_A \alpha_2$ implies $[\alpha_1] = [\alpha_2]$. If this is the case, we have seen that both area forms induce the same orientation, their convex combinations are symplectic, and thus we get an isotopy and in particular a symplectomorphism $(A, \alpha_1) \to (A, \alpha_2)$. We follow [4] in this section.

Let us use the following notation:

$$Z_c^n(M) = \{\omega \in \Omega_c^n(M) \mid d\omega = 0\}$$
$$B_c^n(M) = \{\omega \in \Omega_c^n(M) \mid \omega = d\mu \text{ for some } \mu \in \Omega_c^{n-1}(M)\}.$$

Then we have $\Omega_c^n(M) = Z_c^n(M)/B_c^n(M)$. Next, note that for M a connected manifold of dimension n, the map

$$\int_M : H_c^n(M) \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto \int_M \omega,$$

where $H_c^n(M)$ denotes the compactly supported cohomology of M, is well-defined, linear, and surjective. It is well defined by Stokes' theorem using the same argument as in lemma 2.11, and surjective as we may pick any nonexact form $\omega \in \Omega_c^n(M)$ and multiply it by a suitable cutoff function ρ such that $\int_M \rho\omega = a \neq 0$; then for any c, we have $\int_M \frac{c}{a}\rho\omega = c$.

We will need the following result on the cohomology of the sphere:

Proposition 2.12 *$H^k(S^n) = 0$ for $k \in \{0, n\}$ and $H^k(S^n) = \mathbb{R}$ for $1 \leq k < n$.*

We refer to section 15.10 of [5] for a proof.

This, together with the properties of $\int_M$ discussed above, has an important consequence:

Lemma 2.13 $\int_M : H^n(S^n) \to \mathbb{R}$ *is a linear isomorphism.*

Proof Since S^n is compact, $H^n_c(S^n) = H^n(S^n)$. As $H^n(S^n) = \mathbb{R}$, we can view $\int_M$ as a linear map from $\mathbb{R}$ to $\mathbb{R}$, so as a surjective linear map between vector spaces of the same dimension, it must also be injective. $\square$

Theorem 2.14 (Poincaré Lemma) *The compactly supported cohomology of* $\mathbb{R}^n$ *is* $H^k_c(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n \\ 0, & \textit{else.} \end{cases}$

Proof
Step 1: We have $H^0_c(\mathbb{R}^n) = 0$ since if $\omega \in \Omega^0_c(\mathbb{R}^n)$ is closed, it must be a constant function. The only compactly supported constant function is the trivial constant zero function, however.

Consider next the special case of $H^1_c(\mathbb{R})$. The map

$$\int_{\mathbb{R}} : Z^1_c(\mathbb{R}) \longrightarrow \mathbb{R}, \qquad \omega \longmapsto \int_{\mathbb{R}} \omega$$

is linear, surjective, and vanishes on exact forms, that is, on $B^1_c(\mathbb{R})$. Hence it induces a map from $H^1_c(\mathbb{R})$ to $\mathbb{R}$. We show its kernel is precisely $B^1_c(\mathbb{R})$, which will imply that the induced map is an isomorphism.

So take $fdt \in Z^1_c(\mathbb{R})$ for some $f \in C^\infty_c(\mathbb{R})$ with $\int_{\mathbb{R}} f(t)dt = 0$. Then the function $g(t) = \int_{-\infty}^t f(t)dt$ is smooth, compactly supported and satisfies $dg = fdt$, hence $fdt \in B^1_c(\mathbb{R})$.

Step 2: If $n > 1$, we show $H^1_c(\mathbb{R}^n) = 0$ by identifying $\mathbb{R}^n$ with $S^n \setminus \{p\}$ for some point $p \in S^n$. Thus any $\omega \in \Omega^1_c(\mathbb{R}^n)$ which is closed also defines a closed 1-form in $\Omega^1_c(S^n)$ which vanishes on a neighbourhood U of p. By the preceding proposition, $H^1(S^n) = 0$, so $\omega = d\eta$ for some $\eta \in \Omega^0(S^n) = C^\infty(S^n)$. As $\omega = d\eta = 0$ on U, this implies that η is equal to a constant c on U, so that $\bar{\eta} = \eta - c$ defines a compactly supported function in $\Omega^0_c(S^n \setminus \{p\}) = \Omega^0_c(\mathbb{R}^n)$. Hence $d\bar{\eta} = \omega$ as elements of $\Omega^1_c(\mathbb{R}^n)$.

Step 3: For any $k < n$, the argument is similar: take $\omega \in \Omega^k_c(\mathbb{R}^n)$ closed, which gives rise to a closed form $\omega \in \Omega^k(S^n)$ with support in $S^n \setminus U$, where U can be chosen to be a contractible neighbourhood of p. Then due to $H^k(S^n) = 0$, there is $\eta \in \Omega^{k-1}(S^n)$ such that $d\eta = \omega$. By the other theorem known as Poincaré's lemma, which states that closed forms are locally exact on contractible neighbourhoods, we have that as $d\eta$ vanishes on U and

U is contractible, η, too, is exact on U. Thus on U, we can write $\eta = d\mu$ for some $\mu \in \Omega^{k-2}(S^n)$. Next, pick a cutoff function ρ with

$$\chi_V \leq \rho \leq \chi_U$$

for χ the characteristic function of the appropriate set and $V \subset U$ a closed neighbourhood of p. Then $\bar{\eta} = \eta - d(\rho\mu)$ is a well-defined $(k-1)$-form on S^n that vanishes near p, hence it defines a compactly supported $(k-1)$-form on $\Omega^{k-1}(S^n \setminus U) = \Omega_c^{k-1}(\mathbb{R}^n)$ which satisfies $d\bar{\eta} = d\eta = \omega$.

Step 4: Now for the case $k = n \geq 2$. We know that $\int_M : H_c^n(\mathbb{R}^n) \to \mathbb{R}$ is linear and surjective, so we are left to show injectivity.

Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ such that $\int_M \omega = 0$. We identify again $\mathbb{R}^n$ with $S^n \setminus \{p\}$, so we may consider the embedding $i : \mathbb{R}^n \hookrightarrow S^n$. Then the pushforward $i_*\omega$ is an n-form on S^n, and

$$\int_{S^1} i_*\omega = \int_{\mathbb{R}^n} \omega = 0$$

implies by lemma 2.13 that $i_*\omega = d\eta$ for some $\eta \in \Omega^{n-1}(S^n)$. The remaining argument is the same as in the last case: pick a contractible neighbourhood U of p on which ω vanishes, deduce $\eta = d\mu$ on U for $\mu \in \Omega^{n-2}(S^n)$, and apply a cutoff function ρ to define $\bar{\eta} = \eta - d(\rho\mu)$. Then $\bar{\eta} \in \Omega_c^{n-1}(\mathbb{R}^n)$ and $d\bar{\eta} = \omega$. □

We can now prove the theorem allowing us to conclude the classification:

Theorem 2.15 *Let M an orientable connected manifold of dimension n. Then the map $\int_M : H_c^n(M) \to \mathbb{R}$ defined by*

$$[\omega] \mapsto \int_M \omega$$

is a linear isomorphism. In particular, if $\int_M \omega_0 = \int_M \omega_1$, then $[\omega_0] = [\omega_1] \in H_c^n(M)$.

Proof Again, it remains to be shown that $\int_M$ is injective, that is,

$$\omega \in H_c^n(M) \text{ such that } \int_M \omega = 0 \implies \omega = d\eta$$

for some $\eta \in \Omega_c^{n-1}(M)$. We argue by induction on the minimal number k of open sets required to cover the support of ω by a good cover, that is, a cover of sets $\{U_i\}$ such that each U_i is the domain of a chart σ_i which is a homeomorphism from U_i to $\mathbb{R}^n$. As we are working with compactly supported forms, the minimal number of sets required will always be finite.

If $k = 1$, then ω uniquely defines an n-form $\sigma_*\omega \in \Omega^n(\mathbb{R}^n)$, such that by the Poincaré lemma, $\int_{\mathbb{R}^n} \sigma_*\omega = \int_M \omega = 0$ implies $[\sigma_*\omega] = 0$ and thus $[\omega] = 0$. Suppose now all $\omega' \in \Omega_c^n(M)$ whose support can be covered by $k-1$ good sets and which satisfy $\int_M \omega' = 0$ are exact, and consider $\omega \in \Omega_c^n(M)$ such that $\int_M \omega = 0$ and $\{U_1, \dots, U_k\}$ is a good cover of $\text{supp}(\omega)$.

Let $U := \bigcup_{i=1}^{k-1} U_i$ and $V := U_k$. Pick a partition of unity $\{\rho_U, \rho_V\}$ subordinate to the cover $\{U, V\}$ of $\text{supp}(\omega)$, and define $\omega_U = \rho_U \omega$ and $\omega_V = \rho_V \omega$.

Choose $\omega_0 \in \Omega_c^n(M)$ with support in $U \cap V$ such that

$$\int_M \omega_0 = \int_M \omega_U,$$

which is possible by the same argument as the one used for the surjectivity of $\int_M$. Then $\omega_U - \omega_0$ has support in U, which admits a cover of $k-1$ good chart domains, and $\int_M \omega_U - \omega_0 = 0$, so by hypothesis, there exists $\eta_U \in \Omega_c^{n-1}(M)$ such that

$$\omega_U - \omega_0 = d\eta_U.$$

Using that $0 = \int_M \omega = \int_M \omega_U + \omega_V \iff \int_M \omega_0 = \int_M \omega_0 + \omega_U + \omega_V$ gives that

$$\int_M \omega_0 + \omega_V = \int_M \omega_0 - \omega_U = 0,$$

where $\omega_0 + \omega_V$ has support in V. Hence this form is also exact, so there is $\eta_V \in \Omega_c^{n-1}(M)$ such that

$$\omega_0 + \omega_V = d\eta_V.$$

Thus we have $\omega_U = d\eta_U + \omega_0$ and $\omega_V = d\eta_V - \omega_0$, whence we conclude

$$\omega = \omega_U + \omega_V = d(\eta_U + \eta_V). \qquad \square$$

2.4 Conclusion

Let us quickly recapture what we proved and what we started with: If we have a compact, connected, orientable surface A with any area form α, then $\int_A \alpha \in \mathbb{R}$. We may multiply α by any nonzero scalar λ such that λa remains an area form, proving that the correspondence

$$\begin{aligned} \{\alpha \text{ Area form on A}\} \sqcup \{0\} &\longrightarrow \mathbb{R} \\ \alpha &\longmapsto \int_A \alpha \end{aligned}$$

is surjective. The main statement we proved was that it is also injective up to cohomology: if α_1, α_2 are forms on A such that $\int_A \alpha_1 = \int_A \alpha_2$, then theorem 2.15 gives that $[\alpha_1] = [\alpha_2]$ (since area forms are top-dimensional forms). Moser's theorem subsequently provides us with a symplectomorphism $\varphi : (A, \alpha_1) \to (A, \alpha_2)$.

Thus we have proven theorem 2.1, that we may indeed classify (A, α) by total area. Hence for two surfaces with an area form, (A_1, α_1) and (A_2, α_2), there exists a symplectomorphism

$$\varphi : (A_1, \alpha_1) \longrightarrow (A_2, \alpha_2)$$

if and only if there is a diffeomorphism from A_1 to A_2, and in addition

$$\int_{A_1} \alpha_1 = \int_{A_2} \alpha_2.$$

Chapter 3

Generalization to Hamiltonian Circle Actions

This chapter introduces the notion of hamiltonian actions and has as its central object of study the triple

$$(A, \alpha, \psi),$$

where (A, α) is again a compact, connected surface with area form α, and $\psi : S^1 \to A$ is a hamiltonian action. We will see that along with any hamiltonian action of a Lie group G on a manifold M comes a smooth moment map $\mu : M \to \mathfrak{g}^*$, and thus in the special case where $G = S^1$ acts on the surface A, this can be seen as a map

$$\mu : A \to \mathbb{R}.$$

A being compact and connected, it follows that $\mu(A)$ is a closed interval in $\mathbb{R}$; we will prove in this chapter that it is possible to classify (A, α, ψ) in terms of area viewed as the length of this interval $\ell(\mu(A))$.

We will discuss first the case where ψ is, in addition, an effective action. This allows us to see (A, α, ψ, μ) as a so-called *symplectic toric manifold*, for which there already exists a handy classification theorem by Delzant. After giving the constructive part of the proof, we will first find the symplectic toric manifold corresponding to $\mu(A)$ by following closely Delzant's construction, and then show it is equivalent to a more geometrically intuitive one, namely the sphere S^2 being acted on by rotation with respect to the vertical axis.

In the end, we will treat the case where ψ is not effective and show that this changes very little: in fact, the only difference will turn out to be that $e^{i\theta}$ acts on $A = S^2$ by rotation by $n\theta$ for some $n \in \mathbb{N}$.

3.1 Symplectic Toric Manifolds

We start with explaining the meaning of an action being hamiltonian and the definition of a symplectic toric manifold, proceeding as in [2] in this section.

Definition 3.1 (Hamiltonian action) *Let (M,ω) be a symplectic manifold, G a Lie group with an action $\psi : G \to \mathrm{Diff}(M)$. The action is said to be* ***hamiltonian*** *if there exists a map*

$$\mu : M \to \mathfrak{g}^*$$

such that the following two conditions are satisfied:

- *For each $v \in \mathfrak{g}$, let $\mu^v : M \to \mathbb{R}$, $\mu^v(p) := \langle \mu(p), v \rangle$, the component of μ along v.*
  Let ξ_v denote the fundamental vector field generated by the action, explicitly given by
  $$\xi_v(p) = \frac{d}{dt}\Big|_{t=0} \psi_{\exp(tv)}(p).$$
  In this case,
  $$d\mu^v = -\imath_{\xi_v}\omega.$$
- *The map μ is equivariant with respect to the G-action on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$: for all $g \in G$, we have*
  $$\mu \circ \psi_g = \mathrm{Ad}^*_g \circ \mu.$$

Then the tuple (M,ω,G,μ) is called a ***hamiltonian G-space****.*

Let us note the following consequence of this definition for the action restricted to a Lie subgroup:

Lemma 3.2 *Let G a Lie group and H a closed subgroup of G, and let $\mathfrak{g}$ and $\mathfrak{h}$ denote the respective Lie algebras. Write $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ for the dual map to the inclusion $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$, that is, $i^*(\varphi) = \varphi \circ i$ for $\varphi \in \mathfrak{g}^*$. Suppose (M,ω,G,μ) is a hamiltonian G-space. Then the restriction of the (hamiltonian) G-action to H is hamiltonian with moment map*

$$i^* \circ \mu : M \to \mathfrak{h}^*.$$

Proof For $p \in M$, $v \in \mathfrak{h}$, we have

$$\begin{aligned}(i^* \circ \mu)^v(p) &= \langle i^*\mu(p), v \rangle \\ &= \langle \mu(p), i(v) \rangle \\ &= \mu^{i(v)}(p).\end{aligned}$$

Thus as μ is a moment map, $d(i^* \circ \mu)^v = -\imath_{\xi_{i(v)}}\omega$. But since we know that the exponential map associated to H is the restriction to $\mathfrak{h}$ of the exponential map $\mathfrak{g} \to G$, we see that

$$\xi_{i(v)} = \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(ti(v))} = \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(tv)} = \xi_v,$$

where ψ denotes the action. Thus the fundamental vector field associated to the action restricted to H is just the fundamental vector field ξ_v for v restricted to $\mathfrak{h}$, and hence the first condition is satisfied.
For equivariance, let $g \in H$, $p \in M$, and $v \in \mathfrak{h}$:

$$\begin{aligned}(i^* \circ \mu) \circ \psi_g(p)(v) &= \mu(\psi_g(p))(i(v)) \\ &= \mathrm{Ad}_g^* \circ \mu(p)(i(v)) \\ &= \mathrm{Ad}_g^* \circ (i^* \circ \mu(p))(v).\end{aligned}$$

□

We will be concerned with the case where G is a torus of exactly half the dimension of M, as we are interested in a surface and a hamiltonian S^1-action. In the following, we regard $\mathbb{T}^n = (S^1)^n$ and write elements of $\mathbb{T}^n$ as tuples $[\theta] = (e^{i\theta_1}, \ldots, e^{i\theta_n})$ for $\theta_i \in \mathbb{R}/2\pi\mathbb{Z}$.

We further have $\mathfrak{t}^{\mathfrak{n}} \cong \mathbb{R}^n$, and we can identify $\mathbb{R}^n$ with its dual via the pairing given by the standard inner product, which allows us to see the moment map as a map

$$\mu : M \to \mathbb{R}^n.$$

Definition 3.3 (Symplectic toric manifolds) *Let (M, ω) a compact symplectic manifold of dimension $2n$. If we consider a hamiltonian action of $\mathbb{T}^n$ on M which is, in addition, effective, then for a choice of moment map μ, the hamiltonian $\mathbb{T}^n$-space $(M, \omega, \mathbb{T}^n, \mu)$ is a* ***symplectic toric manifold****.*

In the special case of hamiltonian torus spaces, the definition of a moment map simplifies. The adjoint and coadjoint actions are trivial for a torus as it is abelian; Then a moment map of an action of $\mathbb{T}^n$ is a map $\mu : M \to \mathbb{R}^n$ such that the coordinate functions μ_k satisfy

- $\mathbb{T}^n$-invariance: $\mu_k([\theta] \cdot p) = \mu_k(p)$ for all $[\theta] \in \mathbb{T}^n$ and $p \in M$.
- μ_k is a hamiltonian function for ξ_{e_k} where e_k is the k-th standard basis vector of $\mathbb{R}^n$, that is,
$$d\mu_k = -\imath_{\xi_{e_k}}\omega.$$

From this description of the moment map, it follows that for any $c \in \mathbb{R}^n$, the map $\mu + c$ is also a moment map for the same action, and if we have two moment maps μ and $\tilde{\mu}$, then $d(\mu_k - \tilde{\mu}_k) = 0$ for each k implies that the two moment maps differ by a constant.

We give an example of a hamiltonian action which will be important later on:

Example 3.4 *Let $d \in \mathbb{N}$ and consider the action of $\mathbb{T}^d$ on $\mathbb{C}^d$ by component-wise multiplication:*

$$(e^{i\theta_1}, \ldots, e^{i\theta_d}) \cdot (z_1, \ldots, z_d) := (e^{i\theta_1} z_1, \ldots, e^{i\theta_d} z_d).$$

This action is hamiltonian with moment map

$$\mu(z_1, \ldots, z_d) = \frac{1}{2}(|z_1|^2, \ldots, |z_d|^2) + const.$$

Proof We compute for $v \in \mathbb{R}^n \cong \mathfrak{t}^n$ the fundamental vector field ξ_v at $z = (r_1 e^{i\theta_1}, \ldots, r_d e^{i\theta_d})$:

$$\begin{aligned}
\xi_v(z) &= \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(tv)}(z) \\
&= \frac{d}{dt}\bigg|_{t=0} (e^{itv_1}, \ldots, e^{itv_d}) \cdot z \\
&= \frac{d}{dt}\bigg|_{t=0} (r_1 e^{i(tv_1+\theta_1)}, \ldots, r_d^{i(tv_d+\theta_d)}) \\
&= \sum_{i=1}^{d} \frac{d}{dt}\bigg|_{t=0} (r_i)\partial_{r_i} + \frac{d}{dt}\bigg|_{t=0} (tv_i + \theta_d)\partial_{\theta_i} \\
&= \sum_{i=1}^{d} v_i \partial_{\theta_i}.
\end{aligned}$$

Hence $\xi_{e_k} = \partial_{\theta_k}$. We go on to compute for X a vector field on $\mathbb{C}^d$

$$\begin{aligned}
\imath_{\xi_{e_k}} \omega_0(X) &= \omega_0(\partial_{\theta_k}, X) \\
&= \sum_{i=1}^{d} r_i dr_i(\partial_{\theta_k}) d\theta_i(X) - r_i dr_i(X) d\theta_i(\partial_{\theta_k}) \\
&= -r_k dr_k(X) \\
&= -\tfrac{1}{2} dr_k^2(X).
\end{aligned}$$

Hence the moment map has k-th component $\mu_k(z) = \frac{1}{2} r_k^2$, which proves the claim. □

3.2 Delzant Polytopes

As mentioned above, we will use the image $\mu(A)$ to classify (A, α, ψ). More generally, for symplectic toric manifolds, the image of the manifold by the moment map is always a so-called Delzant polytope, and it is in terms of this type of polytope that symplectic toric manifolds can be classified.

First note that a **polytope** is the convex hull of a set of points in $\mathbb{R}^n$, whereas a **convex polyhedron** is the intersection of a finite number of affine half-spaces in $\mathbb{R}^n$. A theorem due to Weyl and Minkowski states that convex polyhedra coincide with bounded polytopes, see for instance theorem 1.1 in [6].

Definition 3.5 *Let $\Delta \subset \mathbb{R}^n$ be a polytope. A **face** of the polytope is a set of the form*

$$\{x \in \mathbb{R}^n \mid f(x) = c\}$$

for some $c \in \mathbb{R}$ and $f \in (\mathbb{R}^n)^*$ such that $f(x) \geq c$ for all $x \in \Delta$.

*A **vertex** is a 0-dimensional face, an **edge** is a 1-dimensional face, and a **facet** is an $(n-1)$-dimensional face.*

Definition 3.6 *A **Delzant polytope** $\Delta \subset \mathbb{R}^n$ is a polytope satisfying*

- ***Simplicity**: There are n edges meeting at each vertex.*
- ***Rationality**: The edges meeting at a vertex τ are rational in the sense that each edge is of the form $\tau + tu_k$ for $t \geq 0$ and $u_k \in \mathbb{Z}^n$.*
- ***Smoothness**: For each vertex, the edge vectors $u_1, \ldots, u_n$ of edges meeting at this vertex can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Equivalently, this means that if $U = (u_1|\ldots|u_n)$ denotes the $n \times n$-matrix which has u_k as its k-th column, we have $\det(U) = \pm 1$.*

Thus if Δ is a Delzant polytope with d facets and v_i are the primitive inward pointing normal vectors to the facets (where $v_i \in \mathbb{Z}^n$ is primitive if it cannot be written as $v_i = lu_i$ for $l \in \mathbb{Z}$ and $u \in \mathbb{Z}^n$), there are scalars λ_i, $i = 1, \ldots, d$, such that

$$\Delta = \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle \geq \lambda_i,\ i = 1, \ldots, d\}. \tag{3.1}$$

for some $v_i \in \mathbb{R}^n$, $\lambda_i \in \mathbb{R}$ and $d \in \mathbb{N}$.

The following consequence of this definition allows for some more geometrical intuition:

Lemma 3.7 *Let $\Delta \subset \mathbb{R}^n$ a Delzant polytope. Then there are n facets meeting at each vertex.*

Proof Consider any vertex of Δ and let $(u_1, \ldots, u_n)$ a $\mathbb{Z}$-basis of edge vectors incident on this vertex. By the smoothness axiom and a change of basis, we may assume $(u_1, \ldots, u_n)$ is the standard basis.

Then the primitive inward pointing normal vectors to the facets meeting at our vertex are again the standard basis, as orthogonality to the facet determined by the edge vectors $u_1, \ldots, \hat{u}_i, \ldots, u_n$, where the hat operator denotes omission, implies the normal can only be a multiple of u_i. Requiring it to be primitive gives that the inward pointing normal vector must be exactly u_i, which proves that there are n facets meeting at each vertex. □

3.3 Delzant's Classification

In our setting where (A, α) is a compact surface with an area form (and thus in particular a compact symplectic manifold by proposition 2.3), if we require the S^1-action to be effective, then up to a choice of moment map, we are working with a symplectic toric manifold. These are, up to equivalence, which is captured by the notion of equivariant symplectomorphisms, classified by Delzant's theorem.

Definition 3.8 *Let $(M_1, \omega_1, G, \mu_1)$ and $(M_2, \omega_2, G, \mu_2)$ hamiltonian G-spaces and $\varphi : M_1 \to M_2$ a smooth map. Then φ is* ***equivariant*** *if for all $p \in M_1, g \in G$*

$$\varphi(g \cdot p) = g \cdot \varphi(p).$$

Hence an equivariant symplectomorphism between symplectic toric manifolds is a symplectomorphism in the sense of definition 2.2 satisfying $\varphi([\theta] \cdot p) = [\theta] \cdot \varphi(p)$.

Theorem 3.9 (Delzant) *There is a bijective correspondence between*

$$\left\{\begin{matrix}\text{Symplectic toric manifolds} \\ \text{up to equivalence}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Delzant polytopes} \\ \text{up to translation}\end{matrix}\right\}.$$

For a toric manifold $(M^{2n}, \omega, \mathbb{T}^n, \mu)$, the corresponding polytope is given by $\mu(M)$.

For a complete treatment of the proof, we refer to the original paper by Delzant [7]. We prove the surjectivity statement below, following chapter 29 of [1].

Notice that in the case where we consider a hamiltonian S^1-action on a surface, the moment map μ is in particular a smooth map from M to $\mathbb{R}$. Hence the Delzant polytope corresponding to the surface will be an interval; Our goal in the following will be to construct the symplectic toric manifold corresponding to this situation. According to the theorem, this manifold will be unique up to equivariant symplectomorphism. For this, we shall give the general construction of the symplectic toric manifold corresponding to a given Delzant polytope.

The main tool used in the construction is the technique of symplectic reduction, which is a theorem independently proven by Marsden with Weinstein, and Meyer:

Theorem 3.10 (Marsden-Weinstein, Meyer) *Let (M, ω, G, μ) a hamiltonian G-space for a compact Lie group G. Let $i : \mu^{-1}(0) \hookrightarrow M$ denote the inclusion and assume that G acts freely on $\mu^{-1}(0)$. Then*

(a) the orbit space $M_{\text{red}} = \mu^{-1}(0)/G$ is a manifold.

(b) $\text{pr} : \mu^{-1}(0) \to M_{\text{red}}$ is a principal G-bundle, where pr denotes the canonical projection.

(c) there is a symplectic form ω_{red} on M_{red} such that $i^*\omega = \text{pr}^*\omega_{\text{red}}$.

For a proof, see section 23 of [1]. Note that as pr is a surjection, pr^* is injective, and thus ω_{red} is the *unique* two-form on M_{red} satisfying (c).

3.3.1 Delzant's Construction

Let Δ be a Delzant polytope with d facets. We consider $\Delta \subset (\mathbb{R}^n)^*$ for convenience, and consider the normal vectors to the facets to be in $\mathbb{R}^n$. Let $v_i \in \mathbb{Z}^n$, $i = 1, \ldots, d$ be the inward pointing primitive normal vectors to the facets. Then for some $\lambda_i \in \mathbb{R}$, we can write

$$\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i \rangle \geq \lambda_i,\ i = 1, \ldots, n\}.$$

Let e_i denote the standard basis vectors of $\mathbb{R}^n$, and consider

$$\begin{aligned} \tilde{\pi} : \mathbb{R}^d &\to \mathbb{R}^n \\ e_i &\mapsto v_i. \end{aligned}$$

Claim 1: The map $\tilde{\pi}$ is surjective and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$.

Hence $\tilde{\pi}$ induces a surjective map $\pi : \mathbb{R}^d/(2\pi\mathbb{Z}^d) \to \mathbb{R}^n/(2\pi\mathbb{Z}^n)$, such that for $x \in \mathbb{R}^d$, we have $\pi(x + 2\pi\mathbb{Z}^d) = \tilde{\pi}(x) + 2\pi\mathbb{Z}^n$. Identify $\mathbb{R}^k/(2\pi\mathbb{Z}^k)$ with $\mathbb{T}^k$.

Now let $N = \ker \pi$ and $\mathfrak{n}$ the Lie algebra of N. Then N is a closed subgroup of $\mathbb{T}^d$ of dimension $d - n$, and hence itself a torus. Let $i : N \hookrightarrow \mathbb{T}^d$ denote the inclusion and identify the Lie algebras of $\mathbb{T}^d$ and $\mathbb{T}^n$ with $\mathbb{R}^d$ and $\mathbb{R}^n$, respectively. Then we have an exact sequence of tori

$$\mathbb{1} \longrightarrow N \xrightarrow{\ i\ } \mathbb{T}^d \xrightarrow{\ \pi\ } \mathbb{T}^n \longrightarrow \mathbb{1}$$

which induces the exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\ i\ } \mathbb{R}^d \xrightarrow{\ \tilde{\pi}\ } \mathbb{R}^n \longrightarrow 0$$

with the dual sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\ \tilde{\pi}^*\ } (\mathbb{R}^d)^* \xrightarrow{\ i^*\ } \mathfrak{n}^* \longrightarrow 0.$$

Now consider $\mathbb{C}^d$ with its standard symplectic form $\omega_0 = \frac{i}{2}\sum_{k=1}^d z_k \wedge \overline{z}_k$, along with the action of $\mathbb{T}^d$ given by

$$(e^{t_1}, \dots, e^{t_d}) \cdot (z_1, \dots, z_d) = (e^{t_1} z_1, \dots, e^{t_d} z_d).$$

According to example 3.4, this action is hamiltonian with moment map

$$\begin{aligned} \mu : \mathbb{C}^d &\longrightarrow (\mathbb{R}^d)^* \\ (z_1, \dots, z_d) &\longmapsto \frac{1}{2}(|z_1|^2, \dots, |z_d|^2) + \text{const.} \end{aligned}$$

We choose the constant to be $(\lambda_1, \dots, \lambda_d)$. Next, consider the restricted action of $N \subset \mathbb{T}^d$ on $\mathbb{C}^d$. According to lemma 3.2, this action is also hamiltonian with moment map

$$i^* \circ \mu,$$

where $i^* : (\mathfrak{t}^d)^* \to (\mathfrak{n}^d)$ is the dual map to the inclusion $i : \mathfrak{n} \to \mathfrak{t}^d$. Consider $Z = (i^* \circ \mu)^{-1}(0)$. We claim:

Claim 2: Z is compact and N acts freely on Z.

If this is true, the conditions for the Marsden-Meyer-Weinstein theorem are met and we obtain the reduced space $M_\Delta = Z/N$, along with a symplectic form ω_Δ such that if $j : Z \hookrightarrow \mathbb{C}^d$ is the inclusion and $\mathrm{pr} : Z \to M_\Delta$ is the projection, then $\mathrm{pr}^*\omega_\Delta = j^*\omega_0$.

The next section will prove the claims made above and introduce the hamiltonian action making $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ into a symplectic toric manifold.

3.3.2 The Action on the Reduced Space

We start by reviewing the claims made in Delzant's construction.

Claim 1: The map $\tilde{\pi} : \mathbb{R}^d \to \mathbb{R}^n$ is surjective and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$.

Proof The argument is very similar to that of lemma 3.7. Fix a vertex of Δ. By lemma 3.7, there are n facets meeting at this vertex, which are determined by the $n-1$ edge vectors they meet; thus if the facet meets $u_1, \dots, \hat{u}_i, \dots, u_n$, after a change of basis transforming $u_1, \dots, u_n$ into the standard basis, we see that the inward pointing primitive normal vector to this facet is just u_i. Hence the set of primitive inward pointing normal vectors can by this change of basis be assumed to be the standard basis, proving the claim. $\square$

Claim 2: N acts freely on Z, and Z is compact.

Proof To show that Z is compact, it suffices by Heine-Borel to show that Z is closed and bounded. It is clearly closed as it is the preimage of $\{0\}$ by a continuous map, and we show that $\mu(Z) = \tilde{\pi}^*(\Delta) =: \Delta'$:

Lemma 3.11 *Let* $y \in (\mathbb{R}^d)^*$. *Then*

$$y \in \Delta' \iff y \in \mu(Z).$$

Proof y is in $\mu(Z) = \mu\left((i^* \circ \mu)^{-1}(0)\right)$ if and only if

1. y is in the image of μ,
2. $i^* y = 0$.

Using that $\mu(z_1, \ldots, z_d) = \frac{1}{2}(|z_1|^2, \ldots, |z_d|^2) + (\lambda_1, \ldots, \lambda_d)$, we see that the first condition is equivalent to

$$\langle y, e_k \rangle \geq \lambda_k \quad k = 1, \ldots, d,$$

Using the dual exact sequence, the second condition is equivalent to y being in the image of $\tilde{\pi}^*$, that is

$$y = \tilde{\pi}^*(x)$$

for some $x \in (\mathbb{R}^n)^*$. So if $y = \tilde{\pi}^*(x)$, we have

$$\begin{aligned}
\langle y, e_k \rangle \geq \lambda_k,\ \forall k \quad &\iff \quad \langle \tilde{\pi}^*(x), e_k \rangle \geq \lambda_k,\ \forall k \\
&\iff \quad \langle x, \pi(e_k) \rangle \geq \lambda_k,\ \forall k \\
&\iff \quad \langle x, v_k \rangle \geq \lambda_k,\ \forall k \\
&\iff \quad x \in \Delta.
\end{aligned}$$

We conclude $y \in \mu(Z) \iff y \in \tilde{\pi}^*(\Delta) = \Delta'$. □

Note that μ is a proper map, that is, if $C \subset (\mathbb{R}^d)^*$ is compact, then $\mu^{-1}(C)$ is compact. Indeed, if C is compact, then $C = C_1 \times \ldots \times C_d$ for C_k compact subsets of $\mathbb{R}^*$, and thus $\mu_k^{-1}(C_k) = \{2z - \lambda_k \mid |z|^2 \in C_k\}$. This is bounded as C_k is bounded, and hence $\mu^{-1}(C)$ is a product of bounded sets. It is also closed as μ is continuous, so it is compact by Heine-Borel.

Using this and the lemma we just proved, as Δ' is compact and $\mu(Z) = \Delta'$, Z must be bounded, and hence compact.

In order to use the Marsden-Weinstein-Meyer theorem, we still have to prove that N acts freely on Z.

So pick a vertex τ of Δ and let $I = \{k_1, \ldots, k_n\}$ denote the set of indices for the n faces meeting τ. Pick $z \in Z$ such that $\mu(z) = \tilde{\pi}^*(\tau)$, which exists by lemma 3.11. τ being a vertex means that it is characterised (as seen in definition 3.5) by n equations $\langle \tau, v_k \rangle = \lambda_k$ for all $k \in I$. This gives

$$\begin{aligned}
\langle \tau, v_k \rangle = \lambda_k \quad &\iff \quad \langle \tau, \tilde{\pi}(e_k) \rangle = \lambda_k \\
&\iff \quad \langle \tilde{\pi}^*(\tau), e_k \rangle = \lambda_k \\
&\iff \quad \langle \mu(z), e_k \rangle = \lambda_k \\
&\iff \quad \text{the } k\text{-th coordinate of } \mu(z) \text{ is } \lambda_k \\
&\iff \quad \tfrac{1}{2}|z_k|^2 + \lambda_k = \lambda_k \\
&\iff \quad z_k = 0.
\end{aligned}$$

Thus these z are precisely those whose coordinate entries for the coordinates in I are zero and whose other entries are nonzero. We may assume without loss of generality that $I = \{1, \ldots, n\}$. Then the stabilizer of z is

$$(\mathbb{T}^d)_z = \{(e^{i\theta_1}, \ldots, e^{i\theta_n}, 1, \ldots, 1) \in \mathbb{T}^d\}.$$

Letting $(\mathbb{R}^d)_z = \{(x_1, \ldots, x_n, 2\pi\mathbb{Z}, \ldots, 2\pi\mathbb{Z}) \subset \mathbb{R}^d\}$, we have by claim 1 that the restriction $\tilde{\pi} : (\mathbb{R}^d)_z \to \mathbb{R}^n$ maps $e_1, \ldots, e_n$ to a $\mathbb{Z}$-basis $v_1, \ldots, v_n$ of $\mathbb{Z}^n$. Hence projecting to tori, we must have that $\pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$ is bijective as a group homomorphism. Recalling $N = \ker \pi$ for $\pi : \mathbb{T}^d \to \mathbb{T}^n$, we conclude that $N \cap (\mathbb{T}^d)_z = \{1\}$, so $N_z = \{1\}$. We have thus shown that all stabilizers of the action by N at points being mapped to vertices are trivial.

If z' is another point in Z which does not map to a vertex, then $N_{z'}$ is contained in N_z for some z which does map to a vertex because then some of the equalities characterising $\mu(z')$ become inequalities, so the k-th coordinates of $\mu(z')$ do not have to be zero, which puts more restrictions on the stabilizer $(\mathbb{T}^d)_z$. This proves that indeed, all stabilizers are trivial and the action by N is free, proving claim 2. □

Proposition 3.12 *The reduced space Z/N inherits a hamiltonian $\mathbb{T}^n$-action with a moment map μ_Δ such that $\mu_\Delta(Z/N) = \Delta$.*

Proof As in the proof just above, pick $z \in Z$ such that $\mu(z) = \tilde{\pi}^*(\tau)$ for τ a vertex of Δ. We have seen that the restriction $\pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$ is a bijection, so let σ be its inverse. Then σ is also a right-inverse of π on the whole of $\mathbb{T}^d$, so from the exact sequence

$$\mathbb{1} \longrightarrow N \xrightarrow{\;i\;} \mathbb{T}^d \underset{\sigma}{\overset{\pi}{\rightleftharpoons}} \mathbb{T}^n \longrightarrow \mathbb{1}$$

we obtain an isomorphism $(i, \sigma) : N \times \mathbb{T}^n \to \mathbb{T}^d$.

We can now endow Z/N with the action induced from the $\mathbb{T}^n$ factor above. In precise terms, we define for $[\theta] \in \mathbb{T}^n$, $z \in Z$ and $\mathrm{pr} : Z \to Z/N$ the projection

$$[\theta] \cdot \mathrm{pr}(z) := \mathrm{pr}(\sigma([\theta]) \cdot z),$$

where the action on the right hand side is the standard action on $\mathbb{C}^d$ by $\mathbb{T}^d$. If $w = g \cdot z$ for some $g \in N$, then $\sigma([\theta]) \cdot z = \sigma([\theta]) g \cdot w = g \cdot (\sigma([\theta]) \cdot w)$ as $\mathbb{T}^d$ is abelian, so $\sigma([\theta]) \cdot z$ and $\sigma([\theta]) \cdot w$ are in the same N-orbit, hence this action is indeed well-defined. Next, consider the diagram

$$\begin{array}{ccccccc} Z & \xhookrightarrow{\;j\;} & \mathbb{C}^d & \xrightarrow{\;\mu\;} & (\mathbb{R}^d)^* \cong \mathfrak{n}^* \oplus (\mathbb{R}^n)^* & \xrightarrow{\;\sigma^*\;} & (\mathbb{R}^n)^* \\ \big\downarrow{\scriptstyle \mathrm{pr}} & & & & & \nearrow_{\mu_\Delta} & \\ Z/N & & & & & & \end{array}$$

Here, j is the inclusion, and $(\mathbb{R}^d)^* \cong \mathfrak{n}^* \oplus (\mathbb{R}^n)^*$ uses the isomorphism (i, σ) from above at the level of Lie algebra duals. From this we obtain that σ^* is just the projection onto the second factor.

As μ is a moment map for the $\mathbb{T}^d$-action, it is in particular N-invariant, so the composition of horizontal maps in the diagram is constant along N-orbits. Thus we obtain a moment map μ_Δ for the $\mathbb{T}^n$-action such that the diagram commutes, that is

$$\mu_\Delta \circ \mathrm{pr} = \sigma^* \circ \mu \circ j.$$

Finally, we have

$$\mu_\Delta(Z/N) = \mu_\Delta(\mathrm{pr}(Z)) = \sigma^* \circ \mu \circ j(Z).$$

Then recall $\mu(j(Z)) = \mu(Z) = \tilde{\pi}^*(\Delta)$, and $\sigma^* \circ \tilde{\pi}^* = \mathrm{id}$ to obtain

$$\sigma^* \circ \mu \circ j(Z) = \sigma^* \circ \tilde{\pi}^*(\Delta) = \Delta. \qquad \square$$

We have thus constructed from a Delzant polytope Δ the symplectic toric manifold $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$. For more details, for example a proof that $\mu(M)$ is always a Delzant polytope and in addition just the one such that following the construction above, we recover M, see chapter 29 of [1].

3.4 The Case of Surfaces

3.4.1 The Manifold

Let us now return to the setting of a compact, orientable surface A with a hamiltonian S^1-action. As we have seen, a choice of area form α defines a symplectic form on A, and thus for a choice of moment map μ of the hamiltonian action, (A, α, S^1, μ) is a toric manifold. As $\mu : A \to \mathbb{R}$ is smooth, it is in particular continuous, and thus its image $\mu(A)$ is an interval in $\mathbb{R}$, which we can take to be $[0, r]$ for some $r > 0$ since classification in terms of Delzant polytopes is only up to translation.

We construct the toric manifold corresponding to $\Delta = [0, r]$. The primitive inward pointing normal vectors to the facets are in this case $v_1 = v$ and $v_2 = -v$ for $v = e_1 = 1$, the standard basis vector of $\mathbb{R}$. Written in the terms of equation 3.1, the polytope becomes

$$\Delta = \{x \in \mathbb{R} \mid \langle x, -v \rangle \geq -r,\ \langle x, v \rangle \geq 0\},$$

so $\lambda_1 = 0$ and $\lambda_2 = -r$.

The map $\tilde{\pi} : \mathbb{R}^2 \to \mathbb{R}$ maps $e_1 \mapsto v$ and $e_2 \mapsto -v$, so it is given by $\tilde{\pi}(t_1, t_2) = t_1 - t_2$ and thus its kernel is $\mathrm{span}(e_1 + e_2)$. The induced map is thus given by

$$\pi((t_1, t_2) + \mathbb{R}^2/2\pi\mathbb{Z}^2) = \tilde{\pi}(t_1, t_2) + \mathbb{R}/2\pi\mathbb{Z} = (t_1 - t_2) + \mathbb{R}/2\pi\mathbb{Z}.$$

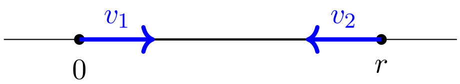


Figure 3.1: Δ with its inward pointing normal vectors

Using the identification $t \sim e^{it}$ for $\mathbb{T} \cong S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, we can write this as

$$\pi(e^{it_1}, e^{it_2}) = e^{it_1}e^{-it_2},$$

or simply, for $(\theta_1, \theta_2) \in \mathbb{T}^2$

$$\pi(\theta_1, \theta_2) = \theta_1\theta_2^{-1}.$$

Thus the kernel of the induced map π is the diagonal subgroup

$$N = \{(\theta, \theta) \in \mathbb{T}^2 \cong S^1 \times S^1 \mid \theta \in [0, 2\pi)\}.$$

We can thus identify N simply with S^1. Its Lie algebra is simply the kernel of $\tilde{\pi}$, which is $\mathfrak{n} = \{x(e_1 + e_2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$, which we can identify with $\mathbb{R}$.

We now compute the dual maps $\tilde{\pi}^*$ and i^*. As $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ are isomorphic under the pairing given by the standard inner product, every element of $(\mathbb{R}^n)^*$ is a map of the form $\langle \cdot, x\rangle$ for some $x \in \mathbb{R}^n$. Hence $\tilde{\pi}^*(\langle \cdot, x\rangle) \in (\mathbb{R}^2)^*$ is given by

$$\tilde{\pi}^*(\langle \cdot, x\rangle)(a, b) = \langle \tilde{\pi}(a, b), x\rangle = \langle a, x\rangle - \langle b, x\rangle = \langle (a, b), (x, -x)\rangle$$

for $(a, b) \in \mathbb{R}^2$. Thus under the identification of $\mathbb{R}^n$ with its dual, this becomes

$$\tilde{\pi}^*(x) = (x, -x)$$

for $x \in \mathbb{R}^*$.

Similarly for i^*, we have for $\langle \cdot, (x_1, x_2)\rangle \in (\mathbb{R}^2)^*$ and $y \in \mathfrak{n}$ that

$$i^*\left(\langle \cdot, (x_1, x_2)\rangle\right)(y) = \langle i(y), (x_1, x_2)\rangle = x_1 y + x_2 y = \langle y, x_1 + x_2\rangle,$$

and hence

$$i^*(x_1, x_2) = x_1 + x_2.$$

This gives us the exact sequences

$$\begin{array}{ccccccccc} \mathbb{1} & \longrightarrow & N & \xrightarrow{\;i\;} & \mathbb{T}^2 & \xrightarrow{\;\pi\;} & S^1 & \longrightarrow & \mathbb{1} \\ & & \theta & \longmapsto & (\theta, \theta) & & & & \\ & & & & (\theta_1, \theta_2) & \longmapsto & \theta_1\theta_2^{-1} & & \end{array}$$

for the sequence of tori,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n} & \xrightarrow{\quad i \quad} & \mathbb{R}^2 & \xrightarrow{\quad \tilde{\pi} \quad} & \mathbb{R} & \longrightarrow & 0 \\ & & x & \longmapsto & (x, x) & & & & \\ & & & & (x_1, x_2) & \longmapsto & x_1 - x_2 & & \end{array}$$

for the sequence of Lie algebras, and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}^* & \xrightarrow{\quad \tilde{\pi}^* \quad} & (\mathbb{R}^2)^* & \xrightarrow{\quad i^* \quad} & \mathfrak{n}^* & \longrightarrow & 0 \\ & & x & \longmapsto & (x, -x) & & & & \\ & & & & (x_1, x_2) & \longmapsto & x_1 + x_2 & & \end{array}$$

for the dual sequence.

As in the general construction, we consider the action of $\mathbb{T}^2$ on $(\mathbb{C}^2, \omega_0)$ given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$$

with moment map

$$\mu(z_1, z_2) = \frac{1}{2}(|z_1|^2, |z_2|^2) + (0, -r).$$

The action of the diagonal group N on $\mathbb{C}^2$ is thus given by $e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$ and, using lemma 3.2 and the definition of i^* computed above, has moment map

$$(i^* \circ \mu)(z_1, z_2) = \frac{1}{2}(|z_1|^2 + |z_2|^2) - r.$$

Its zero level is

$$Z = (i^* \circ \mu)^{-1}(0) = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 2r\},$$

which can be seen to be the (real) sphere $S^3(\sqrt{2r})$ with radius $\sqrt{2r}$. Hence the quotient Z/N is the quotient $S^3/\sim$ by the equivalence relation $(z_1, z_2) \sim (w_1, w_2) \iff (z_1, z_2) = \lambda(w_1, w_2)$ for some $\lambda \in S^1$, which is diffeomorphic to the complex projective space $\mathbb{CP}^1$. This uses the description of $\mathbb{CP}^n$ given as in lemma 3.15, together with the fact that $S^3(\sqrt{2r})$ is diffeomorphic to the unit three-sphere simply by the map sending $z \in S^3$ to $\sqrt{2r}z$.

3.4.2 The Action

Recall that the action of $\mathbb{T}^n$ on Z/N is given by

$$[\theta] \cdot \mathrm{pr}(z) = \mathrm{pr}(\sigma([\theta]) \cdot z),$$

where σ is the inverse of the bijection $\pi : (\mathbb{T}^d)_z \to \mathbb{T}^z$ for $z \in Z$ such that $\mu(z) = \tilde{\pi}^*(\tau)$, and τ is a vertex of Δ.

So for $\Delta = [0, r]$, let us pick $\tau = 0$. Recalling $\tilde{\pi}^*(x) = (x, -x)$, this gives $\tilde{\pi}^*(0) = (0, 0)$. Our expression for μ is

$$\mu(z_1, z_2) = \frac{1}{2}(|z_1|^2, |z_2|^2) + (0, -r),$$

so if $\mu(z_1, z_2) = \tilde{\pi}^*(\tau)$, we must have $z_1 = 0$ and, as $Z = S^3(\sqrt{2r})$, $z_2 = \sqrt{2r}e^{i\theta}$ for any $\theta \in [0, 2\pi)$. Let us choose $z = (0, \sqrt{2r})$. The stabilizer is

$$(\mathbb{T}^2)_z = \{(e^{i\theta}, 1) \in \mathbb{T}^2\},$$

and thus the restriction $\pi : (\mathbb{T}^2)_z \to S^1$ is a bijection. Recall that $\pi(\theta_1, \theta_2) = \theta_1\theta_2^{-1}$, so that its inverse is simply

$$\begin{aligned} \sigma : S^1 &\longrightarrow \mathbb{T}^2 \\ e^{i\theta} &\longmapsto (e^{i\theta}, 1). \end{aligned}$$

Now we can write the action explicitly as

$$e^{i\theta} \cdot [z_0 : z_1] = [e^{i\theta} z_0 : z_1].$$

The moment map is then $\mu_\Delta \circ \mathrm{pr} = \sigma^* \circ \mu \circ j$, so

$$\mu_\Delta([z_0 : z_1]) = \sigma^*((\tfrac{1}{2}|z_0|^2, \tfrac{1}{2}|z_1|^2 - r)) = \tfrac{1}{2}|z_0|^2.$$

Note that as we take $(z_0, z_1) \in S^3(\sqrt{2r})$, $|z_0|^2$ ranges in $[0, 2r]$, so that indeed, the image of $\mathbb{CP}^1$ by μ_Δ is $\Delta = [0, r]$.

3.5 Complex Projective Spaces

In order to determine the symplectic form induced by the symplectic reduction of the zero level Z by N above, we recall some basics related to projective spaces and their natural smooth structure. First, we recall some equivalent ways to define $\mathbb{CP}^n$, and then we see how $\mathbb{CP}^n$ can be obtained by symplectic reduction, thereby endowing it with a natural symplectic form.

Definition 3.13 *The* ***complex projective space*** $\mathbb{CP}^n$ *is given by* $\left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \sim$, *the equivalence relation being defined by*

$$(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$$

for $\lambda \in \mathbb{C}$ *and* $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$.

We denote a point in $\mathbb{CP}^n$ *by*

$$[(z_0, \ldots, z_n)] = [z_0 : \ldots : z_n]$$

for $z_i \in \mathbb{C}$ *and not all* z_i *equal to zero.*

Informally, this describes the set of complex lines in $\mathbb{C}^{n+1}$ *through the origin.*

The following lemma describes the smooth structure on $\mathbb{CP}^n$. For a proof, see for example [8].

Lemma 3.14 *For $i = 0, \dots, n$, define the sets*

$$U_i := \{[z_0 : \dots : z_n] \in \mathbb{CP}^n \mid z_i \neq 0\},$$

together with the maps

$$\begin{aligned} \varphi_i : U_i &\longrightarrow \mathbb{C}^n \cong \mathbb{R}^{2n} \\ [z_0 : \dots : z_n] &\longmapsto \frac{1}{z_i}(z_0, \dots, \hat{z}_i, \dots, z_n), \end{aligned}$$

where $\hat{z}_i$ denotes the omission of z_i.

Then $\{(\varphi_i, U_i) \mid i = 0, \dots, n\}$ is a smooth atlas on $\mathbb{CP}^n$.

It is a standard result that this definition of $\mathbb{CP}^n$ is equivalent to the complex n-sphere, where its antipodal points are identified. Let us phrase this in the following lemma, and refer to [9] for a proof.

Lemma 3.15 *Let S^{2n+1} denote the real unit sphere and identify it with the complex unit sphere $S^n \subset \mathbb{C}^{n+1}$. Define an equivalence relation on S^{2n+1} by*

$$(z_0, \dots, z_n) \sim e^{i\theta}(z_0, \dots, z_n), \quad \theta \in [0, 2\pi).$$

Then $S^{2n+1}/\sim$ is diffeomorphic to $\mathbb{CP}^n$.

Note that this implies that $\mathbb{CP}^n$ is a compact manifold. The above equivalence relation could also be seen as arising from an action of S^1, which will guide us into seeing how $\mathbb{CP}^n$ can be obtained by symplectic reduction. This way, we will obtain a symplectic form by the Marsden-Weinstein-Meyer theorem.

Proposition 3.16 *Consider the action of S^1 on $\mathbb{C}^{n+1}$ given by*

$$e^{i\theta} \cdot (z_0, \dots, z_n) := (e^{i\theta} z_0, \dots, e^{i\theta} z_n).$$

This action is hamiltonian with moment map $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$,

$$\mu(z_0, \dots, z_n) = \frac{1}{2} \sum_{j=0}^{n} |z_j|^2 + const.$$

Proof The standard symplectic form on $\mathbb{C}^{n+1}$ in polar coordinates is $\omega_0 = \sum_{j=0}^n r_j dr_j \wedge d\theta_j$. So for $v \in \mathbb{R} \cong T_1 S^1$ and $z = (z_0, \dots, z_n)$, we compute the fundamental vector field

$$\begin{aligned}\xi_v(z) &= \left.\frac{d}{dt}\right|_{t=0} \exp(tv)\cdot(z_0,\ldots,z_n)\\ &= \left.\frac{d}{dt}\right|_{t=0} (e^{itv}r_0e^{i\theta_0},\ldots,e^{itv}r_ne^{i\theta_n})\\ &= \sum_{j=0}^n \left(\left.\frac{d}{dt}\right|_{t=0} r_j\right)\left.\frac{\partial}{\partial\theta_j}\right|_z + \left.\frac{d}{dt}\right|_{t=0}(\theta_j+tv)\left.\frac{\partial}{\partial r_j}\right|_z\\ &= \sum_{j=0}^n v\left.\frac{\partial}{\partial r_j}\right|_z .\end{aligned}$$

Letting $v = 1$ and X any vector field on $\mathbb{C}^{n+1}$:

$$\begin{aligned}\imath_{\xi_1}\omega_0(X) &= \sum_{j=0}^n r_j dr_j \wedge d\theta_j \left(\sum_{k=0}^n \frac{\partial}{\partial\theta_k}, X\right)\\ &= \sum_{j=0}^n r_j \underbrace{dr_j\left(\sum_{k=0}^n \partial\theta_k\right)}_{=0} d\theta_j(X) - r_j dr_j(X)\underbrace{d\theta_j\left(\sum_{k=0}^n \partial\theta_k\right)}_{=\delta_{jk}}\\ &= -\sum_{j=0}^n r_j dr_j(X)\\ &= -\frac{1}{2}\sum_{j=0}^n dr_j^2(X).\end{aligned}$$

From this and $\mu = \frac{1}{2}\sum_{j=0}^n r_j^2 + const.$, it is immediate that $d\mu = -\imath_{\xi_1}\omega_0$. □

Thus $(\mathbb{C}^{n+1}, \omega_0, S^1, \mu)$ is a hamiltonian S^1-space and S^1 is compact. If we choose the additive constant of the moment map to be $-\frac{1}{2}$, its zero level set is

$$\mu^{-1}(0) = \{z \in \mathbb{C}^{n+1} \mid \frac{1}{2}\sum_{j=0}^n |z_j|^2 - \frac{1}{2} = 0\} = S^{2n+1}.$$

For any $e^{i\theta} \in S^1$, we have that $e^{i\theta}(z_0,\ldots,z_n) = (z_0,\ldots,z_n)$ implies $z_0 = \ldots = z_n = 0$, so the action is free on $\mathbb{C}^{n+1} \setminus \{0\}$ and thus in particular on $\mu^{-1}(0) = S^{2n+1}$. Hence the conditions for Marsden-Weinstein-Meyer are satisfied; the quotient S^{2n+1}/S^1 is a manifold, and the orbit equivalence relation of this quotient is just the same as the equivalence relation from lemma 3.15, hence the reduced space is just $\mathbb{CP}^n$. We take this as the definition of our symplectic form on $\mathbb{CP}^n$:

Definition 3.17 *The* ***Fubini-Study form*** ω_{FS} *on* $\mathbb{CP}^n$ *is the symplectic form induced by symplectic reduction of* $\mathbb{C}^{n+1}$ *by* S^1 *with respect to the hamiltonian action and moment map described in proposition 3.16.*

Equivalently, if $\pi : S^{2n+1} \to \mathbb{CP}^n$ *denotes the projection and* $i : S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ *the inclusion,* ω_{FS} *is the unique symplectic form on* $\mathbb{CP}^n$ *satisfying*

$$\pi^*\omega_{FS} = i^*\omega_0.$$

3.6 The Case of Surfaces - The Area Form

We have defined the symplectic form induced on $\mathbb{CP}^n$ seen as the quotient S^{2n+1}/S^1 to be the Fubini-Study form. Note carefully, however, that both spheres in this quotient are of unit radius, whereas if we recall the moment map we obtained by Delzant's construction,

$$(i^* \circ \mu)(z_1, z_2) = \frac{1}{2}(|z_1|^2 + |z_2|^2) - r,$$

we saw that its zero level set Z is the sphere with radius $\sqrt{2r}$, denoted $S^3(\sqrt{2r})$. During the symplectic reduction, we considered the equivalence relation arising from the subgroup N, which in this case was just S^1 with the action $e^{i\theta} \cdot (z_1, z_2) := (e^{i\theta}z_1, e^{i\theta}z_2)$, whose orbit equivalence relation is precisely the one used in the characterisation of $\mathbb{CP}^n$ as given in lemma 3.15. Let us thus investigate how the radius of the sphere affects the symplectic manifold obtained by reduction.

Consider $S^{2n+1}(a)$ for general $a > 0$ and $n \in \mathbb{N}$. We write as usual $S^{2n+1}(1) = S^{2n+1}$. Then of course, S^{2n+1} is diffeomorphic to $S^{2n+1}(a)$ by

$$\begin{aligned} \tilde{\varphi} : S^{2n+1} &\longrightarrow S^{2n+1}(a) \\ z &\longmapsto az. \end{aligned}$$

This map induces a diffeomorphism φ such that the following diagram commutes:

$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{\tilde{\varphi}} & S^{2n+1}(a) \\ \downarrow & & \downarrow \\ S^{2n+1}/S^1 & \xrightarrow{\varphi} & S^{2n+1}(a)/S^1 \end{array}$$

So both quotients are diffeomorphic to $\mathbb{CP}^n$, but $\tilde{\varphi}$ may not be a symplectomorphism.

Let ω_0 denote the standard symplectic form on $\mathbb{C}^{n+1}$ restricted to $S^{2n+1}(a)$ and S^{2n+1}, respectively. Compute for $z \in S^{2n+1}$ and $u, v \in T_zS^{2n+1}$

$$(\tilde{\varphi}^*\omega_0)_z(u, v) = (\omega_0)_{az}\left(D\tilde{\varphi}(z)[u], D\tilde{\varphi}(z)[v]\right) = (\omega_0)_{az}(au, av),$$

so by bilinearity of ω_0, we see that $\tilde{\varphi}^*\omega_0 = a^2\omega_0$. This implies that the symplectic form obtained by reducing the sphere $S^{2n+1}(a)$ is just a^2 times the symplectic form obtained from reducing S^{2n+1}. So letting ω denote the symplectic form on $S^{2n+1}(a)/S^1$, we have

$$\omega = a^2 \omega_{FS}.$$

Thus in our case where $a = \sqrt{2r}$, we have that the symplectic manifold (A, α) with hamiltonian S^1-action has to be

$$(A, \alpha) = (\mathbb{CP}^1, 2r\, \omega_{FS})$$

with the action given by

$$e^{i\theta} \cdot [z_0 : z_1] = [e^{i\theta} z_0 : z_1],$$

which completes our classification.

3.7 Equivalence to Spheres

In this section, we exhibit an equivariant symplectomorphism from the toric manifold obtained above to one allowing for more geometric intuition. Our strategy is showing that $\mathbb{CP}^1$ is diffeomorphic to S^2, where we adapt an argument from [10], checking what the symplectic form must be on S^2 for our diffeomorphism to become a symplectomorphism, and finally defining the action on S^2 to be such that the symplectomorphism becomes equivariant. We will carry out the proofs for the sphere with unit radius first and generalize in the end.

3.7.1 The Manifold

Proposition 3.18 $\mathbb{CP}^1$ *carrying its standard smooth structure is diffeomorphic to* S^2*, the charts on* S^2 *being given by stereographic projection.*

Proof We first prove that replacing the chart φ_0 from lemma 3.14 by $\bar{\varphi}_0 : U_0 \to \mathbb{C}$, given by

$$\bar{\varphi}_0([z_0 : z_1]) = \overline{z_1 z_0^{-1}},$$

induces the same smooth structure on $\mathbb{CP}^1$. For this, we need to prove that $\{\varphi_0, \varphi_1, \bar{\varphi}_0\}$ is still a smooth atlas, that is, that all transition functions are smooth. Two quick computations show

$$\rho_{0,\bar{0}}(z) = \varphi_0 \circ \bar{\varphi}_0^{-1}(z) = \varphi_0([1 : \bar{z}]) = \bar{z},$$

as well as

$$\rho_{1,\bar{0}}(z) = \varphi_1 \circ \bar{\varphi}_0^{-1}(z) = \varphi_1([1 : \bar{z}]) = \frac{1}{\bar{z}} = \frac{z}{|z|^2}.$$

Essentially the same computation shows $\rho_{1,\bar{0}} = \rho_{\bar{0},1}$ and $\rho_{0,\bar{0}} = \rho_{\bar{0},0}$. Identifying $\mathbb{C} \cong \mathbb{R}^2$ via the canonical isomorphism, the transition maps become

$$\begin{aligned}\rho_{0,\bar{0}}(x_0, x_1) &= (x_0, -x_1)\\ \rho_{1,\bar{0}}(x_0, x_1) &= \left(\frac{x_0}{x_0^2 + x_1^2}, \frac{x_1}{x_0^2 + x_1^2}\right),\end{aligned}$$

both evidently smooth.

Recall at this point the standard smooth structure on S^2: Define $V_N = S^2 \setminus \{x_N\}$ and $V_S = S^2 \setminus \{x_S\}$ for $x_N = (0, 0, 1)$ the north pole and $x_S = (0, 0, -1)$ the south pole. The charts are given by

$$\begin{aligned}\psi_0 : V_N \to \mathbb{R}^2, \qquad \psi_0(x_1, x_2, x_3) &= \frac{1}{1 - x_3}(x_1, x_2)\\ \psi_1 : V_S \to \mathbb{R}^2, \qquad \psi_1(x_1, x_2, x_3) &= \frac{1}{1 + x_3}(x_1, x_2),\end{aligned}$$

and both transition functions are

$$\psi_0 \circ \psi_1^{-1}(x_0, x_1) = \psi_1 \circ \psi_0^{-1}(x_0, x_1) = \left(\frac{x_0}{x_0^2 + x_1^2}, \frac{x_1}{x_0^2 + x_1^2}\right).$$

The reason we swapped φ_0 by $\bar{\varphi}_0$ before was to achieve the same transition functions for both atlases:

$$\varphi_1 \circ \bar{\varphi}_0^{-1} = \psi_1 \circ \psi_0^{-1}. \tag{$*$}$$

This enables us to define the map

$$\begin{aligned}\Psi : \mathbb{CP}^1 &\longrightarrow S^2\\ \varphi_1^{-1}(x_1, x_2) &\longmapsto \psi_1^{-1}(x_1, x_2)\\ \bar{\varphi}_0^{-1}(x_1, x_2) &\longmapsto \psi_0^{-1}(x_1, x_2),\end{aligned}$$

with inverse

$$\begin{aligned}\Psi^{-1} : S^2 &\longrightarrow \mathbb{CP}^1\\ \psi_0^{-1}(x_1, x_2) &\longmapsto \bar{\varphi}_0^{-1}(x_1, x_2)\\ \psi_1^{-1}(x_1, x_2) &\longmapsto \varphi_1^{-1}(x_1, x_2).\end{aligned}$$

Ψ is well-defined because of ($*$): if $\bar{\varphi}_0^{-1}(x_1, x_2) = \varphi_1^{-1}(y_1, y_2)$ for some (x_1, x_2) and (y_1, y_2) in $\mathbb{R}^2$, then this is equivalent to

$$\begin{aligned}(x_1, x_2) &= \bar{\varphi}_0 \circ \varphi_1^{-1}(y_1, y_2)\\ &\overset{*}{=} \psi_0 \circ \psi_1^{-1}(y_1, y_2)\\ \iff \psi_0^{-1}(x_1, x_2) &= \psi_1^{-1}(y_1, y_2).\end{aligned}$$

That is to say that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{\quad\Psi\quad} & S^2 \\ & {\scriptstyle \varphi_i}\searrow \quad \swarrow{\scriptstyle \psi_i} & \\ & \mathbb{R}^2 & \end{array} \tag{3.2}$$

for $i = 0, 1$. As $\Psi = \psi_i^{-1} \circ \varphi_i$ is a composition of diffeomorphisms and the transition functions are smooth, it is a diffeomorphism, too. □

3.7.2 The Area Form

Note that on the sphere S^2 with unit radius, the standard volume form $\mathrm{vol} = dx \wedge dy \wedge dz$ of $\mathbb{R}^3$ induces an area form: $M := B_1(0) = \{v \in \mathbb{R}^3 \mid \|v\| \leq 1\}$ is a manifold with boundary S^2, and restricting the standard volume form gives a volume form on $B_1(0)$. Hence vol induces an area form on $\partial B_1(0)$ given by $\imath_X(\mathrm{vol})$ for X the section of $TM|_{\partial M}$ which sends $p = (x, y, z) \in S^2$ to the outward-pointing normal vector $(x\partial_x + y\partial_y + z\partial_z)_p \in T_p\mathbb{R}^3$. Denoting the form on S^2 by σ, we compute for Y and Z vector fields

$$\begin{aligned}
\sigma(Y, Z) = \imath_X \mathrm{vol}(Y, Z) &= \det \begin{pmatrix} dx(X) & dx(Y) & dx(Z) \\ dy(X) & dy(Y) & dy(Z) \\ dz(X) & dz(Y) & dz(Z) \end{pmatrix} \\
&= dx(X) \cdot dy \wedge dz(Y, Z) - dy(X) \cdot dx \wedge dz(Y, Z) + dz(X) \cdot dx \wedge dy(Y, Z) \\
&= (x dy \wedge dz + y dz \wedge dx + z dx \wedge dy)\,(Y, Z).
\end{aligned}$$

By construction, this is an area form and thus a symplectic form on S^2. To give a more handy characterization, we pull it back to $\mathbb{R}^2$ by the following parametrization:

$$\begin{aligned}
\Phi : \mathbb{R}^2 &\longrightarrow S^2 \\
(\theta, h) &\longmapsto \begin{pmatrix} \sqrt{1-h^2}\cos(\theta) \\ \sqrt{1-h^2}\sin(\theta) \\ h \end{pmatrix}
\end{aligned}$$

for $h \in [-1, 1]$ and $\theta \in [0, 2\pi)$. This can be seen as computing σ in polar coordinates:

$$\begin{aligned}
\Phi^* dx &= \partial_\theta(x \circ \Phi) d\theta + \partial_h(x \circ \Phi) dh \\
&= -\sqrt{1-h^2}\sin(\theta) d\theta - \frac{h}{\sqrt{1-h^2}}\cos(\theta) dh, \\
\Phi^* dy &= \sqrt{1-h^2}\cos(\theta) d\theta - \frac{h}{\sqrt{1-h^2}}\sin(\theta) dh, \\
\Phi^* dz &= dh.
\end{aligned}$$

Thus for the wedge products, we obtain

$$\begin{aligned}
\Phi^* dy \wedge dz &= \sqrt{1-h^2}\cos(\theta) d\theta \wedge dh, \\
\Phi^* dz \wedge dx &= \sqrt{1-h^2}\sin(\theta) d\theta \wedge dh, \\
\Phi^* dx \wedge dy &= h\sin^2(\theta) d\theta \wedge dh - h\cos^2(\theta) dh \wedge d\theta \\
&= h d\theta \wedge dh.
\end{aligned}$$

Inserting this into $\Phi^*\sigma$ gives

$$\begin{aligned}
\Phi^*\sigma &= \left((1-h^2)\cos^2(\theta) + (1-h^2)\sin^2(\theta) + h^2\right) d\theta \wedge dh \\
&= d\theta \wedge dh.
\end{aligned}$$

This shall be our definition for the standard symplectic form on S^2:

Definition 3.19 *We call $\omega_{Eucl} := \sigma$ the **Euclidean symplectic form** on S^2. We will from now on suppress Φ^* from our notation and just write*

$$\omega_{Eucl} = d\theta \wedge dh.$$

So far, we know that $\mathbb{CP}^1$ is diffeomorphic to S^2, and we have natural symplectic forms on each manifold. The next few propositions illustrate the close relation between $(\mathbb{CP}^1, \omega_{FS})$ and (S^2, ω_{Eucl}) as symplectic manifolds. First, we give a concrete description of ω_{FS} on its first coordinate chart:

Lemma 3.20 *Let $\bar{\varphi}_0, U_0$ as in lemma 3.18 the first chart on $\mathbb{CP}^1$, that is, $U_0 = \{[z_0 : z_1] \in \mathbb{CP} \mid z_0 \neq 0\}$ and $\bar{\varphi}_0([z_0 : z_1]) = \overline{z_1 z_0^{-1}}$. On this chart, we have*

$$\omega_{FS} = -\frac{dx \wedge dy}{(1+x^2+y^2)^2}.$$

Denote $\tilde{\omega}_{FS} = \frac{dx\wedge dy}{(1+x^2+y^2)^2}$. Then this is to say that $\bar{\varphi}_0^*(-\tilde{\omega}_{FS}) = \omega_{FS}$.

Proof We use the uniqueness of the induced symplectic form on the reduced space by the Marsden-Weinstein-Meyer theorem 3.10. Recall how we obtained $\mathbb{CP}^n$ as the quotient S^{2n+1}/S^1 and, for $n = 1$, the following diagram:

$$\begin{array}{ccc}
S^3 & \overset{i}{\hookrightarrow} & \mathbb{C}^2 \\
\downarrow \pi & & \\
\mathbb{CP}^1 & \xrightarrow{\bar{\varphi}_0} & \mathbb{C} \cong \mathbb{R}^2
\end{array}$$

Hence $\bar{\varphi}_0^*(-\tilde{\omega}_{FS}) = \omega_{FS}$ if and only if $(\bar{\varphi}_0 \circ \pi)^*\tilde{\omega}_{FS} = -i^*\omega_0$ for ω_0 the standard symplectic form on $\mathbb{C}^2$.

We work in polar coordinates. Then we have $\omega_0 = r_0 dr_0 \wedge d\theta_0 + r_1 dr_1 \wedge d\theta_1$, and for $(z_0, z_1) = (r_0 e^{i\theta_0}, r_1 e^{i\theta_1}) \in S^3$

$$\begin{aligned}\psi(z_0, z_1) &:= (\bar{\varphi}_0 \circ \pi)(z_0, z_1) = \frac{r_1}{r_0} e^{-i(\theta_1 - \theta_0)} \in \mathbb{C} \\ &\simeq \left(\frac{r_1}{r_0} \cos(\theta_0 - \theta_1), \frac{r_1}{r_0} \sin(\theta_0 - \theta_1) \right) = (x, y) \in \mathbb{R}^2.\end{aligned}$$

Then abbreviating $\Delta = \theta_0 - \theta_1$, we have

$$\begin{aligned}\psi^*(dx) &= \partial_{r_0}(x \circ \psi) dr_0 + \partial_{\theta_0}(x \circ \psi) d\theta_0 + \partial_{r_1}(x \circ \psi) dr_1 + \partial_{\theta_1}(x \circ \psi) d\theta_1 \\ &= -\tfrac{r_1}{r_0^2} \cos\Delta dr_0 - \tfrac{r_1}{r_0} \sin\Delta d\theta_0 + \tfrac{1}{r_0} \cos\Delta dr_1 + \tfrac{r_1}{r_0} \sin\Delta d\theta_1 \\ &= \tfrac{1}{r_0} \cos\Delta \left(dr_1 - \tfrac{r_1}{r_0} dr_0 \right) + \tfrac{r_1}{r_0} \sin\Delta \left(d\theta_1 - d\theta_0 \right), \\ \psi^*(dy) &= -\tfrac{r_1}{r_0^2} \sin\Delta dr_0 + \tfrac{r_1}{r_0} \cos\Delta d\theta_0 + \tfrac{1}{r_0} \sin\Delta dr_1 - \tfrac{r_1}{r_0} \cos\Delta d\theta_1 \\ &= \tfrac{r_1}{r_0} \cos\Delta \left(d\theta_0 - d\theta_1 \right) + \tfrac{1}{r_0} \sin\Delta \left(dr_1 - \tfrac{r_1}{r_0} dr_0 \right).\end{aligned}$$

Thus we obtain for the wedge product

$$\begin{aligned}\psi^*(dx \wedge dy) &= \tfrac{r_1}{r_0^2} \cos^2\Delta \left(dr_1 \wedge d\theta_0 - dr_1 \wedge d\theta_1 - \tfrac{r_1}{r_0} dr_0 \wedge d\theta_0 + \tfrac{r_1}{r_0} dr_0 \wedge d\theta_1 \right) \\ &\quad + \tfrac{r_1}{r_0^2} \sin^2\Delta \left(d\theta_1 \wedge dr_1 - \tfrac{r_1}{r_0} d\theta_1 \wedge dr_0 - d\theta_0 \wedge dr_1 + \tfrac{r_1}{r_0} d\theta_0 \wedge dr_0 \right) \\ &= \tfrac{r_1}{r_0^2} \left(dr_1 \wedge d\theta_0 - dr_1 \wedge d\theta_1 - \tfrac{r_1}{r_0} dr_0 \wedge d\theta_0 + \tfrac{r_1}{r_0} dr_0 \wedge d\theta_1 \right).\end{aligned}$$

For the denominator in the expression for $\tilde{\omega}_{FS}$, we have

$$\psi^*(1 + x^2 + y^2)^2 = \left(1 + (x \circ \psi)^2 + (y \circ \psi)^2\right)^2 = (1 + \tfrac{r_1^2}{r_0^2})^2,$$

so that finally,

$$\psi^*(\tilde{\omega}_{FS}) = \frac{r_1 r_0^2}{r_0^2 + r_1^2} \left(dr_1 \wedge d\theta_0 - dr_1 \wedge d\theta_1 - \tfrac{r_1}{r_0} dr_0 \wedge d\theta_0 + \tfrac{r_1}{r_0} dr_0 \wedge d\theta_1 \right). \tag{3.3}$$

This is now a form on $S^3 \subset \mathbb{C}^2$, so the following identities hold:

1. $r_0^2 + r_1^2 = 1$,
2. $r_0 dr_0 + r_1 dr_1 = 0$.

The second identity is obtained by applying the exterior differential to the first one. Using this, 3.3 simplifies to

$$\begin{aligned}\psi^*(\tilde{\omega}_{FS}) &= \underbrace{r_0^2 r_1 dr_1}_{= -r_0^3 dr_0} \wedge d\theta_0 - r_0^2 r_1 dr_1 \wedge d\theta_1 - r_1^2 r_0 dr_0 \wedge d\theta_0 + \underbrace{r_1^2 r_0 dr_0}_{= -r_1^3 dr_1} \wedge d\theta_1 \\ &= r_1 dr_1 \wedge d\theta_1 (-r_0^2 - r_1^2) + r_0 dr_0 \wedge d\theta_0 (-r_0^2 - r_1^2) \\ &= -\omega_0.\end{aligned}$$

□

The following diagram describes our situation:

$$\begin{array}{ccc} (\mathbb{CP}^1, \omega_{FS}) & & (S^2, \omega_{Eucl}) \\ \Big\downarrow \bar{\varphi}_0 & \swarrow \psi_0 & \Big\uparrow \Phi \\ (\mathbb{R}^2, \tilde{\omega}_{FS}) & & [-1,1) \times [0, 2\pi) \end{array}$$

An equally lengthy but mostly identical computation shows that $\varphi_1^* \tilde{\omega}_{FS} = \omega_{FS}$.

As a next step, we will pull back $\tilde{\omega}_{FS}$ to $[-1,1) \times [0,2\pi)$ and see that we essentially obtain ω_{Eucl}.

Proposition 3.21 $\omega_{FS} = \frac{1}{4}\omega_{Eucl}$ *for the Fubini-Study form on* $\mathbb{CP}^1$.

Proof Recall the parametrization of S^2 from the beginning of the section, given by

$$\begin{aligned} \Phi : \mathbb{R}^2 &\longrightarrow S^2 \\ (\theta, h) &\longmapsto \left(\sqrt{1-h^2}\cos(\theta), \sqrt{1-h^2}\sin(\theta), h\right). \end{aligned}$$

Composing this with the stereographic projection

$$\begin{aligned} \psi_0 : S^2 \setminus \{x_N\} &\longrightarrow \mathbb{R}^2 \\ (x,y,z) &\longmapsto \frac{1}{1-z}(x,y) \end{aligned}$$

gives the map

$$\begin{aligned} \rho_0 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (\theta, h) &\longmapsto \begin{pmatrix} \frac{\sqrt{1-h^2}}{1-h}\cos(\theta) \\ \frac{\sqrt{1-h^2}}{1-h}\sin(\theta) \end{pmatrix}. \end{aligned}$$

The rest of the proof is a matter of computig $\rho_0^* \tilde{\omega}_{FS}$. So set $A := \frac{1-h^2}{(1-h)^2} = \frac{1+h}{1-h}$ and note $\sqrt{A} = \frac{\sqrt{1-h^2}}{1-h}$. Then

$$\frac{\partial \sqrt{A}}{\partial h} = \frac{1}{2\sqrt{A}} \frac{1-h+1+h}{(1-h)^2} = \frac{1}{\sqrt{A}(1-h)^2}.$$

Using this, we compute

$$\begin{aligned} \rho_0^* dx &= \frac{\partial}{\partial \theta}(x \circ \rho_0) d\theta + \frac{\partial}{\partial h}(x \circ \rho_0) dh \\ &= -\sqrt{A}\sin(\theta) d\theta + \frac{\cos(\theta)}{\sqrt{A}(1-h)^2} dh, \end{aligned}$$

and similarly

$$\rho_0^* dy = \sqrt{A}\cos(\theta)d\theta + \frac{\sin(\theta)}{\sqrt{A}(1-h)^2}dh.$$

Hence $\rho_0^*(dx \wedge dy) = -\frac{\sin^2(\theta)}{(1-h)^2}d\theta \wedge dh + \frac{\cos^2(\theta)}{(1-h)^2}dh \wedge d\theta = \frac{dh\wedge d\theta}{(1-h)^2}$. We also have

$$\rho_0^*((1+x^2+y^2)^2) = \left(1 + \frac{1-h^2}{(1-h)^2}\right)^2 = \left(\frac{2}{1-h}\right)^2,$$

so inserting this into the expression for $\tilde{\omega}_{FS}$ yields

$$\rho_0^*\tilde{\omega}_{FS} = \frac{(1-h)^2}{4(1-h)^2}dh \wedge d\theta = -\frac{1}{4}\omega_{Eucl}.$$

The sign change occurs as ψ_0 is orientation-reversing. If we now apply $\bar{\varphi}_0^*$, lemma 3.20 tells us that we obtain precisely ω_{FS}. □

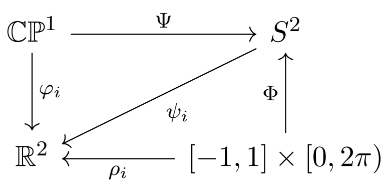
Again, we could carry out a very similar computation for ψ_1 and obtain that $\rho_1^*\tilde{\omega}_{FS} = \frac{1}{4}\omega_{Eucl}$. We chose to work out the case for $\bar{\varphi}_0$ and ψ_0 to emphasise the occurring sign change due to the charts being orientation reversing, which does not happen for φ_1 and ψ_1. Let us summarise these results in the following lemma:

Lemma 3.22 *Let $\bar{\varphi}_0, \varphi_1$ and ψ_0, ψ_1 the charts on $\mathbb{CP}^1$ and S^2, respectively, as in proposition 3.18. Let $\rho_i = \psi_i \circ \Phi$. The following hold:*

1. $\rho_0^*\tilde{\omega}_{FS} = -\frac{1}{4}\omega_{Eucl}$;
2. $\rho_1^*\tilde{\omega}_{FS} = \frac{1}{4}\omega_{Eucl}$;
3. $\bar{\varphi}_0^*\tilde{\omega}_{FS} = -\omega_{FS}$;
4. $\varphi_1^*\tilde{\omega}_{FS} = \omega_{FS}$.

Proof We have proven 1. and 3. in the last two propositions, 2. and 4. are analogous. □

Hence our commutative diagram becomes



The relations from lemma 3.22 then tell us that if we simply equip S^2 with $\frac{1}{4}\omega_{Eucl}$,

$$\Psi^* \tfrac{1}{4}\omega_{Eucl} = \bar{\varphi}_0^*(\rho_0^{-1})^*\frac{d\theta \wedge dh}{4} = \bar{\varphi}_0^*(-\tilde{\omega}_{FS}) = \omega_{FS}$$

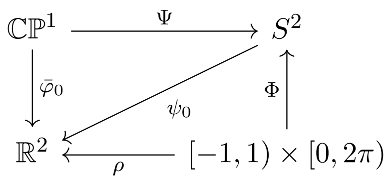
by the previous lemma on $U_0 \subset \mathbb{CP}^1$, and similarly on U_1: Hence Ψ is a symplectomorphism.

3.7.3 The Action

To see what the action we obtained corresponds to on S^2, we must define it such that our symplectomorphism Ψ becomes equivariant (as else the resulting toric manifolds would not be equivalent). That is, we define

$$e^{i\theta} \cdot \Psi([z_0 : z_1]) = \Psi([e^{i\theta} z_0 : z_1]).$$

To see how this acts concretely on S^2, we find an explicit expression for Ψ. We work on the charts $\bar{\varphi}_0$ and ψ_0, but the case for φ_1 and ψ_1 is analogous. Consider the diagram from before:



The diagram commutes by our definition of $\Psi = \psi_0^{-1} \circ \bar{\varphi}_0$ on U_0 and $\rho = \psi_0 \circ \Phi$. We shall regard $\mathbb{R}^2 \cong \mathbb{C}$ here and work in polar coordinates. Recall that

$$\begin{aligned} \rho : [-1,1) \times [0,2\pi) &\longrightarrow \mathbb{R}^2 \\ (h,\theta) &\longmapsto \begin{pmatrix} \sqrt{\frac{1+h}{1-h}}\cos(\theta) \\ \sqrt{\frac{1+h}{1-h}}\sin(\theta) \end{pmatrix} = \sqrt{\frac{1+h}{1-h}} e^{i\theta}, \end{aligned}$$

with inverse

$$re^{i\theta} \longmapsto \left(\frac{r^2-1}{r^2+1},\ \theta \right).$$

Hence as the diagram commutes, $\Psi = \Phi \circ \rho^{-1} \circ \bar{\varphi}_0$, and using $\bar{\varphi}_0([z_0 : z_1]) = \overline{z_1 z_0^{-1}} = \frac{r_1}{r_0} e^{i(\theta_0 - \theta_1)}$, we obtain

$$\begin{aligned} \Psi([z_0 : z_1]) &= \Phi\left(\frac{r_1^2 - r_0^2}{r_1^2 + r_0^2}, \theta_0 - \theta_1 \right) \\ &= \begin{pmatrix} \sqrt{1-R^2}\cos(\theta_0 - \theta_1) \\ \sqrt{1-R^2}\sin(\theta_0 - \theta_1) \\ R \end{pmatrix} \end{aligned}$$

for $R = \frac{r_1^2 - r_0^2}{r_1^2 + r_0^2}$. While a somewhat cumbersome expression, it allows us to easily see how S^1 acts on S^2, namely if $\Psi([z_0 : z_1]) = \Phi(h,\theta)$ is a point in S^2 for some (h,θ), then

$$e^{i\varphi} \cdot \Phi(h,\theta) = \Psi([e^{i\varphi} z_0 : z_1]) = \begin{pmatrix} \sqrt{1-h^2}\cos(\theta_0 + \varphi - \theta_1) \\ \sqrt{1-h^2}\sin(\theta_0 + \varphi - \theta_1) \\ h \end{pmatrix} = \Phi(h, \theta + \varphi),$$

so $[\varphi] \in S^1$ acts on S^2 simply by rotation about the vertical axis by φ.

Let us find the moment map corresponding to this action. Suppressing Φ from our notation, we have $e^{i\varphi} \cdot (h, \theta) = (h, \theta + \varphi)$. The fundamental vector field is, for $v \in \mathbb{R}$,

$$\begin{aligned} \xi_v(h,\theta) &= \frac{d}{dt}\bigg|_{t=0} \exp(tv)\cdot(h,\theta) \\ &= \frac{d}{dt}\bigg|_{t=0} (h, \theta + tv) \\ &= \frac{d}{dt}\bigg|_{t=0} (h)\partial_h + \frac{d}{dt}\bigg|_{t=0} (\theta + tv)\partial_\theta \\ &= v\partial_\theta. \end{aligned}$$

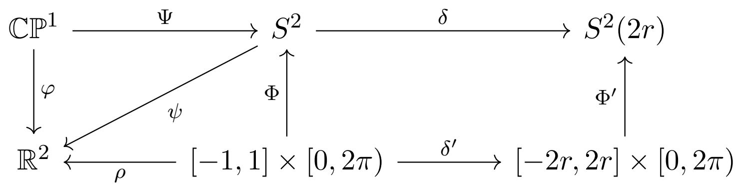
Thus $\xi_1 = \frac{\partial}{\partial\theta}$. We go on to compute for Y a vector field on S^2

$$\begin{aligned} \imath_{\xi_1}(\tfrac{1}{4}\omega_{Eucl})(Y) &= \tfrac{1}{4}(d\theta \wedge dh)\,(\partial_\theta, Y) \\ &= \tfrac{1}{4}\,(d\theta(\partial_\theta)dh(Y) - d\theta(Y)dh(\partial_\theta)) \\ &= \tfrac{1}{4}dh(Y). \end{aligned}$$

For μ a moment map, this is supposed to be $-d\mu(Y)$, so it is immediate that $\mu = -\frac{1}{4}h$ is the moment map we are looking for. Recall that we are considering the classification only up to translation of the Delzant polytope, so that $-\frac{1}{4}h(S^2) = [-\frac{1}{2}, \frac{1}{2}] \neq [0,1]$ is not an issue. By the remarks after definition 3.3, we could consider instead the moment map $-\frac{1}{4}h + \frac{1}{2}$ to recover $[0,1]$ as moment polytope.

3.7.4 Arbitrary Radius

Recall that for (A, ω, S^1, μ) a two-dimensional symplectic toric manifold with moment polytope $[0, r]$, we obtained that it must be $(\mathbb{CP}^1, 2r\omega_{FS}, S^1, \mu_\Delta)$. Thus the previous sections only give an equivariant symplectomorphism to $(S^2, \frac{1}{4}\omega_{Eucl}, S^1, -\frac{1}{4}h)$ for $r = \frac{1}{2}$. To extend this to arbitrary r, we expand the diagram 3.2 again:



Here, δ is dilation $p \mapsto 2rp$ and $\delta'(h, \theta) = (2rh, \theta)$. The map Φ' is given in analogy to Φ by

$$\Phi'(h', \theta) = \begin{pmatrix} \sqrt{(2r)^2 - h^2}\cos(\theta) \\ \sqrt{(2r)^2 - h^2}\sin(\theta) \\ h \end{pmatrix}.$$

The new maps are all diffeomorphisms, and one quickly checks that $\delta \circ \Phi = \Phi' \circ \delta'$. Equip $S^2(2r)$ also with $\frac{1}{4}d\theta \wedge dh'$, where we write (h', θ) for the coordinates in $[-2r, 2r] \times [0, 2\pi)$. By the results for unit radius of the previous sections, linearity of the pullback and the fact that the diagram commutes, it is enough to show that

$$\delta'^* \tfrac{1}{4} d\theta \wedge dh' = 2r \tfrac{1}{4} d\theta \wedge dh$$

to conclude that

$$\delta \circ \Psi : (\mathbb{CP}^1, 2r\omega_{FS}) \longrightarrow (S^2(2r), \tfrac{1}{4}\omega_{Eucl})$$

is a symplectomorphism.

This is quickly verified:

$$\begin{aligned} \delta'^*(dh') &= \partial_\theta(h' \circ \delta')d\theta + \partial_h(h' \circ \delta')dh \\ &= 2r dh; \\ \delta'^*(d\theta) &= d\theta. \end{aligned}$$

Hence $\delta'^*(\frac{1}{4}d\theta \wedge dh') = \frac{r}{2} d\theta \wedge dh$ as desired.

We may now, just as before, define the action of S^1 on $S^2(2r)$ to be such that $\delta \circ \Psi$ is equivariant, whereby we obtain again that

$$e^{i\varphi} \cdot \Phi'(h', \theta) = \Phi'(h', \theta + \varphi)$$

corresponds to rotation of the sphere about the vertical axis. An identical computation to the one in the last section shows that the moment map is again $-\frac{1}{4}h'$. Figure 3.2 illustrates the action on S^2 with its moment map.

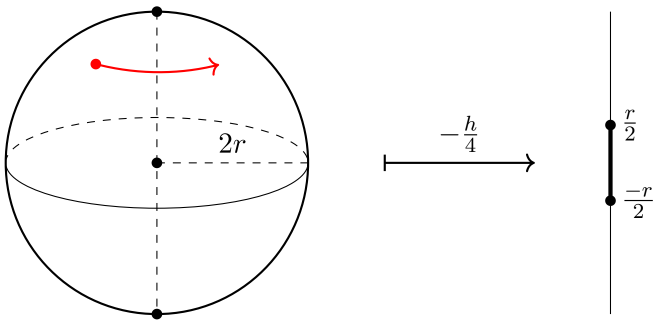


Figure 3.2: S^2 with its hamiltonian action and moment map

3.8 The Non-Effective Case

If our action $\psi : S^1 \to \mathrm{Diff}(A)$ is not effective, the hamiltonian S^1-space (A, α, ψ, μ) for a choice of moment map μ is not a toric manifold, so we cannot use Delzant's classification theorem. ψ not being effective means that the group homomorphism ψ has a kernel, which is always a subgroup of S^1. We adapt the argument in [11] to classify the possible subgroups.

Lemma 3.23 *S^1 has as finite subgroups the n-th roots of unity Γ_n for every $n \in \mathbb{N}$. If $H < S^1$ is an infinite subgroup, it is dense in S^1.*

Proof
Step 1: If $H < S^1$ is nontrivial and of finite order n, it contains an element $g_0 = e^{2\pi i t_0}$, where $t_0 := \min\{t \in (0,1) \mid e^{2\pi i t} \in H\}$. This minimum exists as it is taken over a set of numbers we assume to be finite. Note that this implies that t_0 is rational: we have $g_0^n = e^{2\pi i n t_0} = 1$, implying $nt_0 \in \mathbb{Z}$. We claim g_0 generates H. If not, take any $h \in H$ not in the cyclic subgroup generated by g_0 and write $h = e^{2\pi i x}$. Due to the minimality of t_0, there exists $n \in \mathbb{N}$ such that

$$nt_0 < x < (n+1)t_0,$$

where the inequalities are strict because we assume that h is not in $\langle g_0 \rangle$. This is equivalent to $0 < x - nt_0 < t_0$, but since $e^{2\pi i(x-nt_0)} = h(g_0^{-1})^n \in H$, the inequality poses a contradiction to the minimality of t_0.

Hence if $H < S^1$ is a finite subgroup of order n, it must be the group Γ_n of n-th roots of unity.

Step 2: Assume now that H is infinite and contains only elements of the form $e^{2\pi i r}$ for $r \in \mathbb{Q}$. Any $e^{2\pi i \frac{p}{q}}$ generates a cyclic subgroup of order q, assuming p, q are coprime. Hence this subgroup contains the element $g_0 = e^{\frac{2\pi i}{q}}$. As this subgroup is finite and H is infinite, there exists another element of H $e^{2\pi i \frac{s}{t}}$ which is not contained in the subgroup generated by g_0. From this element, we obtain another generator $g_1 = e^{\frac{2\pi i}{t}}$, and thus also $e^{\frac{2\pi i}{sq}} \in H$. As this still generates only a finite subgroup, we may continue this argument to conclude that there are elements $e^{2\pi i \frac{1}{q}}$ in H for q arbitrarily large. Hence H is in fact dense in S^1.

Step 3: Lastly, if H contains any element $g = e^{2\pi i \theta}$ for θ irrational, we claim H is also dense in S^1. It suffices to show that $\{n\theta \mod 1 \mid n \in \mathbb{Z}\}$ is dense in $[0,1)$ to conclude that g generates a dense cyclic subgroup.

To see this, note first that $n \mapsto n\theta \mod 1$ is injective: If $n\theta \mod 1 = m\theta \mod 1$, there exist integers such that $m\theta - k = n\theta - l$. Unless $m = n$, this gives $\theta = \frac{k-l}{m-n}$, contradicting irrationality.

Then pick any $m \in \mathbb{N}$ and divide $[0,1)$ into m half-open intervals of length $\frac{1}{m}$, explicitly $I_k = [\frac{k}{m}, \frac{k+1}{m})$ for $k \in \{0, \dots, m-1\}$. By injectivity, there must be two distinct $i, j \in \{1, \dots, m+1\}$ that are mapped into the same interval I_k. Thus we have in particular $\frac{k}{m} \leq i\theta \mod 1$ and $j\theta \mod 1 < \frac{k+1}{m}$. But then subtracting these inequalities, we find

$$|(j-i)\theta \mod 1| \leq \frac{1}{m},$$

which gives that $\{n(j-i)\theta \mod 1 \mid n \in \mathbb{Z}\}$ is $\frac{1}{m}$-dense in $[0,1)$. As m was arbitrary, we see that $\{n\theta \mod 1 \mid n \in \mathbb{Z}\}$ is dense in $[0,1)$. □

3.8.1 Reducing to an Effective Action

In the case where $\ker\psi$ is a dense subgroup, smoothness and thus continuity of ψ give that ψ is trivial. The trivial action imposes no constraints on our symplectic manifold, so we are back in the situation of chapter one. There, we concluded that (A_1, α_1) and (A_2, α_2) are symplectomorphic if and only if A_1 and A_2 are diffeomorphic, and their total area is the same.

So let us consider the case where $H = \ker\psi$ is a finite subgroup of S^1, hence it must be Γ_n for some natural number n. As S^1 is abelian, every subgroup is normal, so we may consider the quotient group S^1/H. The action ψ descends to an action $\tilde{\psi}$ of S^1/H on A which is effective:

$$\tilde{\psi}_{gH} := \psi_g$$

is well-defined since if $gh^{-1} \in H$, then $\mathrm{id} = \psi_{gh^{-1}} \iff \psi_g = \psi_h$. It is effective because $\tilde{\psi}_{gH} = \mathrm{id}$ implies $\psi_g = \mathrm{id}$, so $g \in H$ and thus $gH = H$, so $\tilde{\psi}$ is injective.

For $H = \Gamma_n$, the map from $S^1 \to S^1$ sending $g \mapsto g^n$ is a surjective group homomorphism with kernel Γ_n, and thus by the first isomorphism theorem, $S^1/\Gamma_n \cong S^1$. The explicit isomorphism is

$$\begin{aligned} \beta_n : S^1/\Gamma_n &\longrightarrow S^1 \\ g\Gamma_n &\longmapsto g^n. \end{aligned}$$

Thus we obtain an action of S^1 which is also hamiltonian, allowing us to use what we proved in the preceding chapters. To obtain an action of S^1 instead of S^1/Γ_1, we need only compose β_n^{-1} with $\tilde{\psi}$. For $g = e^{2\pi i\theta} \in S^1$, we have $\beta_n^{-1}(g) = e^{\frac{2\pi i\theta}{n}}$. Hence define

$$\begin{aligned} \Psi : S^1 &\longrightarrow \mathrm{Diff}(A) \\ g &\longmapsto \psi_{e^{\frac{2\pi i\theta}{n}}}. \end{aligned}$$

To see that this is hamiltonian, denote by $\tilde{\xi}_1$ the fundamental vector field generated by Ψ, and compute for $p \in A$

$$\begin{aligned}\tilde{\xi}_1(p) &= \frac{d}{dt}\Big|_{t=0} \Psi_{\exp(t)}(p)\\ &= \frac{d}{dt}\Big|_{t=0} \psi_{\exp(\frac{t}{n})}(p)\\ &= \frac{1}{n}\frac{d}{dt}\Big|_{t=0} \psi_{\exp(t)}(p)\\ &= \frac{1}{n}\xi_1(p)\end{aligned}$$

by the chain rule. Now let μ a moment map for the original, non-effective hamiltonian action ψ, and α the area form on A. We see that

$$\begin{aligned}-\imath_{\tilde{\xi}_1}\alpha &= -\imath_{\frac{1}{n}\xi_1}\alpha\\ &= -\frac{1}{n}\imath_{\xi_1}\alpha\\ &= \frac{1}{n}\mu,\end{aligned}$$

so $\frac{1}{n}\mu$ is a moment map for Ψ.

Hence considering the tuple $(A, \alpha, \Psi, \frac{1}{n}\mu)$ for the action Ψ gives by the discussion in the previous chapters that it is $(S^2(2r), \frac{1}{4}\omega_{Eucl}, S^1, -\frac{1}{4}h)$, where S^1 acts by rotation and r is the interval length of the moment polytope by $\frac{1}{n}\mu$. Hence (A, α, ψ, μ) is

$$(S^2(2r), \tfrac{1}{4}\omega_{Eucl}, S^1, -\tfrac{n}{4}h),$$

and the action is given by

$$e^{i\varphi} \cdot (h, \theta) = (h, \theta + n\varphi).$$

This is just rotation around the vertical axis, but "n times more quickly" than before.

Equivalently, in terms of projective space, it is equivariantly symplectomorphic to

$$(\mathbb{CP}^1, 2r\omega_{FS}, S^1, n\mu_\Delta),$$

the action given by

$$e^{i\varphi} \cdot [z_0 : z_1] = [e^{in\varphi}z_0 : z_1].$$

3.9 Conclusion

Let us quickly recapture the content of this chapter. Starting with a connected, compact surface with an area form (A, α), which is additionally endowed with a hamiltonian circle action ψ with moment map μ, we first imposed the further constraint that ψ be effective. This led us to the theory of symplectic toric manifolds and Delzant's classification theorem, letting us conclude that (A, α, ψ) is determined (up to equivariant symplectomorphism) by the interval length of the moment polytope $\mu(A)$, and must be a two-sphere with a multiple of the standard area form, being acted on by rotation around the vertical axis.

If the action is not effective, we have seen that it is either trivial, in which case the conclusion of chapter one is applicable, or the resulting manifold is also (up to equivalence) a two-sphere, where the action is rotation by a fixed integer multiple of the angle. We shall formulate this precisely in the following theorem:

Theorem 3.24 *Let $T_1 := (A_1, \alpha_1, \psi_1, \mu_1)$ and $T_2 := (A_2, \alpha_2, \psi_2, \mu_2)$ be two compact, connected, orientable manifolds endowed with area forms α_i and hamiltonian circle actions $\psi_i : S^1 \to \mathrm{Diff}(A_i)$. Let μ_i the moment map corresponding to the action ψ_i.*

Then there exists an equivariant symplectomorphism between T_1 and T_2 if and only if one of the following conditions is satisfied:

1. *Both actions are effective and $\ell(\mu_1(A_1)) = \ell(\mu_2(A_2))$;*
2. *Neither action is effective, but both are nontrivial and $\ell(\mu_1(A_1)) = \ell(\mu_2(A_2))$, as well as $|\ker \psi_1| = |\ker \psi_2|$;*
3. *Both actions are trivial, A_1 is diffeomorphic to A_2, and $\int_{A_1} \alpha_1 = \int_{A_2} \alpha_2$.*

Proof
Case 1 This case is precisely the content of the injectivity statement of Delzant's classification, so there is nothing left to prove. Let $r = \ell(\mu_i(A_i))$. The work done in section 3.7 shows that T_i is equivalent to $(S^2(2r), \frac{1}{4}\omega_{Eucl}, S^1, -\frac{1}{4}h)$, the action given by rotation.

Case 2 Let $n_i = |\ker \psi_i|$ and $\tilde{\mu}_i$ the moment map for the action on A_i by $S^1/\ker \psi_i$. Let $r_i = \ell(\tilde{\mu}_i(A_i))$. The induced symplectic toric manifolds are equivalent if and only if $r_1 = r_2$ by Delzant's theorem, and by section 3.8, T_i is equivariantly symplectomorphic to

$$(S^2(2r_i), \tfrac{1}{4}\omega_{Eucl}, \psi_i^{\varphi}(\theta, h) = (\theta + n_i\varphi, h), -\tfrac{n_i}{4}h).$$

So for these to be equivalent, we need in addition $n_1 = n_2$. Hence T_1 is equivalent to T_2 if and only if $r_1 = r_2$ and $n_1 = n_2$; Noting that $\ell(\mu_i(A_i)) = n_i r_i$ then gives the claim.

Case 3 Any symplectic manifold (M, ω) can be endowed with the trivial circle action $\psi_\theta = \mathrm{id}$ for all $\theta \in S^1$, so this imposes no additional constraint on our surfaces. The work that was done in chapter 1 applies to the tuples (A_1, α_1) and (A_2, α_2), and the claim is precisely theorem 2.1. $\square$

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